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THE FIRST NEGATIVE COEFFICIENTS OF SYMMETRIC SQUARE L -FUNCTIONS

Y.-K. LAU, J.-Y. LIU & J. WU

ABSTRACT. Let $n_{\text{sym}^2 f}$ be the greatest integer such that $\lambda_{\text{sym}^2 f}(n) \geq 0$ for all $n < n_{\text{sym}^2 f}$ and $(n, N) = 1$, where $\lambda_{\text{sym}^2 f}(n)$ is the n th coefficient of the Dirichlet series representation of the symmetric square L -function $L(s, \text{sym}^2 f)$ associated to a primitive form f of level N and of weight k . In this paper we establish the subconvexity bound: $n_{\text{sym}^2 f} \ll (k^3 N^2)^{40/113}$ where the implied constant is absolute.

1. INTRODUCTION

A classical question in analytic number theory concerns the least quadratic non-residue, see for example, [30, 2, 18, 16] for some investigations. More importantly, along these studies many useful tools were developed, such as the estimates on character sums [2, 7] and the large sieve inequalities [18, 21]. Recently much attention is drawn to GL_2 analogues, and the generalizations include the first negative Hecke eigenvalues [11, 9, 13, 20], the recognition of newforms by values or signs of Hecke eigenvalues [5, 14, 17, 13, 20], etc.

Let $k \geq 2$ be an even integer and $N \geq 1$ be an integer. We denote by $H_k^*(N)$ the set of all primitive cusp forms of weight k and of level N . For each integer $n \geq 1$, let $\lambda_f(n)$ be the Hecke eigenvalue of $f \in H_k^*(N)$ under the Hecke operator T_n . The eigenvalues $\lambda_f(n)$'s are real and verify the Hecke relation:

$$(1.1) \quad \lambda_f(m)\lambda_f(n) = \sum_{\substack{d|(m,n) \\ (d,N)=1}} \lambda_f\left(\frac{mn}{d^2}\right)$$

for all integers $m \geq 1$ and $n \geq 1$. Note that $\lambda_f(1) = 1$. The problem of the first negative Hecke eigenvalues is to evaluate the size of the least integer n_f among all n satisfying

$$(1.2) \quad \lambda_f(n) < 0 \quad \text{and} \quad (n, N) = 1,$$

for instance, to give a good bound for n_f in term of conductor $k^2 N$ of $f \in H_k^*(N)$. This question was firstly studied by Kohnen & Sengupta [11], and subsequently Iwaniec, Kohnen & Sengupta [9] introduced a new method to achieve the ‘‘subconvexity bound’’[†]

$$n_f \ll (k^2 N)^{29/60}.$$

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[†] The convexity bound means the exponent $1/2$ in place of $29/60$, which is an immediate consequence of the convexity bound for Hecke L -function on the critical line.

But interestingly, this bound is obtained without using any subconvexity bound for Hecke L -functions on the critical line. Their method has been refined very recently by Kowalski et al. [13] and by Matomäki [20], and the exponent $29/60$ is improved to $9/20$ and $2/5$ respectively. The refinement of the method of Iwaniec, Kohnen & Sengupta in [13, 20] makes use of the following three ingredients:

- Deligne's result: there is a real number $\theta_f(p) \in [0, \pi]$ such that

$$(1.3) \quad \lambda_f(p) = 2 \cos \theta_f(p);$$

- The Hecke relation for $\lambda_f(p^\nu)$ in the form of

$$(1.4) \quad \lambda_f(p^\nu) = \frac{\sin((\nu + 1)\theta_f(p))}{\sin \theta_f(p)} \quad (p \nmid N, \nu \geq 1);$$

- The respective results for the density of integers without large and small prime factors, and the density of squarefree friable integers coprime with N .

This problem is further extended to higher rank cases. In this direction, Qu [22] obtained a polynomial bound: Let $m \geq 2$ be an integer, and π an irreducible unitary cuspidal representation for $GL_m(\mathbb{A}_{\mathbb{Q}})$ with arithmetic conductor N_π and analytic conductor Q_π . We write $L(s, \pi)$ for the attached automorphic L -function and let $\{\lambda_\pi(n)\}_{n \geq 1}$ be the sequence of coefficients in the Dirichlet series of $L(s, \pi)$ in the half-plane $\Re s > 1$. Assume that the sequence $\{\lambda_\pi(n)\}_{n \geq 1}$ is real, and let n_π be the least integer n such that $\lambda_\pi(n) < 0$.[‡] Qu derived the result [22, Theorem 1.2] that for any $\varepsilon > 0$,

$$(1.5) \quad n_\pi \ll_{m, \varepsilon} Q_\pi^{m/2 + \varepsilon}$$

where the implied constant depends only on m and ε , with her very elegant inequality [22, Lemma 5.3]

$$|\lambda_\pi(p)| + \cdots + |\lambda_\pi(p^m)| \geq 1/m \quad (p \nmid N_\pi).$$

Very recently the exponent $m/2$ in (1.5) has been improved to 1 by Liu, Qu & Wu [19]. These results cover generic cases, but are weaker than the convexity bound when $m \geq 2$. Breaking the convexity is doubtless of deeper interest but no such result for GL_m , $m \geq 3$, is available in the literature.

In this paper we establish a subconvexity bound for a special case of GL_3 - the symmetric square lift of GL_2 forms. To each $f \in H_k^*(N)$ is associated a symmetric square L -function, defined as

$$L(s, \text{sym}^2 f) := \prod_p \left(1 - \frac{\lambda_f(p^2)}{p^s} + \frac{\psi_N(p)\lambda_f(p^2)}{p^{2s}} - \frac{\psi_N(p)}{p^{3s}} \right)^{-1} =: \sum_{n \geq 1} \frac{\lambda_{\text{sym}^2 f}(n)}{n^s}$$

for $\Re s > 1$, where ψ_N denotes the principal character mod N (cf. [24]). Inherited from the construction, $\lambda_{\text{sym}^2 f}(n)$ is real, multiplicative and satisfies

$$(1.6) \quad \lambda_{\text{sym}^2 f}(n) = \sum_{d^2 m = n} \lambda_f(m^2) \quad \text{for } (n, N) = 1.$$

Let us write $n_{\text{sym}^2 f}$ for the least integer n such that

$$(1.7) \quad \lambda_{\text{sym}^2 f}(n) < 0 \quad \text{and} \quad (n, N) = 1.$$

[‡]Here there is a slight difference from (1.2): without the extra condition $(n, N_\pi) = 1$.

By the work of Gelbart & Jacquet [6], there is an irreducible unitary cuspidal representation π for $GL_3(\mathbb{A}_{\mathbb{Q}})$ such that $L(s, \text{sym}^2 f) = L(s, \pi)$. Thus Qu's bound (1.5) with the refinement of [19] reads as

$$(1.8) \quad n_{\text{sym}^2 f} \ll_{\varepsilon} (k^3 N^2)^{1+\varepsilon},$$

where the implied constant depends on ε only. Extending the method of [9, 13, 20], we derive a quite good subconvexity bound for $n_{\text{sym}^2 f}$.

Theorem 1. *Let $k \geq 2$ be an even integer and $N \geq 1$ be an integer. Then for all $f \in H_k^*(N)$, we have*

$$(1.9) \quad n_{\text{sym}^2 f} \ll (k^3 N^2)^{40/113},$$

where the implied constant is absolute.

It is worth to notice that the exponent $40/113$ is smaller than the GL_2 -exponent $2/5$ of Matomäki [20]. The underlying reason seems due to the methodology and the asymmetric distribution of $\lambda_{\text{sym}^2 f}(p)$, for $-1 \leq \lambda_{\text{sym}^2 f}(p) \leq 3$ while $-2 \leq \lambda_f(p) \leq 2$, the sum of $\lambda_{\text{sym}^2 f}(n)$ over squarefree friable n will heuristically bias towards positive more rapidly under the assumption $\lambda_{\text{sym}^2 f}(p) \geq 0$ for small p 's.

Plainly $n_{\text{sym}^2 f} = p^{\nu}$ is a prime power due to the multiplicativity of $\lambda_{\text{sym}^2 f}(n)$. However unlike the least quadratic non-residues, we do not know whether the first negative coefficient of symmetric square L -function is attained at a prime argument (i.e. $\nu = 1$). Let us introduce $n_{f,2}$ for the least prime number $p \nmid N$ such that $\lambda_{\text{sym}^2 f}(p) < 0$. Clearly $n_{\text{sym}^2 f} \leq n_{f,2}$. Under the Grand Riemann Hypothesis for $L(s, \text{sym}^2 f)$, one can show $n_{f,2} \ll (\log(kN))^2$ where the implied constant is absolute. In [13], Kowalski et al. obtained an almost-all result: Let $k \geq 2$ be an even integer and $N \geq 1$ be a squarefree integer. There is a positive absolute constant c such that

$$n_{f,2} \ll \log(kN)$$

for all but except $O(kN e^{-c \log(kN)/\log_2(kN)})$ forms $f \in H_k^*(N)$. Here the implied constants in the \ll and O -symbols are absolute. These conditional and almost all bounds for $n_{f,2}$ also hold for $n_{\text{sym}^2 f}$, since $n_{\text{sym}^2 f} \leq n_{f,2}$.

We end this section with an outline of the method. Similarly to [9, 13, 20], let y be the greatest integer such that

$$(1.10) \quad \lambda_{\text{sym}^2 f}(n) \geq 0 \quad \text{for } n \leq y \quad \text{and} \quad (n, N) = 1,$$

and consider

$$(1.11) \quad S_{\text{sym}^2 f}(y^u) := \sum_{n \leq y^u}^b \lambda_{\text{sym}^2 f}(n),$$

where $\sum^b := \sum_{(n, N)=1} \mu(n)^2$ and $\mu(n)$ is the Möbius function. We shall obtain an estimate for y by comparing the upper and lower bounds for $S_{\text{sym}^2 f}(y^u)$. The former is rather easy, and for the latter, the principle of the methods in [9, 13, 20] is still effective. Nonetheless we need to invoke new identities and new tools in our manipulation. More precisely, with (1.6) and (1.4), we can prove that

$$(1.12) \quad \lambda_{\text{sym}^2 f}(p^{\nu}) = \frac{\sin((\nu+2)\theta_f(p)) \sin((\nu+1)\theta_f(p))}{\sin \theta_f(p) \sin(2\theta_f(p))} \quad (p \nmid N, \nu \geq 1).$$

However using merely this identity and the positivity hypothesis (1.10), we cannot derive directly the required lower bound for $\lambda_{\text{sym}^2 f}(p)$. We must exclude those primes p for which $\lambda_{\text{sym}^2 f}(p^\nu) = 0$ where $1 \leq \nu \leq 4$. (See Lemma 3.1 below for details.) Such primes are few, because it is equivalent to enumerate p with $\lambda_f(p) = \alpha$ for a given algebraic number $\alpha \neq 0$. In fact, it was observed in [12] the sparsity of p where $\lambda_f(p) = \pm 1$. Lemma 2.4 below is a generalization to suit our purpose.

Another technicality is the mean value of a multiplicative function g over friable integers coprime to q :

$$(1.13) \quad \sum_{\substack{n \leq y^u, (n, q) = 1 \\ P(n) \leq y}} g(n),$$

where $P(n)$ denotes the greatest prime factor of the integer n with the convention that $P(1) = 1$. There seems no handy reference in the literature. To this end we prove Lemma 4.2 below, in which the ranges of q, u and y are however rather weak. Much more general and better results will be obtained if one combines the methods in [28, 8, 29] (where the case of $q = 1$ is treated) and in [4] (where $g(n) \equiv 1$). This problem deserves more attention because of its own interest and future applications.

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2. EXCLUDING CERTAIN BAD BEHAVIOR OF HECKE EIGENVALUES

In order to bound $S_{\text{sym}^2 f}(y^u)$ from below, we need a control on small $\lambda_{\text{sym}^2 f}(p)$ which reduces, via (1.12), to remove the “bad” primes p , all contained in the set:

$$(2.1) \quad \mathcal{P}_f := \bigcup_{1 \leq \nu \leq 4} \{p : |\lambda_f(p)| = 2 \cos(\pi/(\nu + 2))\}.$$

A general result of Serre [23, Theorem 15] implies that

$$(2.2) \quad |\mathcal{P}_f| \ll_{f, \delta} \frac{x}{(\log x)^{1+\delta}}$$

for all $\delta < 1/2$ and $x \geq 2$. This bound is non-trivial, but unfortunately not sufficient for our purpose. Instead the unpublished work [12] of Kowalski is fitting more, and we devote this section to its slight generalization. Firstly we invoke a result of Besicovitch, see the lemma of Chandrasekharan in [3, p.204].

Lemma 2.1. (Besicovitch) *Let $a_j \in \mathbb{Z}$ for $j = 1, \dots, r$ and $q_j = a_j p_j$ where p_1, \dots, p_r are distinct primes. Suppose that $(a_j, p_1 \cdots p_r) = 1$.[§] Then*

$$\sqrt{q_j} \notin \mathbb{Q}_j := \mathbb{Q}(\sqrt{q_1}, \dots, \sqrt{q_{j-1}}, \sqrt{q_{j+1}}, \dots, \sqrt{q_r}).$$

Below is a direct consequence.

[§]Remark that $(0, m) = m$.

Lemma 2.2. *Let \mathbb{K} be a finite extension field over \mathbb{Q} . Then there are constants $M_{\mathbb{K}}, N_{\mathbb{K}} \in \mathbb{N}$ such that for any rational prime $p \nmid M_{\mathbb{K}}$ and for any $a \in \mathbb{Z}$ with $(a, pN_{\mathbb{K}}) = 1$, we have $\sqrt{ap} \notin \mathbb{K}$.*

Proof. Let p_1, \dots, p_d be distinct rational primes, and a_1, \dots, a_d be integers satisfy $(a_1 \cdots a_d, p_1 \cdots p_d) = 1$. Then by Lemma 2.1, we see that

$$[\mathbb{Q}(\sqrt{a_1 p_1}, \dots, \sqrt{a_d p_d}) : \mathbb{Q}] = 2^d,$$

and hence, there is an upper bound for the number r for which

$$\mathbb{Q}(\sqrt{a_1 p_1}, \dots, \sqrt{a_r p_r}) \subset \mathbb{K}$$

where p_1, \dots, p_r are distinct rational primes and $(a_1 \cdots a_r, p_1 \cdots p_r) = 1$. Take r to be the maximal value and let a_i, p_i ($i = 1, \dots, r$) be a maximal set. We define

$$N_{\mathbb{K}} = \prod_{1 \leq i \leq r} p_i \quad \text{and} \quad M_{\mathbb{K}} = N_{\mathbb{K}} \prod_{1 \leq i \leq r} a_i.$$

Now, any $p \nmid M_{\mathbb{K}}$ and any $(a, pN_{\mathbb{K}}) = 1$ satisfy $(a_1 \cdots a_r a, p_1 \cdots p_r p) = 1$ and thus $\sqrt{ap} \notin \mathbb{Q}(\sqrt{a_1 p_1}, \dots, \sqrt{a_r p_r})$. If $\sqrt{ap} \in \mathbb{K}$, it would follow

$$\mathbb{Q}(\sqrt{a_1 p_1}, \dots, \sqrt{a_r p_r}, \sqrt{ap}) \subset \mathbb{K}.$$

This contradicts to the maximality of r . \square

Next we deduce the following lemma by the argument in the proof of Lemma 2.1 in [3].

Lemma 2.3. *Let \mathbb{K} be a finite extension field over \mathbb{Q} , and $M_{\mathbb{K}}, N_{\mathbb{K}}$ be the numbers same as in Lemma 2.2. Given any distinct rational primes $p_1, \dots, p_{\ell} \nmid M_{\mathbb{K}}$, we have*

$$\sqrt{np_j} \notin \mathbb{K}_j := \mathbb{K}(\sqrt{p_1}, \dots, \sqrt{p_{j-1}}, \sqrt{p_{j+1}}, \dots, \sqrt{p_{\ell}})$$

for any integer $(n, p_1 \cdots p_{\ell} N_{\mathbb{K}}) = 1$ and any $j = 1, \dots, \ell$.

Proof. When $\ell = 1$, we have $\mathbb{K}_1 = \mathbb{K}$. This reduces to the case in Lemma 2.2, so the statement holds. Assume the induction hypothesis for the case of ℓ distinct primes.

Consider distinct primes $p_1, \dots, p_{\ell+1} \nmid M_{\mathbb{K}}$ and suppose

$$\sqrt{np_{\ell+1}} \in \mathbb{K}(\sqrt{p_1}, \dots, \sqrt{p_{\ell}}) = \mathbb{K}(\sqrt{p_1}, \dots, \sqrt{p_{\ell-1}})(\sqrt{p_{\ell}})$$

where $(n, p_1 \cdots p_{\ell+1} N_{\mathbb{K}}) = 1$. It follows that $\sqrt{np_{\ell+1}} = \alpha + \beta \sqrt{p_{\ell}}$ where $\alpha, \beta \in \mathbb{K}(\sqrt{p_1}, \dots, \sqrt{p_{\ell-1}})$, and consequently,

$$2\alpha\beta\sqrt{p_{\ell}} = np_{\ell+1} - \alpha^2 - \beta^2 p_{\ell} \in \mathbb{K}(\sqrt{p_1}, \dots, \sqrt{p_{\ell-1}}).$$

By the induction assumption, we infer that $\alpha = 0$ or $\beta = 0$, for otherwise we have $\sqrt{p_{\ell}} \in \mathbb{K}(\sqrt{p_1}, \dots, \sqrt{p_{\ell-1}})$.

If $\alpha = 0$, then $\sqrt{np_{\ell+1} p_{\ell}} = \beta p_{\ell} \in \mathbb{K}(\sqrt{p_1}, \dots, \sqrt{p_{\ell-1}})$. As $(np_{\ell+1}, p_1 \cdots p_{\ell} N_{\mathbb{K}}) = 1$, it contradicts to the induction assumption. So $\beta = 0$, and then we have that $\sqrt{np_{\ell+1}} \in \mathbb{K}(\sqrt{p_1}, \dots, \sqrt{p_{\ell-1}})$. But now we apply the induction assumption to the ℓ distinct primes $p_1, \dots, p_{\ell-1}, p_{\ell+1}$, we can infer that $\sqrt{np_{\ell+1}} \notin \mathbb{K}(\sqrt{p_1}, \dots, \sqrt{p_{\ell-1}})$ since $(n, p_1 \cdots p_{\ell-1} p_{\ell+1} N_{\mathbb{K}}) = 1$. Contradiction arises again. Our proof is hence complete. \square

We come to the main result of this section - Lemma 2.4 - which is substantially verbatim from Kowalski [12], in view of his excellent elucidation.

Lemma 2.4. *Let $k \geq 2$ be an even integer and $N \geq 1$ be an integer. There is an absolute constant C such that the inequality*

$$(2.3) \quad |\mathcal{P}_f| \leq \frac{4}{\log 2} \log(kN) + C$$

holds for all $f \in H_k^*(N)$.

Proof. We will need two basic facts on Fourier coefficients of primitive forms, which are essentially due to Shimura [25]:

– The field

$$\mathbb{Q}_f = \mathbb{Q}(a_f(n))_{n \geq 1}$$

is a number field, where $a_f(n) := \lambda_f(n)n^{(k-1)/2}$.

– For any automorphism σ in the Galois group of $\overline{\mathbb{Q}}$ over \mathbb{Q} , the function

$$f^\sigma := \sum_{n \geq 1} \sigma(a_f(n))e^{2\pi inz} \quad (\Im z > 0)$$

is also an element of $H_k^*(N)$. From these two properties, we deduce first that

$$(2.4) \quad [\mathbb{Q}_f : \mathbb{Q}] \leq |H_k^*(N)|$$

Indeed, notice that we have $f^\sigma = f$ if and only if σ is in the subgroup of the Galois group of \mathbb{Q} fixing \mathbb{Q}_f , so that the number of distinct conjugates f is at most the index of this subgroup, or in other words the degree of the extension field \mathbb{Q}_f , while on the other hand there can be no more than $|H_k^*(N)|$ distinct conjugates by the second property.

Now since the Fourier coefficients are real numbers, we have

$$|\lambda_f(p)| = 2 \cos(\pi/(\nu + 2)) \Leftrightarrow a_f(p) = \pm 2 \cos(\pi/(\nu + 2))p^{(k-1)/2}.$$

Since k is even, this implies in either case that $\cos(\pi/(\nu + 2))\sqrt{p} \in \mathbb{Q}_f$. Fix $1 \leq \nu \leq 4$ and write

$$\alpha_\nu = \cos(\pi/(\nu + 2)) \neq 0.$$

Set $\mathbb{K} = \mathbb{Q}(\alpha_\nu)$ and write $M_\nu = M_{\mathbb{K}}$ as defined in Lemma 2.3. Let $p_1 < p_2 < \dots < p_d$ be distinct primes such that $p_i \nmid M_\nu$ and $|\lambda_f(p_i)| = 2\alpha_\nu$. It follows that

$$\mathbb{Q}(\alpha_\nu\sqrt{p_1}, \dots, \alpha_\nu\sqrt{p_d}) \subset \mathbb{Q}_f.$$

Next we claim that

$$(2.5) \quad [\mathbb{Q}(\alpha_\nu\sqrt{p_1}, \dots, \alpha_\nu\sqrt{p_d}) : \mathbb{Q}] \geq 2^d,$$

which is clearly true once

$$\alpha_\nu\sqrt{p_j} \notin \mathbb{Q}(\alpha_\nu\sqrt{p_1}, \dots, \alpha_\nu\sqrt{p_{j-1}}) \quad (j = 1, \dots, d).$$

Plainly,

$$\mathbb{Q}(\alpha_\nu\sqrt{p_1}, \dots, \alpha_\nu\sqrt{p_{j-1}}) \subset \mathbb{K}(\sqrt{p_1}, \dots, \sqrt{p_{j-1}}),$$

but by Lemma 2.3, $\sqrt{p_j} \notin \mathbb{K}(\sqrt{p_1}, \dots, \sqrt{p_{j-1}})$ and neither does $\alpha_\nu\sqrt{p_j}$.

It follows from (2.4) and (2.5) that

$$2^d \leq [\mathbb{Q}_f : \mathbb{Q}] \leq |H_k^*(N)| \ll kN.$$

with an absolute implied constant, we obtain the bound

$$d \leq \frac{1}{\log 2} \log(kN) + O(1).$$

Since there are at most $O(\log M_\nu)$ prime factors of M_ν , the desired bound for $|\mathcal{P}_f|$ follows. \square

Remark. Bruinier & Kohnen [1, Remark 2.3] gave a non-explicit form of (2.3) for the simpler case $|\lambda_f(p)| = 2$. Some interesting applications of (2.3) and (2.2) are given in [10, 15].

3. PROOF OF THEOREM 1

In this section, we prove Theorem 1 by assuming Lemma 3.2 below, whose proof will be given in Sections 4 and 5.

We begin with the lower bounds for $\lambda_{\text{sym}^2 f}(p)$ under the positivity hypothesis (1.10). Define

$$(3.1) \quad N_f := \prod_{p|N} p \times \prod_{p \in \mathcal{P}_f} p,$$

where \mathcal{P}_f is defined as in (2.1). Note that Lemma 2.4 implies

$$(3.2) \quad \omega(N_f) \ll \log(kN)$$

for all $f \in \mathbb{H}_k^*(N)$, where the implied constant is absolute. The symbol $\omega(n)$ denotes the number of distinct prime factors of n with the convention $\omega(1) = 0$.

Lemma 3.1. *Let $k \geq 2$ be even integer and $N \geq 1$ be a positive integer.*

(i) *Formula (1.12) holds for all $f \in \mathbb{H}_k^*(N)$.*

(ii) *Let y be defined as in (1.10) and $1 \leq \nu \leq 4$. Then for $p \leq y^{1/\nu}$ and $p \nmid N_f$, we have*

$$(3.3) \quad \lambda_{\text{sym}^2 f}(p) \geq \kappa_\nu := 3 - 4 \sin^2(\pi/(\nu + 2)).$$

More precisely

$$(3.4) \quad \lambda_{\text{sym}^2 f}(p) \geq \begin{cases} 0 & \text{if } y^{1/2} < p \leq y \text{ and } p \nmid N_f, \\ 1 & \text{if } y^{1/3} < p \leq y^{1/2} \text{ and } p \nmid N_f, \\ (\sqrt{5} + 1)/2 & \text{if } y^{1/4} < p \leq y^{1/3} \text{ and } p \nmid N_f, \\ 2 & \text{if } p \leq y^{1/4} \text{ and } p \nmid N_f. \end{cases}$$

Proof. For $p \nmid N$ and $\nu \geq 1$, from (1.6) and (1.4) we can deduce that

$$\lambda_{\text{sym}^2 f}(p^{2\nu-1}) = \sum_{\ell=1}^{\nu} \lambda_f(p^{4\ell-2}) = \sum_{\ell=1}^{\nu} \frac{\sin((4\ell-1)\theta_f(p))}{\sin \theta_f(p)}.$$

By using the identity $2 \sin x \sin y = \cos(x-y) - \cos(x+y)$, it follows that

$$\begin{aligned} \lambda_{\text{sym}^2 f}(p^{2\nu-1}) &= \sum_{\ell=1}^{\nu} \frac{\cos((4\ell-3)\theta_f(p)) - \cos((4\ell+1)\theta_f(p))}{2 \sin \theta_f(p) \sin(2\theta_f(p))} \\ &= \frac{\cos \theta_f(p) - \cos((4\nu+1)\theta_f(p))}{2 \sin \theta_f(p) \sin(2\theta_f(p))}. \end{aligned}$$

Using the preceding identity again yields

$$\lambda_{\text{sym}^2 f}(p^{2\nu-1}) = \frac{\sin((2\nu+1)\theta_f(p)) \sin(2\nu\theta_f(p))}{\sin \theta_f(p) \sin(2\theta_f(p))}.$$

This proves Part (i), as a similar argument applies to

$$\lambda_{\text{sym}^2 f}(p^{2\nu}) = \frac{\sin((2\nu + 2)\theta_f(p)) \sin((2\nu + 1)\theta_f(p))}{\sin \theta_f(p) \sin(2\theta_f(p))}$$

for $p \nmid N$ and $\nu \geq 1$.

Now we know

$$(3.5) \quad \lambda_{\text{sym}^2 f}(p) = \frac{\sin(3\theta_f(p))}{\sin \theta_f(p)} = 3 - 4 \sin^2 \theta_f(p).$$

In view of the definitions of y and N_f , we have that for $1 \leq \nu \leq 4$ and $p \leq y^{1/\nu}$ with $p \nmid N_f$,

$$\frac{\lambda_{\text{sym}^2 f}(p^j)}{\lambda_{\text{sym}^2 f}(p^{j-1})} = \frac{\sin((j+2)\theta_f(p))}{\sin(j\theta_f(p))} > 0 \quad (1 \leq j \leq \nu),$$

recalling $\lambda_{\text{sym}^2 f}(1) = 1$. The case $j = 1$ implies

$$0 \leq \theta_f(p) < \pi/3 \quad \text{or} \quad 2\pi/3 < \theta_f(p) \leq \pi$$

(as $\theta_f(p) \in [0, \pi]$). Observe that

$$\frac{\sin((2\ell + 1)\theta_f(p))}{\sin \theta_f(p)} = \prod_{j=1}^{\ell} \frac{\lambda_{\text{sym}^2 f}(p^{2j-1})}{\lambda_{\text{sym}^2 f}(p^{2j-2})}$$

and

$$\frac{\sin((2\ell + 2)\theta_f(p))}{\sin(2\theta_f(p))} = \prod_{j=1}^{\ell} \frac{\lambda_{\text{sym}^2 f}(p^{2j})}{\lambda_{\text{sym}^2 f}(p^{2j-1})}.$$

If $0 \leq \theta_f(p) < \pi/3$, then both $\sin \theta_f(p)$ and $\sin(2\theta_f(p)) > 0$. A successive application of the positivity with the last two formulas yields $\sin((\ell + 2)\theta_f(p)) > 0$ for all $1 \leq \ell \leq \nu$, and hence $0 \leq \theta_f(p) < \pi/(\nu + 2)$. In case $2\pi/3 < \theta_f(p) \leq \pi$, we take $\vartheta_f(p) = \pi - \theta_f(p) \in [0, \pi/3]$, then the above formulas hold for $\vartheta_f(p)$ in place of $\theta_f(p)$, with $\sin \vartheta_f(p) > 0$ and $\sin(2\vartheta_f(p)) > 0$. We hence obtain that $(\nu + 1)\pi/(\nu + 2) < \theta_f(p) \leq \pi$.

Thus (3.3) follows with (3.5), and (3.4) is also an immediate consequence, for $\kappa_1 = 0$, $\kappa_2 = 1$, $\kappa_3 = (\sqrt{5} + 1)/2$ and $\kappa_4 = 2$. \square

In view of (3.4) and the fact

$$(3.6) \quad \lambda_{\text{sym}^2 f}(p) = \lambda_f(p^2) = \lambda_f(p)^2 - 1 \geq -1 \quad (p \nmid N),$$

we introduce the auxiliary multiplicative function $h = h_{N_f, y}$ defined as

$$h_{N_f, y}(p) = \begin{cases} -1 & \text{if } p > y \text{ and } p \nmid N_f, \\ 0 & \text{if } y^{1/2} < p \leq y \text{ or } p \mid N_f, \\ 1 & \text{if } y^{1/3} < p \leq y^{1/2} \text{ and } p \nmid N_f, \\ (\sqrt{5} + 1)/2 & \text{if } y^{1/4} < p \leq y^{1/3} \text{ and } p \nmid N_f, \\ 2 & \text{if } p \leq y^{1/4} \text{ and } p \nmid N_f, \end{cases}$$

and $h_{N_f, y}(p^\nu) = 0$ for all p and $\nu \geq 2$. The key to obtain the required lower bound for $S_{\text{sym}^2 f}(y^u)$ is to evaluate the mean value of $h_{N_f, y}(n)$ (as shown in (3.11) below). The size of this mean value is related to the solution of a difference-differential equation.

Let $\kappa \geq 1$ and $\rho_\kappa(t)$ be the unique continuous solution of the difference-differential equation

$$(3.7) \quad \begin{cases} \rho_\kappa(t) = t^{\kappa-1}/\Gamma(\kappa) & (0 \leq t \leq 1), \\ (t^{1-\kappa}\rho_\kappa(t))' = -\kappa t^{-\kappa}\rho_\kappa(t-1) & (t > 1), \end{cases}$$

where $\Gamma(\kappa)$ denotes the gamma function. By [26, Lemma 4.2], if $\kappa \geq 1$, $\rho_\kappa(t)$ is increasing on $[0, t_\kappa]$ and decreasing on $[t_\kappa, \infty)$ where $\max\{1, \kappa - 1\} \leq t_\kappa \leq \kappa$; furthermore, $\rho_\kappa(t) \rightarrow 0$ as $t \rightarrow \infty$. In particular we note that $t_2 = \sqrt{e}$.

Define

$$(3.8) \quad \Pi_{q,\kappa} := \left(\frac{\varphi(q)}{q}\right)^\kappa \prod_{p|q} \left(1 - \frac{1}{p}\right)^\kappa \left(1 + \frac{\kappa}{p}\right),$$

where $\varphi(n)$ is the Euler totient function. We have the following lemma, proven in Section 5.

Lemma 3.2. *With the previous notation, we have*

$$(3.9) \quad \sum_{n \leq y^u} h_{N_f,y}(n) \geq \Pi_{N_f,2} y^u (\log y^{1/4}) \delta(u) \left\{ 1 + O\left(\frac{(\log_2 y)^5}{\log y}\right) \right\}$$

uniformly for

$$(3.10) \quad \frac{4}{3} \leq u \leq \frac{3}{2} \quad \text{and} \quad y \geq (k^3 N^2)^{1/100},$$

where

$$\delta(u) := \delta_1(u) + \delta_2(u) + \delta_3(u) - \delta_4(u),$$

and

$$\delta_1(u) := \rho_2(4u),$$

$$\delta_2(u) := \kappa_3 \int_1^{4/3} \frac{\rho_2(4u-t)}{t} dt + \int_{4/3}^2 \frac{\rho_2(4u-t)}{t} dt,$$

$$\begin{aligned} \delta_3(u) &:= \kappa_3^2 \int_1^{4/3} \frac{dt}{t} \int_t^{4/3} \frac{\rho_2(4u-t-s)}{s} ds \\ &\quad + \kappa_3 \int_1^{4/3} \frac{dt}{t} \int_{4/3}^2 \frac{\rho_2(4u-t-s)}{s} ds + \int_{4/3}^2 \frac{dt}{t} \int_t^2 \frac{\rho_2(4u-t-s)}{s} ds, \end{aligned}$$

$$\begin{aligned} \delta_4(u) &:= \int_4^{4u} \frac{\rho_2(4u-t)}{t} dt + \kappa_3 \int_4^{4u-4/3} \frac{dt}{t} \int_1^{4/3} \frac{4u-t-s}{s} ds \\ &\quad + \int_4^{4u-4/3} \frac{dt}{t} \int_{4/3}^{4u-t} \frac{4u-t-s}{s} ds + \kappa_3 \int_{4u-4/3}^{4u-1} \frac{dt}{t} \int_1^{4u-t} \frac{4u-t-s}{s} ds \end{aligned}$$

with $\kappa_3 = (\sqrt{5} + 1)/2$. The function $\delta(u)$ is decreasing on $[1 + \sqrt{e}/4, 3/2]$ and $\delta(u) > 0$ for all $u < u_0$, where u_0 is the solution to $\delta(u_0) = 0$ in $[1 + \sqrt{e}/4, 3/2]$. We have $u_0 > 113/80$.

Now we are ready to show Theorem 1 with the help of Lemma 3.2. Let us start with the lower bound for $S_{\text{sym}^2 f}(y^u)$. As in [13], we notice that

$$(3.11) \quad S_{\text{sym}^2 f}(y^u) \geq \sum_{n \leq y^u} h_{N_f, y}(n)$$

for all $u < u_0$, provided y is large enough, for instance, $y \geq (k^3 N^2)^{1/100}$ and kN is large enough, which can obviously be assumed in proving Theorem 1.

To see (3.11), let $g_{N_f, y}$ be the multiplicative function defined by the Dirichlet convolution identity $\lambda_{\text{sym}^2 f} = g_{N_f, y} * h_{N_f, y}$. Then $g_{N_f, y}(n) \geq 0$ for all squarefree integers $n \geq 1$ with $(n, N_f) = 1$, since $g_{N_f, y}(p) = \lambda_{\text{sym}^2 f}(p) - h_{N_f, y}(p) \geq 0$ for $p \nmid N_f$. This is easily verified from the definition of $h_{N_f, y}$, (3.4) and (3.6).

According to Lemma 3.2, we have

$$\sum_{n \leq y^u} h_{N_f, y}(n) \geq 0$$

for $u \leq u_0$ and sufficiently large y . But, as $g_{N_f, y}(1) = 1$, we infer that

$$\begin{aligned} S_{\text{sym}^2 f}(y^u) &= \sum_{n \leq y^u} g_{N_f, y}(n) h_{N_f, y}(n) \\ &= \sum_{d \leq y^u} g_{N_f, y}(d) \sum_{m \leq y^u/d} h_{N_f, y}(m) \\ &\geq \sum_{m \leq y^u} h_{N_f, y}(m), \end{aligned}$$

which is (3.11). Then we deduce from Lemma 3.2 the required lower bound

$$(3.12) \quad S_{\text{sym}^2 f}(y^u) \gg \frac{y^u \log y}{\{\log_2(kN)\}^2} \quad (u < u_0),$$

since we have, by (3.8) and (3.2),

$$\Pi_{N_f, 2} \gg \{\log(\omega(N_f) + 3)\}^{-2} \gg \{\log_2(kN)\}^{-2}.$$

Next we establish an upper bound for $S_{\text{sym}^2 f}(y^u)$. For $\Re s > 1$, we have

$$\sum_{n \geq 1} \frac{\lambda_{\text{sym}^2 f}(n)}{n^s} = \prod_{p \nmid N} \left(1 + \frac{\lambda_f(p^2)}{p^s} \right) = L(s, \text{sym}^2 f) G_f(s),$$

where the Dirichlet series of

$$G_f(s) := \prod_{p \mid N} \left(1 - \frac{\lambda_f(p^2)}{p^s} \right) \prod_{p \nmid N} \left(1 - \frac{\lambda_f(p^2)^2 - \lambda_f(p^2)}{p^{2s}} + \frac{\lambda_f(p^2)^2 - 1}{p^{3s}} - \frac{\lambda_f(p^2)}{p^{4s}} \right)$$

converges absolutely and so $G_f(s) \ll_\varepsilon N^\varepsilon$ in the half-plane $\Re s \geq 1/2 + \varepsilon$ and $G_f(s) \ll_\varepsilon 1$ for $\Re s \geq 1 + \varepsilon$ (as $|\lambda_f(p^\nu)| \leq \nu + 1$ by Deligne's inequality).

The Perron formula (cf. [27, Theorem II.2.3]) gives

$$\sum_{n \leq x} \lambda_{\text{sym}^2 f}(n) = \frac{1}{2\pi i} \int_{\kappa - iT}^{\kappa + iT} L(s, \text{sym}^2 f) G_f(s) \frac{x^s}{s} ds + O\left(x^\varepsilon \left(1 + \frac{x}{T}\right)\right)$$

where $\kappa = 1 + \varepsilon$. Using the convexity bound

$$L(s, \text{sym}^2 f) \ll_\varepsilon (k^3 N^2 (|\tau|^3 + 1))^{1/4+\varepsilon} \quad (s = \frac{1}{2} + \varepsilon + i\tau, \tau \in \mathbb{R}),$$

we move the line of integration $\Re s = \kappa$ to $\Re s = \frac{1}{2} + \varepsilon$ and select $T = x^\varepsilon$ to deduce that for $x \leq (k^3 N^2)^{1/2+\varepsilon}$,[‡]

$$S_{\text{sym}^2 f}(x) \ll_\varepsilon (k^3 N^2)^{1/4+\varepsilon} x^{1/2+\varepsilon}.$$

Now, a comparison with (3.12) gives the estimate

$$y \leq (k^3 N^2)^{1/(2u_0)+\varepsilon}.$$

Quoting the lower bound for u_0 from Lemma 3.2, the proof of Theorem 1 is done.

4. MEAN VALUE OF MULTIPLICATIVE FUNCTION OVER FRIABLE INTEGERS COPRIME WITH q

We prepare for the proof of Lemma 3.2. To this end we consider a mean value theorem of the multiplicative function $n \mapsto \mu(n)^2 \kappa^{\omega(n)}$ over friable integers coprime with q , where $\kappa > 0$ is a constant. For $x \geq 1$, $y \geq 2$ and $q \in \mathbb{N}$, define

$$\Xi_{q,\kappa}(x, y) := \sum_{\substack{n \leq x, (n,q)=1 \\ P(n) \leq y}} \mu(n)^2 \kappa^{\omega(n)} \quad \text{and} \quad \Xi_{q,\kappa}(x) := \Xi_{q,\kappa}(x, x).$$

We begin with the treatment of $\Xi_{q,\kappa}(x)$.

Lemma 4.1. *Under the previous notation, there is a positive constant $C = C(\kappa)$ depending only on κ such that we have*

$$\Xi_{q,\kappa}(x) = \frac{\Pi_{q,\kappa}}{\Gamma(\kappa)} x (\log x)^{\kappa-1} \left\{ 1 + O_\kappa \left(\frac{L_q^{\varepsilon\kappa+2}}{\log x} \right) \right\}$$

uniformly for

$$(4.1) \quad q \geq 1 \quad \text{and} \quad x \geq \exp(CL_q^{\varepsilon\kappa+2}),$$

where $\Pi_{q,\kappa}$ is defined as in (3.8) and

$$(4.2) \quad L_q := \log(\omega(q) + 3).$$

Proof. For $\Re s > 1$, we have

$$\sum_{\substack{n=1 \\ (n,q)=1}}^{\infty} \mu(n)^2 \kappa^{\omega(n)} n^{-s} = \prod_{p \nmid q} (1 + \kappa p^{-s}) = \zeta(s)^\kappa G_q(s),$$

where $\zeta(s)$ is the Riemann ζ -function and

$$G_q(s) := \prod_{p|q} (1 - p^{-s})^\kappa \prod_{p \nmid q} (1 - p^{-s})^\kappa (1 + \kappa p^{-s})$$

converges absolutely for $\Re s \geq 1/2 + \varepsilon$ and any $\varepsilon > 0$.

[‡]In [13], the inequality sign of " $x \geq Q^{2\eta+\varepsilon}$ " below (2.1) should be reversed, and Q tacitly means $k^2 N(1 + |t|^2)$.

By the Perron formula (see [27, Theorem II.2.3]), we can write

$$(4.3) \quad \Xi_{q,\kappa}(x) = \frac{1}{2\pi i} \int_{b-iT}^{b+iT} \zeta(s)^2 G_q(s) \frac{x^s}{s} ds + O(\mathcal{R}_1),$$

where $b = 1 + 1/\log x$, $T \geq 3$ and

$$\mathcal{R}_1 := x \sum_{n \geq 1} \frac{\kappa^{\omega(n)}}{n^b (1 + T |\log(x/n)|)}.$$

The implied constant in the O -term is absolute.

The summation of \mathcal{R}_1 over n with $|\log(x/n)| \leq T^{-1/2}$ is

$$\begin{aligned} & \ll_{\kappa} \sum_{|n-x| \leq xT^{-1/2}} \kappa^{\omega(n)} \\ & \ll_{\kappa} \left(\sum_{|n-x| \leq xT^{-1/2}} \kappa^{2\omega(n)} \right)^{1/2} \left(\sum_{|n-x| \leq xT^{-1/2}} 1 \right)^{1/2} \\ & \ll_{\kappa} \frac{x(\log x)^{(\kappa^2-1)/2}}{T^{1/4}}, \end{aligned}$$

and the remaining part of \mathcal{R}_1 , contributed from the sum over n with $|\log(x/n)| > T^{-1/2}$, is

$$\ll_{\kappa} \frac{x}{T^{1/2}} \sum_{|n-x| > xT^{-1/2}} \frac{\kappa^{\omega(n)}}{n^b} \ll_{\kappa} \frac{x(\log x)^{\kappa-1}}{T^{1/2}}.$$

As a result, we have

$$(4.4) \quad \mathcal{R}_1 \ll_{\kappa} \frac{x(\log x)^{c_1(\kappa)}}{T^{1/4}},$$

where and in the sequel, $c_i(\kappa)$ ($i = 1, 2, \dots$) denotes a positive constant depending only on κ .

It remains to evaluate the integral on the right-hand side of (4.3). Let c be a suitable positive constant and

$$\sigma(T) := 1 - c/\log T.$$

Let $r = 1/(2 \log x)$ and assume $1 - r > \sigma(T)$. The truncated Hankel contour Γ is a positively oriented contour formed from the circle $|s - 1| = r$ excluding the point $s = 1 - r$ and joining with the half-segment $[\sigma(T), 1 - r]$ which is traced out twice with respective arguments $+\pi$ and $-\pi$. We apply the residue theorem to the integral over the closed path that consists of the vertical line segments $[b - iT, b + iT]$ and $\mathcal{L}_v^{\pm} := [\sigma(T), \sigma(T) \pm iT]$, two horizontal line segments $\mathcal{L}_h^{\pm} := [\sigma(T) + iT, b \pm iT]$ and the contour Γ .

For $\Re s \geq \sigma(T)$, we have

$$\begin{aligned} |G_q(s)| & \leq \prod_{p|q} (1 + p^{-\sigma(T)})^{\kappa} \\ & \leq \exp \left\{ \kappa \sum_{p \leq p_{\omega(q)}} p^{-\sigma(T)} \right\} \\ & \ll \exp \left\{ \kappa p_{\omega(q)}^{c/\log T} \log_2 p_{\omega(q)} \right\}, \end{aligned}$$

where p_n is the n th prime. Since $p_n \sim n \log n$ by prime number theorem, we have

$$(4.5) \quad |G_q(s)| \ll \exp \left\{ \kappa \exp(2cL_q/\log T) \log L_q \right\} \ll L_q^{e\kappa}$$

provided

$$(4.6) \quad T \geq \exp\{2cL_q\}.$$

Together with the well-known bound $\zeta(s) \ll \log T$ for $s \in \mathcal{L}_h^\pm \cup \mathcal{L}_v^\pm \cup \Gamma$, it follows that

$$\int_{\mathcal{L}_h^\pm \cup \mathcal{L}_v^\pm} \zeta(s)^\kappa G_q(s) \frac{x^s}{s} ds \ll L_q^{e\kappa} \left(\frac{x}{T} + x^{\sigma(T)} \right) (\log T)^{\kappa+1}$$

if (4.6) is satisfied.

By (4.5) and the properties of $\zeta(s)$, we have

$$s^{-1}((s-1)\zeta(s))^\kappa G_q(s) = G_q(1) + O_\kappa(L_q^{e\kappa}|s-1|)$$

for $s \in \Gamma$, under the hypothesis (4.6). The error term contributes a term

$$\begin{aligned} &\ll L_q^{e\kappa} \int_\Gamma |(s-1)^{1-\kappa} x^s| |ds| \\ &\ll L_q^{e\kappa} \int_{\sigma(T)}^{1-r} (1-\sigma)^{1-\kappa} x^\sigma d\sigma + x^{1+r} r^{2-\kappa} \\ &\ll_\kappa L_q^{e\kappa} x (\log x)^{\kappa-2}. \end{aligned}$$

By [27, Corollary II.5.2.1], we get from $G_q(1)$ the main term,

$$\frac{G_q(1)}{2\pi i} \int_\Gamma (s-1)^{-\kappa} x^s ds = \frac{G_q(1)}{\Gamma(\kappa)} x (\log x)^{\kappa-1} \{1 + O_\kappa(e^{-c(\log x)/\log T})\}.$$

Combining them gives

$$\begin{aligned} \frac{1}{2\pi i} \int_{\kappa-iT}^{\kappa+iT} \zeta(s)^\kappa G_q(s) \frac{x^s}{s} ds &= \frac{G_q(1)}{\Gamma(\kappa)} x (\log x)^{\kappa-1} \{1 + O(e^{-c(\log x)/\log T})\} \\ &\quad + O_\kappa \left(L_q^{e\kappa} \left(\frac{x}{T} + x^{\sigma(T)} \right) (\log T)^{\kappa+1} + L_q^{e\kappa} x (\log x)^{\kappa-2} \right) \end{aligned}$$

under the hypothesis (4.6).

Inserting into (4.3) with (4.4), we obtain that

$$(4.7) \quad \Xi_{q,\kappa}(x) = \frac{G_q(1)}{\Gamma(\kappa)} x (\log x)^{\kappa-1} + O_\kappa(\mathcal{R}_2)$$

where

$$\mathcal{R}_2 := \frac{G_q(1)x(\log x)^{\kappa-1}}{e^{c(\log x)/\log T}} + L_q^{e\kappa} \left(\frac{x}{T^{1/4}} + x^{\sigma(T)} \right) (\log T)^{c_2(\kappa)} + L_q^{e\kappa} x (\log x)^{\kappa-2}$$

if (4.6) holds.

It is easy to see that

$$\Pi_{q,\kappa} = G_q(1) \gg L_q^{-2}.$$

We take

$$T = \exp \left\{ c_3(\kappa) (\log x)^{1/2} \right\},$$

then the condition (4.6) holds valid since $x \geq \exp(CL_q^2)$. Moreover we can easily see that

$$(4.8) \quad \mathcal{R}_2 \ll_{\kappa} L_q^{-2} x (\log x)^{\kappa-1} \frac{L_q^{e\kappa+2}}{\log x} \ll_{\kappa} \Pi_{q,\kappa} x (\log x)^{\kappa-1} \frac{L_q^{e\kappa+2}}{\log x}$$

uniformly for q and x verifying (4.1). The required result follows from (4.8) into (4.7). \square

The next lemma plays a key role in the proof of Lemma 3.2. As mentioned in the introduction, we do not make effort to widen the ranges of the parameters involved.

Lemma 4.2. *Let $\kappa \geq 1$ and $U > 1$ be two fixed constants. For some suitable constant $C = C(\kappa, U)$ depending only on κ and U , we have*

$$(4.9) \quad \Xi_{q,\kappa}(y^u, y) = \Pi_{q,\kappa} y^u (\log y)^{\kappa-1} \rho_{\kappa}(u) \left\{ 1 + O_{\kappa,U} \left(\frac{L_q^{e\kappa+2} (\log_2 y)^{\delta_{\kappa,1}}}{\log y} \right) \right\}$$

uniformly for

$$(4.10) \quad q \geq 1, \quad y \geq \exp(2CL_q^{e\kappa+2}), \quad U^{-1} \leq u \leq U,$$

where $\Pi_{q,\kappa}$, L_q and $\rho_{\kappa}(u)$ are defined as in (3.8), (4.2) and (3.7), respectively, and

$$\delta_{\kappa,1} := \begin{cases} 1 & \text{if } \kappa = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. If $U^{-1} \leq u \leq 1$, we have $\Xi_{q,\kappa}(y^u, y) = \Xi_{q,\kappa}(y^u)$. Thus Lemma 4.1 gives us immediately the required asymptotic formula since $\rho_{\kappa}(u) = u^{\kappa-1}/\Gamma(\kappa)$.

Next we suppose that $1 \leq u \leq 2$. Write

$$(4.11) \quad \Xi_{q,\kappa}(y^u, y) = \Xi_{q,\kappa}(y^u) - \kappa \sum_{\substack{y < p \leq y^u \\ p \nmid q}} \Xi_{q,\kappa}(y^u/p).$$

With the help of Lemma 4.1, we have

$$\Xi_{q,\kappa}(y^u) = \Pi_{q,\kappa} y^u (\log y)^{\kappa-1} \frac{u^{\kappa-1}}{\Gamma(\kappa)} \left\{ 1 + O_{\kappa} \left(\frac{L_q^{e\kappa+2}}{\log y} \right) \right\}$$

and so

$$\begin{aligned} & \sum_{y < p \leq y^u e^{-CL_q^{e\kappa+2}}} \Xi_{q,\kappa}(y^u/p) \\ &= \sum_{y < p \leq y^u e^{-CL_q^{e\kappa+2}}} \Pi_{q,\kappa} \frac{y^u \{\log(y^u/p)\}^{\kappa-1}}{\Gamma(\kappa)p} \left\{ 1 + O \left(\frac{L_q^{e\kappa+2}}{\log(y^u/p)} \right) \right\}. \end{aligned}$$

The O -terms are absorbed in the O -term of (4.9) by partial integration with the prime number theorem and the fact that $\rho_{\kappa}(u) \gg_{\kappa} 1$ uniformly for $1 \leq u \leq 2$. The

main term is

$$\begin{aligned} & \sum_{y < p \leq y^u e^{-CL_q^{\epsilon\kappa+2}}} \Pi_{q,\kappa} \frac{y^u (\log p)^{\kappa-1}}{p} \rho_\kappa \left(\frac{\log(y^u/p)}{\log p} \right) \\ &= \Pi_{q,\kappa} y^u (\log y^u)^{\kappa-1} \int_1^u \frac{\rho_\kappa(t-1)}{t^\kappa} dt \left\{ 1 + O \left(\frac{L_q^{\epsilon\kappa+2}}{\log y} \right) \right\}. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} & \left(\sum_{\substack{y < p \leq y^u \\ p|q}} + \sum_{\substack{y^u e^{-CL_q^{\epsilon\kappa+2}} < p \leq y^u}} \right) \Xi_{q,\kappa}(y^u/p) \\ & \ll y^u \left(\sum_{\substack{y < p \leq y^u \\ p|q}} + \sum_{\substack{y^u e^{-CL_q^{\epsilon\kappa+2}} < p \leq y^u}} \right) \frac{\{\log(y^u/p)\}^{\kappa-1}}{p} \\ & \ll_\kappa y^u (\log y)^{\kappa-1} \left(\frac{L_q^{\epsilon\kappa+2}}{\log y} \right)^\kappa \end{aligned}$$

which is admissible, for $\log y \gg L_q^{\epsilon\kappa+2}$. Inserting these estimates into (4.11) and noticing that

$$\rho_\kappa(u) = u^{\kappa-1} \left(\frac{1}{\Gamma(\kappa)} - \kappa \int_1^u \frac{\rho_\kappa(t-1)}{t^\kappa} dt \right) \quad (1 \leq u \leq 2),$$

we find that the asymptotic formula (4.9) holds uniformly for $q \geq 1$, $y \geq \exp(CL_q^{\epsilon\kappa+2})$ and $1 \leq u \leq 2$. Recursively we get the result for the general case $1 \leq u \leq U$. \square

5. PROOF OF LEMMA 3.2

Lastly we complete the postponed proof of Lemma 3.2, and there are two assertions.

5.1. Proof of (3.9). According to the definition of $h_{N_f,y}$ after (3.6), we have

$$(5.1) \quad \sum_{n \leq y^u} h_{N_f,y}(n) = \sum_{\substack{n \leq y^u \\ P(n) \leq \sqrt{y}}} h_{N_f,y}(n) - \sum_{\substack{y < p \leq y^u \\ p \nmid N_f}} \sum_{n \leq y^u/p} h_{N_f,y}(n)$$

for all u and y satisfying (3.10).

With the Buchstab identity, it follows that

$$\sum_{\substack{n \leq y^u \\ P(n) \leq \sqrt{y}}} h_{N_f,y}(n) = \Xi_{N_f,2}(y^u, y^{1/4}) + \left(\sum_{\substack{y^{1/4} < p \leq y^{1/3} \\ p \nmid N_f}} \kappa_3 + \sum_{\substack{y^{1/3} < p \leq y^{1/2} \\ p \nmid N_f}} \right) \sum_{\substack{n \leq y^u/p \\ P(n) < p}} h_{N_f,y}(n).$$

Repeating this procedure, we obtain

$$\sum_{\substack{n \leq y^u \\ P(n) \leq \sqrt{y}}} h_{N_f,y}(n) \geq S_1 + S_2 + S_3,$$

where

$$\begin{aligned}
S_1 &:= \Xi_{N_f,2}(y^u, y^{1/4}), \\
S_2 &:= \left(\sum_{\substack{y^{1/4} < p \leq y^{1/3} \\ p \nmid N_f}} \kappa_3 + \sum_{\substack{y^{1/3} < p \leq y^{1/2} \\ p \nmid N_f}} \right) \Xi_{N_f,2}\left(\frac{y^u}{p}, y^{1/4}\right), \\
S_3 &:= \left(\sum_{\substack{y^{1/4} < p_1 < p_2 \leq y^{1/3} \\ p_i \nmid N_f}} \kappa_3^2 + \sum_{\substack{y^{1/4} < p_1 \leq y^{1/3} < p_2 \leq y^{1/2} \\ p_i \nmid N_f}} \kappa_3 + \sum_{\substack{y^{1/3} < p_1 < p_2 \leq y^{1/2} \\ p_i \nmid N_f}} \right) \Xi_{N_f,2}\left(\frac{y^u}{p_1 p_2}, y^{1/4}\right).
\end{aligned}$$

In view of (3.2), we have

$$L_{N_f} = \log(\omega(N_f) + 3) \ll \log_2(kN) \ll \log_2 y,$$

since $y \geq (k^3 N^2)^{1/100}$. Thus $y \geq \exp(CL_{N_f}^{2e+2})$ provided $k^3 N^2$ is suitably large. So we can apply Lemma 4.2 with $q = N_f$ and $\kappa = 2$ to write

$$(5.2) \quad S_1 = \rho_2(4u) C_{N_f}(y, u),$$

where

$$C_{N_f}(y, u) := \Pi_{N_f,2} y^u (\log y^{1/4}) \left\{ 1 + O\left(\frac{(\log_2 y)^5}{\log y}\right) \right\}.$$

Similarly, by Lemma 4.2 with $q = N_f$, we have

$$S_2 = C_{N_f}(y, u) \left(\sum_{\substack{y^{1/4} < p \leq y^{1/3} \\ p \nmid N_f}} \kappa_3 + \sum_{\substack{y^{1/3} < p \leq y^{1/2} \\ p \nmid N_f}} \right) \frac{1}{p} \rho_2\left(\frac{\log(y^u/p)}{\log y^{1/4}}\right).$$

Integration by parts with the prime number theorem yields

$$\left(\sum_{y^{1/4} < p \leq y^{1/3}} \kappa_3 + \sum_{y^{1/3} < p \leq y^{1/2}} \right) \frac{1}{p} \rho_2\left(\frac{\log(y^u/p)}{\log y^{1/4}}\right) = \delta_2(u) \left\{ 1 + O\left(\frac{1}{\log y}\right) \right\}.$$

Trivially we have the estimate

$$\sum_{\substack{y^{1/4} < p \leq y^{1/2} \\ p \nmid N_f}} \frac{1}{p} \rho_2\left(\frac{\log(y^u/p)}{\log y^{1/4}}\right) \ll \frac{\log(kN)}{y^{1/4}} \ll \frac{(\log_2 y)^5}{\log y},$$

and in summary,

$$(5.3) \quad S_2 = \delta_2(u) C_{N_f}(y, u).$$

Similarly we prove that

$$(5.4) \quad S_3 = \delta_3(u) C_{N_f}(y, u).$$

The treatment of the double sum in (5.1) is even simpler. For u, y verifying (3.10), we have

$$\begin{aligned} \sum_{y < p \leq y^u} \sum_{n \leq y^u/p} h_{N_f, y}(n) &\leq \sum_{y < p \leq y^u} \Xi_{N_f, 2} \left(\frac{y^u}{p}, y^{1/4} \right) \\ &+ \left(\sum_{\substack{y < p_1 \leq y^{u-1/3} \\ y^{1/4} < p_2 \leq y^{1/3}}} \kappa_3 + \sum_{\substack{y < p_1 \leq y^{u-1/3} \\ y^{1/3} < p_2 \leq y^u/p_1}} + \sum_{\substack{y^{u-1/3} < p_1 \leq y^{u-1/4} \\ y^{1/4} < p_2 \leq y^u/p_1}} \kappa_3 \right) \Xi_{N_f, 2} \left(\frac{y^u}{p_1 p_2} \right). \end{aligned}$$

The previous argument applies and we get that

$$(5.5) \quad \sum_{y < p \leq y^u} \sum_{n \leq y^u/p} h_{N_f, y}(n) \leq \delta_4(u) C_{N_f}(y, u).$$

Inserting (5.2), (5.3), (5.4) and (5.5) into (5.1), we get the desired inequality in (3.9).

5.2. Study of $\delta(u)$. To facilitate the numerical computation, we put

$$v := 4u, \quad \tilde{\delta}_i(v) := \delta_i(v/4) \quad \text{and} \quad \tilde{\delta}(v) = \delta(v/4).$$

Thus we have

$$\tilde{\delta}(v) = \tilde{\delta}_1(v) + \tilde{\delta}_2(v) + \tilde{\delta}_3(v) - \tilde{\delta}_4(v) \quad (16/3 \leq v \leq 6).$$

After some standard manipulations with the change of variables, the interchange of integrals and integration by parts, we deduce that

$$\begin{aligned} \tilde{\delta}_1(v) &= \rho_2(v), \\ \tilde{\delta}_2(v) &= \int_{v-2}^{v-4/3} \frac{\rho_2(t)}{v-t} dt + \kappa_3 \int_{v-4/3}^{v-1} \frac{\rho_2(t)}{v-t} dt, \\ \tilde{\delta}_3(v) &= \int_{v-4}^{v-10/3} \frac{\rho_2(t)}{v-t} \log \left(\frac{2}{v-2-t} \right) dt + \int_{v-3}^{v-2} \frac{\rho_2(t)}{v-t} \log(v-1-t) dt \\ &+ \int_{v-10/3}^{v-3} \frac{\rho_2(t)}{v-t} \left\{ \kappa_3 \log \left(\frac{2}{v-2-t} \frac{4/3}{v-4/3-t} \right) + \log \left(\frac{v-4/3-t}{4/3} \right) \right\} dt \\ &+ (\kappa_3^2 - 1) \int_{v-8/3}^{v-7/3} \frac{\rho_2(t)}{v-t} \log \left(\frac{4/3}{v-4/3-t} \right) dt, \\ \tilde{\delta}_4(v) &= v \log \left(\frac{v}{v-1} \right) - 1 + \int_1^{4/3} \frac{(3 - \kappa_3)t - (2 - \kappa_3)t \log t - (2 - \kappa_3)}{v-t} dt \\ &+ \int_{4/3}^{v-4} \frac{(2 + (\kappa_3 - 1) \log(4/3))t - t \log t - (2 + \kappa_3)/3}{v-t} dt. \end{aligned}$$

Next we show that all summands on right-hand side of $\tilde{\delta}_i(v)$ ($1 \leq i \leq 3$) are decreasing on $[4 + \sqrt{e}, 6]$. The proofs are quite similar, so we only consider, as an example, the third summand in the expression of $\tilde{\delta}_3(v)$. Denote this term by $\tilde{\delta}_{3,3}(v)$ and define

$$F_{3,3}(t, v) := \frac{1}{v-t} \left\{ \kappa_3 \log \left(\frac{2}{v-2-t} \frac{4/3}{v-4/3-t} \right) + \log \left(\frac{v-4/3-t}{4/3} \right) \right\}.$$

Noticing

$$\frac{\partial F_{3,3}}{\partial v}(t, v) = -\frac{\partial F_{3,3}}{\partial t}(t, v),$$

we infer that

$$\begin{aligned} \tilde{\delta}'_{3,3}(v) &= \left[\rho_2(t) F_{3,3}(t, v) \right]_{v-10/3}^{v-3} - \int_{v-10/3}^{v-3} \rho_2(t) \frac{\partial F_{3,3}}{\partial t}(t, v) dt \\ &= \int_{v-10/3}^{v-3} \rho_2'(t) F_{3,3}(t, v) dt \\ &< 0, \end{aligned}$$

since $\rho_2'(t) \leq 0$ for $t \geq \sqrt{e}$ and $F_{3,3}(t, v) > 0$ for $v - 10/3 \leq t \leq v - 3$.

Consequently we have

$$\begin{aligned} \tilde{\delta}'_4(v) &= \log\left(\frac{v}{v-1}\right) + \int_1^{4/3} \frac{1 - (2 - \kappa_3) \log t}{v-t} dt \\ &\quad + \int_{4/3}^{v-4} \frac{1 + (\kappa_3 - 1) \log(4/3) - \log t}{v-t} dt \\ &> 0. \end{aligned}$$

Hence $\tilde{\delta}(v)$ is decreasing on $[4 + \sqrt{e}, 6]$.

Using MAPLE, we check that $\tilde{\delta}(40/113) > 0.002 \dots$. Thus $\tilde{\delta}(\tilde{u}_0) = 0$ with $\tilde{u}_0 > 113/20 > 4 + \sqrt{e}$, and we have $\delta(u_0) = 0$ with $u_0 = \tilde{u}_0/4 > 113/80$.

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