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THE PRIME NUMBER THEOREM IN SHORT INTERVALS FOR AUTOMORPHIC L -FUNCTIONS

Y. QU & J. WU

ABSTRACT. In this paper, we generalize Selberg's normal density theorem in short intervals [12] and Heath-Brown's refinement [2] to the case of automorphic L -functions of $GL_m(\mathbb{A}_{\mathbb{Q}})$.

1. INTRODUCTION

The well known Legendre conjecture states that there is at least one prime number between n^2 and $(n+1)^2$ for each positive integer n . A related problem is the existence of primes in short intervals. Denote, as usual, by $\zeta(s)$ the Riemann zeta-function, and define the von Mangoldt function $\Lambda(n)$ by

$$-\frac{\zeta'}{\zeta}(s) = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} \quad (\sigma > 1),$$

where $s = \sigma + i\tau$. Then

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^\nu \text{ with } \nu \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Write

$$\psi(x) := \sum_{n \leq x} \Lambda(n).$$

It is known that, under the Riemann Hypothesis (RH in brief) for $\zeta(s)$,

$$\psi(x) = x + O(x^{1/2}(\log x)^2) \quad (x \geq 2).$$

From this we immediately deduce that, under RH,

$$(1.1) \quad \psi(x + h(x)) - \psi(x) \sim h(x) \quad (x \rightarrow \infty)$$

for any increasing functions $h(x) \leq x$ satisfying

$$\frac{h(x)}{x^{1/2}(\log x)^2} \rightarrow \infty \quad \text{as } x \rightarrow \infty.$$

It seems be an interesting problem to determine how short $h(x)$ can be. According to Cramér's model, we could take $h(x)/(\log x)^2 \rightarrow \infty$ as $x \rightarrow \infty$. In 1943, Selberg

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[12] partially confirmed this by showing that under RH asymptotic relationship

$$(1.2) \quad \int_1^X |\psi(x+h(x)) - \psi(x) - h(x)|^2 dx = o(h(X)^2 X) \quad (X \rightarrow \infty)$$

holds for any increasing functions $h(x) \leq x$ satisfying

$$(1.3) \quad \frac{h(x)}{(\log x)^2} \rightarrow \infty \quad \text{as } x \rightarrow \infty.$$

It shows that, under RH, (1.1) holds for almost all $x \geq 2$ provided (1.3) is satisfied.

In order to better understand the connection between the distribution of zeros of $\zeta(s)$ and that of primes, Montgomery [9] introduced the pair correlation function

$$(1.4) \quad F_T(X) := \sum_{0 < \gamma_1, \gamma_2 \leq T} W(\gamma_1 - \gamma_2) e(X(\gamma_1 - \gamma_2)),$$

where

$$W(u) := \frac{4}{4 + u^2}, \quad e(u) := e^{2\pi i u},$$

and γ runs over imaginary parts $\Im m \rho$ of the nontrivial zeros ρ of $\zeta(s)$ (counted according to multiplicity). Assuming RH and that

$$(1.5) \quad F_T\left(\frac{\log x}{2\pi}\right) \ll T \log T$$

uniformly for $x(\log x)^{-3} \leq T \leq x$, Heath-Brown [2] showed that (1.2) holds for any increasing functions $h(x) \leq x$ satisfying

$$(1.6) \quad \frac{h(x)}{\log x} \rightarrow \infty \quad \text{as } x \rightarrow \infty.$$

In this paper, we shall investigate analogues of (1.2) for automorphic L -functions. Let us fix our notation first. To each irreducible unitary cuspidal representation $\pi = \otimes \pi_p$ of $GL_m(\mathbb{A}_{\mathbb{Q}})$ with $m \geq 2$, one can attach a global L -function

$$(1.7) \quad L(s, \pi) = \prod_{p < \infty} L_p(s, \pi_p)$$

converging for $\sigma > 1$ (see [5]), where local factors are given by

$$(1.8) \quad L_p(s, \pi_p) = \prod_{j=1}^m (1 - \alpha_{\pi}(p, j) p^{-s})^{-1}.$$

The complete L -function $\Phi(s, \pi)$ is defined by

$$(1.9) \quad \Phi(s, \pi) = L(s, \pi) L_{\infty}(s, \pi_{\infty}),$$

where $N_{\pi} \geq 1$ is an integer called the arithmetic conductor of π , and

$$(1.10) \quad L_{\infty}(s, \pi_{\infty}) := \left(\frac{N_{\pi}}{\pi^m}\right)^{s/2} \prod_{j=1}^m \Gamma\left(\frac{s + \mu_{\pi}(j)}{2}\right)$$

is the Archimedean local factor. Here $\{\alpha_{\pi}(p, j)\}_{j=1}^m$ and $\{\mu_{\pi}(j)\}_{j=1}^m$ are complex numbers associated with π_p and π_{∞} , respectively, according to the Langlands correspondence. Good bounds for these local parameters are of fundamental importance

for the study of automorphic L -functions. Thanks to the work of Luo-Rudnick-Sarnak [8], it is known that

$$(1.11) \quad \begin{cases} |\alpha_\pi(p, j)| \leq p^\theta & \text{if } \pi \text{ is unramified at } p, \\ |\Re \mu_\pi(j)| \leq \theta & \text{if } \pi \text{ is unramified at } \infty, \end{cases}$$

with $\theta = 1/2 - 1/(m^2 + 1)$. The Generalized Ramanujan Conjecture (GRC in brief) asserts that (1.11) should hold with $\theta = 0$. It also follows from work of Shahidi [13, 14, 15, 16] that the complete L -function $\Phi(s, \pi)$ has an analytic continuation to the whole complex plane and satisfies the functional equation

$$(1.12) \quad \Phi(s, \pi) = \varepsilon_\pi \Phi(1 - s, \tilde{\pi}),$$

where ε_π is the root number satisfying $|\varepsilon_\pi| = 1$, and $\tilde{\pi}$ is the representation contra-gradient to π . Iwaniec and Sarnak [4] introduced the analytic conductor of π . It is a function over the reals given by

$$Q_\pi(t) = N_\pi \prod_{j=1}^m (3 + |t + \mu_\pi(j)|),$$

which puts together all the important parameters for π . The quantity

$$Q_\pi = Q_\pi(0) = N_\pi \prod_{j=1}^m (3 + |\mu_\pi(j)|)$$

is also important, and it is named the conductor of π .

Similarly to the classical case, we define $\Lambda_\pi(n)$ by taking logarithmic differentiation in (1.7)

$$(1.13) \quad -\frac{L'}{L}(s, \pi) = \sum_{n=1}^{\infty} \frac{\Lambda_\pi(n)}{n^s} \quad (\sigma > 1).$$

With the help of (1.8), it is easy to see that

$$(1.14) \quad \Lambda_\pi(n) = \begin{cases} \sum_{j=1}^m \alpha_\pi(p, j)^\nu \log p & \text{if } n = p^\nu \text{ with } \nu \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

The prime number theorem for $L(s, \pi)$ concerns the asymptotic behavior of the counting function

$$(1.15) \quad \psi(x, \pi) := \sum_{n \leq x} \Lambda_\pi(n).$$

This problem was first studied by Liu & Ye [7] and Qu [10, 11]. In particular Qu [11] proved that, under the Generalized Riemann Hypothesis (GRH in brief) for $L(s, \pi)$, we have

$$(1.16) \quad \int_1^X |\psi(x + h(x), \pi) - \psi(x, \pi)|^2 dx = o(h(X)^2 X) \quad (X \rightarrow \infty)$$

for any increasing functions $h(x) \leq x$ satisfying

$$\frac{h(x)}{x^\theta (\log x)^2} \rightarrow \infty \quad \text{as } x \rightarrow \infty,$$

where θ is given by (1.11).

The first aim of this paper is to improve the above result by removing x^θ , which offers an exact generalization of Selberg's (1.2) and (1.3) to automorphic L -functions.

thm 1. *Let π be an irreducible unitary cuspidal representation of $GL_m(\mathbb{A}_\mathbb{Q})$ with $m \geq 2$. Assume GRH for $L(s, \pi)$. Then for $X \geq 2$ we have*

$$(1.17) \quad \int_1^X |\psi(x+h(x), \pi) - \psi(x, \pi)|^2 dx \ll h(X)X \log^2(Q_\pi X) + \left(\frac{\log Q_\pi}{\log X}\right)^4$$

for any increasing functions $h(x) \leq x$, where the implied constant depends only on m . In particular, (1.16) holds for any increasing functions $h(x) \leq x$ satisfying

$$(1.18) \quad \frac{h(x)}{(\log x)^2} \rightarrow \infty \quad \text{as } x \rightarrow \infty.$$

Our second aim of this paper is to consider the analogue of Heath-Brown's (1.2) and (1.6). Similar to (1.4), we can also define

$$F_T^\pi(X) := \sum_{0 < \gamma_1, \gamma_2 \leq T} W(\gamma_1 - \gamma_2) e(X(\gamma_1 - \gamma_2)),$$

where γ runs over imaginary parts $\Im m \rho$ of the nontrivial zeros ρ of $L(s, \pi)$ (counted according to multiplicity).

thm 2. *Let π be an irreducible unitary cuspidal representation of $GL_m(\mathbb{A}_\mathbb{Q})$ with $m \geq 2$. Assume GRH for $L(s, \pi)$, and*

$$(1.19) \quad F_T^\pi \left(\frac{\log X}{2\pi} \right) \ll T \log(Q_\pi T)$$

uniformly for $T \leq (X \log X)^2$. Then for $X \geq 2$ we have

$$(1.20) \quad \int_1^X |\psi(x+h(x), \pi) - \psi(x, \pi)|^2 dx \ll h(X)X \log(Q_\pi X) + X \log^2(Q_\pi X) + \left(\frac{\log Q_\pi}{\log X}\right)^4$$

for any increasing functions $h(x) \leq x$, where the implied constant depends only on m . In particular, (1.16) holds for any increasing functions $h(x) \leq x$ satisfying

$$(1.21) \quad \frac{h(x)}{\log x} \rightarrow \infty \quad \text{as } x \rightarrow \infty.$$

Theorem 1 (resp. Theorem 2) shows that, under GRH (resp. under GRH and (1.19)) for almost all $x \geq 2$, we have

$$\psi(x+h(x), \pi) - \psi(x, \pi) = o(h(x)) \quad (x \rightarrow \infty),$$

provided (1.18) (resp. (1.21)) is satisfied. It means that the sequence $\{\Lambda_\pi(n)\}_{n \geq 1}$ changes signs (different from the classical case $\{\Lambda(n)\}_{n \geq 1}$). Very recently, Liu, Qu & Wu [6] showed that if $\Lambda_\pi(n)$ is real for all $n \geq 1$, then there is some n satisfying

$$n \ll_{m, \varepsilon} Q_\pi^{1+\varepsilon}$$

such that $\Lambda_\pi(n) < 0$. The implied constant depends only on m and ε . In particular, this result is true for any self-contragredient irreducible unitary cuspidal representation π for $GL_m(\mathbb{A}_\mathbb{Q})$ with trivial central character.

The main new ideas for proving Theorems 1 and 2 are a delicate application of Iwaniec-Kowalski's mean value estimate (cf. (3.3) below) and an explicit formula in a more precise form adapted to our purpose (cf. Lemma 3.1 below).

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2. PRELIMINARY LEMMAS

In view of (1.10) and the fact that $L(s, \pi)$ and $\Phi(s, \pi)$ are entire, it is not difficult to see that the trivial zeros of $L(s, \pi)$ and the poles of $L_\infty(1-s, \tilde{\pi}_\infty)$ are

$$(2.1) \quad \mu := -2n - \mu_\pi(j) \text{ for } n = 0, 1, 2, \dots; j = 1, \dots, m,$$

$$(2.2) \quad P_{n,j} := 2n + 1 + \mu_{\tilde{\pi}}(j) \text{ for } n = 0, 1, 2, \dots; j = 1, \dots, m,$$

respectively. As in [7], we let $\mathbb{C}(m)$ denote the complex plane with the discs

$$|s - P_{n,j}| < (8m)^{-1} \text{ for } n = 0, 1, 2, \dots; j = 1, \dots, m$$

excluded. Thus, for any $s \in \mathbb{C}(m)$, the quantity $(1 - s + \mu_{\tilde{\pi}}(j))/2$ is away from all poles of $\Gamma(s)$ by at least $(16m)^{-1}$. For $j = 1, \dots, m$, denote by $\beta(j)$ the fractional part of $\Re \mu_{\tilde{\pi}}(j)$. In addition, let $\beta(0) = 0$ and $\beta(m+1) = 1$. Then all $\beta(j) \in [0, 1]$, and hence there exist $\beta(j_1), \beta(j_2)$ such that $\beta(j_2) - \beta(j_1) \geq 1/(3m)$ and there is no $\beta(j)$ lying between $\beta(j_1)$ and $\beta(j_2)$. Consequently, for all $n = 0, 1, 2, \dots$, the strips

$$(2.3) \quad S_{-n} := \{s \in \mathbb{C} : -n + \beta(j_1) + (8m)^{-1} \leq \Re s \leq -n + \beta(j_2) - (8m)^{-1}\}$$

are subsets of $\mathbb{C}(m)$.

The following assertions (i) and (ii) are Lemma 4.3(d) and Lemma 4.4 of [7], respectively.

Lemma 2.1. *Let π be an irreducible unitary cuspidal representation of $GL_m(\mathbb{A}_\mathbb{Q})$ with $m \geq 2$.*

(i) *For $T \geq 2$, there exists τ_T with $T \leq \tau_T \leq T + 1$ such that*

$$-\frac{L'}{L}(\sigma \pm i\tau_T, \pi) \ll \log^2(Q_\pi T) \quad (|\sigma| \leq 2).$$

(ii) *If s is in some strip S_{-n} as in (2.3) with $n \geq 2$, then*

$$-\frac{L'}{L}(s, \pi) \ll 1.$$

The implied constants depend only on m .

The next lemma is about the distribution of zeros of $L(s, \pi)$. For its proof, one is referred to Lemma 4.3 of Liu & Ye [7], or Theorem 5.8 of Iwaniec & Kowalski [3].

Lemma 2.2. *Let π be an irreducible unitary cuspidal representation of $GL_m(\mathbb{A}_{\mathbb{Q}})$ with $m \geq 2$. All the nontrivial zeros of $\Phi(s, \pi)$ are in the critical strip $0 \leq \sigma \leq 1$. Let $N(T, \pi)$ be the number of its nontrivial zeros within the rectangular $0 \leq \sigma \leq 1$ and $|\tau| \leq T$. Then*

$$(2.4) \quad N(T, \pi) \ll T \log(Q_{\pi} T)$$

and

$$(2.5) \quad N(T+1, \pi) - N(T, \pi) \ll \log(Q_{\pi} T),$$

where the implied constants are absolute.

3. AN EXPLICIT FORMULA

Explicit formulae of different forms were established by many authors. In particular, under GRC, explicit formulae for general L -functions were proved in [3, (5.53)]. The explicit formula below is unconditional, and plays a key role in the proofs of Theorems 1 and 2.

Lemma 3.1. *Let π be an irreducible unitary cuspidal representation of $GL_m(\mathbb{A}_{\mathbb{Q}})$ with $m \geq 2$, and $A > 0$. Then, for $x \geq 2$ and $2 \leq T \leq x^A$, we have*

$$(3.1) \quad \psi(x, \pi) = - \sum_{|\gamma| \leq T} \frac{x^{\rho}}{\rho} - \sum_{\substack{\kappa' < \lambda < \kappa \\ |\nu| \leq T}} \frac{x^{\mu}}{\mu} - \frac{L'}{L}(0, \pi) + O(R_{\pi}(x, T)),$$

where

$$R_{\pi}(x, T) := \sum_{|n-x| \leq x/\sqrt{T}} |\Lambda_{\pi}(n)| + \frac{x(\log Q_{\pi}) \log(Q_{\pi} x)}{\sqrt{T}} + \frac{x \log^2(Q_{\pi} x)}{T} + \frac{\log T}{x},$$

$-2 < \kappa' < -1$, $\kappa = 1 + 1/\log x$, and μ (resp. ρ) goes over the trivial zeros $\mu = \lambda + i\nu$ (resp. the nontrivial zeros $\rho = \beta + i\gamma$) of $L(s, \pi)$. The implied constant depends only on A and m .

Proof. Since the series (1.13) converges absolutely for $\sigma > 1$, we can apply the Perron formula [17, Theorem II.2.2] with $\kappa = 1 + 1/\log x$, so that

$$(3.2) \quad \psi(x, \pi) = \frac{1}{2\pi i} \int_{\kappa-iT}^{\kappa+iT} -\frac{L'}{L}(s, \pi) \frac{x^s}{s} ds + O\left(x^{\kappa} \sum_{n=1}^{\infty} \frac{|\Lambda_{\pi}(n)|}{n^{\kappa}(1+T|\log(x/n)|)}\right).$$

In order to treat the O -term, we split the sum into two parts according to

$$|x-n| \leq x/\sqrt{T} \quad \text{or} \quad |x-n| > x/\sqrt{T}.$$

By the Cauchy-Schwarz inequality, it follows that

$$\begin{aligned} \sum_{|x-n| > x/\sqrt{T}} \frac{|\Lambda_{\pi}(n)|}{n^{\kappa}(1+T|\log(x/n)|)} &\ll \frac{1}{\sqrt{T}} \sum_{n=1}^{\infty} \frac{|\Lambda_{\pi}(n)|}{n^{\kappa}} \\ &\ll \frac{1}{\sqrt{T}} \left(\sum_{n=1}^{\infty} \frac{|\Lambda_{\pi}(n)|^2}{n^{\kappa}} \right)^{1/2} \left(\sum_{n=1}^{\infty} \frac{1}{n^{\kappa}} \right)^{1/2}. \end{aligned}$$

According to [3, (5.48)], we have

$$(3.3) \quad \sum_{n \leq u} |\Lambda_\pi(n)|^2 \ll m^2 u \log^2(Q_\pi u) \quad (u \geq 1),$$

where the implied constant is absolute. Thus a simple integration by parts gives us

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{|\Lambda_\pi(n)|^2}{n^\kappa} &= \int_{1-}^{\infty} \frac{1}{u^\kappa} d\left(\sum_{n \leq u} |\Lambda_\pi(n)|^2\right) \\ &\ll_m \int_1^{\infty} \frac{\log^2(Q_\pi u)}{u^\kappa} du \\ &\ll_m \frac{\log^2 Q_\pi}{\kappa - 1} + \frac{1}{(\kappa - 1)^3}. \end{aligned}$$

Similarly but more easily, we have

$$\sum_{n=1}^{\infty} \frac{1}{n^\kappa} \ll \frac{1}{\kappa - 1}.$$

Combining these estimates, we can find that

$$(3.4) \quad x^\kappa \sum_{|x-n| > x/\sqrt{T}} \frac{|\Lambda_\pi(n)|}{n^\kappa (1 + T |\log(x/n)|)} \ll \frac{x(\log x) \log(Q_\pi x)}{\sqrt{T}}.$$

Next, we shall evaluate the integral on the right-hand side of (3.2). For this purpose, we shift the contour of integration to the left. Choose κ' with $-2 < \kappa' < -1$ such that the vertical line $\sigma = \kappa'$ is contained in the strip $S_{-2} \subset \mathbb{C}(m)$; this is guaranteed by the structure of $\mathbb{C}(m)$. Without loss of generality, let $T \geq 2$ be a large number such that T and $-T$ can be taken as the τ_T in Lemma 2.1(i). Now we consider the contour $\mathcal{L}_1 \cup \mathcal{L}_2 \cup \mathcal{L}_3$ with

$$\mathcal{L}_1 := [\kappa' - iT, \kappa - iT], \quad \mathcal{L}_2 := [\kappa' - iT, \kappa' + iT], \quad \mathcal{L}_3 := [\kappa' + iT, \kappa + iT].$$

By Lemma 2.2 and (2.1), certain nontrivial zeros $\rho = \beta + i\gamma$ and trivial zeros $\mu = \lambda + i\nu$ of $L(s, \pi)$, as well as $s = 0$ are passed by the shifting of the contour.

Computing the residues, we have

$$(3.5) \quad \begin{aligned} \frac{1}{2\pi i} \int_{\kappa - iT}^{\kappa + iT} -\frac{L'}{L}(s, \pi) \frac{x^s}{s} ds &= -\sum_{|\gamma| \leq T} \frac{x^\rho}{\rho} - \sum_{\substack{\kappa' < \lambda < \kappa \\ |\nu| \leq T}} \frac{x^\mu}{\mu} - \frac{L'}{L}(0, \pi) \\ &\quad - \frac{1}{2\pi i} \int_{\mathcal{L}_1 \cup \mathcal{L}_2 \cup \mathcal{L}_3} -\frac{L'}{L}(s, \pi) \frac{x^s}{s} ds. \end{aligned}$$

The integral on \mathcal{L}_1 can be estimated by Lemma 2.1(i) as

$$\begin{aligned} \frac{1}{2\pi i} \int_{\mathcal{L}_1} -\frac{L'}{L}(s, \pi) \frac{x^s}{s} ds &\ll \int_{\kappa'}^{\kappa} \log^2(Q_\pi T) \frac{x^\sigma}{T} d\sigma \\ &\ll \frac{x \log^2(Q_\pi T)}{T}, \end{aligned}$$

and the same upper bound also holds for the integral on \mathcal{L}_3 . By Lemma 2.1(ii), then

$$\begin{aligned} \frac{1}{2\pi i} \int_{\mathcal{L}_2} -\frac{L'}{L}(s, \pi) \frac{x^s}{s} ds &\ll \int_{-T}^T \frac{x^{\kappa'}}{|\tau|+1} dt \\ &\ll \frac{\log T}{x}. \end{aligned}$$

Therefore, (3.5) becomes

$$\begin{aligned} \frac{1}{2\pi i} \int_{\kappa-iT}^{\kappa+iT} -\frac{L'}{L}(s, \pi) \frac{x^s}{s} ds &= -\sum_{|\gamma| \leq T} \frac{x^\rho}{\rho} - \sum_{\substack{\kappa' < \lambda < \kappa \\ |\nu| \leq T}} \frac{x^\mu}{\mu} - \frac{L'}{L}(0, \pi) \\ &\quad + O\left(\frac{x(\log x) \log(Q_\pi x)}{\sqrt{T}} + \frac{x \log^2(Q_\pi T)}{T} + \frac{\log T}{x}\right). \end{aligned}$$

Inserting the above formula and (3.4) into (3.2), we obtain the required result. \square

4. GALLAGHER LEMMA AND PROOF OF THEOREM 1

Our main tool is the following lemma of Gallagher [1, Lemma 1].

Lemma 4.1. *Let $U > 0$ and $\delta = \vartheta/U$ with $0 < \vartheta < 1$, and let*

$$S(u) := \sum_{\nu} c(\nu) e^{2\pi i \nu u}$$

be absolutely convergent, where the coefficients $c(\nu) \in \mathbb{C}$, and the frequencies of ν run over an arbitrary sequence of real numbers. Then

$$\int_{-U}^U |S(u)|^2 du \ll_{\vartheta} U^2 \int_{-\infty}^{+\infty} \left| \sum_{t < \nu \leq t+\delta} c(\nu) \right|^2 dt.$$

Now we prove Theorem 1. Let $10^4 \leq X \leq x \leq 2X$, and take $T = (X \log X)^2$ in the explicit formula (3.1) of Lemma 3.1. Since the length of the interval $(x - x/(X \log X), x + x/(X \log X)]$ is $2x/(X \log X) \leq 1/2$, this interval contains at most one integer; we denote this possible integer by n_x . Thus our explicit formula becomes

$$\begin{aligned} \psi(x, \pi) &= -\sum_{|\gamma| \leq (X \log X)^2} \frac{x^\rho}{\rho} - \sum_{\substack{\kappa' < \lambda < \kappa \\ |\nu| \leq (X \log X)^2}} \frac{x^\mu}{\mu} - \frac{L'}{L}(0, \pi) \\ &\quad + O\left(|\Lambda_\pi(n_x)| + \log(Q_\pi X) + \frac{(\log Q_\pi)^2}{X(\log X)^2}\right), \end{aligned}$$

where the implied constant depends only on m . From this we can write

$$\psi(x+h, \pi) - \psi(x, \pi) = A + B + O\left(C + \log(Q_\pi X) + \frac{(\log Q_\pi)^2}{X(\log X)^2}\right),$$

where $h \leq 2X \leq 2x$ and

$$\begin{aligned} A &:= - \sum_{|\gamma| \leq (X \log X)^2} \frac{(x+h)^\rho - x^\rho}{\rho}, \\ B &:= - \sum_{\substack{\kappa' < \lambda < \kappa \\ |\nu| \leq (X \log X)^2}} \frac{(x+h)^\mu - x^\mu}{\mu}, \\ C &:= |\Lambda_\pi(n_{x+h})| + |\Lambda_\pi(n_x)|. \end{aligned}$$

Clearly,

$$(4.1) \quad \int_X^{2X} |\psi(x+h, \pi) - \psi(x, \pi)|^2 dx \ll \int_X^{2X} (|A|^2 + |B|^2 + |C|^2) dx + X \log^2(Q_\pi X) + \frac{(\log Q_\pi)^4}{X(\log X)^4}.$$

We start from A . In A , we split the sum over $|\gamma|$ at T , with $4 \leq T \leq (X \log X)^2$ a parameter that will be specified later, and define

$$S_1(y) := \sum_{|\gamma| \leq T} y^{i\gamma} \quad \text{and} \quad S_2(y) := \sum_{T < |\gamma| \leq (X \log X)^2} \frac{y^{i\gamma}}{\rho}.$$

Under GRH, the sum in A runs over the nontrivial zeros $\rho = 1/2 + i\gamma$ of $L(s, \pi)$ with $|\gamma|$ up to $(X \log X)^2$. Thus we can write

$$\begin{aligned} A &= - \sum_{|\gamma| \leq T} \frac{(x+h)^\rho - x^\rho}{\rho} - \sum_{T < |\gamma| \leq (X \log X)^2} \frac{(x+h)^\rho - x^\rho}{\rho} \\ &= - \sum_{|\gamma| \leq T} \int_x^{x+h} y^{\rho-1} dy - \sum_{T < |\gamma| \leq (X \log X)^2} \frac{(x+h)^{1/2+i\gamma} - x^{1/2+i\gamma}}{\rho} \\ &= - \int_x^{x+h} \frac{S_1(y)}{y^{1/2}} dy - (x+h)^{1/2} S_2(x+h) + x^{1/2} S_2(x) \\ &=: A_1 + A_2 + A_3, \end{aligned}$$

say. By the Cauchy-Schwarz inequality,

$$|A_1|^2 \leq h \int_x^{x+h} \frac{|S_1(y)|^2}{y} dy.$$

In view of $h \leq 2X$, the contribution from $|A_1|^2$ is estimated as

$$\int_X^{2X} |A_1|^2 dx \ll h \int_X^{2X} \left(\int_x^{x+h} \frac{|S_1(y)|^2}{y} dy \right) dx \ll h^2 \int_X^{4X} \frac{|S_1(y)|^2}{y} dy.$$

Changing variable $y = Xe^{2\pi u}$ and applying Lemma 4.1 and (2.5) of Lemma 2.2, it follows that

$$\begin{aligned}
\int_X^{2X} |A_1|^2 dx &\ll h^2 \int_0^{(\log 2)/\pi} \left| \sum_{|\gamma| \leq T} X^{i\gamma} e^{2\pi i \gamma u} \right|^2 du \\
(4.2) \qquad &\ll h^2 \int_{-\infty}^{+\infty} \left(\sum_{|\gamma| \leq T, t < \gamma \leq t+1} 1 \right)^2 dt \\
&\ll h^2 \int_0^T \left(\sum_{t < \gamma \leq t+1} 1 \right)^2 dt \\
&\ll h^2 T \log^2(Q_\pi T).
\end{aligned}$$

The contribution from $|A_2|^2$ can be estimated as

$$\begin{aligned}
\int_X^{2X} |A_2|^2 dx &\ll X^2 \int_X^{4X} \frac{|S_2(x)|^2}{x} dx \\
(4.3) \qquad &= X^2 \int_0^{(\log 2)/\pi} \left| \sum_{T < |\gamma| \leq (X \log X)^2} \frac{X^{i\gamma}}{\rho} e^{2\pi i \gamma u} \right|^2 du \\
&\ll X^2 \int_{-\infty}^{+\infty} \left(\sum_{T < |\gamma| \leq (X \log X)^2, t < \gamma < t+1} \frac{1}{|\gamma|} \right)^2 dt \\
&\ll X^2 \int_{T-1}^{(X \log X)^2} \left(\sum_{t < \gamma \leq t+1} \frac{1}{|\gamma|} \right)^2 dt.
\end{aligned}$$

By using (2.5) and (2.4) of Lemma 2.2, a simple integration by parts gives us

$$\sum_{t < \gamma \leq t+1} \frac{1}{|\gamma|} = \int_t^{t+1} \frac{1}{u} dN(u, \pi) \ll \frac{\log(Q_\pi t)}{t}.$$

Thus

$$(4.4) \qquad \int_X^{2X} |A_2|^2 dx \ll X^2 \int_{T-1}^{(X \log X)^2} \frac{\log^2(Q_\pi t)}{t^2} dt \ll \frac{X^2 \log^2(Q_\pi T)}{T}.$$

Similarly, after taking $x + h = y$, we have

$$(4.5) \qquad \int_X^{2X} |A_3|^2 dx \ll \frac{X^2 \log^2(Q_\pi T)}{T}.$$

We conclude from (4.2), (4.4) and (4.5) that

$$(4.6) \qquad \int_X^{2X} |A|^2 dx \ll h^2 T \log^2(Q_\pi T) + \frac{X^2 \log^2(Q_\pi T)}{T}.$$

For the mean-value of $|B|^2$, we apply (2.1) and (1.11), to get

$$\begin{aligned}
 \int_X^{2X} |B|^2 dx &= \int_X^{2X} \left| \sum_{\substack{\kappa' < \lambda < \kappa \\ |\nu| \leq (X \log X)^2}} \frac{(x+h)^\mu - x^\mu}{\mu} \right|^2 dx \\
 (4.7) \qquad &\ll \int_X^{2X} \left(\sum_{\substack{\kappa' < \lambda < \kappa \\ |\nu| \leq (X \log X)^2}} x^{\lambda-1} h \right)^2 dx \\
 &\ll \int_X^{2X} (x^{\theta-1} h)^2 dx \\
 &\ll h^2.
 \end{aligned}$$

It remains to estimate the contribution of $|C|^2$. We have

$$\begin{aligned}
 \int_X^{2X} |C|^2 dx &= \int_X^{2X} (|\Lambda_\pi(n_{x+h})| + |\Lambda_\pi(n_x)|)^2 dx \\
 &\ll \sum_{j=[X]}^{[2X]} \int_j^{j+1} (|\Lambda_\pi(n_{x+h})|^2 + |\Lambda_\pi(n_x)|^2) dx.
 \end{aligned}$$

Since $h(x)$ is increasing and $h(x) \leq x$, we have trivially, for $j \leq x \leq j+1$, that

$$j-1 \leq n_{x+h(x)} \leq 2(j+2), \quad j-1 \leq n_x \leq j+2.$$

Thus,

$$(4.8) \quad \int_X^{2X} |C|^2 dx \ll \sum_{j=[X]-1}^{3[2X]} |\Lambda_\pi(j)|^2 \ll X \log^2(Q_\pi X),$$

by applying (3.3).

Finally inserting (4.6), (4.7) and (4.8) to (4.1), and taking $T = X/h(2X)$, we find

$$\int_X^{2X} |\psi(x+h, \pi) - \psi(x, \pi)|^2 dx \ll h(2X) X \log^2(Q_\pi X) + \frac{(\log Q_\pi)^4}{X (\log X)^4}$$

for any increasing function $h(x)$ satisfying $1 \leq h(x) \leq x$. A splitting-up argument then gives the required inequality (1.17). This completes the proof of Theorem 1.

5. PAIR CORRELATION OF ZEROS AND PROOF OF THEOREM 2

The proof of Theorem 2 is very similar to that of Theorem 1. The only difference is to estimate the contribution of $|A_i|^2$ with the help of hypothesis (1.19) instead of Gallagher's lemma and Lemma 2.2. We retain the notation in Section 4. According to the first line of (4.2), we have

$$\int_X^{2X} |A_1|^2 dx \ll h^2 \int_0^{(\log 2)/\pi} \left| \sum_{|\gamma| \leq T} X^{i\gamma} e^{2\pi i \gamma u} \right|^2 du.$$

In view of the trivial inequality $e^{-4\pi|u|} \gg 1$ ($0 \leq u \leq 1$) and the classical formula

$$2\pi \int_{-\infty}^{+\infty} e^{-4\pi|u|+2\pi i t u} du = W(t),$$

we can deduce

$$\begin{aligned}
\int_X^{2X} |A_1|^2 dx &\ll h^2 \int_0^{(\log 2)/\pi} e^{-4\pi|u|} \left| \sum_{|\gamma| \leq T} X^{i\gamma} e^{2\pi i \gamma u} \right|^2 du \\
&\ll h^2 \int_{-\infty}^{+\infty} e^{-4\pi|u|} \left| \sum_{0 < \gamma \leq T} X^{i\gamma} e^{2\pi i \gamma u} \right|^2 du \\
(5.1) \quad &\ll h^2 \sum_{0 < \gamma_1, \gamma_2 \leq T} X^{i(\gamma_1 - \gamma_2)} \int_{-\infty}^{+\infty} e^{-4\pi|u|} e^{2\pi i(\gamma_1 - \gamma_2)u} du \\
&\ll h^2 \sum_{0 < \gamma_1, \gamma_2 \leq T} X^{i(\gamma_1 - \gamma_2)} W(\gamma_1 - \gamma_2) \\
&= h^2 F_T^\pi \left(\frac{\log X}{2\pi} \right).
\end{aligned}$$

Assuming (1.19), we have

$$(5.2) \quad \int_X^{2X} |A_1|^2 dx \ll h^2 T \log(Q_\pi T).$$

Next we estimate the contribution of $|A_2|^2$. By partial summation, it follows that

$$\begin{aligned}
\sum_{T < \gamma \leq (X \log X)^2} \frac{X^{i\gamma}}{\rho} e^{2\pi i \gamma u} &= \int_T^{(X \log X)^2} \frac{1}{1/2 + it} d \left(\sum_{\gamma \leq t} X^{i\gamma} e^{2\pi i \gamma u} \right) \\
&= \sum_{\gamma \leq (X \log X)^2} \frac{X^{i\gamma} e^{2\pi i \gamma u}}{1/2 + i(X \log X)^2} - \sum_{\gamma \leq T} \frac{X^{i\gamma} e^{2\pi i \gamma u}}{1/2 + iT} \\
&\quad + i \int_T^{(X \log X)^2} \sum_{\gamma \leq t} X^{i\gamma} e^{2\pi i \gamma u} \frac{dt}{(1/2 + it)^2}.
\end{aligned}$$

By the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
&\left(\int_T^{(X \log X)^2} \left| \sum_{\gamma \leq t} X^{i\gamma} e^{2\pi i \gamma u} \right| \frac{dt}{t^2} \right)^2 \\
&\ll \int_T^{(X \log X)^2} \frac{dt}{t(\log(1 + t/T))^2} \int_T^{(X \log X)^2} \left| \sum_{\gamma \leq t} X^{i\gamma} e^{2\pi i \gamma u} \right|^2 \frac{(\log(1 + t/T))^2}{t^3} dt \\
&\ll \int_T^{(X \log X)^2} \left| \sum_{\gamma \leq t} X^{i\gamma} e^{2\pi i \gamma u} \right|^2 \frac{(\log(1 + t/T))^2}{t^3} dt.
\end{aligned}$$

Thus

$$\begin{aligned} \left| \sum_{T < \gamma \leq (X \log X)^2} \frac{X^{i\gamma}}{\rho} e^{2\pi i \gamma u} \right|^2 &\ll \frac{1}{X^4} \left| \sum_{\gamma \leq (X \log X)^2} X^{i\gamma} e^{2\pi i \gamma u} \right|^2 + \frac{1}{T^2} \left| \sum_{\gamma \leq T} X^{i\gamma} e^{2\pi i \gamma u} \right|^2 \\ &\quad + \int_T^{(X \log X)^2} \left| \sum_{\gamma \leq t} X^{i\gamma} e^{2\pi i \gamma u} \right|^2 \frac{(\log(1 + t/T))^2}{t^3} dt. \end{aligned}$$

In view of the first two lines of (4.3) and the estimate above, we can write

$$\begin{aligned} \int_X^{2X} |A_2|^2 dx &\ll X^2 \left\{ \frac{1}{X^4} \int_0^{(\log 2)/\pi} \left| \sum_{\gamma \leq (X \log X)^2} X^{i\gamma} e^{2\pi i \gamma u} \right|^2 du \right. \\ &\quad + \frac{1}{T^2} \int_0^{(\log 2)/\pi} \left| \sum_{\gamma \leq T} X^{i\gamma} e^{2\pi i \gamma u} \right|^2 du \\ &\quad \left. + \int_T^{(X \log X)^2} \left(\int_0^{(\log 2)/\pi} \left| \sum_{\gamma \leq t} X^{i\gamma} e^{2\pi i \gamma u} \right|^2 du \right) \frac{(\log(1 + t/T))^2}{t^3} dt \right\}. \end{aligned}$$

From this, a similar argument to (5.1) allows us to deduce

$$\begin{aligned} &\int_X^{2X} |A_2|^2 dx \\ &\ll X^2 \left(\frac{1}{T} + \int_T^{(X \log X)^2} \frac{(\log(1 + t/T))^2}{t^2} dt \right) \sup_{T \leq t \leq (X \log X)^2} \frac{1}{t} F_t^{\text{Auto}} \left(\frac{\log X}{2\pi} \right) \\ &\ll \frac{X^2}{T} \sup_{T \leq t \leq (X \log X)^2} \frac{1}{t} F_t^{\text{Auto}} \left(\frac{\log X}{2\pi} \right). \end{aligned}$$

Assuming (1.19), it follows that

$$\int_X^{2X} |A_2|^2 dx \ll \frac{X^2}{T} \sup_{T \leq t \leq (X \log X)^2} \log(Q_\pi t) \ll \frac{X^2 \log(Q_\pi X)}{T}.$$

The same estimate also holds for $\int_X^{2X} |A_3|^2 dx$.

From these conclusions and (5.2), we get that

$$(5.3) \quad \int_X^{2X} |A|^2 dx \ll h^2 T \log(Q_\pi T) + \frac{X^2 \log(Q_\pi X)}{T}.$$

Finally, inserting (5.3), (4.7), and (4.8) to (4.1), and taking $T = Xh(2X)^{-1}$, we find that

$$\int_X^{2X} |\psi(x+h, \pi) - \psi(x, \pi)|^2 dx \ll h(2X)X \log(Q_\pi X) + X \log^2(Q_\pi X) + \frac{(\log Q_\pi)^4}{X(\log X)^4}$$

for any increasing functions $h(x)$ satisfying $1 \leq h(x) \leq x$. A splitting-up argument then gives the required inequality (1.20). This completes the proof of Theorem 2.

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