



**HAL**  
open science

## On a divisor problem related to the Epstein Zeta-Function, II

Guangshi Lü, Jie Wu, Wenguang Zhai

► **To cite this version:**

Guangshi Lü, Jie Wu, Wenguang Zhai. On a divisor problem related to the Epstein Zeta-Function, II. *Journal of Number Theory*, 2011, 132, pp.1734-1742. hal-01277992

**HAL Id: hal-01277992**

**<https://hal.science/hal-01277992>**

Submitted on 24 Feb 2016

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# ON A DIVISOR PROBLEM RELATED TO THE EPSTEIN ZETA-FUNCTION, II

GUANGSHI LÜ, JIE WU & WENGUANG ZHAI

ABSTRACT. Recently by the theory of modular forms and the Riemann zeta-function, Lü improved the estimates for the error term in a divisor problem related to the Epstein zeta-function established by Sankaranarayanan. In this short note, we are able to further sharpen some results of Sankaranarayanan and of Lü, and to establish corresponding  $\Omega$ -estimates.

## 1. INTRODUCTION

For a positive definite quadratic form  $Q(\mathbf{y}) = Q(y_1, \dots, y_\ell)$  in  $\ell \geq 2$  variables with integral coefficients, we can write it in Siegel's notation as

$$Q(\mathbf{y}) = \frac{1}{2} \mathbf{A}[\mathbf{y}] = \frac{1}{2} \mathbf{y}^t \mathbf{A} \mathbf{y} = \sum_{i < j} a_{ij} y_i y_j + \frac{1}{2} \sum_i a_{ii} y_i^2,$$

where  $\mathbf{y}^t$  is the transpose of  $\mathbf{y}$ , and the matrix  $\mathbf{A} = (a_{ij})$  has integral entries which are even on the diagonal, i.e.,  $a_{ii} \equiv 0 \pmod{2}$  for  $0 \leq i \leq \ell$ . Then the corresponding Epstein zeta-function is initially defined by the Dirichlet series

$$(1.1) \quad Z_Q(s) := \sum_{\substack{y_1 \in \mathbb{Z} \\ \dots \\ y_\ell \in \mathbb{Z} \\ (y_1, \dots, y_\ell) \neq (0, \dots, 0)}} \dots \sum_{y_\ell \in \mathbb{Z}} Q(y_1, \dots, y_\ell)^{-s}$$

for  $\Re s > \ell/2$ . We can also rewrite it as, in the same region,

$$Z_Q(s) = \sum_{n \geq 1} a_n n^{-s},$$

where  $a_n$  is the number of the solutions of the equation  $Q(\mathbf{y}) = n$  with  $\mathbf{y} \in \mathbb{Z}^\ell$ . It is known that  $Z_Q(s)$  has an analytic continuation to the whole complex plane  $\mathbb{C}$  with only a simple pole at  $s = \ell/2$ , and satisfies a functional equation of Riemann type

$$(d^{1/\ell}/2\pi)^s \Gamma(s) Z_Q(s) = (d^{1-1/\ell}/2\pi)^{\ell/2-s} \Gamma(\ell/2 - s) Z_{\bar{Q}}(\ell/2 - s) \quad (s \in \mathbb{C}),$$

where  $d$  is the discriminant of  $Q$  and  $\bar{Q}(\mathbf{y}) := \frac{1}{2} \mathbf{y}^t (d\mathbf{A}^{-1}) \mathbf{y}$  (cf. [9]).

---

*Date:* July 30, 2012.

*2000 Mathematics Subject Classification.* 11F30, 11F11, 11F66.

*Key words and phrases.* Epstein zeta-function, divisor problem, modular form.

The work of Guangshi Lü is supported in part by key project of the National Natural Science Foundation of China (Grant No. 11031004), Shandong Province Natural Science Foundation (Grant No. ZR2009AM007), and IIFSDU. Wenguang Zhai is supported by the National Natural Science Foundation of China (Grant No. 10771127).

If we write for any integer  $k \geq 1$ ,

$$Z_Q(s)^k = \sum_{n \geq 1} a_k(n) n^{-s},$$

then

$$a_k(n) = \sum_{n_1 \cdots n_k = n} a_{n_1} \cdots a_{n_k}.$$

In particular  $a_1(n) = a_n$ . It seems interesting to study the asymptotic behavior of the sum  $\sum_{n \leq x} a_k(n)$ . It is easy to show that its main term is

$$\operatorname{Res}_{s=\ell/2} (Z_Q(s)^k x^s s^{-1}) = x^{\ell/2} P_k(\log x),$$

where  $P_k(t)$  is a polynomial in  $t$  of degree  $k-1$ . Then the real hard work is to study the error term

$$(1.2) \quad \Delta_k^*(Q, x) := \sum_{n \leq x} a_k(n) - x^{\ell/2} P_k(\log x).$$

In 1912, Landau [7] proved that for  $\ell = 2$ ,  $\Delta_1^*(Q, x) \ll x^{1/3+\varepsilon}$ , where and throughout this paper  $\varepsilon$  denotes an arbitrarily small positive constant. Landau's method can also be applied to treat the general case. In fact his method implies that for  $k \geq 1$  and  $\ell \geq 2$ ,

$$\Delta_k^*(Q, x) \ll x^{\ell/2 - \ell/(k\ell+1) + \varepsilon}.$$

Later Chandrasekharan and Narasimhan [1] were able to delete the  $\varepsilon$  in the exponent of  $x$ . In [9], Sankaranarayanan improved these classical results by showing that for  $k \geq 2$  and  $\ell \geq 3$ ,

$$(1.3) \quad \Delta_k^*(Q, x) \ll x^{\ell/2 - 1/k + \varepsilon}.$$

Recently inspired by Iwaniec's book [5], Lü [8] was able to improve (1.3) for the quadratic forms of level one (see [5, Chapter 11]). These quadratic forms are defined by  $Q(\mathbf{y}) = \frac{1}{2} \mathbf{A}[\mathbf{y}]$  with  $\operatorname{diag}(\mathbf{A}) = \operatorname{diag}(\mathbf{A}^{-1}) \equiv 0 \pmod{2}$ , where  $\operatorname{diag}(\mathbf{A})$  denotes the set of entries on the diagonal of the matrix  $\mathbf{A}$ . Moreover we have that  $\det(\mathbf{A}) = 1$ ,  $\mathbf{A}$  is equivalent to  $\mathbf{A}^{-1}$ , and the number of variables satisfies  $\ell \equiv 0 \pmod{8}$ . Denote by  $\mathcal{Q}_\ell$  the set of quadratic forms of level one with  $\ell$  variables. For  $Q \in \mathcal{Q}_\ell$ , we have (see [5, (11.32)] or [8, Lemma 2.1])

$$a_n = A_\ell \sigma_{\ell/2-1}(n) + a_f(n, Q) \quad (n \geq 1),$$

where

$$A_\ell := \frac{(2\pi)^{\ell/2}}{\zeta(\ell/2)\Gamma(\ell/2)}, \quad \sigma_k(n) = \sum_{d|n} d^k,$$

$\zeta(s)$  is the Riemann zeta-function,  $\Gamma(s)$  is the Gamma function and  $a_f(n, Q)$  is the  $n$ th Fourier coefficient of a cusp form  $f(z, Q)$  of weight  $\ell/2$  with respect to the full modular group  $\operatorname{SL}(2, \mathbb{Z})$ . Thus

$$(1.4) \quad Z_Q(s) = A_\ell \zeta(s - \ell/2 + 1) \zeta(s) + L(s, f) \quad (\Re s > \ell/2),$$

where  $L(s, f)$  is the Hecke  $L$ -function associated with  $f(z, Q)$ . According to Deligne's well known work [2], we have

$$(1.5) \quad |a_f(n, Q)| \leq n^{(\ell/2-1)/2} \tau(n),$$

where  $\tau(n)$  is the divisor function. With the help of these properties, Lü proved, by complex integration method, a better estimate than Sankaranarayanan's (1.3) for all  $k \geq 3$  and  $8 \mid \ell$ . For  $r \geq 0$ , the  $r$ -dimensional divisor function  $\tau_r(n)$  is defined by

$$\zeta(s)^r = \sum_{n \geq 1} \tau_r(n) n^{-s} \quad (\Re s > 1).$$

The  $r$ -dimensional divisor problem is concerned with the estimate of the error term

$$(1.6) \quad \Delta_r(x) := \sum_{n \leq x} \tau_r(n) - \operatorname{Res}_{s=1} (\zeta(s)^r x^s s^{-1}) = \sum_{n \leq x} \tau_r(n) - x G_r(\log x),$$

where  $G_r(t)$  is a polynomial of degree  $r - 1$  if  $r \geq 1$  and  $G_0(t) \equiv 0$ . It is known that

$$(1.7) \quad \Delta_r(x) \ll x^{\theta_r + \varepsilon} \quad (x \geq 2)$$

where

$$(1.8) \quad \theta_0 = 0, \quad \theta_1 = 0, \quad \theta_2 = 131/416, \quad \theta_3 = 43/96$$

and

$$(1.9) \quad \theta_r = \begin{cases} (3r-4)/(4r) & \text{if } 4 \leq r \leq 8, \\ 35/54 & \text{if } r = 9, \\ 41/60 & \text{if } r = 10, \\ 7/10 & \text{if } r = 11, \\ (r-2)/(r+2) & \text{if } 12 \leq r \leq 25, \\ (r-1)/(r+4) & \text{if } 26 \leq r \leq 50, \\ (31r-98)/(32r) & \text{if } 51 \leq r \leq 57, \\ (7r-34)/(7r) & \text{if } r \geq 58. \end{cases}$$

(The case of  $r = 0, 1$  is trivial. See [3] for  $r = 2$ , [6] for  $r = 3$  and [4, Theorem 12.3] for  $r \geq 4$ .) Lü's result (see [8, Theorem 1.2]) can be stated as follows

$$(1.10) \quad \Delta_k^*(Q, x) \ll \begin{cases} x^{\ell/2-1/2+\varepsilon} & \text{if } k = 3, \\ x^{\ell/2-1+\theta_k+\varepsilon} & \text{if } k \geq 4. \end{cases}$$

In this short note, we can further improve Sankaranarayanan's (1.3) with  $k = 2$  and Lü's (1.10) with  $k = 3$ .

**Theorem 1.** *Let  $k \geq 2$  and  $8 \mid \ell$ . Then for any quadratic form  $Q(\mathbf{y}) \in \mathcal{Q}_\ell$ , we have*

$$\Delta_k^*(Q, x) \ll x^{\ell/2-1+\theta_k+\varepsilon},$$

where  $\theta_k$  is the exponent in (1.7).

For comparison, we note

$$\ell/2 - 1 + \theta_k = \begin{cases} \ell/2 - 1/2 - 0.185\dots & \text{if } k = 2, \\ \ell/2 - 1/2 - 0.052\dots & \text{if } k = 3, \end{cases}$$

which are better than (1.3) with  $k = 2$  and (1.10), respectively.

For  $k = 2$  or  $3$ , we also can establish  $\Omega$ -type result.

**Theorem 2.** *Let  $2 \leq k \leq 8$  and  $8 \mid \ell$ . If there is a positive constant  $\delta$  such that*

$$(1.11) \quad \theta_r \leq (k-1)/(2k) - \delta \quad (0 \leq r \leq k-1),$$

then for any quadratic form  $Q(\mathbf{y}) \in \mathcal{Q}_\ell$  and  $\varepsilon > 0$ , we have

$$(1.12) \quad \Delta_k^*(Q, x) = \Omega \left( x^{\ell/2-1+(k-1)/(2k)} (\log x)^{(k-1)/(2k)} (\log_2 x)^{\beta_k} (\log_3 x)^{-\gamma_k-\varepsilon} \right)$$

where  $\beta_k := (k^{(2k)/(k+1)} - 1)(k+1)/(2k)$  and  $\gamma_k := (3k-1)/(4k)$ .

In particular (1.12) holds unconditionally for  $k = 2$  or  $3$ .

Our method is different from [8]. First we shall establish relations between  $\Delta_k(x)$  and  $\Delta_k^*(Q, x)$  and then deduce Theorems 1 and 2 from known  $O$ -type and  $\Omega$ -type estimates for  $\Delta_k(x)$ .

## 2. PRELIMINARY LEMMAS

This section is devoted to establish three preliminary lemmas, which will be needed in the proof of Theorems 1 and 2.

**Lemma 2.1.** *For any  $\varepsilon > 0$ , we have*

$$\int_1^x \Delta_r(t) dt \ll_{r,\varepsilon} x^{1+\delta_r+\varepsilon} \quad (x \geq 1),$$

where

$$(2.1) \quad \delta_r := \begin{cases} 1/2 - 1/r & \text{if } r = 2, 4, 6, 8, \\ 1/2 - 1/(r+1) & \text{if } r = 1, 3, 5, 7. \end{cases}$$

*Proof.* By Perron's formula [11, Theorem II.2.3], we obtain, with  $b := 1 + 1/\log x$ ,

$$(2.2) \quad \int_0^x \Delta_r(u) du = \frac{1}{2\pi i} \int_{b-i\infty}^{b+i\infty} F_r(s; x) ds - \int_0^x u G_r(\log u) du.$$

where  $b := 1 + 1/\log x$  and  $F_r(s; x) := \zeta(s)^r x^{s+1} / \{s(s+1)\}$ .

Let  $\max\{1 - 6/r, 0\} < a < 1$ . By using the classical estimate

$$\zeta(s) \ll (|t| + 2)^{\max\{(1-\sigma)/3, 0\}} \log(|t| + 2),$$

we deduce that for all  $\varepsilon > 0$  and  $T > 0$ ,

$$\int_{a \leq \sigma \leq b, |\tau|=T} |F_r(s; x)| |ds| \ll (x^2 T^{-2} + x^{1+a} T^{-2+\max\{(1-a)r/3, 0\}}) (\log T)^r,$$

and

$$\int_{\sigma=b, |t| \geq T} |F_r(s; x)| |ds| \ll x^2 T^{-1} (\log T)^r.$$

Using the preceding estimates and shifting the line of integration from  $\sigma = b$  to  $\sigma = a$ , the residue theorem implies that

$$\begin{aligned} \frac{1}{2\pi i} \int_{b-i\infty}^{b+i\infty} F_r(s; x) ds &= \frac{1}{2\pi i} \int_{b-iT}^{b+iT} F_r(s; x) ds + O\left(\frac{x^2}{T} (\log T)^r\right) \\ &= \int_0^x u G_r(u) du + \frac{1}{2\pi i} \int_{a-iT}^{a+iT} F_r(s; x) ds \\ &\quad + O\left(\frac{x^2}{T} (\log T)^r + \frac{x^{1+a}}{T^{2-\max\{(1-a)r/3, 0\}}} (\log T)^r\right), \end{aligned}$$

where we have used the relation

$$\operatorname{Res}_{s=1}(F_r(s; x)) = \int_0^x u G_r(u) du.$$

Making  $T \rightarrow \infty$  and inserting the formula obtained into (2.2), we find that

$$\begin{aligned} \int_1^x \Delta_r(u) du &= \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} F_r(s; x) ds \\ &\ll x^{1+a} \int_{-\infty}^{+\infty} \frac{|\zeta(a+it)|^r}{(|t|+1)^2} dt. \end{aligned}$$

When  $r = 2, 4, 6, 8$ , the last integral is convergent for any  $a > 1/2 - 1/r \geq \max\{1 - 6/r, 0\}$  (see [4, Lemma 13.1 and Theorem 13.4]). For  $r = 2k - 1$  ( $1 \leq k \leq 4$ ), we have

$$\int_{-\infty}^{+\infty} \frac{|\zeta(a+it)|^r}{(|t|+1)^2} dt \leq \left\{ \int_{-\infty}^{+\infty} \frac{|\zeta(a+it)|^{2(k-1)}}{(|t|+1)^2} dt \right\}^{1/2} \left\{ \int_{-\infty}^{+\infty} \frac{|\zeta(a+it)|^{2k}}{(|t|+1)^2} dt \right\}^{1/2} < \infty$$

provided  $a > 1/2 - 1/(2k) = 1/2 - 1/(r+1) \geq \max\{1 - 6/r, 0\}$ . This completes the proof.  $\square$

**Lemma 2.2.** *For  $r \geq 0$ , we have*

$$(2.3) \quad \sum_{n \leq x} \tau_r(n) n^{\ell/2-1} = x^{\ell/2} G_r^*(\log x) + x^{\ell/2-1} \Delta_r(x) + O(x^{\ell/2-1+\delta_r}),$$

where  $G_r^*(t)$  is a polynomial of degree  $r - 1$  with the convention that  $G_0^*(t) \equiv 0$  and the constant  $\delta_r \geq 0$  is given by (2.1). In particular

$$(2.4) \quad \sum_{n \leq x} \tau_r(n) n^{\ell/2-1} = x^{\ell/2} G_r^*(\log x) + O(x^{\ell/2-1+\theta_r}).$$

*Proof.* With the help of (1.6) and Lemma 2.1, a simple partial summation yields

$$\begin{aligned} \sum_{n \leq x} \tau_r(n) n^{\ell/2-1} &= \int_1^x t^{\ell/2-1} (t G_r(\log t))' dt + \int_{1-}^x t^{\ell/2-1} d\Delta_r(t) \\ &= x^{\ell/2} G_r^*(\log x) + x^{\ell/2-1} \Delta_r(x) + O(x^{\ell/2-1+\delta_r}). \end{aligned}$$

This completes the proof.  $\square$

In order to state our third lemma, it is necessary to introduce some notation. By (1.4), we can write, for  $\Re s > \ell/2$ ,

$$\begin{aligned} Z_Q(s)^k &= \sum_{0 \leq r \leq k} A_\ell^r C_k^r \zeta(s)^r L(s, f)^{k-r} \zeta(s - \ell/2 + 1)^r, \\ \zeta(s - \ell/2 + 1)^k &= \sum_{0 \leq r \leq k} A_\ell^{-k} C_k^r (-1)^{k-r} \zeta(s)^{-k} L(s, f)^{k-r} Z_Q(s)^r. \end{aligned}$$

These imply that

$$(2.5) \quad a_k(n) = \sum_{0 \leq r \leq k} A_\ell^r C_k^r \sum_{dm=n} b_{k,r}(d) \tau_r(m) m^{\ell/2-1},$$

$$(2.6) \quad \tau_k(n) n^{\ell/2-1} = \sum_{0 \leq r \leq k} (-1)^{k-r} A_\ell^{-k} C_k^r \sum_{dm=n} c_{k,r}(d) a_r(m),$$

where  $b_{k,r}$  and  $c_{k,r}$  are defined by the relation

$$\zeta(s)^r L(s, f)^{k-r} = \sum_{n \geq 1} b_{k,r}(n) n^{-s}, \quad \zeta(s)^{-k} L(s, f)^{k-r} = \sum_{n \geq 1} c_{k,r}(n) n^{-s},$$

for  $\Re s > \ell/2$ .

**Lemma 2.3.** *Let  $j \geq 0$ ,  $k \geq 2$ ,  $0 \leq r \leq k$ ,  $8 \mid \ell$  and  $\theta > (\ell + 2)/4$ . Then for any quadratic form  $Q(\mathbf{y}) \in \mathcal{Q}_\ell$  and  $d_{k,r} = b_{k,r}$  or  $c_{k,r}$ , we have*

$$(2.7) \quad \sum_{n \leq x} \frac{|d_{k,r}(n)|}{n^\theta} \ll_{j,\ell,\theta} 1 \quad (x \geq 2),$$

$$(2.8) \quad \sum_{n \leq x} \frac{d_{k,r}(n) (\log n)^j}{n^\theta} = C_f(j, k, r, \theta) + O(x^{-\theta + (\ell+2)/4 + \varepsilon}) \quad (x \geq 2),$$

where  $C_f(j, k, r, \theta)$  is a constant.

*Proof.* By the definition of  $b_{k,r}$  and  $c_{k,r}$ , we have

$$\begin{aligned} b_{k,r}(n) &= \sum_{d_1 \cdots d_r m_1 \cdots m_{k-r} = n} a_f(Q, m_1) \cdots a_f(Q, m_{k-r}), \\ c_{k,r}(n) &= \sum_{d_1 \cdots d_k m_1 \cdots m_{k-r} = n} \mu(d_1) \cdots \mu(d_k) a_f(Q, m_1) \cdots a_f(Q, m_{k-r}). \end{aligned}$$

We treat only the case of  $b_{k,r}$  and the latter is completely similar. With the help of the Deligne inequality (1.5), we have

$$\begin{aligned} \sum_{n \leq x} |b_{k,r}(n)| &\leq \sum_{d \leq x} \tau_r(d) \sum_{m \leq x/d} \tau_{2(k-r)}(m) m^{(\ell-2)/4} \\ &\leq \sum_{d \leq x} \tau_r(d) (x/d)^{\ell/4+1/2} (\log x)^{2k-2r-1} \\ &\ll_{j,\ell} x^{(\ell+2)/4} (\log x)^{2k-2r-1} \quad (0 \leq r \leq k). \end{aligned}$$

From this, a simple partial integration allows us to deduce (2.7) and (2.8).  $\square$

## 3. PROOF OF THEOREM 1

By (2.5), (2.4) of Lemma 2.2 and (2.7) of Lemma 2.3, it follows that

$$\sum_{n \leq x} a_k(n) = x^{\ell/2} \sum_{0 \leq r \leq k} C_k^r A_\ell^r \sum_{d \leq x} \frac{b_{k,r}(d)}{d^{\ell/2}} G_r^*(\log(x/d)) + O(x^{\ell/2-1+\theta_k+\varepsilon}).$$

Since  $\ell/2 > (\ell + 2)/4$ , (2.8) of Lemma 2.3 implies that

$$\sum_{0 \leq r \leq k} C_k^r A_\ell^r \sum_{d \leq x} \frac{b_{k,r}(d)}{d^{\ell/2}} G_r^*(\log(x/d)) = P_k(\log x) + O(x^{1/2-\ell/4+\varepsilon}).$$

Inserting it into the preceding formula, we get the required result.  $\square$

## 4. PROOF OF THEOREM 2.

From (2.6) and (1.2), we can deduce that

$$\begin{aligned} \sum_{n \leq x} \tau_k(n) n^{\ell/2-1} &= x^{\ell/2} G_k^*(\log x) + O(x^{(\ell+2)/4+\varepsilon}) \\ &\quad + \sum_{0 \leq r \leq k} (-1)^{k-j} A_\ell^{-k} C_k^r \sum_{d \leq x} c_{k,r}(d) \Delta_r^*(Q, x/d), \end{aligned}$$

where we have used the following estimate

$$\sum_{0 \leq r \leq k} (-1)^{k-j} A_\ell^{-k} C_k^r \sum_{d \leq x} c_{k,r}(d) (x/d)^{\ell/2} P_r(\log(x/d)) = x^{\ell/2} G_k^*(\log x) + O(x^{(\ell+2)/4+\varepsilon}).$$

Comparing with (2.3) of Lemma 2.2 yields

$$x^{\ell/2-1} \Delta_k(x) = \sum_{0 \leq r \leq k} (-1)^{k-r} A_\ell^{-k} C_k^r \sum_{d \leq x} c_{k,r}(d) \Delta_r^*(Q, x/d) + O(x^{\ell/2-1+\delta_k}).$$

Under hypothesis (1.11), by (2.5), (2.3) of Lemma 2.2 and (2.7) of Lemma 2.3 we have

$$\begin{aligned} \Delta_r^*(Q, x) &\ll x^{\ell/2-1+r/\{2(r+1)\}-\delta+\varepsilon} \\ &\ll x^{\ell/2-1+(k-1)/(2k)-\delta/2} \end{aligned}$$

for  $0 \leq r \leq k-1$ . Inserting into the preceding formula and using (2.7), we can deduce

$$(4.1) \quad x^{\ell/2-1} \Delta_k(x) = A_\ell^{-k} \sum_{d \leq x} c_{k,k}(d) \Delta_k^*(Q, x/d) + O(x^{\ell/2-1+(k-1)/(2k)-\delta/2}).$$

On the other hand, according to Soundararajan [10], we have, for any  $k \geq 2$ ,

$$(4.2) \quad \Delta_k(x) = \Omega((x \log x)^{(k-1)/(2k)} (\log_2 x)^{\beta_k} (\log_3 x)^{-\gamma_k}).$$

Now on noting (2.7) of Lemma 2.3, the first assertion of Theorem 2 follows from (4.1) and (4.2).

Finally in view of (1.8), it is easy check that the hypothesis (1.11) is satisfied when  $k = 2$  or  $3$ . Therefore (1.12) holds unconditionally for these two values of  $k$ .

This completes the proof of Theorem 2.



## REFERENCES

- [1] K. Chandrasekharan and R. Narasimhan, *Functional equations with multiple gamma factors and the average order of arithmetical functions*, Ann. of Math. **76** (1962), 93–136.
- [2] P. Deligne, *La Conjecture de Weil*, Inst. Hautes Études Sci. Publ. Math. **43** (1974), 29–39.
- [3] M. N. Huxley, *Integer points, exponential sums and the Riemann zeta function*, in: Number theory for the millennium, II (Urbana, IL, 2000), 275–290, A K Peters, Natick, MA, 2002.
- [4] A. Ivić, *The Riemann zeta-function*, John Wiley & Sons, New York, Chichester, Brisbane, Toronto, Singapore, 1985.
- [5] H. Iwaniec, *Topics in classical automorphic forms*, Graduate Studies in Mathematics, vol. 17, American Mathematical Society, Providence, Rhode Island, 1997.
- [6] G. Kolesnik, *On the estimation of multiple exponential sums*, in: Recent progress in analytic number theory, Vol. 1 (Durham, 1979), Academic Press (London-New York, 1981), 231–246.
- [7] E. Landau, *Über die Anzahl der Gitterpunkte in gewissen Bereichen*, Göttinger Nachr. **18** (1912), 687–773.
- [8] G. S. Lü, *On a divisor problem related to the Epstein zeta-function*, Bull. London Math. Soc. **42** (2010), 267–274.
- [9] A. Sankaranarayanan, *On a divisor problem related to the Epstein zeta-function*, Arch. Math. **65** (1995), 303–309.
- [10] K. Soundararajan, *Omega results for the divisor and circle problems*, Int. Math. Res. Not. **36** (2003), 1987–1998.
- [11] G. Tenenbaum, *Introduction to analytic and probabilistic number theory*, Translated from the second French edition (1995) by C. B. Thomas. Cambridge Studies in Advanced Mathematics, 46. Cambridge University Press, Cambridge, 1995, xvi+448 pp.

SCHOOL OF MATHEMATICS, SHANDONG UNIVERSITY, JINAN, SHANDONG 250100, CHINA  
*E-mail address:* gslv@sdu.edu.cn

INSTITUT ELIE CARTAN NANCY, CNRS, UNIVERSITÉ HENRI POINCARÉ (NANCY 1), INRIA,  
 BOULEVARD DES AIGUILLETES, B.P. 239, 54506 VANDŒUVRE-LÈS-NANCY, FRANCE  
*E-mail address:* wujie@iecn.u-nancy.fr

DEPARTMENT OF MATHEMATICS, CHINA UNIVERSITY OF MINING AND TECHNOLOGY, BEI-  
 JING 100083, CHINA  
*E-mail address:* zhaiwg@hotmail.com