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# Dynamics of Goldring's $w$ -function \*

Yong-Gao Chen<sup>a</sup>, Ying Shi<sup>b</sup> & Jie Wu<sup>c</sup>

<sup>a</sup>School of Mathematical Sciences and Institute of Mathematics,

Nanjing Normal University, Nanjing 210046, P. R. of China

ygchen@njnu.edu.cn

<sup>b</sup>Department of Mathematics, Nanjing Traditional Chinese Medicine University,

Nanjing 210046, P. R. of China

teddy1101@163.com

<sup>c</sup>Institut Elie Cartan Nancy (IECN), CNRS, Nancy-Université, INRIA,

Boulevard des Aiguillettes, B.P. 239, 54506 Vandœuvre-lès-Nancy, France

wujie@iecn.u-nancy.fr

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## Abstract

Let  $\mathcal{A}_3$  be the set of all positive integers  $pqr$ , where  $p, q, r$  are primes such that at least two of them are not equal. Denote by  $P(n)$  the largest prime factor of  $n$ . For  $n = pqr \in \mathcal{A}_3$ , define  $w(n) := P(p+q)P(p+r)P(q+r)$ . In 2006, Wushi Goldring proved that for any  $n \in \mathcal{A}_3$ , there exists an  $i$  such that

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$w^i(n) \in \{20, 98, 63, 75\}$ , where  $w^0(n) = n$  and  $w^i(n) = w(w^{i-1}(n))$  ( $i \geq 1$ ). If  $w(m) = n$ , then  $m$  is called a **parent** of  $n$ . Let  $\mathcal{B}_3$  be the set of all positive integers  $pq^2$  of  $\mathcal{A}_3$ . In this paper, we study the function  $w$  extensively. For example, one of our results is that there exist infinitely many  $n \in \mathcal{B}_3$  which have at least  $n^{1.1886}$  parents in  $\mathcal{B}_3$ . Several open problems are posed.

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## 1 Introduction

Let  $\mathcal{P}$  be the set of all (rational) positive primes. For any positive integer  $n$  let  $P(n)$  denote the largest prime factor of  $n$  with the convention  $P(1) = 1$ . Let

$$\mathcal{A}_3 = \{pqr \mid p, q, r \in \mathcal{P}\} \setminus \{p^3 \mid p \in \mathcal{P}\}.$$

Recently Goldring [4] introduced his  $w$ -function on  $\mathcal{A}_3$ , which is defined by

$$w(n) := P(p+q)P(p+r)P(q+r) \quad (n = pqr \in \mathcal{A}_3),$$

and investigated its dynamics. Here and in the sequel, the letters  $p, q, r$  and  $s$  denote prime numbers. According to [4, Lemma 2.1], we have  $w(n) \in \mathcal{A}_3$  for all  $n \in \mathcal{A}_3$ . Thus we can consider iteration of  $w$ . For every integer  $i \geq 0$ , write

$$w^0(n) := n, \quad w^i(n) := w(w^{i-1}(n)) \quad (i = 1, 2, \dots).$$

The  $w$ -orbit of  $n$  is denoted by

$$\mathcal{W}(n) := [n, w(n), \dots, w^i(n), \dots].$$

For example, one can verify that

$$w(20) = 98, \quad w(98) = 63, \quad w(63) = 75, \quad w(75) = 20.$$

Interestingly, Goldring [4, Theorem 1.1] proved that for every  $n \in \mathcal{A}_3$ , there exists an integer  $i$  such that  $w^i(n) \in \{20, 98, 63, 75\}$ . Denoting by  $\text{ind}(n)$  the smallest such integer  $i$ . We call  $\text{ind}(n)$  the *periodicity index* of  $n$ . Goldring [4] proved that

$\text{ind}(n) \leq 4(\pi(P(n)) - 3)$ , and posed several conjectures related to  $w(n)$ . Two of them are

**Conjecture A** ([4, Conjecture 2.9]). We have  $\text{ind}(n) = O(\log \pi(P(n)))$ .

**Conjecture B** ([4, Conjecture 2.10]). *There are subsets in  $\mathcal{A}_3$  of arbitrarily large periodicity index.*

Chen and Shi [2] proved Conjecture B and  $\text{ind}(n) = O((\log P(n))^2)$ .

Let  $n \in \mathcal{A}_3$  and  $\mathcal{S} \subset \mathcal{A}_3$ . By a parent of  $n \in \mathcal{A}_3$  in  $\mathcal{S}$ , we mean a positive integer  $m \in \mathcal{S}$  such that  $w(m) = n$ . We also call  $m$  a  $\mathcal{S}$ -parent of  $n$ . Let  $\mathcal{B}_3 = \{p^2q : p \neq q, p, q \in \mathcal{P}\}$  and  $\mathcal{C}_3 = \mathcal{A}_3 \setminus \mathcal{B}_3$ . Goldring [4] proved that there exist infinitely many elements of  $\mathcal{B}_3$  that have at least seven  $\mathcal{B}_3$ -parents and posed the following conjecture:

**Conjecture C** [4, Conjecture 2.16]). *Every element of  $\mathcal{A}_3$  (respectively  $\mathcal{B}_3$ ) has infinitely many  $\mathcal{C}_3$ -parents (respectively  $\mathcal{B}_3$ ).*

Chen and Shi [3] proved that for any positive integer  $k$  there exist infinitely many elements of  $\mathcal{B}_3$  that have at least  $k$  parents in  $\mathcal{B}_3$  and there exist infinitely many elements of  $\mathcal{B}_3$  that have no parents in  $\mathcal{B}_3$ .

Define

$$N_{\mathcal{S}}(n) := |\{m \in \mathcal{S} : w(m) = n\}|,$$

Recently Jia [6, Theorem 3] established more precise results:

- There is an element  $n = pq^2$  of  $\mathcal{B}_3$  with  $x < p \leq 2x$  and  $\sqrt{x} \log x < q \leq 2\sqrt{x} \log x$  such that

$$N_{\mathcal{B}_3}(n) \gg x^{1/2}(\log x)^{-2} \gg n^{1/4}(\log n)^{-5/2},$$

provided  $x$  is sufficiently large.

- There exists an element  $n = qr^2$  of  $\mathcal{B}_3$  with  $q \leq 4x$  and  $\sqrt{x} \log x < r \leq 2\sqrt{x} \log x$  such that

$$N_{\mathcal{C}_3}(n) \gg x(\log x)^{-4} \gg n^{1/2}(\log n)^{-5},$$

provided  $x$  is sufficiently large.

- There is an element  $n = qrs$  of  $\mathcal{C}_3$  with  $q \leq 4x$  and  $\sqrt{x} \log x < r, s \leq 2\sqrt{x} \log x$  such that

$$N_{\mathcal{C}_3}(n) \gg x(\log x)^{-4} \gg n^{1/2}(\log n)^{-5},$$

provided  $x$  is sufficiently large.

Here we establish some stronger results.

**Theorem 1.** (i) *Let  $B \geq -1$ . As  $x \rightarrow \infty$ , we have*

$$(1.1) \quad \#\{n \leq x : n \in \mathcal{B}_3, N_{\mathcal{B}_3}(n) \gg n^{1/3}(\log n)^B\} \gg x^{2/3}(\log x)^{-B-3}.$$

(ii) *There exist infinitely many  $n \in \mathcal{B}_3$  such that  $N_{\mathcal{B}_3}(n) \geq n^{1.1886}$ .*

**Theorem 2.** *For any  $\varepsilon > 0$ , as  $x \rightarrow \infty$ , we have*

$$(1.2) \quad \#\{n \leq x : n \in \mathcal{B}_3, N_{\mathcal{C}_3}(n) \geq 1\} \gg_{\varepsilon} x^{1/3}(\log x)^{-2-\varepsilon},$$

$$(1.3) \quad \#\{n \leq x : n \in \mathcal{B}_3, N_{\mathcal{C}_3}(n) \gg n^{1/2}(\log n)^{-2-\varepsilon}\} \gg \varepsilon \log \log x,$$

where the implied constant in (1.3) is absolute.

**Theorem 3.** *For any  $\varepsilon > 0$ , as  $x \rightarrow \infty$ , we have*

$$(1.4) \quad \#\{n \leq x : n \in \mathcal{C}_3, N_{\mathcal{C}_3}(n) \gg n^{1/2}(\log n)^{-2-\varepsilon}\} \gg (\varepsilon \log \log x)^2,$$

$$(1.5) \quad \#\{n \leq x : n \in \mathcal{C}_3, N_{\mathcal{C}_3}(n) \gg n^{1/2-\varepsilon}\} \gg (\varepsilon \log x)^2,$$

where the implied constants are absolute.

In order to improve Goldring's bound to

$$\text{ind}(n) \ll (\log P(n))^2 \quad (n \in \mathcal{A}_3),$$

Chen & Shi [2] proved their [2, Lemma 4], which is stated as follows: for each  $n \in \mathcal{A}_3$  there is a positive integer  $i = i(n)$  such that

$$1 \leq i \leq \log(P(n) + 6) + 2 \quad \text{and} \quad P(w^i(n)) \leq \frac{15}{16}P(n) + 6.$$

In Section 5 and 6, we study the distribution of sequences  $\{P(w(n))/P(n)\}_{n \in \mathcal{A}_3}$  and  $\{w(n)/n\}_{n \in \mathcal{A}_3}$  and establish the following result.

**Theorem 4.** *We have*

$$(1.6) \quad \liminf_{\substack{n \rightarrow \infty \\ n \in \mathcal{A}_3}} \frac{\log P(w(n))}{\log P(n)} \leq 0.2962,$$

$$(1.7) \quad \liminf_{\substack{n \rightarrow \infty \\ n \in \mathcal{A}_3}} \frac{\log w(n)}{\log n} \leq 0.5924,$$

$$(1.8) \quad \limsup_{\substack{n \rightarrow \infty \\ n \in \mathcal{A}_3}} \frac{\log w(n)}{\log n} \geq 1.354.$$

From Theorem 4, we can deduce the following corollary.

**Corollary 5.** *We have*

$$(1.9) \quad \liminf_{\substack{n \rightarrow \infty \\ n \in \mathcal{A}_3}} \frac{P(w(n))}{P(n)} = 0 \quad \text{and} \quad \limsup_{\substack{n \rightarrow \infty \\ n \in \mathcal{A}_3}} \frac{P(w(n))}{P(n)} = 1,$$

$$(1.10) \quad \liminf_{\substack{n \rightarrow \infty \\ n \in \mathcal{A}_3}} \frac{w(n)}{n} = 0 \quad \text{and} \quad \limsup_{\substack{n \rightarrow \infty \\ n \in \mathcal{A}_3}} \frac{w(n)}{n} = \infty.$$

Introduce the notation

$$(1.11) \quad A_3(x) := |\mathcal{A}_3 \cap [1, x]|.$$

**Theorem 6.** *For all  $\alpha \in (\frac{1}{2}, 1]$ , we have, as  $x \rightarrow \infty$ ,*

$$(1.12) \quad \#\{n \leq x : n \in \mathcal{A}_3, P(w(n)) \leq \alpha P(n)\} \sim A_3(x).$$

*When  $\alpha \in (0, \frac{1}{2}]$ , we have, as  $x \rightarrow \infty$ ,*

$$(1.13) \quad \#\{n \leq x : n \in \mathcal{A}_3, P(w(n)) \leq \alpha P(n)\} \gg \alpha^2 A_3(x),$$

*where the implied constant is absolute.*

In view of Corollary 5, it is natural to raise the following problems.

**Problem 7.** *Is the set  $\{P(w(n))/P(n) : n \in \mathcal{A}_3\}$  dense in  $[0, 1]$ ?*

**Problem 8.** *Is the set  $\{w(n)/n : n \in \mathcal{A}_3\}$  dense in  $[0, \infty)$ ?*

Theorem 6 shows that the density

$$D_{\mathcal{A}_3}(\alpha) := \lim_{x \rightarrow \infty} \frac{1}{A_3(x)} \sum_{\substack{n \leq x, n \in \mathcal{A}_3 \\ P(w(n)) \leq \alpha P(n)}} 1$$

exists for  $\frac{1}{2} < \alpha \leq 1$  and is equal to 1. Naturally we want to know the other case.

**Problem 9.** Does  $D_{\mathcal{A}_3}(\alpha)$  exist for  $\alpha \in (0, \frac{1}{2}]$ ? If it exists, what is its value?

The lower bound (1.1) of Theorem 1 is probably far from optimal. On the other hand, Chen & Shi [3, Theorem 4] proved that there exist infinitely many elements of  $\mathcal{B}_3$  which have no  $\mathcal{B}_3$ -parent. It is very natural to ask

**Problem 10.** Does the set  $\{n \in \mathcal{B}_3 : n \text{ has no } \mathcal{B}_3\text{-parent}\}$  have zero density?

In view of [4, Conjecture 2.16], it is interesting to know if there exist elements  $n \in \mathcal{B}_3$  such that  $N_{\mathcal{B}_3}(n) = \infty$ . As Chen & Shi [3, Remark] indicated, this assertion is equivalent to the following conjecture: For any nonzero integer  $a$  there are infinitely many primes  $p$  such that  $P(a+p)$  takes the same value. This conjecture seems very difficult. Here we propose a slightly easier question.

**Problem 11.** Let  $q$  be a large prime number. Are there infinitely many primes  $p$  such that  $P(p+2) = q$ ?

If the answer to this problem is affirmative, then  $N_{\mathcal{B}_3}(2q^2) = \infty$  (since  $w(4p) = 2q^2$ ). In Section 7, we shall give a heuristic argument for Problem 11.

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## 2 Proof of Theorem 1

First we need to prove a preliminary lemma, which is of independent interest.

**Lemma 2.1.** For two coprime positive integers  $a$  and  $d$  with  $1 \leq a \leq d-1$ , denote by  $\pi(x, d, a)$  the number of primes  $p \leq x$  such that  $p \equiv a \pmod{d}$ . Let  $B_1 \geq 0$  and

$P = Q(\log Q)^{-B_1}$ . Then we have

$$(2.1) \quad \sum_{\substack{Q < q \leq 2Q \\ \pi(Q^2, q, q-p) \geq Q/(8 \log Q)}} \sum_{p \leq P} 1 \sim \frac{PQ}{\log P \log Q} \sim \frac{Q^2}{(\log Q)^{B_1+2}},$$

$$(2.2) \quad \sum_{\substack{Q < q \leq 2Q \\ N_{\mathcal{E}_3}(pq^2) \geq Q/(8 \log Q)}} \sum_{p \leq P} 1 \sim \frac{PQ}{\log P \log Q} \sim \frac{Q^2}{(\log Q)^{B_1+2}},$$

as  $Q \rightarrow \infty$ .

*Proof.* According to [7, Theorem 7.12], for  $1 \leq Q \leq x$  and any  $A > 0$  we have

$$(2.3) \quad \sum_{d \leq Q} \sum_{\substack{1 \leq a \leq d \\ (a,d)=1}} |E(x, d, a)|^2 = Qx \log Q + O_A \left( Qx + \frac{x^2}{(\log x)^A} \right),$$

where

$$(2.4) \quad E(x, d, a) := \sum_{\substack{p \leq x \\ p \equiv a \pmod{d}}} \log p - \frac{x}{\varphi(d)}$$

and  $\varphi(q)$  is the Euler function. Here the implied constant depends only on  $A$ .

Clearly (2.3) with the choice of  $x = Q^2$  implies

$$(2.5) \quad \sum_{Q < q \leq 2Q} \sum_{p \leq P} |E(Q^2, q, q-p)|^2 \leq \sum_{Q < q \leq 2Q} \sum_{\substack{1 \leq a \leq q \\ (a,q)=1}} |E(Q^2, q, a)|^2 \\ \ll_A \frac{Q^4}{(\log Q)^A}$$

for all  $Q \geq 3$ .

Introduce

$$S(Q) := \{(q, p) : Q < q \leq 2Q, p \leq P\}, \\ E(Q) := \{(q, p) \in S(Q) : |E(Q^2, q, q-p)| \geq Q^2/(2\varphi(q))\}.$$

From the inequality (2.5), we easily deduce

$$|E(Q)| \left( \frac{Q^2}{2(2Q-1)} \right)^2 \ll_A \frac{Q^4}{(\log Q)^A}.$$

Thus

$$(2.6) \quad |E(Q)| \ll_A \frac{Q^2}{(\log Q)^A}.$$



For all  $(q, p) \in S(Q) \setminus E(Q)$ , we have

$$|E(Q^2, q, q-p)| < \frac{Q^2}{2\varphi(q)}.$$

From this we deduce

$$\sum_{\substack{p' \leq Q^2 \\ p' \equiv q-p \pmod{q}}} \log p' > \frac{Q^2}{2\varphi(q)}$$

for all  $(q, p) \in S(Q) \setminus E(Q)$ . Thus

$$\pi(Q^2, q, q-p) \geq \frac{Q}{8 \log Q}$$

for all  $(q, p) \in S(Q) \setminus E(Q)$ . Taking  $A = B_1 + 3$  in (2.6), we can write

$$\begin{aligned} \sum_{Q < q \leq 2Q} \sum_{p \leq P} 1 &= \sum_{\substack{Q < q \leq 2Q \\ |E(Q^2, q, q-p)| < Q^2/(2\varphi(q))}} \sum_{p \leq P} 1 + O\left(\frac{Q^2}{(\log Q)^{B_1+3}}\right) \\ (2.7) \quad &\leq \sum_{\substack{Q < q \leq 2Q \\ \pi(Q^2, q, q-p) \geq Q/(8 \log Q)}} \sum_{p \leq P} 1 + O\left(\frac{Q^2}{(\log Q)^{B_1+3}}\right) \\ &\leq \sum_{Q < q \leq 2Q} \sum_{p \leq P} 1 + O\left(\frac{Q^2}{(\log Q)^{B_1+3}}\right). \end{aligned}$$

Now the required asymptotic formula (2.1) follows from (2.7) and the prime number theorem.

Now we prove (2.2). For each pair  $(q, p)$  with  $Q < q \leq 2Q, p \leq P$  and  $\pi(Q^2, q, q-p) \geq Q/(8 \log Q)$ , there are at least  $Q/(8 \log Q)$  prime numbers  $p_i \leq Q^2$  such that  $p_i \equiv q-p \pmod{q}$ . Writting  $p_i = \ell_i q + q-p$ , we must have  $1 \leq \ell_i \leq q-1$  for  $i = 1, \dots, \lceil Q/(8 \log Q) \rceil$ . Thus

$$w(p^2 p_i) = pP(p+p_i)^2 = pP((\ell_i+1)q)^2 = pq^2$$

and

$$N_{\mathcal{B}_3}(pq^2) \geq \frac{Q}{8 \log Q}.$$

From these and (2.1), we deduce that

$$\begin{aligned}
\frac{Q^2}{(\log Q)^{B_1+2}} &\sim \sum_{\substack{Q < q \leq 2Q \\ \pi(Q^2, q, q-p) \geq Q/(8 \log Q)}} \sum_{p \leq P} 1 \\
&\leq \sum_{\substack{Q < q \leq 2Q \\ N_{\mathcal{B}_3}(pq^2) \geq Q/(8 \log Q)}} \sum_{p \leq P} 1 \\
&\leq \sum_{Q < q \leq 2Q} \sum_{p \leq P} 1 \\
&\sim \frac{Q^2}{(\log Q)^{B_1+2}}.
\end{aligned}$$

This is equivalent to (2.2).  $\square$

Now we are ready to prove Theorem 1.

The inequality (1.1) is an immediate consequence of (2.2) by writing  $n = pq^2$  and  $B_1 = 3B + 3$ .

Next we prove the second assertion. According to [1, Theorem 1], for any given prime  $q$ , there are two positive constants  $c_0 = c_0(q)$  and  $x_0 = x_0(q)$  such that

$$\sum_{p \leq x, P(p+q) \leq x^{0.2961}} 1 > \frac{x}{(\log x)^{c_0}} \quad (x \geq x_0).$$

Hence there exists a prime  $r \leq x^{0.2961}$  such that

$$\sum_{p \leq x, P(p+q)=r} 1 \geq \frac{x}{\pi(x^{0.2961})(\log x)^{c_0}} > \frac{0.2x^{0.7039}}{(\log x)^{c_0-1}} \quad (x \geq x_0).$$

Let  $p_1 < \dots < p_k \leq x$  with  $P(p_i + q) = r$  ( $1 \leq i \leq k = [0.2x^{0.7039}/(\log x)^{c_0-1}]$ ). Then  $w(p_i q^2) = qr^2$  for  $i = 1, 2, \dots, k$  and  $qr^2 \leq qx^{0.5922} < x^{0.5922} \log x$ . Thus for  $n_q = qr^2$  we have

$$N_{\mathcal{B}_3}(n_q) \geq k > 0.1x^{0.7039}/(\log x)^{c_0-1} > n_q^{1.1886}.$$

Since there are infinitely many prime numbers  $q$ , there are infinitely many  $n \in \mathcal{B}_3$  such that  $N_{\mathcal{E}_3}(n) \geq n^{1.1886}$ .

### 3 Proof of Theorem 2

As before, we first prove a preliminary lemma, which is an analogue of Lemma 2.1.

**Lemma 3.1.** For two coprime positive integers  $a$  and  $d$  with  $1 \leq a \leq d - 1$ , denote by  $\pi(x, d, a)$  the number of primes  $p \leq x$  such that  $p \equiv a \pmod{d}$ . Then, for any  $B > 3$ , we have

$$(3.1) \quad \sum_{\substack{Q < q \leq 2Q \\ \pi(Q(\log Q)^B, q, q-p) \gg (\log Q)^{B-1}}} \sum_{p \leq Q} 1 \sim \frac{Q^2}{(\log Q)^2},$$

$$(3.2) \quad \sum_{\substack{Q < q \leq 2Q \\ \exists r \leq Q(\log Q)^B \text{ such that } N_{\mathcal{E}_3}(q^2 r) \geq 1}} \sum_{p \leq Q} 1 \sim \frac{Q^2}{(\log Q)^2},$$

as  $Q \rightarrow \infty$ .

*Proof.* We shall prove only (3.2), since (3.1) is very similar to (2.1).

By using (3.1), for each  $(q, p)$  counted in the left-hand side of (3.1), there are  $c(\log Q)^{B-1}$  prime numbers  $p_i \leq Q(\log Q)^B$  such that  $p_i \equiv q - p \pmod{q}$ . Writting  $p_i = \ell_i q + q - p$ , we must have  $1 \leq \ell_i \leq (\log Q)^B < Q \leq q - 1$  for  $i = 1, \dots, [c(\log Q)^{B-1}]$ . Thus

$$w(pp_i p_j) = P(p + p_i)P(p + p_j)P(p_i + p_j) = q^2 P(p_i + p_j).$$

Clearly  $p \neq p_i$  and  $P(p_i + p_j) \neq q$  for all  $1 \leq i, j \leq [c(\log Q)^{B-1}]$ . Taking  $r = P(p_i + p_j)$ , we have

$$N_{\mathcal{E}_3}(q^2 r) \geq 1.$$

This and (3.1) imply our required result.  $\square$

Now we are ready to prove Theorem 2.

From (3.2), we easily deduce that

$$\sum_{\substack{Q < q \leq 2Q \\ \exists r \leq Q(\log Q)^B \text{ such that } N_{\mathcal{E}_3}(q^2 r) \geq 1}} 1 \gg \frac{Q}{\log Q}$$

and

$$\sum_{\substack{n \leq x, n \in \mathcal{B}_3 \\ N_{\mathcal{E}_3}(n) \geq 1}} 1 \geq \sum_{\substack{0.5x^{1/3}(\log x)^{-B/3} < q \leq x^{1/3}(\log x)^{-B/3} \\ \exists r \leq x^{1/3}(\log x)^{2B/3} \text{ such that } N_{\mathcal{E}_3}(q^2 r) \geq 1}} 1 \gg \frac{x^{1/3}}{(\log x)^{(B+3)/3}}.$$

This proves the inequality (1.2) by taking  $B = 3\varepsilon + 3$ .

Next we shall prove the estimate (1.3). The method of proof is the same as in [6, Theorem 2]. The new ingredient is that we add a process of summation, which allows us to get the lower bound  $\gg \log \log x$ .

For  $k \geq 1$  and  $2^{k+1} \leq x^{1/2}/(\log x)^7$ , we consider the sum

$$(3.3) \quad \begin{aligned} \sigma_k &:= \sum_{\substack{x < p_1 \leq 2x \\ 2^k x^{1/2} < P(p_1+p_2) = P(p_1+p_3) \leq 2^{k+1} x^{1/2}}} \sum_{x < p_2 \leq 2x} \sum_{x < p_3 \leq 2x} \log p_2 \log p_3 \\ &= \sum_{x < p_1 \leq 2x} \sum_{2^k x^{1/2} < r \leq 2^{k+1} x^{1/2}} \left( \sum_{\substack{x < p_2 \leq 2x \\ P(p_1+p_2) = r}} \log p_2 \right)^2. \end{aligned}$$

Since for all  $x < p_1 \leq 2x$  and  $2^k x^{1/2} < r \leq 2^{k+1} x^{1/2}$ , we have

$$x < p_2 \leq 2x \text{ and } P(p_1 + p_2) = r \Leftrightarrow x < p_2 \leq 2x \text{ and } p_2 \equiv -p_1 \pmod{r},$$

we can write

$$\sum_{\substack{x < p_2 \leq 2x \\ P(p_1+p_2) = r}} \log p_2 = \frac{x}{\varphi(r)} + E^*(x, r, -p_1),$$

where

$$(3.4) \quad E^*(x, r, -p_1) = E(2x, r, -p_1) - E(x, r, -p_1)$$

and  $E(x, d, a)$  is defined as in (2.4). Thus

$$(3.5) \quad \sigma_k = \sigma_k^{(1)} + 2\sigma_k^{(2)} + \sigma_k^{(3)},$$

where

$$\begin{aligned} \sigma_k^{(1)} &:= \sum_{x < p_1 \leq 2x} \sum_{2^k x^{1/2} < r \leq 2^{k+1} x^{1/2}} \left( \frac{x}{\varphi(r)} \right)^2, \\ \sigma_k^{(2)} &:= \sum_{x < p_1 \leq 2x} \sum_{2^k x^{1/2} < r \leq 2^{k+1} x^{1/2}} \frac{x}{\varphi(r)} E^*(x, r, -p_1), \\ \sigma_k^{(3)} &:= \sum_{x < p_1 \leq 2x} \sum_{2^k x^{1/2} < r \leq 2^{k+1} x^{1/2}} |E^*(x, r, -p_1)|^2. \end{aligned}$$

By the prime number theorem, it is easy to see that

$$(3.6) \quad \sigma_k^{(1)} \geq \left( \frac{x}{2^{k+1} x^{1/2}} \right)^2 \sum_{x < p_1 \leq 2x} \sum_{2^k x^{1/2} < r \leq 2^{k+1} x^{1/2}} 1 \gg \frac{2^{-k} x^{5/2}}{(\log x)^2} \left\{ 1 + O\left( \frac{1}{\log x} \right) \right\}$$

uniformly for

$$(3.7) \quad k \geq 1 \quad \text{and} \quad 2^k \leq x^{1/2}/(\log x)^7.$$

By the simple inequality

$$|E^*(x, r, -p_1)|^2 \leq 2(|E(2x, r, -p_1)|^2 + |E(x, r, -p_1)|^2)$$

and formula (2.3) with  $A = 4$ , we have

$$(3.8) \quad \begin{aligned} \sigma_k^{(3)} &= \sum_{2^k x^{1/2} < r \leq 2^{k+1} x^{1/2}} \sum_{1 \leq a < r} \sum_{\substack{x < p_1 \leq 2x \\ p_1 \equiv -a \pmod{r}}} |E^*(x, r, a)|^2 \\ &\ll \sum_{2^k x^{1/2} < r \leq 2^{k+1} x^{1/2}} \sum_{1 \leq a < r} \frac{x}{r} |E^*(x, r, a)|^2 \\ &\ll \frac{2^{-k} x^{5/2}}{(\log x)^4}, \end{aligned}$$

uniformly for  $x$  and  $k$  satisfying (3.7).

By the Cauchy-Schwarz inequality and the following simple bound

$$\sum_{x < p_1 \leq 2x} \sum_{2^k x^{1/2} < r \leq 2^{k+1} x^{1/2}} 1 \ll \frac{2^k x^{3/2}}{(\log x)^2}$$

uniformly for  $x$  and  $k$  satisfying (3.7), it follows that

$$(3.9) \quad \begin{aligned} |\sigma_k^{(2)}|^2 &\ll \left( 2^{-k} x^{1/2} \sum_{x < p_1 \leq 2x} \sum_{2^k x^{1/2} < r \leq 2^{k+1} x^{1/2}} |E^*(x, r, -p_1)| \right)^2 \\ &\ll \frac{2^{-k} x^{5/2}}{(\log x)^2} \sum_{x < p_1 \leq 2x} \sum_{2^k x^{1/2} < r \leq 2^{k+1} x^{1/2}} |E^*(x, r, -p_1)|^2 \\ &= \frac{2^{-k} x^{5/2}}{(\log x)^2} \sigma_k^{(3)} \ll \frac{2^{-2k} x^5}{(\log x)^6} \end{aligned}$$

uniformly for  $x$  and  $k$  satisfying (3.7).

Inserting (3.6)-(3.9) into (3.5), we find that

$$(3.10) \quad \sigma_k \gg \frac{2^{-k} x^{5/2}}{(\log x)^2} \left\{ 1 + O\left(\frac{1}{\log x}\right) \right\}$$

uniformly for  $x$  and  $k$  satisfying (3.7).

Since

$$\sum_{x < p_1 \leq 2x} \sum_{x < p_2 \leq 2x} \sum_{\substack{x < p_3 \leq 2x \\ p_1 p_2 p_3 \notin \mathcal{C}_3}} \log p_2 \log p_3 \ll x^2 \log x,$$

the inequality (3.10) yields immediately

$$\begin{aligned}
(3.11) \quad \sigma_k^* &:= \sum_{\substack{x < p_1 \leq 2x \\ 2^k x^{1/2} < P(p_1+p_2) = P(p_1+p_3) \leq 2^{k+1} x^{1/2} \\ p_1 p_2 p_3 \in \mathcal{C}_3}} \sum_{x < p_2 \leq 2x} \sum_{x < p_3 \leq 2x} \log p_2 \log p_3 \\
&\gg \frac{2^{-k} x^{5/2}}{(\log x)^2} \left\{ 1 + O\left(\frac{1}{\log x}\right) \right\}
\end{aligned}$$

uniformly for  $x$  and  $k$  satisfying (3.7). On the other hand, we can write

$$\sigma_k^* = \sum_{q \leq 4x} \sum_{\substack{2^k x^{1/2} < r \leq 2^{k+1} x^{1/2} \\ qr^2 \in \mathcal{B}_3}} \sigma_k(q, r)$$

with

$$\sigma_k(q, r) := \sum_{\substack{x < p_1 \leq 2x \\ P(p_2+p_3)=q, P(p_1+p_2)=P(p_1+p_3)=r \\ p_1 p_2 p_3 \in \mathcal{C}_3}} \sum_{x < p_2 \leq 2x} \sum_{x < p_3 \leq 2x} \log p_2 \log p_3.$$

Since we have

$$\sum_{q \leq 4x} \sum_{\substack{2^k x^{1/2} < r \leq 2^{k+1} x^{1/2} \\ qr^2 \in \mathcal{B}_3}} 1 \ll \frac{2^k x^{3/2}}{(\log x)^2}$$

uniformly for  $x$  and  $k$  satisfying (3.7), the inequality (3.11) guarantees that there is at least a couple  $(q, r)$  satisfying  $q \leq 4x$  and  $2^k x^{1/2} < r \leq 2^{k+1} x^{1/2}$  such that

$$\sigma_k(q, r) \gg \frac{x}{2^{2k}} \gg \frac{x}{(\log x)^{2\varepsilon}} \quad (2^k \leq (\log x)^\varepsilon).$$

In other words, for each integer  $k$  with  $2^k \leq (\log x)^\varepsilon$ , there is at least a couple  $(q, r)$  with  $q \leq 4x$  and  $2^k x^{1/2} < r \leq 2^{k+1} x^{1/2}$  such that  $N_{\mathcal{C}_3}(qr^2) \gg x/(\log x)^{2+2\varepsilon}$ . Thus

$$\begin{aligned}
\sum_{\substack{n \leq 4x^2 (\log x)^{2\varepsilon}, n \in \mathcal{B}_3 \\ N_{\mathcal{C}_3}(n) \gg x/(\log x)^{2+2\varepsilon}}} 1 &\geq \sum_{\substack{qr^2 \leq 4x^2 (\log x)^{2\varepsilon}, qr^2 \in \mathcal{B}_3 \\ q \leq 4x, 2^k x^{1/2} \leq r \leq x^{1/2} (\log x)^\varepsilon \\ N_{\mathcal{C}_3}(qr^2) \gg x/(\log x)^{2+2\varepsilon}}} 1 \\
&\geq \sum_{k \leq \varepsilon \log \log x} \sum_{\substack{qr^2 \leq 4x^2 (\log x)^{2\varepsilon}, qr^2 \in \mathcal{B}_3 \\ q \leq 4x, 2^k x^{1/2} \leq r \leq 2^{k+1} x^{1/2} \\ N_{\mathcal{C}_3}(qr^2) \gg x/(\log x)^{2+2\varepsilon}}} 1 \\
&\geq \varepsilon \log \log x - 1.
\end{aligned}$$

This implies (1.3) by replacing  $4x^2(\log x)^{2\varepsilon}$  with  $x$ .

## 4 Proof of Theorem 3

Since the proof is similar to (1.3), we shall point out the principal lines only.

For  $2^{j+1} \leq x^{1/2}/(\log x)^7$  and  $2^{k+1} \leq x^{1/2}/(\log x)^7$  with  $j \geq 1$  and  $k \geq 1$ , consider the sum

$$(4.1) \quad \sigma_{j,k} := \sum_{\substack{x < p_1 \leq 2x \\ 2^j x^{1/2} < P(p_1+p_2) \leq 2^{j+1} x^{1/2}}} \sum_{\substack{x < p_2 \leq 2x \\ 2^k x^{1/2} < P(p_1+p_3) \leq 2^{k+1} x^{1/2}}} \sum_{x < p_3 \leq 2x} \log p_2 \log p_3.$$

Similar to (3.5), we have

$$(4.2) \quad \sigma_{j,k} = \sigma_{j,k}^{(1)} + \sigma_{j,k}^{(2)} + \sigma_{j,k}^{(3)} + \sigma_{j,k}^{(4)},$$

where

$$\begin{aligned} \sigma_{j,k}^{(1)} &:= \sum_{x < p_1 \leq 2x} \sum_{2^j x^{1/2} < q \leq 2^{j+1} x^{1/2}} \frac{x}{\varphi(q)} \sum_{2^k x^{1/2} < r \leq 2^{k+1} x^{1/2}} \frac{x}{\varphi(r)}, \\ \sigma_{j,k}^{(2)} &:= \sum_{x < p_1 \leq 2x} \sum_{2^j x^{1/2} < q \leq 2^{j+1} x^{1/2}} \frac{x}{\varphi(q)} \sum_{2^k x^{1/2} < r \leq 2^{k+1} x^{1/2}} E^*(x, r, -p_1), \\ \sigma_{j,k}^{(3)} &:= \sum_{x < p_1 \leq 2x} \sum_{2^j x^{1/2} < q \leq 2^{j+1} x^{1/2}} E^*(x, q, -p_1) \sum_{2^k x^{1/2} < r \leq 2^{k+1} x^{1/2}} \frac{x}{\varphi(r)}, \\ \sigma_{j,k}^{(4)} &:= \sum_{x < p_1 \leq 2x} \sum_{2^j x^{1/2} < q \leq 2^{j+1} x^{1/2}} E^*(x, q, -p_1) \sum_{2^k x^{1/2} < r \leq 2^{k+1} x^{1/2}} E^*(x, r, -p_1), \end{aligned}$$

and  $E^*(x, r, -p_1)$  is defined in (3.4).

By the prime number theorem, it is easy to see that

$$(4.3) \quad \sigma_{j,k}^{(1)} \geq (\log 2)^2 \frac{x^3}{(\log x)^3} \left\{ 1 + O\left(\frac{1}{\log x}\right) \right\}$$

uniformly for

$$(4.4) \quad j, k \geq 1, \quad 2^j \leq x^{1/2}/(\log x)^7 \quad \text{and} \quad 2^k \leq x^{1/2}/(\log x)^7.$$

Similar to  $\sigma_k^{(2)}$ , with the help of (2.3) with  $A = 6$  we easily prove that

$$(4.5) \quad \sigma_{j,k}^{(i)} \ll \frac{x^3}{(\log x)^4} \quad (i = 2, 3)$$

uniformly for  $x, j$  and  $k$  satisfying (4.4).

By applying the Cauchy-Schwarz inequality two times, it follows that

$$\begin{aligned}
|\sigma_{j,k}^{(4)}|^2 &\leq \sum_{x < p_1 \leq 2x} \sum_{2^j x^{1/2} < q \leq 2^{j+1} x^{1/2}} |E^*(x, q, -p_1)|^2 \times \\
&\quad \times \sum_{x < p'_1 \leq 2x} \sum_{2^j x^{1/2} < q' \leq 2^{j+1} x^{1/2}} \left( \sum_{2^k x^{1/2} < r \leq 2^{k+1} x^{1/2}} E^*(x, r, -p'_1) \right)^2 \\
&\leq \sum_{2^j x^{1/2} < q' \leq 2^{j+1} x^{1/2}} \sum_{2^k x^{1/2} < r' \leq 2^{k+1} x^{1/2}} \sigma_j^{(3)} \sigma_k^{(3)}.
\end{aligned}$$

Thus (3.8) implies that

$$(4.6) \quad |\sigma_{j,k}^{(4)}|^2 \ll \frac{2^j x^{1/2}}{\log x} \frac{2^k x^{1/2}}{\log x} \frac{2^{-j} x^{5/2}}{(\log x)^4} \frac{2^{-k} x^{5/2}}{(\log x)^4} \ll \frac{x^6}{(\log x)^{10}}$$

uniformly for  $x, j$  and  $k$  satisfying (4.4).

Inserting (4.3)-(4.6) into (4.2), we find that

$$(4.7) \quad \sigma_{j,k} \geq (\log 2)^2 \frac{x^3}{(\log x)^3} \left\{ 1 + O\left(\frac{1}{\log x}\right) \right\}$$

uniformly for  $x, j$  and  $k$  satisfying (4.4).

In order to remove  $p_1, p_2, p_3$  in  $\sigma_{j,k}$  such that  $w(p_1 p_2 p_3) \notin \mathcal{C}_3$ , we consider the subsums of  $\sigma_{j,k}$ :

$$\begin{aligned}
\sigma'_{j,k} &:= \sum_{\substack{x < p_1 \leq 2x \\ 2^j x^{1/2} < P(p_1+p_2) \leq 2^{j+1} x^{1/2}}} \sum_{\substack{x < p_2 \leq 2x \\ P(p_1+p_2) = P(p_1+p_3)}} \sum_{\substack{x < p_3 \leq 2x \\ 2^k x^{1/2} < P(p_1+p_3) \leq 2^{k+1} x^{1/2}}} \log p_2 \log p_3, \\
\sigma''_{j,k} &:= \sum_{\substack{x < p_1 \leq 2x \\ 2^j x^{1/2} < P(p_1+p_2) \leq 2^{j+1} x^{1/2}}} \sum_{\substack{x < p_2 \leq 2x \\ P(p_1+p_2) = P(p_2+p_3)}} \sum_{\substack{x < p_3 \leq 2x \\ 2^k x^{1/2} < P(p_1+p_3) \leq 2^{k+1} x^{1/2}}} \log p_2 \log p_3, \\
\sigma'''_{j,k} &:= \sum_{\substack{x < p_1 \leq 2x \\ 2^j x^{1/2} < P(p_1+p_2) \leq 2^{j+1} x^{1/2}}} \sum_{\substack{x < p_2 \leq 2x \\ P(p_1+p_3) = P(p_2+p_3)}} \sum_{\substack{x < p_3 \leq 2x \\ 2^k x^{1/2} < P(p_1+p_3) \leq 2^{k+1} x^{1/2}}} \log p_2 \log p_3.
\end{aligned}$$



For  $\sigma'_{j,k}$ , we must have  $j = k$  and

$$\begin{aligned}
\sigma'_{j,k} &= \sum_{2^j x^{1/2} < q \leq 2^{j+1} x^{1/2}} \sum_{x < p_1 \leq 2x} \sum_{\substack{x < p_2 \leq 2x \\ P(p_1+p_2)=q}} \sum_{\substack{x < p_3 \leq 2x \\ P(p_1+p_3)=q}} \log p_2 \log p_3 \\
&\leq 4(\log x)^2 \sum_{2^j x^{1/2} < q \leq 2^{j+1} x^{1/2}} \sum_{x < p_1 \leq 2x} \sum_{\substack{x < p_2 \leq 2x \\ p_2 \equiv -p_1 \pmod{q}}} \sum_{\substack{x < p_3 \leq 2x \\ p_3 \equiv -p_1 \pmod{q}}} 1 \\
&\leq 16(\log x)^2 \sum_{2^j x^{1/2} < q \leq 2^{j+1} x^{1/2}} \sum_{x < p_1 \leq 2x} \frac{x^2}{q^2} \\
&\leq 20x^{5/2}(\log x)^2.
\end{aligned}$$

The same bound holds for  $\sigma''_{j,k}$  and  $\sigma'''_{j,k}$ . These estimates allow us to remove the  $p_1, p_2, p_3$  in  $\sigma_{j,k}$  such that  $w(p_1 p_2 p_3) \notin \mathcal{C}_3$ :

$$\begin{aligned}
(4.8) \quad \sigma_{j,k}^* &:= \sum_{\substack{x < p_1 \leq 2x \\ 2^j x^{1/2} < P(p_1+p_2) \leq 2^{j+1} x^{1/2}}} \sum_{\substack{x < p_2 \leq 2x \\ 2^k x^{1/2} < P(p_1+p_3) \leq 2^{k+1} x^{1/2}}} \sum_{\substack{x < p_3 \leq 2x \\ w(p_1 p_2 p_3) \in \mathcal{C}_3}} \log p_2 \log p_3 \\
&\geq (\log 2)^2 \frac{x^3}{(\log x)^3} \left\{ 1 + O\left(\frac{1}{\log x}\right) \right\}
\end{aligned}$$

uniformly for  $x, j$  and  $k$  satisfying (4.4). On the other hand, we can write

$$\sigma_{j,k}^* = \sum_{q \leq 4x} \sum_{2^j x^{1/2} < r \leq 2^{j+1} x^{1/2}} \sum_{\substack{2^k x^{1/2} < s \leq 2^{k+1} x^{1/2} \\ qrs \in \mathcal{C}_3}} \sigma_{j,k}(q, r, s)$$

with

$$\sigma_{j,k}(q, r, s) := \sum_{\substack{x < p_1 \leq 2x \\ P(p_2+p_3)=q}} \sum_{\substack{x < p_2 \leq 2x \\ P(p_1+p_2)=r}} \sum_{\substack{x < p_3 \leq 2x \\ P(p_1+p_3)=s}} \log p_2 \log p_3.$$

Since we have

$$\sum_{q \leq 4x} \sum_{2^j x^{1/2} < r \leq 2^{j+1} x^{1/2}} \sum_{\substack{2^k x^{1/2} < s \leq 2^{k+1} x^{1/2} \\ qrs \in \mathcal{C}_3}} 1 \ll \frac{2^{j+k} x^2}{(\log x)^3}$$

uniformly for  $x, j$  and  $k$  satisfying (4.4). The inequality (4.8) guarantees that there is at least a triple  $(q, r, s)$  with  $q \leq 4x$ ,  $2^j x^{1/2} < r \leq 2^{j+1} x^{1/2}$  and  $2^k x^{1/2} < s \leq 2^{k+1} x^{1/2}$  such that

$$\sigma_{j,k}(q, r, s) \gg \frac{x}{2^{j+k}} \gg \begin{cases} x/(\log x)^{2\varepsilon} & \text{for } 2^j \leq (\log x)^\varepsilon \text{ and } 2^k \leq (\log x)^\varepsilon, \\ x^{1-2\varepsilon} & \text{for } 2^j \leq x^\varepsilon \text{ and } 2^k \leq x^\varepsilon. \end{cases}$$

In other words, for each couple  $(j, k)$  with  $2^j \leq (\log x)^\varepsilon$  and  $2^k \leq (\log x)^\varepsilon$  (resp.  $2^j \leq x^\varepsilon$  and  $2^k \leq x^\varepsilon$ ), there is at least a triple  $(q, r, s)$  with  $q \leq 4x$ ,  $2^j x^{1/2} < r \leq 2^{j+1} x^{1/2}$  and  $2^k x^{1/2} < s \leq 2^{k+1} x^{1/2}$  such that  $N_{\mathcal{C}_3}(qrs) \gg x/(\log x)^{2+2\varepsilon}$  (resp.  $x^{1-3\varepsilon}$ ). Thus

$$\begin{aligned}
\sum_{\substack{n \leq 4x^2(\log x)^{2\varepsilon}, n \in \mathcal{C}_3 \\ N_{\mathcal{C}_3}(n) \gg x/(\log x)^{2+2\varepsilon}}} 1 &\geq \sum_{\substack{qrs \leq 4x^2(\log x)^{2\varepsilon}, qrs \in \mathcal{C}_3 \\ q \leq 4x, 2x^{1/2} \leq r \leq x^{1/2}(\log x)^\varepsilon, 2x^{1/2} \leq s \leq x^{1/2}(\log x)^\varepsilon \\ N_{\mathcal{C}_3}(qrs) \gg x/(\log x)^{2+2\varepsilon}}} 1 \\
&\geq \sum_{j \leq \varepsilon \log \log x} \sum_{k \leq \varepsilon \log \log x} \sum_{\substack{qrs \leq 4x^2(\log x)^{2\varepsilon}, qrs \in \mathcal{C}_3 \\ q \leq 4x, 2^j x^{1/2} \leq r \leq 2^{j+1} x^{1/2}, 2^k x^{1/2} \leq s \leq 2^{k+1} x^{1/2} \\ N_{\mathcal{C}_3}(qrs) \gg x/(\log x)^{2+2\varepsilon}}} 1 \\
&\geq (\varepsilon \log \log x)^2 - 1.
\end{aligned}$$

This implies (1.4) by replacing  $4x^2(\log x)^{2\varepsilon}$  with  $x$ . The inequality (1.5) can be proved in the same way.

## 5 Proofs of Theorem 4 and Corollary 5

According to [1, Theorem 1], there are two positive constants  $c_1$  and  $x_0$  such that

$$(5.1) \quad \sum_{p \leq x, P(p+2) \leq x^{0.2961}} 1 \gg \frac{x}{(\log x)^{c_1}} \quad (x \geq x_0).$$

Since

$$\begin{aligned}
\sum_{p \leq x, P(p+2) \leq x^{0.2961}} 1 &= \sum_{x^{0.2961/0.2962} < p \leq x, P(p+2) \leq x^{0.2961}} 1 + O(x^{0.2961/0.2962}) \\
&\leq \sum_{p \leq x, P(p+2) \leq p^{0.2962}} 1 + O(x^{0.2961/0.2962}),
\end{aligned}$$

the inequality (5.1) implies

$$\sum_{p \leq x, P(p+2) \leq p^{0.2962}} 1 \gg \frac{x}{(\log x)^{c_1}} \quad (x \geq x_0).$$

For such  $p$ , we have

$$\begin{aligned}
\frac{\log P(w(4p))}{\log P(4p)} &= \frac{\log P(p+2)}{\log p} \leq 0.2962, \\
\frac{\log w(4p)}{\log(4p)} &= \frac{\log(2P(p+2)^2)}{\log(4p)} \leq \frac{0.5924 \log p + \log 2}{\log(4p)}.
\end{aligned}$$

These imply (1.6) and (1.7).

By applying [1, Theorem 2], there are infinitely many prime numbers  $p_i$  with  $P(p_i + 2) \geq p_i^{0.677}$ . Thus

$$\frac{\log w(4p_i)}{\log(4p_i)} = \frac{\log(2P(p_i + 2)^2)}{\log(4p_i)} \geq \frac{1.354 \log p_i + \log 2}{\log p_i + \log 4}.$$

Hence (1.8) follows immediately. This proves Theorem 4.

Clearly Theorem 4 implies Corollary 5 except for the second formula of (1.9). In order to prove it, we first notice that  $P(w(n)) \leq P(n) + 2$  for all  $n \in \mathcal{A}_3$ . Thus

$$(5.2) \quad \limsup_{\substack{n \rightarrow \infty \\ n \in \mathcal{A}_3}} \frac{P(w(n))}{P(n)} \leq 1.$$

On the other hand, according to the Green-Tao Theorem on arithmetic progressions in primes [5], for any integer  $k \geq 2$ , there are two positive integers  $a = a(k)$  and  $d = d(k)$  such that  $a + id$  is prime for  $i = 0, \dots, 2^k$ . Put

$$p := a + 2^k d, \quad q := a + (2^k - 2)d, \quad r := a$$

and

$$\begin{aligned} p' &:= \frac{p + q}{2} = a + (2^k - 1)d, \\ q' &:= \frac{p + r}{2} = a + 2^{k-1}d, \\ r' &:= \frac{q + r}{2} = a + (2^{k-1} - 1)d. \end{aligned}$$

Obviously  $w(pqr) = p'q'r'$  and  $P(w(pqr)) = p'$ . Thus

$$(5.3) \quad \lim_{k \rightarrow \infty} \frac{P(w(pqr))}{P(pqr)} = \lim_{k \rightarrow \infty} \frac{p'}{p} = 1.$$

Now the second relation of (1.9) follows from (5.2) and (5.3).

## 6 Proof of Theorem 6

We need to establish two preliminary lemmas.

**Lemma 6.1.** *With the notation of (1.11) ( $B_3(x)$  and  $C_3(x)$  can be defined similarly), we have, for  $x \rightarrow \infty$ ,*

$$(6.1) \quad A_3(x) \sim \frac{x}{2 \log x} (\log \log x)^2,$$

$$(6.2) \quad B_3(x) \sim B_3 \frac{x}{\log x} \quad \text{with} \quad B_3 := \sum_p \frac{1}{p^2},$$

$$(6.3) \quad C_3(x) \sim \frac{x}{2 \log x} (\log \log x)^2.$$

*Proof.* By the prime number theorem, we can write

$$B_3(x) = \sum_{p^2 q \leq x, p \neq q} 1 \sim \sum_{p \leq (x/2)^{1/2}} \frac{x}{p^2 \log(x/p^2)}.$$

On the other hand, we have

$$\begin{aligned} \sum_{\log x < p \leq (x/2)^{1/2}} \frac{x}{p^2 \log(x/p^2)} &\ll \sum_{p > \log x} \frac{x}{p^2} \\ &\ll \frac{x}{(\log x) \log \log x}, \\ \sum_{p \leq \log x} \frac{x}{p^2 \log(x/p^2)} &\sim \frac{x}{\log x} \sum_{p \leq \log x} \frac{1}{p^2} \\ &\sim B_3 \frac{x}{\log x}. \end{aligned}$$

Inserting these into the preceding formula, we get (6.2).

According to a classic result of Landau (see [8, Chapter II.6]), we have

$$\sum_{\substack{n \leq x \\ \Omega(n)=3}} 1 \sim \frac{x}{2 \log x} (\log \log x)^2 \quad (x \rightarrow \infty).$$

From this and (6.2), we immediately deduce (6.3), since

$$C_3(x) = \sum_{\substack{n \leq x \\ \Omega(n)=3}} 1 - \sum_{\substack{n \leq x \\ \Omega(n)=3, \omega(n)=2}} 1 - \sum_{\substack{n \leq x \\ \Omega(n)=3, \omega(n)=1}} 1.$$

Finally (6.1) follows from (6.2) and (6.3), since  $A_3(x) = B_3(x) + C_3(x)$ .  $\square$

**Lemma 6.2.** For  $k \in \mathbb{N}$ , define

$$\mathcal{C}_3(k) := \{pqr : r < q < p, q \equiv r \pmod{2^k}, p \equiv -r \pmod{2^k}\}.$$

For any  $\varepsilon \in (0, \frac{1}{2^{k+1}})$ , we have, as  $x \rightarrow \infty$ ,

$$(6.4) \quad \sum_{\substack{n \leq x, n \in \mathcal{C}_3(k) \\ P(w(n)) \leq (1+\varepsilon)2^{-k}P(n)}} 1 \sim \sum_{n \leq x, n \in \mathcal{C}_3(k)} 1.$$

Further we have

$$(6.5) \quad \sum_{n \leq x, n \in \mathcal{C}_3(1)} 1 \sim \frac{x}{2 \log x} (\log \log x)^2,$$

$$(6.6) \quad \sum_{n \leq x, n \in \mathcal{C}_3(k)} 1 \gg \frac{A_3(x)}{2^{2k}},$$

where the implied constant in the  $\gg$ -symbol is absolute.

*Proof.* For any  $\varepsilon \in (0, \frac{1}{2^{k+1}})$ , we have

$$\begin{aligned} \sum_{r \leq x^{1/3}} \sum_{q \leq (x/r)^{1/2}} \sum_{p \leq q/\varepsilon} 1 &\ll_{\varepsilon} \sum_{r \leq x^{1/3}} \sum_{q \leq (x/r)^{1/2}} \frac{q}{\log q} \\ &\ll_{\varepsilon} \sum_{r \leq x^{1/3}} \frac{x}{r \log^2(x/r)} \\ &\ll_{\varepsilon} \frac{x \log \log x}{(\log x)^2}. \end{aligned}$$

Thus we can write

$$\begin{aligned} \sum_{n \leq x, n \in \mathcal{C}_3(k)} 1 &= \sum_{r \leq x^{1/3}} \sum_{\substack{r < q \leq (x/r)^{1/2} \\ q \equiv r \pmod{2^k}}} \sum_{\substack{q < p \leq x/(qr) \\ p \equiv -r \pmod{2^k}}} 1 \\ &= \sum_{r \leq x^{1/3}} \sum_{\substack{r < q \leq (x/r)^{1/2} \\ q \equiv r \pmod{2^k}}} \sum_{\substack{q/\varepsilon < p \leq x/(qr) \\ p \equiv -r \pmod{2^k}}} 1 + O_{\varepsilon} \left( \frac{x \log \log x}{(\log x)^2} \right). \end{aligned}$$

For each  $(p, q, r)$  counted in the last triple sums, we have obviously that

$$\begin{aligned} P(p+q) &\leq \frac{p+q}{2^k} \leq \frac{1+\varepsilon}{2^k} p, \\ P(p+r) &\leq \frac{p+r}{2^k} \leq \frac{1+\varepsilon}{2^k} p, \\ P(q+r) &\leq 2q \leq 2\varepsilon p \leq \frac{1+\varepsilon}{2^k} p. \end{aligned}$$

Thus

$$P(w(pqr)) = \max\{P(p+q), P(p+r), P(q+r)\} \leq (1+\varepsilon)2^{-k}p$$

and

$$\begin{aligned} \sum_{n \leq x, n \in \mathcal{C}_3(k)} 1 &\leq \sum_{r \leq x^{1/3}} \sum_{\substack{r < q \leq (x/r)^{1/2} \\ P(w(pqr)) \leq (1+\varepsilon)2^{-k}P(pqr)}} \sum_{q/\varepsilon < p \leq x/(qr)} 1 + O_\varepsilon\left(\frac{x \log \log x}{(\log x)^2}\right) \\ &\leq \sum_{r \leq x^{1/3}} \sum_{\substack{r < q \leq (x/r)^{1/2} \\ P(w(pqr)) \leq (1+\varepsilon)2^{-k}P(pqr)}} \sum_{q < p \leq x/(qr)} 1 + O_\varepsilon\left(\frac{x \log \log x}{(\log x)^2}\right) \\ &= \sum_{\substack{n \leq x, n \in \mathcal{C}_3(k) \\ P(w(n)) \leq (1+\varepsilon)2^{-k}P(n)}} 1 + O_\varepsilon\left(\frac{x \log \log x}{(\log x)^2}\right). \end{aligned}$$

This implies (6.4).

The asymptotic formula (6.5) is an immediate consequence of (6.3), since

$$\begin{aligned} \sum_{n \leq x, n \in \mathcal{C}_3} 1 &= \sum_{r \leq x^{1/3}} \sum_{r < q \leq (x/r)^{1/2}} \sum_{q < p \leq x/(qr)} 1 \\ &= \sum_{3 \leq r \leq x^{1/3}} \sum_{r < q \leq (x/r)^{1/2}} \sum_{q < p \leq x/(qr)} 1 + O\left(\frac{x \log \log x}{\log x}\right) \\ &= \sum_{n \leq x, n \in \mathcal{C}_3(1)} 1 + O\left(\frac{x \log \log x}{\log x}\right). \end{aligned}$$

In order to show the lower bound of (6.6), we begin by the following trivial inequality

$$\sum_{\substack{n \leq x \\ n \in \mathcal{C}_3(k)}} 1 \geq \sum_{r \leq x^{1/6}} \sum_{\substack{r < q \leq x^{1/3} \\ q \equiv r \pmod{2^k}}} \sum_{\substack{x^{1/3} < p \leq x/(qr) \\ p \equiv -r \pmod{2^k}}} 1.$$

Then we apply the prime number (in arithmetic progressions) theorem to write

$$\begin{aligned}
\sum_{\substack{n \leq x \\ n \in \mathcal{C}_3(k)}} 1 &\gg \frac{x}{2^k \log x} \sum_{r \leq x^{1/6}} \sum_{r < q \leq x^{1/3}, q \equiv r \pmod{2^k}} \frac{1}{qr} \\
&\gg \frac{x}{2^{2k} \log x} \sum_{r \leq x^{1/6}} \frac{1}{r} \{ \log \log x^{1/3} - \log \log r + O(1) \} \\
&\gg \frac{x}{2^{2k} \log x} \left\{ (\log \log x^{1/3})(\log \log x^{1/6}) - \frac{1}{2}(\log \log x^{1/6})^2 + O(\log \log x) \right\} \\
&\gg \frac{1}{2^{2k}} \frac{x}{\log x} (\log \log x)^2 \\
&\gg \frac{1}{2^{2k}} A_3(x).
\end{aligned}$$

This completes the proof.  $\square$

Now we are ready to prove Theorem 6.

For any  $\alpha \in (0, 1]$ , we can choose  $k$  and  $\varepsilon$  such that  $(1 + \varepsilon)2^{-k} < \alpha \leq 2^{-k+1}$ . In particular for  $\alpha \in (\frac{1}{2}, 1]$ , we can take  $k = 1$ . Thus

$$\begin{aligned}
\sum_{\substack{n \leq x, n \in \mathcal{A}_3 \\ P(w(n)) \leq \alpha P(n)}} 1 &\geq \sum_{\substack{n \leq x, n \in \mathcal{C}_3(k) \\ P(w(n)) \leq \alpha P(n)}} 1 \\
&\geq \sum_{\substack{n \leq x, n \in \mathcal{C}_3(k) \\ P(w(n)) \leq (1+\varepsilon)2^{-k} P(n)}} 1.
\end{aligned}$$

The asymptotic formula (1.12) follows from (6.4) with  $k = 1$ , (6.5), (6.1) and the trivial inclusion relation  $\mathcal{C}_3 \subset \mathcal{A}_3$ . The inequality (1.13) can be obtained in the same way (i.e. by replacing (6.5) with (6.6)).

## 7 A heuristic argument on Problem 11

The aim of this section is to present an heuristic proof of

$$(7.1) \quad \sum_{p \leq x, P(p+2)=q} 1 \rightarrow \infty \quad (x \rightarrow \infty),$$

where  $q$  is a large fixed prime number.

Since  $P(n+2) = q$  is equivalent to  $n+2 = q\ell$  with  $P(\ell) \leq q$ , we have

$$\sum_{n \leq x, P(n+2)=q} 1 = \sum_{\ell \leq (x+2)/q, P(\ell) \leq q} 1 = \Psi((x+2)/q, q),$$

where

$$\Psi(x, y) := \sum_{n \leq x, P(n) \leq y} 1.$$

According to [8, Theorem III.5.2], we have

$$\log \Psi((x+2)/q, q) = Z \left\{ 1 + O\left( \frac{1}{\log q} + \frac{1}{\log \log((x+2)/q)} \right) \right\}$$

uniformly for  $x+2 \geq q^2 \geq 10000$ , where

$$Z := \frac{\log((x+2)/q)}{\log q} \log \left( 1 + \frac{q}{\log((x+2)/q)} \right) + \frac{q}{\log q} \log \left( 1 + \frac{\log((x+2)/q)}{q} \right).$$

From this we deduce

$$Z \geq \frac{q}{2 \log q} \log \log x$$

and

$$(7.2) \quad \Psi((x+2)/q, q) \geq (\log x)^{q/(4 \log q)}$$

for  $x \geq x_0(q)$ , provided the constants  $q$  and  $x_0(q)$  are suitably large.

According to Cramér's model [9, Section 3.2] it seems reasonable to assume that

$$(7.3) \quad \sum_{p \leq x, P(p+2)=q} 1 \approx \sum_{n \leq x, P(n+2)=q} \frac{1}{\log n} \quad (x \rightarrow \infty).$$

Thus (7.2) implies (under the hypothesis (7.3))

$$\sum_{p \leq x, P(p+2)=q} 1 \gg (\log x)^{q/(4 \log q) - 1} \rightarrow \infty,$$

as  $x \rightarrow \infty$ .

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