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# EXTRAPOLATION OF $L^p$ MAXIMAL REGULARITY FOR SECOND ORDER CAUCHY PROBLEMS

RALPH CHILL AND SEBASTIAN KRÓL

ABSTRACT. If the second order problem  $\ddot{u} + B\dot{u} + Au = f$  has  $L^p$ -maximal regularity for some  $p \in (1, \infty)$ , then it has  $\mathbb{E}_w$ -maximal regularity for every rearrangement invariant Banach function space  $\mathbb{E}$  with Boyd indices  $p_{\mathbb{E}}, q_{\mathbb{E}} \in (1, \infty)$  and for every Muckenhoupt weight  $w \in A_{p_{\mathbb{E}}}$ .

## 1. INTRODUCTION

Motivated by the study of partial differential equations arising in physical models such as the motion of strongly damped (visco-) elastic materials or waves, the notion of  $L^p$ -maximal regularity for the abstract linear second order problem

$$\ddot{u} + B\dot{u} + Au = f \text{ on } \mathbb{R}_+, \quad u(0) = \dot{u}(0) = 0, \quad (1)$$

was introduced and studied in [13]. Here  $A$  and  $B$  are two closed linear operators on a Banach space  $X$ , with dense domains  $D_A$  and  $D_B$ , respectively. We say that the problem (1) has  $L^p$ -maximal regularity, if for every  $f \in L^p_{loc}(\mathbb{R}_+; X)$  the problem (1) admits a unique (strong) solution

$$u \in W^{2,p}_{loc}(\mathbb{R}_+; X) \text{ such that } B\dot{u}, Au \in L^p_{loc}(\mathbb{R}_+; X).$$

Strong solution means that  $u(0) = \dot{u}(0) = 0$  and the differential equation (1) is satisfied almost everywhere.

This definition of  $L^p$ -maximal regularity is similar to that of  $L^p$ -maximal regularity for the first order problem  $\dot{u} + Au = f$ , and is closely related to the abstract notion of maximal regularity studied first by Da Prato & Grisvard [19], and then also by Amann [2], Acquistapace & Terreni [1], Dore & Venni [22], and Labbas & Terreni [29].

It has been shown in [14] that  $L^p$ -maximal regularity for the second order problem is independent of  $p$  in the following sense: if the problem (1) has  $L^p$ -maximal regularity for some  $p \in (1, \infty)$ , then it has  $L^p$ -maximal regularity for every  $p \in (1, \infty)$ . This parallels the fact that  $L^p$ -maximal regularity of first order problems is independent of  $p$ ; see for example, De Simon [21] in the case of Hilbert spaces, and Sobolevskii [35], Cannarsa & Vespri [11], Hieber [26] in the general case. Both extrapolation results, for the first order equation and for the second order equation, use weak  $(1, 1)$  or  $L^\infty - BMO$  estimates for appropriate singular integral operators and the Marcinkiewicz interpolation theorem; see

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Benedek, Calderón & Panzone [6]. Recently, in [12], Chill & Fiorenza have shown that  $L^p$ -maximal regularity for the first order problem actually implies  $\mathbb{E}_w$ -maximal regularity for every rearrangement invariant Banach function space with Boyd indices  $p_{\mathbb{E}}, q_{\mathbb{E}} \in (1, \infty)$  and every Muckenhoupt weight  $w \in A_{p_{\mathbb{E}}}$ . Here, we show the corresponding result for the second order problem. For the definition of the weighted, rearrangement invariant Banach function spaces, Boyd indices and Muckenhoupt weights, see Section 2.

**Theorem 1.** *Assume that the second order problem (1) has  $L^p$ -maximal regularity for some  $p \in (1, \infty)$ . Then it has  $\mathbb{E}_w$ -maximal regularity for every rearrangement invariant Banach function space  $\mathbb{E}$  with Boyd indices  $p_{\mathbb{E}}, q_{\mathbb{E}} \in (1, \infty)$ , and for every Muckenhoupt weight  $w$  in the class  $A_{p_{\mathbb{E}}}$ . This means that for every  $f \in \mathbb{E}_{w,loc}(\mathbb{R}_+; X)$  the problem (1) admits a unique solution  $u \in W_{loc}^{2,1}(\mathbb{R}_+; X)$  satisfying*

$$u, \dot{u}, \ddot{u}, B\dot{u}, Au \in \mathbb{E}_{w,loc}(\mathbb{R}_+; X).$$

We point out that it is not clear whether this theorem can be deduced from the corresponding result for the first order problem. In fact, it is in general not clear how to reduce the complete second order problem (1) to a first order problem: we are not aware of a canonical phase space and an appropriate operator matrix, apart from some special examples of operators  $B$  and  $A$ .

We also point out that the general extrapolation theorem for singular integral operators with operator-valued kernels, Theorem 4.3 in [12], which goes back to Rubio de Francia, Ruiz & Torrea [34], and which was used in the proof of the extrapolation of maximal regularity for the first order problem, can not directly be applied for the second order problem (1). In fact, we are not able to prove the first standard condition for the kernels which are associated with the singular integral operators appearing in the second order problem.

We therefore state and prove a new extrapolation theorem for singular integral operators with operator-valued kernels, which also slightly improves the result by Rubio de Francia, Ruiz & Torrea [34] in the case of weighted Lebesgue spaces (see Theorem 7 in Section 5 below).

At the end of this note, in Section 7, we describe three abstract examples and two examples of partial differential equations to which our main result applies.

## 2. REARRANGEMENT INVARIANT BANACH FUNCTION SPACES

We refer the reader primarily to [7] for the background on rearrangement invariant Banach function spaces.

Throughout, let  $X$  be a Banach space with norm  $|\cdot|_X$ , and let  $\mathbb{E}$  denote a rearrangement invariant Banach function space over  $(\mathbb{R}, dt)$ . Recall that, by Luxemburg's representation theorem [7, Theorem 4.10, p.62], there exists a rearrangement invariant Banach function space  $\bar{\mathbb{E}}$  over  $(\mathbb{R}_+, dt)$  such that for every scalar, measurable function  $f$  on  $\mathbb{R}$ ,  $f \in \mathbb{E}$  if and only if  $f^* \in \bar{\mathbb{E}}$ , where  $f^*$  stands for the decreasing rearrangement of  $f$ . In this case  $\|f\|_{\mathbb{E}} = \|f^*\|_{\bar{\mathbb{E}}}$  for every  $f \in \mathbb{E}$ .

Following [30], we define the *lower* and *upper Boyd indices* respectively by

$$p_{\mathbb{E}} = \lim_{t \rightarrow \infty} \frac{\log t}{\log h_{\mathbb{E}}(t)} = \sup_{1 < t < \infty} \frac{\log t}{\log h_{\mathbb{E}}(t)} \quad \text{and}$$

$$q_{\mathbb{E}} = \lim_{t \rightarrow 0^+} \frac{\log t}{\log h_{\mathbb{E}}(t)} = \inf_{0 < t < 1} \frac{\log t}{\log h_{\mathbb{E}}(t)},$$

where  $h_{\mathbb{E}}(t) = \|D_t\|_{\mathcal{L}(\overline{\mathbb{E}})}$  and  $D_t : \overline{\mathbb{E}} \rightarrow \overline{\mathbb{E}}$  ( $t > 0$ ) is the *dilation operator* defined by

$$D_t f(s) = f(s/t), \quad 0 < t < \infty, \quad f \in \overline{\mathbb{E}}.$$

One always has  $1 \leq p_{\mathbb{E}} \leq q_{\mathbb{E}} \leq \infty$ , see for example [7, Proposition 5.13, p.149], where the Boyd indices are defined as the reciprocals with respect to our definitions. Moreover, if  $p_{\mathbb{E}} > 1$ , then, for every  $p \in [1, p_{\mathbb{E}})$ ,

$$\mathbb{E} \subseteq L_{loc}^p$$

and the embedding is continuous (see [30, Proposition 2.b.3]).

Let  $w$  be a *weight*, that is, a positive, locally integrable function on  $\mathbb{R}$ . Then we can associate with  $\mathbb{E}$  and  $w$  a rearrangement invariant Banach function space over  $(\mathbb{R}, wdt)$  as follows

$$\mathbb{E}_w = \{f : \mathbb{R} \rightarrow \mathbb{C} \text{ measurable} : f_w^* \in \overline{\mathbb{E}}\},$$

and its norm is  $\|f\|_{\mathbb{E}_w} = \|f_w^*\|_{\overline{\mathbb{E}}}$ , where  $f_w^*$  denotes the decreasing rearrangement of  $f$  with respect to  $wdt$ . Special emphasis is in the following given to the *Muckenhoupt weights*. A weight  $w$  belongs to the Muckenhoupt class  $A_p$  ( $1 < p < \infty$ ) if

$$\sup_{a < b} \left( \frac{1}{b-a} \int_a^b w \right) \left( \frac{1}{b-a} \int_a^b w^{1-p'} \right)^{p-1} =: [w]_{A_p} < \infty,$$

and it belongs to the Muckenhoupt class  $A_1$  if

$$Mw \leq Cw \text{ for some constant } C \geq 0.$$

Here,  $M$  is the Hardy-Littlewood maximal operator. One can show that if  $\mathbb{E}$  is a rearrangement invariant Banach function space with lower Boyd index  $p_{\mathbb{E}} > 1$  and if  $w \in A_{p_{\mathbb{E}}}$ , then

$$\mathbb{E}_w \subseteq L_{loc}^1,$$

and the embedding is continuous.

Examples of rearrangement invariant Banach function spaces are the  $L^p$  spaces ( $1 \leq p < \infty$ ), the Lorentz spaces  $L^{p,q}$  ( $1 \leq p, q < \infty$ ), the Orlicz spaces  $L^{\Phi}$ . The Boyd indices can be computed explicitly for many examples of concrete rearrangement invariant Banach function spaces, see e.g. [7, Chapter 4]. The weighted spaces  $L_w^p$  as defined above coincide with the usual weighted Lebesgue spaces  $L^p(wdt)$ , that is, the  $L^p$  spaces with respect to the weighted Lebesgue measure  $wdt$ . A similar statement holds for the weighted Orlicz spaces, see [12, Section 2].

As in Curbera, García-Cuerva, Martell & Pérez [18], we define the vector-valued versions  $\mathbb{E}_w(X)$  of the rearrangement invariant Banach function spaces  $\mathbb{E}_w$  in the following way:

$$\mathbb{E}_w(X) := \{f : \mathbb{R} \rightarrow X \text{ measurable} : |f|_X \in \mathbb{E}_w\},$$

and its norm is  $\|f\|_{\mathbb{E}_w(X)} = \||f|_X\|_{\mathbb{E}_w}$ .

In the sequel, we also consider the function space  $\mathbb{E}_{w,loc}(\mathbb{R}_+; X)$  defined by

$$\mathbb{E}_{w,loc}(\mathbb{R}_+; X) := \{u : \mathbb{R}_+ \rightarrow X : u|_{[0,\tau]} \in \mathbb{E}_w(0, \tau; X) \text{ for every } \tau > 0\},$$

where  $\mathbb{E}_w(0, \tau; X)$  ( $\tau > 0$ ) denotes the space of all functions from  $[0, \tau]$  into  $X$  such that their extensions by 0 to functions on  $\mathbb{R}$  belong to  $\mathbb{E}_w(X)$ . As usual, one can identify  $\mathbb{E}_w(0, \tau; X)$  with a subspace of  $\mathbb{E}_w(X)$ .

## 3. INITIAL VALUE PROBLEM

Troughout this section, let  $\mathbb{E}$  be a rearrangement invariant Banach function space over  $(\mathbb{R}, dt)$  and let  $w$  be a weight such that  $\mathbb{E}_w \subseteq L^1_{loc}$ . Given two Banach spaces  $Y$  and  $Z$  continuously embedded into a Banach space  $X$ , we define the *maximal regularity space*

$$W_{loc}^{2, \mathbb{E}_w}(\mathbb{R}_+; X, Y, Z) := \{u \in W_{loc}^{2,1}(\mathbb{R}_+; X) : u \in \mathbb{E}_{w,loc}(\mathbb{R}_+; Z), \dot{u} \in \mathbb{E}_{w,loc}(\mathbb{R}_+; Y), \\ \text{and } \ddot{u} \in \mathbb{E}_{w,loc}(\mathbb{R}_+; X)\},$$

which becomes in a natural way a Fréchet space. Moreover, we define the *trace space*

$$(X, Y, Z)_{\mathbb{E}_w} := \left\{ (u(0), \dot{u}(0)) \in X \times X : u \in W_{loc}^{2, \mathbb{E}_w}(\mathbb{R}_+; X, Y, Z) \right\}.$$

The trace space becomes a normed space for the quotient norm

$$|(u_0, u_1)|_{(X, Y, Z)_{\mathbb{E}_w}} := \inf \{ \|u\chi_{(0,1)}\|_{\mathbb{E}_w(Z)} + \|\dot{u}\chi_{(0,1)}\|_{\mathbb{E}_w(Y)} + \|\ddot{u}\chi_{(0,1)}\|_{\mathbb{E}_w(X)} : \\ u \in W_{loc}^{2, \mathbb{E}_w}(\mathbb{R}_+; X, Y, Z) \text{ and } (u(0), \dot{u}(0)) = (u_0, u_1) \}.$$

One can show that this space is complete in the unweighted case, that is, when  $w = 1$ , and it is also complete for many other concrete weights, for example, for weights  $w_\alpha(t) = |t|^\alpha$  ( $t \in \mathbb{R}$ ), which belong to  $A_p$  if  $-1 < \alpha < p - 1$ . We do not know whether the trace space is always complete.

As in [13, Lemma 6.1], one shows that the trace space  $(X, Y, Z)_{\mathbb{E}}$  is the product of two Banach spaces which are continuously embedded into  $X$ .

**Lemma 2** (Decomposition of the trace space). *There exist two Banach spaces  $(X, Y, Z)_{\mathbb{E}}^0$  and  $(X, Y, Z)_{\mathbb{E}}^1$  which are continuously embedded into  $X$  such that*

$$(X, Y, Z)_{\mathbb{E}} = (X, Y, Z)_{\mathbb{E}}^0 \times (X, Y, Z)_{\mathbb{E}}^1,$$

and such that the coordinate projections  $(X, Y, Z)_{\mathbb{E}} \rightarrow (X, Y, Z)_{\mathbb{E}}^i$  ( $i = 0, 1$ ) are continuous.

*Proof.* Let  $(u_0, u_1) \in (X, Y, Z)_{\mathbb{E}}$ . By definition of the trace space, there exists an element  $u \in W_{loc}^{2, \mathbb{E}}(\mathbb{R}_+; X, Y, Z)$  such that  $(u(0), \dot{u}(0)) = (u_0, u_1)$ . Set  $v(t) := u(t/2)$  for  $t \in \mathbb{R}_+$ . Then, by considering the decreasing rearrangement of  $v$  one easily checks that  $v \in W_{loc}^{2, \mathbb{E}}(\mathbb{R}_+; X, Y, Z)$ . Hence,  $(v(0), \dot{v}(0)) = (u_0, u_1/2) \in (X, Y, Z)_{\mathbb{E}}$ . By linearity, this implies  $(0, u_1) = 2[(u_0, u_1) - (u_0, u_1/2)] \in (X, Y, Z)_{\mathbb{E}}$  and  $(u_0, 0) = (u_0, u_1) - (0, u_1) \in (X, Y, Z)_{\mathbb{E}}$ . In other words,  $(X, Y, Z)_{\mathbb{E}} = (X, Y, Z)_{\mathbb{E}}^0 \times (X, Y, Z)_{\mathbb{E}}^1$  for two vector spaces  $(X, Y, Z)_{\mathbb{E}}^0, (X, Y, Z)_{\mathbb{E}}^1 \subseteq X$ . These two spaces can be turned into Banach spaces by considering the natural quotient norms on them; for example,

$$|u_0|_{(X, Y, Z)_{\mathbb{E}}^0} := |(u_0, 0)|_{(X, Y, Z)_{\mathbb{E}}} \quad (u_0 \in (X, Y, Z)_{\mathbb{E}}^0)$$

and similarly for  $(X, Y, Z)_{\mathbb{E}}^1$ . Finally, since  $(X, Y, Z)_{\mathbb{E}}^0$  and  $(X, Y, Z)_{\mathbb{E}}^1$  are closed subspaces of  $(X, Y, Z)_{\mathbb{E}}$ , the continuity of the coordinate projections follows from the closed graph theorem.  $\square$

The following existence and uniqueness theorem was proven in the case  $\mathbb{E} = L^p$  in [13, Theorem 2.1] under the additional assumption that  $D_A$  is continuously embedded into  $D_B$ ; without the domain restriction, see [14, Theorem 2.1].

**Lemma 3** (Initial value problem). *Suppose that (1) has  $\mathbb{E}_w$ -maximal regularity. Then, for every  $(u_0, u_1) \in (X, D_B, D_A)_{\mathbb{E}_w}$  there exists a unique solution  $u \in W_{loc}^{2, \mathbb{E}_w}(\mathbb{R}_+; X, D_B, D_A)$  of the initial value problem*

$$\ddot{u} + B\dot{u} + Au = 0 \text{ on } \mathbb{R}_+, \quad u(0) = u_0, \dot{u}(0) = u_1. \quad (2)$$

*Proof. Existence.* Suppose that (1) has  $\mathbb{E}_w$ -maximal regularity, and let initial values  $(u_0, u_1) \in (X, D_B, D_A)_{\mathbb{E}_w}$  be given. By definition of the trace space, there exists  $v \in W_{loc}^{2, \mathbb{E}_w}(\mathbb{R}_+; X, D_B, D_A)$  such that  $(v(0), \dot{v}(0)) = (u_0, u_1)$ . By  $\mathbb{E}_w$ -maximal regularity, there exists a unique solution  $v_0 \in W_{loc}^{2, \mathbb{E}_w}(\mathbb{R}_+; X, D_B, D_A)$  of the inhomogeneous problem

$$\ddot{v}_0 + B\dot{v}_0 + Av_0 = \ddot{v} + B\dot{v} + Av \text{ on } \mathbb{R}_+, \quad v_0(0) = \dot{v}_0(0) = 0;$$

note that the right-hand side belongs to  $\mathbb{E}_{w, loc}(\mathbb{R}_+; X)$  by the regularity of  $v$ . Then  $u := v - v_0 \in W_{loc}^{2, \mathbb{E}_w}(\mathbb{R}_+; X, D_B, D_A)$  is a solution of the initial value problem (2).

*Uniqueness* follows from the uniqueness for the inhomogeneous problem (1).  $\square$

By the preceding lemma, and by Lemma 2, it is possible to define a *sine family* associated with the second order problem. It is, as Theorem 6 below shows, the solution family associated with the initial value problem (2) with  $u_0 = 0$ . In a similar way one could define the *cosine family* associated with the initial value problem (2). Note, however, that these definitions of sine and cosine family as *solution families* differs from the usual definitions in the literature where for example the cosine family is defined through a functional equation (see, for example, [3, Sections 3.14-16], where this theory considers the problem (1) with  $B = 0$ ).

The following result exhibits the regularity of the solutions of the initial value problem (2) and of a particular inhomogeneous problem (1) by invoking an idea used in the proof of [13, Proposition 2.2]; see also [14, Theorem 3.1].

**Theorem 4** (Regularity of solutions of the initial value problem with constant inhomogeneity). *Assume that the problem (1) has  $\mathbb{E}$ -maximal regularity. Let  $(u_0, u_1) \in (X, D_B, D_A)_{\mathbb{E}}$ ,  $x \in X$ , and let  $u \in W_{loc}^{2, \mathbb{E}}(\mathbb{R}_+; X, D_B, D_A)$  be the unique solution of the initial value problem*

$$\ddot{u} + B\dot{u} + Au = x \text{ on } \mathbb{R}_+, \quad u(0) = u_0, \dot{u}(0) = u_1, \quad (3)$$

*which exists by assumption of  $\mathbb{E}$ -maximal regularity and by Lemma 3. Then  $u$  is analytic on  $(0, \infty)$  with values in  $D_A$ , and  $\dot{u}$  is analytic on  $(0, \infty)$  with values in  $D_B$ . Moreover, if for  $k \in \mathbb{N}$  one defines*

$$u_k(t) := t^k u^{(k)}(t), \quad t \in \mathbb{R}_+,$$

*then  $u_k \in W_{loc}^{2, \mathbb{E}}(\mathbb{R}_+; X, D_B, D_A)$ , and there exists an increasing function  $C_k : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ , which is independent of the data, such that, for every  $t > 0$ ,*

$$\begin{aligned} \|u_k \chi_{(0,t)}\|_{\mathbb{E}(D_A)} + \|\dot{u}_k \chi_{(0,t)}\|_{\mathbb{E}(D_B)} + \|\ddot{u}_k \chi_{(0,t)}\|_{\mathbb{E}(X)} &\leq \\ &\leq C_k(t) [\|(u_0, u_1)|_{(X, D_B, D_A)_{\mathbb{E}}}\| + \|\chi_{(0,t)}\|_{\mathbb{E}[X]}]. \end{aligned}$$

*Proof.* Fix an arbitrary  $\tau > 0$ . Set

$$\begin{aligned} W^{2, \mathbb{E}}(0, \tau; X, D_B, D_A) &:= \{u \in W^{2,1}(0, \tau; X) : u \in \mathbb{E}(0, \tau; D_A), \dot{u} \in \mathbb{E}(0, \tau; D_B), \\ &\text{and } \ddot{u} \in \mathbb{E}(0, \tau; X)\}. \end{aligned}$$

We consider the operator

$$\begin{aligned} G : (-1, 1) \times W^{2, \mathbb{E}}(0, \tau; X, D_B, D_A) &\rightarrow \mathbb{E}(0, \tau; X) \times (X, D_B, D_A)_{\mathbb{E}}, \\ (\lambda, v) &\mapsto (\ddot{v} + (1 + \lambda)B\dot{v} + (1 + \lambda)^2Av - (1 + \lambda)^2x, \\ &\quad v(0) - u_0, \dot{v}(0) - (1 + \lambda)u_1). \end{aligned}$$

The operator  $G$  is clearly analytic (see [37] for the definition of an analytic function between two Banach spaces). For  $\lambda \in (-1, 1)$  we put  $u^\lambda(t) := u(t + \lambda t)$ . Then  $u^\lambda \in$

$W^{2,\mathbb{E}}(0, \tau; X, D_B, D_A)$ ,  $u^0 = u$ , and  $G(\lambda, u^\lambda) = 0$  for every  $\lambda \in (-1, 1)$ . Moreover, the partial derivative

$$\begin{aligned} \frac{\partial G}{\partial v}(0, u) : W^{2,\mathbb{E}}(0, \tau; X, D_B, D_A) &\rightarrow \mathbb{E}(0, \tau; X) \times (X, D_B, D_A)_{\mathbb{E}}, \\ v &\mapsto (\ddot{v} + B\dot{v} + Av, v(0), \dot{v}(0)) \end{aligned}$$

is boundedly invertible by  $\mathbb{E}$ -maximal regularity and Lemma 3. Hence, by the implicit function theorem, [37, Theorem 4.B, p. 150], there exists  $\varepsilon > 0$ , a neighbourhood  $U$  of  $u$  in  $W^{2,\mathbb{E}}(0, \tau; X, D_B, D_A)$ , and an analytic function  $g : (-\varepsilon, \varepsilon) \rightarrow U$  such that

$$\{(\lambda, v) \in (-\varepsilon, \varepsilon) \times U : G(\lambda, v) = 0\} = \{(\lambda, g(\lambda)) : \lambda \in (-\varepsilon, \varepsilon)\}.$$

From this we obtain  $g(\lambda) = u^\lambda$ , and hence the function  $\lambda \rightarrow u^\lambda$  is analytic in  $(-\varepsilon, \varepsilon)$ . In particular, the derivatives  $\frac{d^k}{d\lambda^k} u^\lambda|_{\lambda=0}$  exist in  $W^{2,\mathbb{E}}(0, \tau; X, D_B, D_A)$  for every  $k \in \mathbb{N}$ . One easily checks that  $\frac{d^k}{d\lambda^k} u^\lambda|_{\lambda=0}$  coincides with the function  $u_k$  defined in the statement, so that one part of the claim is proved. The regularity of  $u$  and  $\dot{u}$  is an easy consequence of this part.

Finally, by the closed graph theorem, for every  $k \in \mathbb{N}$  and every  $\tau > 0$  the operator  $X \times (X, D_B, D_A)_{\mathbb{E}} \rightarrow W^{2,\mathbb{E}}(0, \tau; X, D_B, D_A)$ , which maps every  $(x, u_0, u_1)$  to the restriction of  $u_k$  to the interval  $(0, \tau)$ , is a bounded, linear operator. Its norm, which we denote by  $C_k(\tau)$  is, for fixed  $k \in \mathbb{N}$ , clearly increasing in  $\tau$ .  $\square$

**Remark 5.** By a more detailed analysis one can show that, for fixed  $k \in \mathbb{N}$ ,  $C_k(\tau)$  grows at most exponentially in  $\tau$  (compare with the proof of [13, Proposition 2.2]).

#### 4. SINE FAMILY AND REPRESENTATION OF SOLUTIONS

The following theorem should be partially compared with [13, Proposition 2.2].

**Theorem 6** (Sine family, representation of solutions). *Let  $\mathbb{E}$  be a rearrangement invariant Banach function space with upper Boyd index  $q_{\mathbb{E}} < \infty$ . Assume that the problem (1) has  $\mathbb{E}$ -maximal regularity. Then there exists a norm-continuous function  $S : \mathbb{R}_+ \rightarrow \mathcal{L}(X)$ , which we call the sine family associated with problem (1), and which has the following properties:*

- (a) *The function  $S$  is analytic on  $(0, \infty)$  with values in  $\mathcal{L}(X, D_A \cap D_B)$ .*
- (b) *For every  $u_1 \in (X, D_B, D_A)_{\mathbb{E}}^1$  the orbit  $S(\cdot)u_1$  is the unique solution of the initial value problem (2) with  $u_0 = 0$ .*
- (c) *For every  $f \in \mathbb{E}_{loc}(\mathbb{R}_+; X)$  the convolution  $S * f$  is the unique solution of the inhomogeneous problem (1).*
- (d) *If  $\mathbb{E} = L^p$  for some  $p \in (1, \infty)$ , then we have the following local, second standard conditions: for every  $\tau > 0$  there exists  $C \geq 0$  such that for every  $0 < 2s < t \leq \tau$ ,*

$$\begin{aligned} |AS(t-s) - AS(t)|_{\mathcal{L}(X)} &\leq C \frac{s^{\frac{1}{p}}}{t^{1+\frac{1}{p}}}, \\ |BS(t-s) - BS(t)|_{\mathcal{L}(X)} &\leq C \frac{s^{\frac{1}{p}}}{t^{1+\frac{1}{p}}}, \text{ and} \\ |\dot{S}(t-s) - \dot{S}(t)|_{\mathcal{L}(X)} &\leq C \frac{s^{\frac{1}{p}}}{t^{1+\frac{1}{p}}}. \end{aligned}$$

*Proof.* For every  $x \in X$  and every  $t \geq 0$  we put  $\mathcal{S}(t)x := u(t)$ , where  $u$  is the unique solution of (3) with initial values  $u_0 = 0$  and  $u_1 = 0$ . The function  $\mathcal{S} : \mathbb{R}_+ \rightarrow \mathcal{L}(X)$  thus defined is strongly continuous. Moreover, by Theorem 4, for every  $x \in X$  the functions  $\mathcal{S}(\cdot)x$  and  $\dot{\mathcal{S}}(\cdot)x$  are analytic on  $(0, \infty)$  with values in  $D_A$  and  $D_B$ , respectively. Taking into account that  $\mathcal{S}(0) = 0$ , the function  $\mathcal{S}$  is thus strongly analytic on  $(0, \infty)$  with values in  $\mathcal{L}(X, D_A \cap D_B)$ . By the uniform boundedness principle, it is therefore uniformly analytic with values in the same operator space. The estimates from Theorem 4 (for  $k = 0, 1, 2$ ) further imply

$$\begin{aligned} & \|\mathcal{S}(\cdot)x\chi_{(0,\tau)}\|_{\mathbb{E}(X)} + \|A\mathcal{S}(\cdot)x\chi_{(0,\tau)}\|_{\mathbb{E}(X)} + \|\dot{\mathcal{S}}(\cdot)x\chi_{(0,\tau)}\|_{\mathbb{E}(X)} + \\ & \quad + \|B\dot{\mathcal{S}}(\cdot)x\chi_{(0,\tau)}\|_{\mathbb{E}(X)} + \|\ddot{\mathcal{S}}(\cdot)x\chi_{(0,\tau)}\|_{\mathbb{E}(X)} \leq C_0(\tau) \|\chi_{(0,\tau)}\|_{\mathbb{E}} |x|_X, \end{aligned}$$

$$\begin{aligned} & \|t\dot{\mathcal{S}}(\cdot)x\chi_{(0,\tau)}\|_{\mathbb{E}(X)} + \|tA\dot{\mathcal{S}}(\cdot)x\chi_{(0,\tau)}\|_{\mathbb{E}(X)} + \|t\ddot{\mathcal{S}}(\cdot)x\chi_{(0,\tau)}\|_{\mathbb{E}(X)} + \\ & \quad + \|tB\ddot{\mathcal{S}}(\cdot)x\chi_{(0,\tau)}\|_{\mathbb{E}(X)} + \|t\mathcal{S}^{(3)}(\cdot)x\chi_{(0,\tau)}\|_{\mathbb{E}(X)} \leq C_1(\tau) \|\chi_{(0,\tau)}\|_{\mathbb{E}} |x|_X, \end{aligned}$$

and

$$\begin{aligned} & \|t^2\ddot{\mathcal{S}}(\cdot)x\chi_{(0,\tau)}\|_{\mathbb{E}(X)} + \|t^2A\ddot{\mathcal{S}}(\cdot)x\chi_{(0,\tau)}\|_{\mathbb{E}(X)} + \|t^2\mathcal{S}^{(3)}(\cdot)x\chi_{(0,\tau)}\|_{\mathbb{E}(X)} + \\ & \quad + \|t^2B\mathcal{S}^{(3)}(\cdot)x\chi_{(0,\tau)}\|_{\mathbb{E}(X)} + \|t^2\mathcal{S}^{(4)}(\cdot)x\chi_{(0,\tau)}\|_{\mathbb{E}(X)} \leq C_2(\tau) \|\chi_{(0,\tau)}\|_{\mathbb{E}} |x|_X. \end{aligned}$$

These estimates (first line) imply that  $\mathcal{S}$  is continuously differentiable at 0 with values in  $\mathcal{L}(X)$ . We now set  $S(t) := \dot{\mathcal{S}}(t)$  for every  $t \geq 0$ . Then the function  $S$  is continuous on  $\mathbb{R}_+$  with values in  $\mathcal{L}(X)$ , and analytic on  $(0, \infty)$  with values in  $\mathcal{L}(X, D_A \cap D_B)$  (property (a)).

Let  $u$  be the unique solution of (2) with  $u_0 = 0$  and  $u_1 \in (X, D_B, D_A)_{\mathbb{E}}^1$ . Then the primitive  $v(t) := \int_0^t u(s) ds$  is solution of the same problem, but with homogeneous initial values, and for constant right-hand side  $u_1$ . By uniqueness,  $v = \mathcal{S}(\cdot)u_1$ , and thus  $u = S(\cdot)u_1$  (property (b)).

Next, let  $f \in C_c^1((0, \infty); X)$ , and put

$$u = S * f = \mathcal{S} * \dot{f}.$$

Then the above estimates for  $\mathcal{S}$  (first line) imply that  $u \in W_{loc}^{2, \mathbb{E}}(\mathbb{R}_+; X, D_B, D_A)$ , and clearly  $u(0) = \dot{u}(0) = 0$ . Moreover, the definition of  $\mathcal{S}$  implies that

$$\ddot{u} + B\dot{u} + Au = (\ddot{\mathcal{S}} + B\dot{\mathcal{S}} + A\mathcal{S}) * \dot{f} = 1 * \dot{f} = f.$$

Hence, for every  $f \in C_c^1((0, \infty); X)$ , the convolution  $S * f$  is the solution of the inhomogeneous problem (1). By  $\mathbb{E}$ -maximal regularity and by the closed graph theorem, the mapping  $\mathbb{E}_{loc}(\mathbb{R}_+; X) \rightarrow W_{loc}^{2, \mathbb{E}}(\mathbb{R}_+; X, D_B, D_A)$ , which maps to every right-hand side  $f \in \mathbb{E}_{loc}(\mathbb{R}_+; X)$  the unique solution  $u \in W_{loc}^{2, \mathbb{E}}(\mathbb{R}_+; X, D_B, D_A)$  of (1), is bounded. By the preceding arguments, this mapping is given by the convolution operator associated with the sine family, at least on  $C_c^1((0, \infty); X)$ . Since, by [30, Proposition 2.b.3],  $L^1 \cap L^q_{\mathbb{E}}$  is continuously embedded in  $\mathbb{E}$ ,  $C_c^1((0, \infty); X)$  is dense in  $\mathbb{E}_{loc}(\mathbb{R}_+; X)$ , and the claim (c) follows.

Assume finally that  $\mathbb{E} = L^p$  for some  $p \in (1, \infty)$ . Then the above estimates (third line) imply in particular, for every  $x \in X$ ,

$$\begin{aligned} & \|t^2 A \dot{S}(\cdot)x\chi_{(0,\tau)}\|_{L^p(X)} + \|t^2 B \ddot{S}(\cdot)x\chi_{(0,\tau)}\|_{L^p(X)} + \|t^2 S^{(3)}(\cdot)x\chi_{(0,\tau)}\|_{L^p(X)} \leq \\ & \leq C_2(\tau) \tau^{\frac{1}{p}} |x|_X. \quad (4) \end{aligned}$$



Then, for  $\tau \geq t > s > 0$ , and  $x \in X$ , we have, on using (4) and Hölder's inequality

$$\begin{aligned} |AS(t-s)x - AS(t)x|_X &= \left| \int_{t-s}^t A\dot{S}(r)x \, dr \right|_X \\ &\leq \left| \int_{t-s}^t r^{-2} r^2 A\dot{S}(r)x \, dr \right|_X \\ &\leq \left( \int_{t-s}^t r^{-2p'} \, dr \right)^{\frac{1}{p'}} \left( \int_0^t |r^2 A\dot{S}(r)x|_X^p \, dr \right)^{\frac{1}{p}} \\ &\leq \left( \int_{t-s}^t r^{-2p'} \, dr \right)^{\frac{1}{p'}} C_2(\tau) t^{\frac{1}{p}} |x|_X. \end{aligned}$$

If in addition  $2s < t$ , then we may estimate the integral on the right-hand side roughly in order to obtain

$$\begin{aligned} |AS(t-s) - AS(t)|_{\mathcal{L}(X)} &\leq s^{\frac{1}{p'}} (t-s)^{-2} C_2(\tau) t^{\frac{1}{p}} \\ &\leq 4C_2(\tau) \frac{s^{\frac{1}{p'}}}{t^{1+\frac{1}{p'}}}. \end{aligned}$$

This yields the first claim in (d). The other estimates are obtained in a similar way.  $\square$

## 5. EXTRAPOLATION OF SINGULAR INTEGRAL OPERATORS

We shall see in Section 6 that Theorem 1 follows from Theorem 6 and a general extrapolation result for singular integral operators with operator-valued kernel, Theorem 7 below.

In the sequel, a measurable function  $K : \mathbb{R} \times \mathbb{R} \rightarrow \mathcal{L}(X)$  is called a *kernel* if  $K(t, \cdot) \in L^1_{loc}(\mathbb{R} \setminus \{t\}; \mathcal{L}(X))$  for every  $t \in \mathbb{R}$ . We say that a bounded linear operator  $T$  on  $L^p(X)$  ( $p \in (1, \infty)$ ) is a *singular integral operator* if there exists a kernel  $K$  such that

$$Tf(t) = \int_{\mathbb{R}} K(t, s)f(s) \, ds$$

for every  $f \in L^\infty(\mathbb{R}; X)$  with compact support, and every  $t \notin \text{supp } f$ .

Following the terminology from [34, Definition 1.1, Part III], we say that a kernel  $K$  satisfies the condition  $(D_r)$  for some  $1 \leq r \leq \infty$ , if there exists a constant  $C_r \geq 0$  such that

$$\sum_{m=1}^{\infty} 2^{m/r'} \left( \int_{S_m(s,z)} |K(t,s) - K(t,z)|_{\mathcal{L}(X)}^r \, dt \right)^{1/r} \leq \frac{C_r}{|s-z|^{1/r'}} \text{ for every } s, z \in \mathbb{R}, s \neq z, \quad (D_r)$$

where  $S_m(s, z) := \{t \in \mathbb{R} : 2^m |s-z| < |t-z| \leq 2^{m+1} |s-z|\}$  ( $m \in \mathbb{N}$ ). For  $r = \infty$  this condition is understood in the usual way, that is,

$$\sum_{m=1}^{\infty} 2^m \sup_{t \in S_m(s,z)} |K(t,s) - K(t,z)|_{\mathcal{L}(X)} \leq \frac{C_\infty}{|s-z|} \text{ for every } s, z \in \mathbb{R}, s \neq z. \quad (D_\infty)$$

Moreover, we say that a kernel  $K$  satisfies the condition  $(D'_r)$  for some  $1 \leq r \leq \infty$ , if the adjoint function  $K'$  given by  $K'(t, s) := K(s, t)$  ( $t, s \in \mathbb{R}$ ) is a kernel which satisfies the condition  $(D_r)$ , say, for a constant  $C'_r \geq 0$ . Note that if a kernel  $K$  satisfies the condition  $(D_r)$  for some  $1 \leq r \leq \infty$ , then it satisfies also the condition  $(D_q)$  for every  $1 \leq q \leq r$ .

**Theorem 7.** *Let  $T$  be a singular integral operator associated with a kernel  $K$  satisfying the conditions  $(D_1)$  and  $(D'_r)$  for every  $1 \leq r < \infty$ . Then, for every rearrangement invariant Banach function space  $\mathbb{E}$  with Boyd indices  $p_{\mathbb{E}}, q_{\mathbb{E}} \in (1, \infty)$ , and for every Muckenhoupt weight  $w \in A_{p_{\mathbb{E}}}$ ,  $T$  extrapolates to a bounded linear operator on  $\mathbb{E}_w(X)$ .*

Theorem 7 may be considered as a refined version of [34, Theorems 1.2 and 1.3, Part III]. It extends also [18, Theorem 2.3(ii)], where the classical Boyd theorem [8, Theorem 1] is extended to Calderón-Zygmund operators with standard kernels. Note carefully that the assumptions on the kernel in Theorem 7 are asymmetric, and in each coordinate weaker than the standard conditions.

More precisely, we shall see below that the conclusion of Theorem 7 for  $\mathbb{E} = L^p$ ,  $p \in (1, \infty)$ , is an immediate consequence of [34, Theorems 1.2 and 1.3, Part III], and the well-known property of the Muckenhoupt weights, which says that if  $w \in A_p$  ( $p \in (1, \infty)$ ), then  $w \in A_{p-\varepsilon}$  for some  $\varepsilon > 0$ . Note also that in the scalar case (that is,  $X = \mathbb{C}$ ), by the classical Boyd theorem, the extension of  $T$  to all weighted Lebesgue spaces  $L^p_w(X)$  ( $p \in (1, \infty)$  and  $w \in A_p$ ) is sufficient to interpolate  $T$  on  $\mathbb{E}_w(X)$ , for every rearrangement invariant Banach function space  $\mathbb{E}$  with Boyd indices  $p_{\mathbb{E}}, q_{\mathbb{E}} \in (1, \infty)$  and every Muckenhoupt weight  $w \in A_{p_{\mathbb{E}}}$ . Indeed, this follows from the above mentioned property of the Muckenhoupt weights and the fact that  $A_p \subseteq A_q$  if  $1 \leq p < q$ .

It seems not to be clear how one should generalise the basic ingredient of Boyd's theorem, that is, the Calderón inequality (see [10, Theorem 8]), to the case of Banach space valued functions. Hence, for the proof of Theorem 7 we shall adapt an extension of the Rubio de Francia algorithm of extrapolation from  $A_p$ -weights; see the proof of [17, Theorem 4.10, p.76], which was also described in [16, Section 5] and [18]. (Note that the proof of [17, Theorem 4.10, p. 76] is based on the classical Boyd theorem.)

The proof of Theorem 7 follows in principle the ideas of the proofs of [34, Theorems 1.2 and 1.3] and [17, Theorem 4.10, p.76], however, it requires a more detailed analysis of the basic ingredients used therein; in particular, we need to have precise estimates for the constants appearing in the various inequalities.

We do not repeat all details here, but for the convenience of the reader, we provide the main supplementary observations to be made.

*Proof of Theorem 7.* Suppose that  $T$  is a bounded operator on  $L^p(X)$  for some  $p \in (1, \infty)$ , associated with a kernel  $K$ . At first note that [34, Theorems 1.2 and 1.3, Part III], in particular, show that the operator  $T$  extends to a bounded operator from  $L^1(X)$  into *weak*- $L^1(X)$ , and to a bounded operator on  $L^q(X)$  for every  $q \in (1, \infty)$ . Note also that the norms of these extensions, which we shall again denote by  $T$ , depend only on  $q$ , the constants  $C_1$  and  $C'_1$  from the conditions  $(D_1)$  and  $(D'_1)$ , and the norm of  $T$  as an operator on  $L^p(X)$ .

Moreover, the proof of [34, Theorem 1.3 (c), Part III] yields

$$M^\#(|Tf|_X)(t) \leq C_{q,r} M(|f|_X^\beta)(t)^{1/\beta} \quad (t \in \mathbb{R}) \quad (5)$$

for every  $f \in L^\infty(X)$  with compact support and every  $q, r \in (1, \infty)$ , where  $\beta := \max(q, r')$  and  $C_{q,r} := 2^{1/q} \|T\|_{\mathcal{L}(L^q(X))} + C'_r$ , and  $M^\#$  denotes the sharp maximal operator.

Recall the weighted version of the Fefferman-Stein inequality, which in particular says that for every  $p \in (1, \infty)$  and every Muckenhoupt weight  $w \in A_p$  there exists a constant  $C_{p,w} > 0$  which depends only on  $p$  and  $[w]_{A_p}$ , such that

$$\int_{\mathbb{R}} M_d f(t)^p w(t) dt \leq C_{p,w} \int_{\mathbb{R}} M^\# f(t)^p w(t) dt \quad (f \in L^p \cap L^p_w), \quad (6)$$

where  $M_d$  is the dyadic maximal operator; see [27, Theorem, p.41], or [24, Theorem 2.20, Chapter IV]. We emphasize here that the constant  $C_{p,w}$  on the right-hand side of this inequality is not given explicitly in literature, but it can be obtained from a detailed analysis of the constants involved in the results which are used in the proof of (6), that the constant  $C_{p,w}$  is bounded from above if  $w$  varies in a subset of  $A_p$  on which  $[w]_{A_p}$  is uniformly bounded. For the convenience of the reader we sketch the proof of this statement. Moreover, we refer the reader to [24, Chapter IV] and [23, Chapter 7] for recent expositions of the results involved in the proof of (6), which originally come from [15], and [32], [33].

By a more detailed analysis of the proof of the weighted Fefferman-Stein inequality, see [27, Theorem, p.41] or [24, Theorem 2.20, Chapter IV], one can show that for every  $p \in (1, \infty)$  and every Muckenhoupt weight  $w \in A_p$ , the constant  $C_{p,w}$  in (6) satisfies the estimate

$$C_{p,w} \leq 2^{2p+2} (2^{p(p+1)} C^p)^{1/\delta},$$

where the constants  $C \geq 0$  and  $\delta > 0$  come from the  $A_\infty$ -condition for  $w$ , that is,

$$\frac{w(S)}{w((a,b))} \leq C \left( \frac{|S|}{b-a} \right)^\delta \quad \text{for every } -\infty < a < b < \infty \text{ and every measurable } S \subseteq [a,b].$$

The proof of the  $A_\infty$ -condition for a Muckenhoupt weight  $w \in A_p$  (see [33], or [34, Theorem 2.9, Chapter IV]) shows that one may choose

$$\delta = \frac{\varepsilon}{1+\varepsilon} \quad \text{and} \quad C = \tilde{C},$$

where  $\varepsilon$  and  $\tilde{C}$  are the constants in the reverse Hölder inequality for  $w$ , that is, in the inequality

$$\left( \frac{1}{b-a} \int_a^b w(t)^{1+\varepsilon} dt \right)^{\frac{1}{1+\varepsilon}} \leq \tilde{C} \frac{1}{b-a} \int_a^b w(t) dt \quad \text{for every } -\infty < a < b < \infty. \quad (7)$$

However, a closer analysis of the proof of the reverse Hölder inequality (see for example [15, Theorem IV], or [34, Lemma 2.5, Chapter IV]) shows that the constants  $\varepsilon$  and  $\tilde{C}$  can be chosen in the following way:

$$\tilde{C} = \left( 1 + \frac{(2/\alpha)^\varepsilon}{1 - (2/\alpha)^\varepsilon \beta} \right)^{\frac{1}{1+\varepsilon}} \quad \text{and} \quad 0 \leq \varepsilon < \log \beta / \log(\alpha/2), \quad (8)$$

where  $\alpha \in (0, 1)$  is arbitrary and  $\beta = 1 - (1 - \alpha)^p / [w]_{A_p}$ . In particular, if for  $p \in (1, \infty)$  we take  $\alpha = 1 - 2^{-1/p}$ , it is easily seen that for every  $0 \leq \varepsilon \leq \varepsilon_w := \log \left( 1 - \frac{1}{4[w]_{A_p}} \right) / \log(\frac{\alpha}{2})$  we have

$$\left( \frac{1}{b-a} \int_a^b w(t)^{1+\varepsilon} dt \right)^{\frac{1}{1+\varepsilon}} \leq (5[w]_{A_p})^{\frac{1}{1+\varepsilon}} \frac{1}{b-a} \int_a^b w(t) dt \quad \text{for every } -\infty < a < b < \infty. \quad (9)$$

Combining these observations, we easily obtain our claim on the dependence of the constant  $C_{p,w}$  in (6).

Recall that for every  $p \in (1, \infty)$  and every Muckenhoupt weight  $w \in A_p$ , the norm of the Hardy-Littlewood maximal operator  $M$  – considered as an operator on  $L_w^p$  – may be estimated by  $C_p [w]_{A_p}^{p'/p}$ , where the constant  $C_p$  depends only upon  $p$ ; see [9]. Moreover, if one sets  $w_k := \inf\{w, k\}$  ( $k \geq 1$ ), then one can show that  $w_k \in A_p$  and  $[w_k]_{A_p} \leq 2^p [w]_{A_p}$  for every  $k \geq 1$ .

We now show that  $T$  extends to a bounded operator on  $L_w^p(X)$  for every  $p \in (1, \infty)$  and every Muckenhoupt weight  $w \in A_p$ . Fix  $p \in (1, \infty)$  and  $w \in A_p$ . By [15, Lemma 2], there exists  $\varepsilon > 0$  such that  $w \in A_{p-\varepsilon}$ . Of course,  $|Tf|_X \in L^p \cap L_{w_k}^p$  for every  $k \geq 1$  and every  $f \in L^\infty(X)$  with compact support. Note that, for every  $f \in L_{loc}^1$ ,  $|f| \leq M_d f$  almost everywhere on  $\mathbb{R}$ ; see for example [23, Theorem 2.10]. Therefore, combining (6) and (5) for  $q = p - \varepsilon$  and  $r = q'$ , we obtain, for every  $f \in L^\infty(X)$  with compact support,

$$\begin{aligned} \int_{\mathbb{R}} |Tf(t)|_X^p w_k(t) dt &\leq \int_{\mathbb{R}} M_d(|Tf|_X)(t)^p w_k(t) dt \\ &\leq C_{p, w_k} \int_{\mathbb{R}} M^\#(|Tf|_X)(t)^p w_k(t) dt \\ &\leq C_{q, r} C_{p, w_k} \int_{\mathbb{R}} M(|f|_X^q)(t)^{p/q} w_k(t) dt \\ &\leq C_{q, r} C_{p, w_k} C_{p/q} [w_k]_{A_{p/q}}^{(p/q)'} \int_{\mathbb{R}} |f(t)|_X^p w_k(t) dt \\ &\leq C_{q, r, p, w} \int_{\mathbb{R}} |f(t)|_X^p w(t) dt, \end{aligned}$$

where  $C_{q, r, p, w} := C_{q, r} C_{p/q} \sup_{k \geq 1} C_{p, w_k} [w_k]_{A_{p/q}}^{(p/q)'}$ . Letting  $k \rightarrow \infty$  in the above inequalities, we thus obtain, for every  $p \in (1, \infty)$ , for every Muckenhoupt weight  $w \in A_p$ , and for every  $f \in L^\infty(X)$  with compact support,

$$\int_{\mathbb{R}} |Tf(t)|_X^p w(t) dt \leq C_{q, r, p, w} \int_{\mathbb{R}} |f(t)|_X^p w(t) dt, \quad (10)$$

where  $C_{q, r, p, w}$  is independent on  $f$ . Since the space of all functions in  $L^\infty(X)$  with compact support is dense in  $L_w^p(X)$ ,  $T$  extends to a bounded operator on  $L_w^p(X)$  as we claimed.

We are now in a position to adapt the extrapolation techniques from  $A_p$  weights; see for example the proof of [17, Theorem 4.10, p. 76]. Fix  $\mathbb{E}$  and  $w \in A_{p_{\mathbb{E}}}$  as in the assumption. It follows from [30, Proposition 2.b.3] that  $L_w^p \cap L_w^q \subseteq \mathbb{E}_w$  for every  $1 < p < p_{\mathbb{E}} \leq q_{\mathbb{E}} < q < \infty$ . In particular, by (10), we obtain

$$|Tf|_X \in \mathbb{E}_w \quad (11)$$

for every  $f \in L^\infty(X)$  with compact support.

Let  $\mathbb{E}'_w$  be the associate space of  $\mathbb{E}_w$ , see [7, Definition 2.3, p. 9]. Let  $\mathcal{R} = \mathcal{R}_w : \mathbb{E}_w \rightarrow \mathbb{E}_w$  and  $\mathcal{R}' = \mathcal{R}'_w : \mathbb{E}'_w \rightarrow \mathbb{E}'_w$  be defined by

$$\begin{aligned} \mathcal{R}h(t) &= \sum_{j=0}^{\infty} \frac{M^j h(t)}{2^j \|M\|_{\mathcal{L}(\mathbb{E}_w)}^j}, \quad 0 \leq h \in \mathbb{E}_w, \text{ and} \\ \mathcal{R}'h(t) &= \sum_{j=0}^{\infty} \frac{S^j h(t)}{2^j \|S\|_{\mathcal{L}(\mathbb{E}'_w)}^j}, \quad 0 \leq h \in \mathbb{E}'_w, \end{aligned}$$

where  $Sh := M(hw)/w$  for  $h \in \mathbb{E}'_w$ . As in the proof of [17, Theorem 4.10, p. 76] the following statements are easily verified:

(i) For every positive  $h \in \mathbb{E}_w$  one has

$$\begin{aligned} h &\leq \mathcal{R}h \text{ and } \|\mathcal{R}h\|_{\mathbb{E}_w} \leq 2\|h\|_{\mathbb{E}_w}, \text{ and} \\ \mathcal{R}h &\in A_1 \text{ with } [\mathcal{R}h]_{A_1} \leq 2\|M\|_{\mathcal{L}(\mathbb{E}_w)}. \end{aligned}$$

(ii) For every positive  $h \in \mathbb{E}'_w$  one has

$$\begin{aligned} h &\leq \mathbb{R}'h \text{ and } \|\mathcal{R}'h\|_{\mathbb{E}'_w} \leq 2\|h\|_{\mathbb{E}'_w}, \text{ and} \\ (\mathcal{R}'h)w &\in A_1 \text{ with } [(\mathcal{R}'h)w]_{A_1} \leq 2\|S\|_{\mathcal{L}(\mathbb{E}'_w)}. \end{aligned}$$

The last lines in (i) and (ii) follow from the estimates  $M(\mathcal{R}h) \leq 2\|M\|_{\mathcal{L}(\mathbb{E}_w)}\mathcal{R}h$  and  $M((\mathcal{R}'h)w) \leq 2\|S\|_{\mathcal{L}(\mathbb{E}'_w)}(\mathcal{R}'h)w$ , respectively, which in turn follow from the definitions of  $\mathcal{R}$  and  $\mathcal{R}'$ .

Now fix  $1 < p < \infty$ . For every  $g \in L^\infty(X)$  with compact support and every positive  $h \in \mathbb{E}'_w$ , set  $w_{g,h} := (\mathcal{R}|g|_X)^{1-p}(\mathcal{R}'h)w$ . By the so-called reverse factorization (or by Hölder's inequality; see for example [23, Proposition 7.2]), and by properties (i) and (ii), we obtain that  $w_{g,h} \in A_p$  and

$$[w_{g,h}]_{A_p} \leq [(\mathcal{R}|g|_X)]_{A_1}^{p-1} [(\mathcal{R}'h)w]_{A_1} \leq 2^p \|M\|_{\mathcal{L}(\mathbb{E}_w)}^{p-1} \|S\|_{\mathcal{L}(\mathbb{E}'_w)}. \quad (12)$$

We claim that there exist constants  $C_0, \varepsilon_0 > 0$  such that

$$w_{g,h} \in A_{p-\varepsilon_0} \quad \text{and} \quad [w_{g,h}]_{A_{p-\varepsilon_0}} \leq C_0 \quad (13)$$

for every  $g \in L^\infty(X)$  with compact support and every positive  $h \in \mathbb{E}'_w$ .

Indeed, note that if  $\mathcal{V} \subseteq A_p$  is a family of Muckenhoupt weights which is bounded in the sense that  $\sup_{v \in \mathcal{V}} [v]_{A_p} = C' < \infty$ , then by (9), every weight  $v \in \mathcal{V}$  satisfies the reverse Hölder inequality (7) with constants  $\varepsilon = \log\left(1 - \frac{1}{4C'}\right) / \log\left(\frac{1-2^{-1/p}}{2}\right)$  and  $\tilde{C} = 5C'$ . Note also that the proof of [15, Lemma 2] shows that, if  $v \in A_p$  satisfies the reverse Hölder inequality (7) with constants  $\varepsilon$  and  $\tilde{C}$ , then for  $\tilde{\varepsilon} := (p-1)\frac{\varepsilon}{1+\varepsilon}$ ,  $v \in A_{p-\tilde{\varepsilon}}$  and  $[v]_{A_{p-\tilde{\varepsilon}}} \leq \tilde{C}^{p-1}[v]_{A_p}$ . Therefore our claim is a consequence of (12).

Applying the Fefferman-Stein inequality (6), the inequality (5) for  $q = p - \varepsilon_0$  and  $r = q'$ , and the estimates (12) and (13), one can easily show that there exists a constant  $C \geq 0$  such that

$$\int_{\mathbb{R}} |Tf(t)|_X^p w_{g,h}(t) dt \leq C \int_{\mathbb{R}} |f(t)|_X^p w_{g,h}(t) dt \quad (14)$$

for every  $f, g \in L^\infty(X)$  with compact support and every positive  $h \in \mathbb{E}'_w$ , where the constant  $C$  is independent on  $f, g$  and  $h$ . Now, using (14), we can simply follow the corresponding idea in the proof of [17, Theorem 4.10] to obtain

$$\int_{\mathbb{R}} |Tf(t)|_X h(t) w(t) dt \leq 4C \|f\|_{\mathbb{E}_w(X)} \|h\|_{\mathbb{E}'_w}$$

for every  $h \in \mathbb{E}'_w$  and every positive  $f \in L^\infty(X)$  with compact support. Therefore, the desired bounded extension of  $T$  on  $\mathbb{E}_w(X)$  is now a consequence of [7, Theorem 2.7, Chapter I] and of the fact that the space of all functions in  $L^\infty(X)$  with compact support is dense in  $\mathbb{E}_w(X)$ .  $\square$

**Remark 8.** (a) Recall that a bounded operator  $T$  on  $L^p(X)$  ( $p \in (1, \infty)$ ) is called a *singular integral operator of convolution type* if there exists a function  $K \in L^1_{loc}(\mathbb{R} \setminus \{0\}; \mathcal{L}(X))$  such that

$$Tf(t) = \int_{\mathbb{R}} K(t-s)f(s) ds$$

for every  $f \in L^\infty(\mathbb{R}; X)$  with compact support, and every  $t \notin \text{supp } f$ .

In other words,  $T$  is a singular integral operator associated with a translation-invariant kernel  $\tilde{K}(t, s) := K(t-s)$ . Note that  $\tilde{K}$  satisfies the conditions  $(D'_r)$  and  $(D_r)$  if and only

if  $K$  satisfies the corresponding condition  $(D_r)$ , which in this situation of a translation-invariant kernel means that there exists a constant  $C_r \geq 0$  such that

$$\sum_{m=1}^{\infty} 2^{m/r'} \left( \int_{2^m|s| < |t| \leq 2^{m+1}|s|} |K(t-s) - K(t)|_{\mathcal{L}(X)}^r dt \right)^{1/r} \leq \frac{C_r}{|s|^{1/r'}} \text{ for every } s \in \mathbb{R} \setminus \{0\}. \quad (D_r)$$

Note that  $(D_\infty)$  for a function  $K \in L^1_{loc}(\mathbb{R} \setminus \{0\}; \mathcal{L}(X))$  is weaker than, for example, the classical Lipschitz or second standard condition, which is the condition that there exist constants  $C, \delta > 0$  such that

$$|K(t-s) - K(t)|_{\mathcal{L}(X)} \leq C \frac{|s|^\delta}{|t|^{1+\delta}} \quad \text{for every } s, t \in \mathbb{R} \text{ with } 2|s| < |t|.$$

(b) Theorem 7 is sharp in the following sense: for every  $r \in [1, \infty)$  there exists a singular integral operator  $T$  of convolution type, associated with a kernel  $K$  satisfying the condition  $(D_r)$ , for which the conclusion of Theorem 7 does not hold for  $\mathbb{E} = L^p$  ( $1 \leq p < r'$ ) and  $w_\theta(t) := |t|^\theta$  ( $t \in \mathbb{R}$ ), with  $-1 < \theta < -p/r'$ , see [31, Theorem 3.2].

In the following we derive from Theorem 7 an extrapolation theorem which is particularly adapted to convolution operators on the positive half-line, and thus, for example, to the second order Cauchy problem. We say that a continuous, linear operator  $T$  on  $L^p_{loc}(\mathbb{R}_+; X)$  ( $p \in (1, \infty)$ ) is a *singular integral operator* if there exists a measurable function  $K : \{(t, s) \in \mathbb{R}_+ \times \mathbb{R}_+ : t > s\} \rightarrow \mathcal{L}(X)$  such that  $K(t, \cdot) \in L^1_{loc}([0, t]; \mathcal{L}(X))$  for every  $t > 0$ , and such that

$$Tf(t) = \int_0^t K(t, s)f(s)ds$$

for every  $f \in L^\infty(\mathbb{R}_+; X)$  with compact support, and every  $t \notin \text{supp } f$ .

As above, we say that  $T$  is associated with a kernel  $K$ . Moreover, we say that  $T$  is a *singular integral operator of convolution type* if it is a singular integral operator, and if  $T$  commutes with the right-translations, that is  $TR(t) = R(t)T$  for every  $t \in \mathbb{R}_+$ ; here,  $(R(t))_{t \geq 0}$  is the right-translation semigroup on  $L^p_{loc}(\mathbb{R}_+; X)$ . If  $T$  is a singular integral operator of convolution type, then there exists a kernel  $K \in L^1_{loc}((0, \infty); \mathcal{L}(X))$  such that

$$Tf(t) = \int_0^t K(t-s)f(s)ds$$

for every  $f \in L^\infty(\mathbb{R}_+; X)$  with compact support, and every  $t \notin \text{supp } f$ .

We say that such a function  $K$  satisfies the condition  $(D_{r,loc})$  ( $r \in [1, \infty]$ ), if for every  $\tau > 0$  the function  $K\chi_{(0, \tau)}$ , considered as a function on  $\mathbb{R}$ , satisfies the condition  $(D_r)$ .

**Corollary 9.** *Let  $T$  be a singular integral operator of convolution type on  $L^p_{loc}(\mathbb{R}_+; X)$  ( $p \in (1, \infty)$ ), associated with a kernel  $K$  satisfying the condition  $(D_{r,loc})$  for every  $r \in [1, \infty)$ . Then for every rearrangement invariant Banach function space  $\mathbb{E}$  with Boyd indices  $p_{\mathbb{E}}, q_{\mathbb{E}} \in (1, \infty)$ , and for every Muckenhoupt weight  $w \in A_{p_{\mathbb{E}}}$ ,  $T$  extrapolates to a continuous linear operator on  $\mathbb{E}_{w,loc}(\mathbb{R}_+; X)$ .*

*Proof.* Let  $\tau > 0$  be arbitrary. By assumption on the kernel,  $K\chi_{[\tau, \infty)} \in L^1_{loc}(\mathbb{R}_+; \mathcal{L}(X))$ . Hence, the convolution operator  $S : L^p_{loc}(\mathbb{R}_+; X) \rightarrow L^p_{loc}(\mathbb{R}_+; X)$  given by

$$Sf := (K\chi_{[\tau, \infty)}) * f \quad (f \in L^p_{loc}(\mathbb{R}_+; X))$$

is well-defined, linear, and continuous. Moreover, it commutes with the right-translations. It then follows that the operator  $R := T - S$  is a singular integral operator of convolution

type on  $L^p_{loc}(\mathbb{R}_+; X)$ , associated with the kernel  $K\chi_{(0,\tau)}$ . Since this kernel has compact support, and since  $R$  commutes with right-translations,  $R$  leaves the space  $L^p(\mathbb{R}_+; X)$  invariant. By the closed graph theorem,  $R$  is continuous on  $L^p(\mathbb{R}_+; X)$ . Using this continuity and again the fact that  $R$  commutes with right-translations, it is easy to see that  $R$  can be extended to a singular integral operator on  $L^p(X) = L^p(\mathbb{R}; X)$  for the same kernel  $K\chi_{(0,\tau)}$ ; using that  $R$  commutes with translations, one first extends  $R$  to the space of functions in  $L^p(X)$  which are supported in a right-half line, and then extends  $R$  to  $L^p(X)$  by continuity. The fact, that  $K$  satisfies the conditions  $(D_{r,loc})$  for every  $r < \infty$  implies that the kernel  $K\chi_{(0,\tau)}$ , considered as a function on  $\mathbb{R}$ , satisfies the conditions  $(D_r)$  for every  $r < \infty$ . Hence, by Theorem 7, for every rearrangement invariant Banach function space  $\mathbb{E}$  with Boyd indices  $p_{\mathbb{E}}, q_{\mathbb{E}} \in (1, \infty)$  and every Muckenhoupt weight  $w \in A_{p_{\mathbb{E}}}$ , the operator  $R$  extends to a bounded linear operator on  $\mathbb{E}_w(X)$ . Using the definition of  $R$ , this implies that  $T$  extends to a bounded, linear operator on  $\mathbb{E}_w(0, \tau; X)$ . Since  $\tau > 0$  was arbitrary, we thus finally obtain the claim.  $\square$

## 6. EXTRAPOLATION OF MAXIMAL REGULARITY - PROOF OF THEOREM 1

*Proof of Theorem 1.* Suppose that the problem (1) has  $L^p$ -maximal regularity. Then, by the closedness of the operators  $A$  and  $B$ , and by the closed graph theorem, the operator  $R : L^p_{loc}(\mathbb{R}_+; X) \rightarrow W^{2,L^p}_{loc}(\mathbb{R}_+; X, D_B, D_A)$  which assigns to each right-hand side  $f \in L^p_{loc}(\mathbb{R}_+; X)$  the unique solution  $u = Rf \in W^{2,L^p}_{loc}(\mathbb{R}_+; X, D_B, D_A)$  of the problem (1) is continuous. It is easy to see that this operator is *causal* in the sense that if  $f = g$  on some interval  $(0, \tau)$ , then  $Rf = Rg$  on the same interval.

The continuity of  $R$  is equivalent to the continuity of the linear operators  $R_i$  on  $L^p_{loc}(\mathbb{R}_+; X)$  ( $i = 0, \dots, 4$ ) which are defined as follows:

$$\begin{aligned} R_0 f &= Rf, & R_3 f &= ARf, \\ R_1 f &= (Rf); & R_4 f &= B(Rf); \\ R_2 f &= (Rf); & & (f \in L^p_{loc}(\mathbb{R}_+; X)). \end{aligned}$$

By Theorem 6, the operator  $R_0$  (respectively,  $R$ ) is given by convolution with the sine function  $(S(t))_{t \geq 0}$ , that is,

$$\begin{aligned} R_0 f(t) &= \int_0^t S(t-s)f(s) ds \\ &\text{for every } f \in L^p_{loc}(\mathbb{R}_+; X) \text{ and every } t \in \mathbb{R}_+. \end{aligned}$$

Since the sine function is uniformly continuous with values in  $\mathcal{L}(X)$  (Theorem 6), it is easy to see that  $R_0$  extends to a bounded linear operator on  $\mathbb{E}_{w,loc}(\mathbb{R}_+; X)$ .

The operator  $R_3$  is the composition of the operator  $R_0$  and the multiplication by  $A$ . In particular,

$$\begin{aligned} R_3 f(t) &= A \int_0^t S(t-s)f(s) ds = \int_0^t AS(t-s)f(s) ds \\ &\text{for every } f \in L^\infty(\mathbb{R}_+; X) \text{ with compact support, and every } t \notin \text{supp } f. \end{aligned}$$

The fact that we may interchange integration and multiplication by  $A$ , at least for functions  $f \in L^\infty(\mathbb{R}_+; X)$  with compact support and  $t \notin \text{supp } f$ , follows from the regularity of the sine family (Theorem 6 (a)). In particular,  $R_3$  is a singular integral operator associated with the kernel  $K(t) = AS(t)$ . Clearly, the operator  $R_3$  commutes also with right-translations. Moreover, it follows easily from Theorem 6 (d), that the kernel  $K$  satisfies the condition  $(D_{\infty,loc})$ . Hence, by Corollary 9,  $R_3$  extends to a bounded linear operator on  $\mathbb{E}_{w,loc}(\mathbb{R}_+; X)$ .

The case of the operators  $R_2$  and  $R_4$  is treated similarly, still using the regularity stated in Theorem 6 (d). From the boundedness of the operators  $R_0$  and  $R_2$  follows the boundedness of the operator  $R_1$  by interpolation.

Finally, we show that for every  $f \in \mathbb{E}_{w,loc}(\mathbb{R}_+; X)$  the function  $u = Rf$  is the unique strong solution of (1), which belongs to  $W_{loc}^{2,\mathbb{E}_w}(\mathbb{R}_+; X, D_B, D_A)$ . In fact, for existence, we have to show that for the extensions of the operators  $R_i$  we have  $R_1 f = (R_0 f)'$ ,  $R_2 f = (R_0 f)''$  etc., as above, and that  $R_2 f + R_3 f + R_4 f = f$  for every  $f \in \mathbb{E}_{w,loc}(\mathbb{R}_+; X)$ . However, for  $f \in L_{loc}^p \cap \mathbb{E}_{w,loc}(\mathbb{R}_+; X)$ , these equalities are clear, and for general  $f \in \mathbb{E}_{w,loc}(\mathbb{R}_+; X)$  the equalities follow from an approximation argument, which uses also the fact that  $\mathbb{E}_{w,loc}(\mathbb{R}_+; X)$  is continuously embedded into  $L_{loc}^1(\mathbb{R}_+; X)$ , and that the operators  $A$  and  $B$  are closed.  $\square$

## 7. APPLICATIONS

**A model problem in  $L^q$  spaces.** There are, as of now, only a few results that ensure that the problem (1) has  $L^p$ -maximal regularity for some / every  $p \in (1, \infty)$ . In our first example, a particular model problem which was already considered in [13, Section 4],  $L^p$ -maximal regularity can be shown by using the characterization of  $L^p$ -maximal regularity in terms of Fourier multipliers and by an application of the Mihlin-Weis multiplier theorem (see [13, Theorems 3.1 and 4.1]). For the different notions which appear in the following theorem, namely *sectorial operator*, *fractional power*, *UMD space*, *bounded  $H^\infty$  functional calculus*, *bounded  $RH^\infty$  functional calculus*, we refer to [13, Section 4], but also to Haase [25], Kalton & Weis [28] and Weis [36].

**Corollary 10.** *Suppose that  $X$  is a UMD space and suppose that  $A$  is a sectorial operator which admits a bounded  $RH^\infty$  functional calculus of angle  $\beta \in [0, \pi)$ . Let  $\varepsilon \in [\frac{1}{2}, 1]$ ,  $\alpha > 0$ , and assume that one of the following cases is satisfied:*

- (a)  $\varepsilon = \frac{1}{2}$ ,  $\alpha \geq 2$ , and  $\beta \in (0, \pi)$ .
- (b)  $\varepsilon = \frac{1}{2}$ ,  $\alpha \in (0, 2)$ , and  $\beta \in (0, \pi - 2 \arctan \frac{\sqrt{4-\alpha^2}}{\alpha})$ .
- (c)  $\varepsilon \in (\frac{1}{2}, 1]$ ,  $\alpha > 0$  and  $\beta \in (0, \frac{\pi}{2\varepsilon})$ .

Then the problem

$$\ddot{u} + \alpha A^\varepsilon \dot{u} + Au = f \text{ in } \mathbb{R}_+, \quad u(0) = \dot{u}(0) = 0 \quad (15)$$

has  $\mathbb{E}_w$ -maximal regularity for every rearrangement invariant Banach function space  $\mathbb{E}$  with Boyd indices  $p_{\mathbb{E}}, q_{\mathbb{E}} \in (1, \infty)$  and every Muckenhoupt weight  $w \in A_{p_{\mathbb{E}}}$ .

*Proof.* This corollary is an immediate consequence of [13, Theorem 4.1] – which says that the problem (15) has  $L^p$ -maximal regularity for every  $p \in (1, \infty)$  – and of Theorem 1.  $\square$

We illustrate this corollary by a particular second order PDE. For simplicity, we consider a one-dimensional problem. Following Chill & Srivastava [13, Section 5.3], we consider the linear equation of an elastic Euler-Bernoulli beam with intermediate damping

$$\begin{cases} u_{tt} - u_{txx} + u_{xxxx} = f & \text{in } \mathbb{R}_+ \times (0, \pi), \\ u(t, 0) = u(t, \pi) = u_{xx}(t, 0) = u_{xx}(t, \pi) = 0 & \text{for every } t \in \mathbb{R}_+, \\ u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x) & \text{for every } x \in (0, \pi). \end{cases} \quad (16)$$

**Corollary 11.** *Let  $q \in (1, \infty)$ , let  $\mathbb{E}$  be a rearrangement invariant Banach function space  $\mathbb{E}$  with Boyd indices  $p_{\mathbb{E}}, q_{\mathbb{E}} \in (1, \infty)$ , and let  $w \in A_{p_{\mathbb{E}}}$  be a Muckenhoupt weight. Put  $D_A = \{u \in W^{4,p}(0, \pi) : u(0) = u(\pi) = u_{xx}(0) = u_{xx}(\pi) = 0\}$ . Then for every  $f \in \mathbb{E}_{w,loc}(\mathbb{R}_+; L^q(0, \pi))$*



and every  $(u_0, u_1) \in (L^q(0, 1), W^{2,q} \cap W_0^{1,q}(0, \pi), D_A)_{\mathbb{E}_w}$  the problem (16) admits a unique solution  $u \in W_{loc}^{2,\mathbb{E}_w}(\mathbb{R}_+; L^q(0, 1), W^{2,q} \cap W_0^{1,q}(0, \pi), D_A)$ . This means that  $u$  satisfies the partial differential equation almost everywhere and the initial conditions and the boundary conditions in the classical sense, and in particular that

$$u, u_{tt}, u_{txx}, u_{xxxx} \in \mathbb{E}_{w,loc}(\mathbb{R}_+; L^q(0, \pi)).$$

**A variational second order equation in Hilbert space.** In the following variational setting, we drop the assumption  $B = \alpha A^\varepsilon$  from the previous example. Let  $V$  and  $H$  be two separable Hilbert spaces such that  $V$  embeds densely and continuously into  $H$ . We identify  $H$  with its dual  $H'$  so that  $H$  is also densely and continuously embedded into  $V'$ .

**Corollary 12.** *Let  $A$  and  $B$  be two linear, maximal monotone, symmetric, not necessarily commuting operators from  $V$  to  $V'$ . Then for every rearrangement invariant Banach function space  $\mathbb{E}$  with Boyd indices  $p_{\mathbb{E}}, q_{\mathbb{E}} \in (1, \infty)$  and every Muckenhoupt weight  $w \in A_{p_{\mathbb{E}}}$  the problem (1) has  $\mathbb{E}_w$ -maximal regularity.*

*Proof.* By a result from Dautray & J.-L. Lions [20, Theorem 1, p.558], the problem (1) has  $L^2$ -maximal regularity in  $V'$ . The claim thus follows from Theorem 1.  $\square$

The fact that in the variational setting above the problem (1) has  $L^2$ -maximal regularity was proved in [20] by the Faedo-Galerkin method and a priori estimates. The proof thus heavily depends on the Hilbert space setting and it does not imply  $L^p$ -maximal regularity for  $p$  different from 2. We point out that the conditions on  $A$  and  $B$  can be considerably relaxed; for the precise assumptions, see [20].

In order to illustrate this corollary we consider an example which is similar to [20, Examples 2 and 3, p.582-583]. Let  $\Omega \subseteq \mathbb{R}^N$  be an open set ( $N \geq 1$ ), and let  $a_{ij}, b_{ij} \in L^\infty(\Omega)$  ( $1 \leq i, j \leq N$ ) be coefficients such that, for some  $\eta > 0$ ,

$$\begin{aligned} a_{ij} &= a_{ji} \text{ and } b_{ij} = b_{ji} \text{ for every } 1 \leq i, j \leq N, \\ \sum_{i,j} a_{ij}(x) \xi_i \xi_j &\geq \eta |\xi|^2 \text{ and} \\ \sum_{i,j} b_{ij}(x) \xi_i \xi_j &\geq \eta |\xi|^2 \text{ for every } x \in \Omega, \xi \in \mathbb{R}^N. \end{aligned}$$

Consider the second order problem

$$\begin{cases} \partial_t^2 u + \sum_{i,j} \partial_j (b_{ij} \partial_i u) + \sum_{i,j} \partial_j (a_{ij} \partial_i u) = f & \text{in } \mathbb{R}_+ \times \Omega, \\ u = 0 & \text{in } \mathbb{R}_+ \times \partial\Omega, \\ u(0, \cdot) = 0, \partial_t u(0, \cdot) = 0 & \text{in } \Omega. \end{cases} \quad (17)$$

In this example,  $V = H_0^1(\Omega)$ , and  $A$  and  $B$  are divergence form, elliptic operators associated with coefficients  $a_{ij}$  and  $b_{ij}$ , respectively.

**Corollary 13.** *Let  $\mathbb{E}$  be a rearrangement invariant Banach function space with Boyd indices  $p_{\mathbb{E}}, q_{\mathbb{E}} \in (1, \infty)$  and let  $w \in A_{p_{\mathbb{E}}}$  be a Muckenhoupt weight. Then, for every  $f \in \mathbb{E}_{w,loc}(\mathbb{R}_+; H^{-1}(\Omega))$  and every  $(u_0, u_1) \in (H^{-1}(\Omega), H_0^1(\Omega), H_0^1(\Omega))_{\mathbb{E}_w}$  the problem (17) admits a unique solution  $u \in W_{loc}^{2,\mathbb{E}_w}(\mathbb{R}_+; H^{-1}(\Omega), H_0^1(\Omega), H_0^1(\Omega))$ . This means that  $u$  satisfies the partial differential equation and the initial conditions and the boundary conditions*

in the weak sense, and in particular that

$$u, \partial_t u \in \mathbb{E}_{w,loc}(\mathbb{R}_+; H_0^1(\Omega)) \text{ and}$$

$$\partial_t^2 u, \partial_j(b_{ij}\partial_i\partial_t u), \partial_j(a_{ij}\partial_i u) \in \mathbb{E}_{w,loc}(\mathbb{R}_+; H^{-1}(\Omega)) \text{ for every } 1 \leq i, j \leq N.$$

**Non-autonomous problems.** By perturbation arguments based on the Neumann series, it is also possible to prove  $\mathbb{E}_w$  maximal regularity for non-autonomous second order problems, that is, problems in which the operators  $A$  and  $B$  may depend on time. The following corollary is stated without proof. We refer the reader to [4] and [5] for corresponding  $L^p$ -maximal regularity results and examples of their applications.

**Corollary 14.** *Let  $X, D_B, D_A$  be three Banach spaces, such that  $D_B$  and  $D_A$  are continuously embedded into  $X$ . Let  $A \in C([0, \tau], \mathcal{L}(D_A, X))$  and  $B \in C([0, \tau], \mathcal{L}(D_B, X))$  be two operator-valued functions. Assume that for every  $t_0 \in [0, \tau]$  the operators  $B(t_0)$  and  $A(t_0)$  are closed, and that the autonomous problem*

$$\ddot{u} + B(t_0)\dot{u} + A(t_0)u = f \text{ in } \mathbb{R}_+, \quad u(0) = \dot{u}(0) = 0,$$

has  $L^p$ -maximal regularity for some fixed  $p \in (1, \infty)$ . Then, for every rearrangement invariant Banach function space  $\mathbb{E}$  with Boyd indices  $p_{\mathbb{E}}, q_{\mathbb{E}} \in (1, \infty)$  and for every Muckenhoupt weight  $w \in A_{p_{\mathbb{E}}}$  the non-autonomous problem

$$\ddot{u} + B(t)\dot{u} + A(t)u = f \text{ on } [0, \tau], \quad u(0) = \dot{u}(0) = 0$$

has  $\mathbb{E}_w$ -maximal regularity. This means that for every  $f \in \mathbb{E}_w(0, \tau; X)$  the above non-autonomous problem admits a unique strong solution  $u \in W^{2, \mathbb{E}_w}(0, \tau; X, D_B, D_A)$ .

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R. CHILL, INSTITUT FÜR ANALYSIS, FACHRICHTUNG MATHEMATIK, TU DRESDEN, 01062 DRESDEN, GERMANY

*E-mail address:* ralph.chill@tu-dresden.de

S. KRÓL, FACULTY OF MATHEMATICS AND COMPUTER SCIENCE, NICOLAUS COPERNICUS UNIVERSITY, UL. CHOPINA 12/18, 87-100 TORUŃ, POLAND

*E-mail address:* sebastian.krol@mat.umk.pl