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Covariant bi-differential operators on matrix space

Jean-Louis Clerc

January 25, 2016

Abstract

A family of bi-differential operators from $C^\infty(\text{Mat}(m, \mathbb{R}) \times \text{Mat}(m, \mathbb{R}))$ into $C^\infty(\text{Mat}(m, \mathbb{R}))$ which are covariant for the projective action of the group $SL(2m, \mathbb{R})$ on $\text{Mat}(m, \mathbb{R})$ is constructed, generalizing both the *transvectants* and the *Rankin-Cohen brackets* (case $m = 1$).

Résumé

On construit une famille d'opérateurs bi-différentiels de $C^\infty(\text{Mat}(m, \mathbb{R}) \times \text{Mat}(m, \mathbb{R}))$ dans $C^\infty(\text{Mat}(m, \mathbb{R}))$ qui sont covariants pour l'action projective du groupe $SL(2m, \mathbb{R})$ sur $\text{Mat}(m, \mathbb{R})$. Dans le cas $m = 1$, cette construction fournit une nouvelle approche des *transvectants* et des *crochets de Rankin-Cohen*.

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Introduction

Let $X = Gr(m, 2m, \mathbb{R})$ the Grassmannian of m -planes in \mathbb{R}^{2m} , and consider the projective action of the group $SL(2m, \mathbb{R})$ on X , given for $g \in G$ and $p \in X$ by $g.p = \{gv, v \in p\}$. Choose an origin o and let P be the stabilizer of o in G . The group P is a maximal parabolic subgroup and $X \sim G/P$. The characters $\chi_{\lambda, \epsilon}$ of P are indexed by $(\lambda, \epsilon) \in \mathbb{C} \times \{\pm\}$. For $(\lambda, \epsilon) \in \mathbb{C} \times \{\pm\}$, let $\pi_{\lambda, \epsilon}$ be the corresponding representation induced from P , realized on the space $\mathcal{E}_{\lambda, \epsilon}$ of smooth sections of the line bundle $E_{\lambda, \epsilon} = X \times_{P, \chi_{\lambda, \epsilon}} \mathbb{C}$ (degenerate principal series). For the purpose of this paper, it is more convenient to work with the *noncompact realization* of $\pi_{\lambda, \epsilon}$ on a space $\mathcal{H}_{\lambda, \epsilon}$ of smooth functions on $V = \text{Mat}(m, \mathbb{R})$.

The *Knapp-Stein intertwining operators* form a meromorphic family (in λ) of operators which intertwines $\pi_{\lambda, \epsilon}$ and $\pi_{2m-\lambda, \epsilon}$ (in our notation). In the

non compact picture, for generic λ , the corresponding operators, denoted by $J_{\lambda,\epsilon}$ are convolution operators on V by certain tempered distributions. The properties of this family of operators are presented in section 3 and are mostly consequences of the theory of *local zeta functions* and their functional equation on (the simple real Jordan algebra) V . Incidentally, the results for $\epsilon = -1$ seem to be new, at least in the present form.

Let $(\lambda, \epsilon), (\mu, \eta) \in \mathbb{C} \times \{\pm\}$ and consider the tensor product $\pi_{\lambda,\epsilon} \otimes \pi_{\mu,\eta}$, realized (after completion) on a space $\mathcal{H}_{(\lambda,\epsilon),(\mu,\eta)}$ of smooth functions on $V \times V$. Because of the *covariance property* (see (9)) of the kernel $k(x, y) = \det(x - y)$ under the diagonal action of G on $V \times V$, the multiplication M by $\det(x - y)$ intertwines $\pi_{\lambda,\epsilon} \otimes \pi_{\mu,\eta}$ and $\pi_{\lambda-1,-\epsilon} \otimes \pi_{\mu-1,-\eta}$ (Proposition 4.1).

Let $(\lambda, \epsilon), (\mu, \eta) \in \mathbb{C} \times \{\pm\}$ and consider the following diagram

$$\begin{array}{ccc} \mathcal{H}_{(\lambda,\epsilon),(\mu,\eta)} & \xrightarrow{\quad ? \quad} & \mathcal{H}_{(\lambda+1,-\epsilon),(\mu+1,-\eta)} \\ \downarrow J_{\lambda,\epsilon} \otimes J_{\mu,\eta} & & \downarrow J_{\lambda+1,-\epsilon} \otimes J_{\mu+1,-\eta} \\ \mathcal{H}_{(2m-\lambda,\epsilon),(2m-\mu,\eta)} & \xrightarrow{\quad M \quad} & \mathcal{H}_{(2m-\lambda-1,-\epsilon),(2m-\mu-1,-\eta)} \end{array}$$

The main result of the paper is a (rather explicit) construction of a *differential operator* on $V \times V$ which completes the diagram (Theorem 4.1). The proof uses the Fourier transform on V and some delicate calculation specific to the matrix space V , based in particular on *Bernstein-Sato's identities* for $(\det x)^s$ (section 2). Up to some normalization factors, this yields a family of differential operators $D_{\lambda,\mu}$ with polynomial coefficients on $V \times V$, covariant w.r.t. $(\pi_{\lambda,\epsilon} \otimes \pi_{\mu,\eta}, \pi_{\lambda+1,-\epsilon} \otimes \pi_{\mu+1,-\eta})$. Their expression does not depend on ϵ and η , and the family depends holomorphically on (λ, μ) . See also Theorem 4.2 for a formulation of the same result in the compact picture.

From this result, it is then easy to construct families of projectively covariant bi-differential operators from $C^\infty(V \times V)$ into $C^\infty(V)$. For any integer k , define

$$B_{\lambda,\mu;2k} = \text{res} \circ D_{\lambda+k,\mu+k} \circ \cdots \circ D_{\lambda+1,\mu+1} \circ D_{\lambda,\mu}$$

where res is the restriction map from $V \times V$ to the diagonal $\text{diag}(V \times V) \sim V$. Clearly, $B_{\lambda,\mu;k}$ is G -covariant w.r.t. $(\pi_{\lambda,\epsilon} \otimes \pi_{\mu,\eta}, \pi_{\lambda+\mu+2k,\epsilon\eta})$. For k fixed, the family depends holomorphically on λ, μ and is generically non trivial.

For $m = 1$, there is another classical construction of such projectively covariant bi-differential operators. The Ω -process, a cornerstone in classical invariant theory leads to the construction of the *transvectants*, which are covariant bi-differential operators for special values of the parameters λ and μ connected to the *finite-dimensional representations* of

$G = SL(2, \mathbb{R})$. The *Rankin-Cohen brackets*, much used in the theory of modular forms, are other examples of such covariant bi-differential operators, for special values of (λ, μ) connected to the *holomorphic discrete series* of $SL(2, \mathbb{R})$.

In case $m = 1$, it has been observed later (see e.g. [13]) that the Ω -process can be extended to general (λ, μ) , yielding both the transvectants and the Rankin-Cohen brackets as special cases. As computations are easy when $m = 1$, the present construction can be shown to coincide with the approach through the Ω -process, and the operators $B_{\lambda, \mu; k}$ for special values of (λ, μ) , essentially coincide with the transvectants or the Rankin-Cohen brackets. For another related but different point of view see [9] (specially section 9) and the expository notes [10]. The situation where $m \geq 2$ is further commented in section 6.

The striking fact that the operator $D_{\lambda, \mu}$, although obtained by composing non-local operators, is a *differential* operator (hence local) was already observed in another geometric context, namely for conformal geometry on the sphere S^d , $d \geq 3$ (see [2], [5]). It seems reasonable to conjecture that similar results are valid for any (real or complex) simple Jordan algebra and its conformal group (see [1] for analysis on these spaces).

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1 The degenerate principal series for $Gr(m, 2m, \mathbb{R})$

Let $X = Gr(m, 2m; \mathbb{R})$ be Grassmannian of m -dimensional vector subspaces of \mathbb{R}^{2m} . The group $G = SL(2m, \mathbb{R})$ acts transitively on X .

Let $(\epsilon_1, \epsilon_2, \dots, \epsilon_{2m})$ be the standard basis of \mathbb{R}^{2m} and let

$$p_0 = \bigoplus_{j=m+1}^{2m} \mathbb{R}\epsilon_j, \quad p_\infty = \bigoplus_{j=1}^m \mathbb{R}\epsilon_j.$$

The stabilizer of p_0 in G is the parabolic subgroup P given by

$$P = \left\{ \begin{pmatrix} a & 0 \\ c & d \end{pmatrix}, a, d \in GL(m, \mathbb{R}), \det a \det d = 1 \right\},$$

and $X \sim G/P$.

Two subspaces p and q in X are said to be *transverse* if $p \cap q = \{0\}$, and this relation is denoted by $p \pitchfork q$. Let $\mathcal{O} = \{p \in X, p \pitchfork p_\infty\}$. Then \mathcal{O} is a dense open subset of X . Any subspace p transverse to p_∞ can be realized as the graph of some linear map $x : p_0 \rightarrow p_\infty$, and vice versa. More explicitly, any $p \in \mathcal{O}$ can be realized as

$$p = p_x = \left\{ \begin{pmatrix} x\xi \\ \xi \end{pmatrix}, \xi \in \mathbb{R}^m \right\},$$

where ξ is interpreted as a column vector in \mathbb{R}^m and x is viewed as an element of $V = \text{Mat}(m, \mathbb{R})$.

Let $g \in G$ and $x \in V$. The element $g \in G$ is said to be *defined at x* if $g \cdot p_x \in \mathcal{O}$ and then $g(x)$ is defined by $p_{g(x)} = g \cdot p_x$. More explicitly, if $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then

$$g \cdot p_x = \left\{ \begin{pmatrix} (ax+b)\xi \\ (cx+d)\xi \end{pmatrix}, \xi \in \mathbb{R}^m \right\},$$

so that g is defined at x iff $(cx+d)$ is invertible, and then

$$g(x) = (ax+b)(cx+d)^{-1}.$$

Define $\alpha : G \times V \rightarrow \mathbb{R}$ by

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \alpha(g, x) = \det(cx+d). \quad (1)$$

The following elementary calculation is left to the reader.

Lemma 1.1. *Let $g, g' \in G$ and $x \in V$, and assume that g' is defined at x and g is defined at $g'(x)$. Then gg' is defined at x and*

$$\alpha(gg', x) = \alpha(g, g'(x))\alpha(g', x). \quad (2)$$

The map $x \mapsto p_x$ is a homeomorphism of V onto \mathcal{O} . The reciprocal of this map $\kappa : \mathcal{O} \rightarrow V$ is a local chart, thereafter called the *principal chart*. For any $g \in G$, let $\mathcal{O}_g = g^{-1}(\mathcal{O})$ and $\kappa_g : \mathcal{O}_g \rightarrow V$ defined by $\kappa_g = \kappa \circ g$. Then $(\mathcal{O}_g, \kappa_g)_{g \in G}$ is an atlas for X .

Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$. Then

$$V_g := \kappa(\mathcal{O}_g \cap \mathcal{O}) = \{x \in V, \det(cx + d) \neq 0\},$$

and the change of coordinates between the charts \mathcal{O} and \mathcal{O}_g is given by

$$V_g \ni x \mapsto g(x) = (ax + b)(cx + d)^{-1}.$$

The group P admits the Langlands decomposition $P = L \ltimes N$, where

$$L = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}, \det a \det d = 1 \right\}, \quad N = \left\{ t_v = \begin{pmatrix} \mathbf{1}_m & 0 \\ v & \mathbf{1}_m \end{pmatrix}, v \in V \right\}.$$

The group L acts on V by

$$l = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}, \quad l(x) = axd^{-1}.$$

Let

$$\overline{N} = \left\{ \overline{n}_y = \begin{pmatrix} \mathbf{1}_m & y \\ 0 & \mathbf{1}_m \end{pmatrix}, y \in V \right\} \sim V$$

be the opposite unipotent subgroup. The subgroup \overline{N} acts on V by translations, i.e. $\overline{n}_y(x) = x + y$ for $y \in V$.

Let $\iota = \begin{pmatrix} 0 & \mathbf{1}_m \\ -\mathbf{1}_m & 0 \end{pmatrix}$ be the *inversion*. It is defined on the open set V^\times of invertible matrices and acts by $\iota(x) = -x^{-1}$. Its differential $D\iota(x)$ is given by $V \ni u \mapsto D\iota(x)u = x^{-1}ux^{-1}$.

It is a well-known result that G is generated by L, \overline{N} and ι (a special case of a theorem valid for the *conformal group* of a simple (real or complex) Jordan algebra).

An element $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$ belongs to $\overline{N}P$ iff $\det d \neq 0$ and then the following *Bruhat decomposition* holds

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \mathbf{1}_m & bd^{-1} \\ 0 & \mathbf{1}_m \end{pmatrix} \begin{pmatrix} a - bd^{-1}c & 0 \\ c & d \end{pmatrix}. \quad (3)$$

Let χ be the character of P defined by

$$P \ni p = \begin{pmatrix} a & 0 \\ c & d \end{pmatrix}, \quad \chi(p) = \det a = (\det d)^{-1}. \quad (4)$$

Lemma 1.2. *Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G, x \in V$ and assume that g is defined at x .*

- i) the differential $Dg(x)$ belongs to L*
- ii) $\chi(Dg(x)) = \alpha(g, x)^{-1}$*
- iii) the Jacobian of g at x is equal to*

$$j(g, x) = \chi(Dg(x))^{2m} = \alpha(g, x)^{-2m}. \quad (5)$$

Proof. By elementary calculation, the statements are verified for elements of N, L and for ι . As these elements generate G , the conclusion follows by using the cocycle relations satisfied by $\alpha(g, x)$ (see (2)) and by $\chi(Dg(x))$ or $j(g, x)$ as consequences of the chain rule. \square

Let $\lambda \in \mathbb{C}$ and $\epsilon \in \{\pm\}$. For $t \in \mathbb{R}^*$ let $t^{\lambda, \epsilon}$ be defined by

$$t \mapsto \begin{cases} |t|^\lambda & \text{if } \epsilon = + \\ \text{sgn}(t)|t|^\lambda & \text{if } \epsilon = - \end{cases}$$

The map $t \mapsto t^{\lambda, \epsilon}$ is a smooth character of \mathbb{R}^* , and any smooth character is of this form.

Let $\chi^{\lambda, \epsilon}$ be the character of P defined by

$$\chi^{\lambda, \epsilon}(p) = \chi(p)^{\lambda, \epsilon}.$$

Let $E_{\lambda, \epsilon}$ be the line bundle over $X = G/P$ associated to the character $\chi^{\lambda, \epsilon}$ of P . Let $\mathcal{E}_{\lambda, \epsilon}$ be the space of smooth sections of $E_{\lambda, \epsilon}$. Then G acts on $\mathcal{E}_{\lambda, \epsilon}$ by the natural action on the sections of $E_{\lambda, \epsilon}$ and gives rise to a representation $\pi_{\lambda, \epsilon}$ of G on $\mathcal{E}_{\lambda, \epsilon}$.

A smooth section of $E_{\lambda, \epsilon}$ can be realized as a smooth function F on G which satisfies

$$F(gp) = \chi(p^{-1})^{\lambda, \epsilon} F(g).$$

To each such function F , associate its restriction to \overline{N} , which can be viewed as a function f on V defined for $y \in V$ by

$$f(y) = F(\overline{n}_y) = F\left(\begin{pmatrix} \mathbf{1}_m & y \\ 0 & \mathbf{1}_m \end{pmatrix}\right).$$

Using the Bruhat decomposition (3), the function F can be recovered from f as

$$F\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = (\det d)^{\lambda, \epsilon} f(bd^{-1}) .$$

The formula is valid for $g \in \overline{N}P$ and extends by continuity to all of G .

This yields the realization of $\pi_{\lambda, \epsilon}$ in the *noncompact picture*, namely for $g \in G$, such that $g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$\begin{aligned} \pi_{\lambda, \epsilon}(g)f(y) &= (\det(cy + d)^{-1})^{\lambda, \epsilon} f((ay + b)(cy + d)^{-1}) \\ &= \alpha(g^{-1}, y)^{-\lambda, \epsilon} f(g^{-1}(y)) . \end{aligned}$$

In the noncompact picture, the representation $\pi_{\lambda, \epsilon}$ is defined on the image $\mathcal{H}_{\lambda, \epsilon}$ of $\mathcal{E}_{\lambda, \epsilon}$ by the principal chart. The local expression of an element of $\mathcal{H}_{\lambda, \epsilon}$ is a function $f \in C^\infty(V)$. For $g \in G$, the function $x \mapsto (\alpha(g, x)^{-1})^{-\lambda, \epsilon} f(g(x))$ is a priori defined on the (dense open) subset \mathcal{O}_g of V . Hence a (rather nasty) characterization of the space is as follows : a smooth function f on V belongs to $\mathcal{H}_{\lambda, \epsilon}$ if and only if,

$$\forall g \in G, \quad x \mapsto (\alpha(g, x)^{-1})^{-\lambda, \epsilon} f(g(x)) \text{ extends as a } C^\infty \text{ function on } V . \quad (6)$$

Let $(\lambda, \epsilon), (\mu, \eta) \in \mathbb{C} \times \{\pm\}$, and let $\pi_{\lambda, \epsilon} \boxtimes \pi_{\mu, \eta}$ be the corresponding product representation of $G \times G$. The space of the representation $\mathcal{E}_{(\lambda, \epsilon), (\mu, \eta)}$ (after completion) is the space of smooth sections of the fiber bundle $E_{\lambda, \epsilon} \boxtimes E_{\mu, \eta}$ over $X \times X$. For the non-compact realization, observe that $\mathcal{O}^2 = \mathcal{O} \times \mathcal{O}$ is an open dense set in $X \times X$. For any $g \in G$, let \mathcal{O}_g^2 be the image of \mathcal{O}^2 under the *diagonal* action of g^{-1} , i.e. $\mathcal{O}_g^2 = \{g(x), g(y), x \in \mathcal{O}, y \in \mathcal{O}\}$. Then the family $(\mathcal{O}_g^2, g \in G)$ is a covering of $X \times X$. Using the corresponding atlas, the local expressions in the principal chart $\kappa \otimes \kappa : \mathcal{O}^2 \rightarrow V \times V$ of $\mathcal{E}_{(\lambda, \epsilon), (\mu, \eta)}$ is the space $\mathcal{H}_{(\lambda, \epsilon), (\mu, \eta)}$ of C^∞ functions f on $V \times V$ such that, for any $g \in G$

$$\alpha(g, x)^{-\lambda, \epsilon} f(g(x), g(y)) \alpha(g, y)^{-\mu, \eta} \text{ extends as a } C^\infty \text{ function on } V \times V . \quad (7)$$

The group $G \times G$ acts on $\mathcal{H}_{(\lambda, \epsilon), (\mu, \eta)}$ by

$$(\pi_\lambda \boxtimes \pi_\mu)(g_1, g_2)f(x, y) = \alpha(g_1^{-1}, x)^{-\lambda, \epsilon} \alpha(g_2^{-1}, y)^{-\mu, \eta} f(g_1^{-1}(x), g_2^{-1}(y)) \quad (8)$$

Lemma 1.3. *Let $g \in G, x, y \in V$ such that g is defined at x and at y . Then*

$$\det(g(x) - g(y)) = \alpha(g, x)^{-1} \det(x - y) \alpha(g, y)^{-1} . \quad (9)$$

Proof. If $g \in \overline{N}$, g acts by translations on V and hence (9) is trivial. If $g = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$, then $g(x) - g(y) = a(x - y)d^{-1}$, $\alpha(g, x) = \alpha(g, y) = \det a^{-1} \det d$ and (9) is easily verified. When $g = \iota$, then

$$\det(-x^{-1} + y^{-1}) = \det(x^{-1}(x - y)y^{-1}) = \det x^{-1} \det(x - y) \det y^{-1}$$

$$\forall v \in V, \quad D\iota(x)v = x^{-1}vx^{-1}, \quad \alpha(\iota, x) = \det x$$

and (9) follows easily. The cocycle property (2) satisfied by α and the fact that G is generated by \overline{N} , L and ι imply (9) in full generality. \square

Proposition 1.1. *The function $k(x, y) = \det(x - y)$ belongs to $\mathcal{H}_{(-1, -), (-1, -)}$ and is invariant under the diagonal action of G .*

Proof. Let $x, y \in V$ and $g \in G$ defined at x and y . (9) implies

$$\alpha(g, x)k(g(x), g(y))\alpha(g, y) = k(x, y)$$

which shows that k belongs to $\mathcal{H}_{(-1, -), (1, -)}$ by the criterion (7). Further apply (8) for $g_1 = g_2 = g$ to get the invariance of k under the diagonal action of G . \square

2 Some functional identities in $\text{Mat}(m, \mathbb{C})$ and $\text{Mat}(m, \mathbb{R})$

Let $(\mathbb{E}, \langle \cdot, \cdot \rangle)$ be a complex finite dimensional Hilbert space. To any holomorphic polynomial p on \mathbb{E} , associate the holomorphic differential operator $p\left(\frac{\partial}{\partial z}\right)$ defined by

$$p\left(\frac{\partial}{\partial z}\right)e^{(z, \xi)} = p(\bar{\xi})e^{(z, \xi)}.$$

Let e_1, e_2, \dots, e_n is an orthonormal basis, with corresponding coordinates z_1, z_2, \dots, z_n . For $I = (i_1, i_2, \dots, i_n)$ a n -tuple of integers, set

$$z^I = z_1^{i_1} z_2^{i_2} \dots z_n^{i_n}, \quad \left(\frac{\partial}{\partial z}\right)^I = \left(\frac{\partial}{\partial z_1}\right)^{i_1} \left(\frac{\partial}{\partial z_2}\right)^{i_2} \dots \left(\frac{\partial}{\partial z_n}\right)^{i_n}.$$

Let $p(z) = \sum_{|I| \leq N} a_I z^I$ be a holomorphic polynomial on \mathbb{E} . Then

$$p\left(\frac{\partial}{\partial z}\right) = \sum_{|I| \leq N} a_I \left(\frac{\partial}{\partial z}\right)^I.$$

Let $(E, \langle \cdot, \cdot \rangle)$ be a finite dimensional Euclidean vector space. To any polynomial p on E associate the differential operator $p\left(\frac{\partial}{\partial x}\right)$ such that

$$p\left(\frac{\partial}{\partial x}\right)e^{\langle x, \xi \rangle} = p(\xi)e^{\langle x, \xi \rangle}.$$

Lemma 2.1. *Let $(\mathbb{E}, (\cdot, \cdot))$ be a complex finite dimensional Hilbert space, and let $(E, \langle \cdot, \cdot \rangle)$ be a real form of \mathbb{E} such that*

$$\forall x, y \in E, \quad (x, y) = \langle x, y \rangle .$$

Let p be a holomorphic polynomial on \mathbb{E} . Let \mathcal{O} be an open subset of \mathbb{E} such that $\omega = \mathcal{O} \cap E \neq \emptyset$. Let f be a holomorphic function f on \mathcal{O} . Then for $x \in \omega$

$$p\left(\frac{\partial}{\partial z}\right)f(x) = p\left(\frac{\partial}{\partial x}\right)f|_{\omega}(x) . \quad (10)$$

Now let $\mathbb{E} = \text{Mat}(m, \mathbb{C}) = \mathbb{V}$ with the inner product $(z, w) = \text{tr } zw^*$. The restriction of this inner product to the real form $E = \text{Herm}(m, \mathbb{C})$ is equal to

$$\langle x, y \rangle = \text{tr } xy^* = \text{tr } xy = \text{tr } y^t x^t = \text{tr } \overline{yx} = \overline{\text{tr } xy} = \overline{\text{tr } xy^*} = \overline{\langle x, y \rangle}$$

and conditions of Lemma 10 are satisfied. Denote by $\Omega_m \subset E$ the open cone of positive-definite Hermitian matrices.

Let $k \in \{1, 2, \dots, m\}$. For $z \in \mathbb{V}$, let $\Delta_k(z)$ be the principal minor of order k of the matrix z . Let $\Delta_k^c(z)$ be the $(m - k)$ anti-principal minor of z . Both Δ_k and Δ_k^c are holomorphic polynomials on \mathbb{V} .

Let \mathbb{V}^\times be the set of invertible matrices in \mathbb{V} . Let $z_0 \in \mathbb{V}^\times$. Choose a local determination of $\ln \det z$ on a neighborhood of z_0 , and, for $s \in \mathbb{C}$ define $(\det z)^s = e^{s \ln \det z}$ accordingly. Any other local determination of $\ln \det z$ is of the form $\ln \det z + 2ik\pi$ for some $k \in \mathbb{Z}$, and the associated local determination of $(\det z)^s$ is given by $e^{2ik\pi s}(\det z)^s$.

Recall the *Pochhammer's symbol*, for $s \in \mathbb{C}, n \in \mathbb{N}$

$$(s)_0 = 1, \quad (s)_1 = s, \dots, \quad (s)_n = s(s+1) \dots (s+n-1), \dots$$

Proposition 2.1. *For any $z \in \mathbb{V}^\times$ and for any local determination of $\ln \det$ in a neighborhood of z*

$$\Delta_k\left(\frac{\partial}{\partial z}\right)(\det z)^s = (s)_k \Delta_k^c(z) (\det z)^{s-1} \quad (11)$$

Proof. Let $z_0 \in \mathbb{V}^\times$. Choose an open neighborhood \mathcal{V} of z contained in \mathbb{V}^\times which is simply connected and such that $\mathcal{V} \cap \Omega_m \neq \emptyset$. On Ω_m , let $x > 0$ so that $\text{Ln } \det z$ (where Ln is the principal determination of the logarithm on $\mathbb{C} \setminus (-\infty, 0]$) is an appropriate determination of $\ln \det z$ in a neighborhood of Ω_m , which can be analytically continued to \mathcal{V} and used for defining $(\det z)^s$ on \mathcal{V} . For $x \in \Omega_m$, the identity

$$\Delta_k\left(\frac{\partial}{\partial x}\right)(\det x)^s = (s)_k \Delta_k^c(x) (\det x)^{s-1}$$

holds. It is a special case of [6] Proposition VII.1.6 for the simple Euclidean Jordan algebra $Herm(m, \mathbb{C})$. By Lemma 2.1, (11) is satisfied for $z \in \mathcal{V} \cap Herm(m, \mathbb{C})$. As both sides of (11) are holomorphic functions, (11) yields everywhere on \mathcal{V} . But if (11) is valid for *some* local determination of $\ln \det z$ it is valid for *any* local determination. \square

There is a real version of these identities.

Proposition 2.2. *The following identity holds for $x \in V^\times$*

$$\Delta_k \left(\frac{\partial}{\partial x} \right) (\det x)^{s, \epsilon} = (s)_k \Delta_k^c(x) (\det x)^{s-1, -\epsilon}. \quad (12)$$

Proof. Let $x \in V^\times$ and assume first that $\det x > 0$. In a neighbourhood of x in \mathbb{V}^\times choose $\text{Ln}(\det z)$ as a local determination of $\ln(\det z)$. Then $(\det x)^s = |\det x|^s$ and hence, using Lemma 2.1 and (11)

$$\Delta_k \left(\frac{\partial}{\partial x} \right) |\det x|^s = (s)_k \Delta_k^c(x) |\det x|^{s-1}.$$

Next assume that $\det x < 0$. In a neighborhood of x in \mathbb{V}^\times choose $\text{Ln}(-\det z) + i\pi$ as a local determination of $\ln(\det z)$. Then $(\det x)^s = e^{is\pi} |\det x|^s$, so that, using again Lemma 2.1 and (11)

$$e^{is\pi} \Delta_k \left(\frac{\partial}{\partial x} \right) |\det x|^s = e^{i(s-1)\pi} (s)_k \Delta_k^c(x) |\det x|^{s-1}.$$

The identity (12) follows. \square

Let $a = (a_{ij})$ be a $m \times m$ matrix with real or complex entries a_{ij} . Let I and J be two subsets of $\{1, 2, \dots, m\}$ both of cardinality k , $0 \leq k \leq m$. After deleting the $m - k$ rows (resp. the $m - k$ columns) corresponding to the indices not in I (resp. not in J), the determinant of the $k \times k$ remaining matrix is the *minor* associated to (I, J) and will be denoted by $\Delta_{I,J}(a)$. For $k = 0$, i.e. $I = J = \emptyset$, by convention $\Delta_{\emptyset, \emptyset}(a) = 1$. For $k = m$, $I = J = \{1, 2, \dots, m\}$, $\Delta_{I,J}(a) = \det a$.

For $I = \{i_1 < i_2 < \dots < i_k\}$, let $|I| = i_1 + i_2 + \dots + i_k$. Also denote by I^c the complement of I in $\{1, 2, \dots, m\}$, which is a subset of cardinality $m - k$. Recall the following elementary result.

Lemma 2.2. *Let $I = \{i_1 < i_2 < \dots < i_k\}$ be a subset of $\{1, 2, \dots, m\}$ of cardinality k . Let $I^c = \{i'_1 < i'_2 < \dots < i'_{m-k}\}$. The permutation σ_I defined by*

$$\sigma_I(1) = i_1, \dots, \sigma_I(k) = i_k, \quad \sigma_I(k+1) = i'_1, \dots, \sigma_I(m) = i'_{m-k}$$

has signature equal to $\epsilon(\sigma_I) = (-1)^{|I|}$.

The next lemma is a variation on (and a consequence of) the previous lemma.

Lemma 2.3. *Let $I = \{i_1 < i_2 < \dots < i_k\}$, $J = \{j_1 < j_2 < \dots < j_k\}$ be two subsets of $\{1, 2, \dots, m\}$ both of cardinality k . Let*

$$I^c = \{i'_1 < i'_2 < \dots < i'_{m-k}\}, \quad J^c = \{j'_1 < j'_2 < \dots < j'_{m-k}\}.$$

The permutation $\sigma = \sigma_{I,J}$ given by

$$\sigma(i_1) = j_1, \dots, \sigma(i_k) = j_k, \quad \sigma(i'_1) = j'_1, \dots, \sigma(i'_{m-k}) = j'_{m-k}$$

has signature $\epsilon(I, J) := \epsilon(\sigma_{I,J}) = (-1)^{|I|+|J|}$.

A permutation σ such that $\sigma(I) = J$ can be written in a unique way as $\sigma = (\tau \vee \tau_c) \circ \sigma_{I,J}$, where τ is a permutation of J and τ_c is a permutation of J^c , and $\tau \vee \tau_c$ is the permutation of $\{1, 2, \dots, m\}$ which coincides on J with τ and on J^c with τ_c .

Proposition 2.3. *Let $I, J \subset \{1, 2, \dots, n\}$ of equal cardinality k . Then, for $x \in \mathbb{V}^\times$*

$$\partial(\Delta_{I,J})(\Delta^s)(x) = \epsilon(I, J)(s)_k \Delta_{I^c, J^c}(x) \Delta(x)^{s-1}. \quad (13)$$

Proof. By permuting rows and columns properly, the minor $\Delta_{I,J}$ becomes the k -th principal minor and Δ^{I^c, J^c} becomes the $m-k$ anti-principal minor, up to a sign. Hence (13) is a consequence of (11) and Lemma 2.2. \square

Proposition 2.4. *Let f, g be two smooth functions defined on \mathbb{V} . Then*

$$\det\left(\frac{\partial}{\partial x}\right)(fg) = \sum_{\substack{I, J \subset \{1, 2, \dots, m\} \\ \#I = \#J}} \epsilon(I, J) \Delta_{I,J}\left(\frac{\partial}{\partial x}\right)f \Delta_{I^c, J^c}\left(\frac{\partial}{\partial x}\right)g \quad (14)$$

Proof. For $\sigma \in \mathfrak{S}_m$

$$\frac{\partial^m}{\partial a_{1\sigma(1)} \partial a_{2\sigma(2)} \dots \partial a_{m\sigma(m)}}(fg) = \sum_{I \subset \{1, 2, \dots, m\}} \left(\prod_{i \in I} \frac{\partial}{\partial a_{i\sigma(i)}} \right) f \left(\prod_{i \in I^c} \frac{\partial}{\partial a_{i\sigma(i)}} \right) g.$$

Now, given $I \subset \{1, 2, \dots, m\}$,

$$\sum_{\sigma \in \mathfrak{S}_m} = \sum_{\substack{J \subset \{1, 2, \dots, m\} \\ \#J = \#I}} \sum_{\substack{\sigma \in \mathfrak{S}_m \\ \sigma(I) = J}}$$

so that

$$\begin{aligned}
\partial(\Delta)(fg) &= \sum_{\sigma \in \mathfrak{S}_m} \epsilon(\sigma) \sum_{I \subset \{1, 2, \dots, m\}} \left(\prod_{i \in I} \frac{\partial}{\partial a_{i\sigma(i)}} \right) f \left(\prod_{i \in I^c} \frac{\partial}{\partial a_{i\sigma(i)}} \right) g \\
&= \sum_{I \subset \{1, 2, \dots, m\}} \sum_{\substack{J \subset \{1, 2, \dots, m\} \\ \#I = \#J}} \sum_{\substack{\sigma \in \mathfrak{S}_m \\ \sigma(I) = J}} \epsilon(\sigma) \left(\prod_{i \in I} \frac{\partial}{\partial a_{i\sigma(i)}} \right) f \left(\prod_{i \in I^c} \frac{\partial}{\partial a_{i\sigma(i)}} \right) g.
\end{aligned}$$

Let

$$\begin{aligned}
I &= \{i_1 < i_2 < \dots < i_k\}, & J &= \{j_1 < j_2 < \dots < j_k\} \\
I^c &= \{i'_1 < i'_2, \dots < i'_{m-k}\}, & J^c &= \{j'_1 < j'_2, \dots < j'_{m-k}\}.
\end{aligned}$$

As noted after the proof of Lemma 2.3, a permutation σ such that $\sigma(I) = J$ can be written in a unique way as

$$\sigma = (\tau \vee \tau_c) \circ \sigma_{I,J}$$

where $\tau \in \mathfrak{S}(J)$, $\tau_c \in \mathfrak{S}(J^c)$. Hence

$$\begin{aligned}
&\sum_{\substack{\sigma \in \mathfrak{S}_m \\ \sigma(I) = J}} \epsilon(\sigma) \left(\prod_{i \in I} \frac{\partial}{\partial a_{i\sigma(i)}} \right) f \left(\prod_{j \in I^c} \frac{\partial}{\partial a_{j\sigma(j)}} \right) g \\
&= \epsilon(I, J) \sum_{\tau \in \mathfrak{S}(J)} \sum_{\tau_c \in \mathfrak{S}(J^c)} \epsilon(\tau) \epsilon(\tau_c) \frac{\partial^k f}{\partial a_{i_1 \tau(j_1)} \dots \partial a_{i_k \tau(j_k)}} \frac{\partial^{m-k} g}{\partial a_{i'_1 \tau_c(j'_1)} \dots \partial a_{i'_{m-k} \tau_c(j'_{m-k})}} \\
&= \epsilon(I, J) (\Delta_{I,J} \left(\frac{\partial}{\partial x} \right) f) \Delta_{I^c, J^c} \left(\frac{\partial}{\partial x} \right) g.
\end{aligned}$$

Formula (14) follows by summing over I and J . \square

There is a similar *relative* result, allowing to compute $\Delta_{I,J}(fg)$ for I, J two subsets of $\{1, 2, \dots, m\}$, both of cardinality $k \leq m$. Let

$$I = \{i_1 < i_2 < \dots < i_k\}, \quad J = \{j_1 < j_2 < \dots < j_k\}.$$

A subset $P \subset I$ (resp. $Q \subset J$) of cardinality $l \leq k$ can be uniquely written as

$$P = \{i_{p_1} < i_{p_2}, \dots < i_{p_l}\}, \quad \text{resp. } Q = \{j_{q_1}, j_{q_2}, \dots, j_{q_l}\}.$$

Set

$$\epsilon(P : I, Q : J) = (-1)^{p_1 + p_2 + \dots + p_l} (-1)^{q_1 + q_2 + \dots + q_l}.$$

Proposition 2.5. *Let I, J be two subsets of $\{1, 2, \dots, m\}$, both of cardinality $k \leq m$. Let f, g be two smooth functions defined on \mathbb{V} . Then*

$$\Delta_{I,J} \left(\frac{\partial}{\partial x} \right) (fg) = \sum_{\substack{P \subset I \\ Q \subset J \\ \#P = \#Q}} \epsilon(P : I, Q : J) \Delta_{P,Q} \left(\frac{\partial}{\partial x} \right) f \Delta_{I \setminus P, J \setminus Q} \left(\frac{\partial}{\partial x} \right) g \quad (15)$$

Proof. In order to calculate the left hand side of (15), it is possible to “freeze” all variables x_{ij} for $(i, j) \notin I \times J$. For $x \in \mathbb{V}$, let

$$\mathbb{V}_{I,J}^x = \left\{ z = \begin{pmatrix} & \\ & z_{ij} \end{pmatrix} \in \text{Mat}(m, \mathbb{C}), z_{ij} = x_{ij} \text{ for } (i, j) \notin I \times J \right\}.$$

Then $\mathbb{V}_{I,J}^x \sim \text{Mat}(k, \mathbb{C})$. Now to compute the left hand side of (15) at x , apply (14) to the restrictions of f and g to $\mathbb{V}_{I,J}^x$. \square

Proposition 2.6. *Let $s, t \in \mathbb{C}$. Then, for $f \in C^\infty(\mathbb{V} \times \mathbb{V})$ and $x, y \in \mathbb{V}$, such that $x, y - x \in \mathbb{V}^\times$*

$$\det \left(\frac{\partial}{\partial x} \right) \left(\det(x)^s \det(y - x)^t f(x, y) \right) = \det(x)^{s-1} \det(y - x)^{t-1} (E_{s,t} f)(x, y) \quad (16)$$

where $E_{s,t}$ is the differential operator on $\mathbb{V} \times \mathbb{V}$ given by

$$E_{s,t} f(x, y) = \sum_{k=0}^m \sum_{\substack{I, J \subset \{1, 2, \dots, m\} \\ \#I = \#J = k}} p_{I,J}(x, y; s, t) \Delta_{I^c, J^c} \left(\frac{\partial}{\partial x} \right) f(x, y)$$

where, for I, J of cardinality k

$$p_{I,J}(x, y; s, t) = \sum_{0 \leq l \leq k} (-1)^l (s)_{(k-l)} (t)_l \sum_{\substack{P \subset I, Q \subset J \\ \#P = \#Q = l}} \epsilon(P : I, Q : J) \Delta_{I^c \cup P, J^c \cup Q}(x) \Delta_{P^c, Q^c}(y - x).$$

Proof. Using (14), the statement is equivalent to
for any $I, J \subset \{1, 2, \dots, n\}$, $\#I = \#J = k$,

$$\epsilon(I, J) \det(x)^{-s+1} \det(y-x)^{-t+1} \Delta_{I,J} \left(\frac{\partial}{\partial x} \right) \left(\det(x)^s \det(y-x)^t \right)$$

a priori defined for $x \in \mathbb{V}^\times, y-x \in \mathbb{V}^\times$ extends as a polynomial in (x, y) equal to $p_{I,J}(x, y; s, t)$.

Use (15) to obtain

$$\begin{aligned} & \Delta_{I,J} \left(\frac{\partial}{\partial x} \right) (\det x)^s (\det(y-x))^t \\ &= \sum_{l=0}^k \sum_{\substack{P \subset I, Q \subset J \\ \#P = \#Q = l}} \epsilon(P : I, Q : J) \Delta_{I \setminus P, J \setminus Q} \left(\frac{\partial}{\partial x} \right) (\det x)^s \Delta_{P,Q} \left(\frac{\partial}{\partial x} \right) (\det(y-x))^t \end{aligned}$$

By (13),

$$\det(x)^{-s+1} \Delta_{I \setminus P, J \setminus Q} \left(\frac{\partial}{\partial x} \right) (\det x)^s = \epsilon(I \setminus P, J \setminus Q) (s)_{k-l} \Delta_{I^c \cup P, J^c \cup Q}(x) .$$

Moreover, as any constant coefficients differential operator, $\Delta_{K,L}(\frac{\partial}{\partial x})$ commutes to translations, so that again by (13)

$$(\det(y-x)^{-t+1} \Delta_{P,Q} \left(\frac{\partial}{\partial x} \right) (\det(y-x))^t = \epsilon(P, Q) (-1)^l (t)_l \Delta_{P^c, Q^c}(y-x) .$$

Next, as $|I \setminus P| + |P| = |I|$ and $|J \setminus Q| + |Q| = |J|$

$$\epsilon(P, Q) \epsilon(I \setminus P, J \setminus Q) = \epsilon(I, J) .$$

It remains to gather all formulæ to finish the proof of Proposition 2.6. \square

Let p be a polynomial on \mathbb{V} , and let q be the polynomial on $\mathbb{V} \times \mathbb{V}$ given by $q(x, y) = p(x-y)$. Let f be a function on $\mathbb{V} \times \mathbb{V}$. Let g be the function on $\mathbb{V} \times \mathbb{V}$ defined by $g(u, v) = f(u, v-u)$ or equivalently $g(x, x+y) = f(x, y)$. Then

$$\left(q \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right) f \right) (x, y) = \left(p \left(\frac{\partial}{\partial u} \right) g \right) (x, x+y) . \quad (17)$$

In the sequel, for commodity reason, the operator $q(\frac{\partial}{\partial x}, \frac{\partial}{\partial y})$ will be denoted by $p(\frac{\partial}{\partial x} - \frac{\partial}{\partial y})$

Proposition 2.7. *Let $s, t \in \mathbb{C}$. For any smooth function on $\mathbb{V} \times \mathbb{V}$ and for $x, y \in \mathbb{V}^\times$*

$$\det \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y} \right) \left((\det x)^s (\det y)^t f \right) (x, y) = (\det x)^{s-1} (\det y)^{t-1} F_{s,t} f(x, y) \quad (18)$$

where $F_{s,t}$ is the differential operator on $\mathbb{V} \times \mathbb{V}$ given by

$$F_{s,t} f(x, y) = \sum_{k=0}^m \sum_{\substack{I, J \subset \{1, 2, \dots, m\} \\ \#I = \#J = k}} q_{I,J}(x, y; s, t) \Delta_{I^c, J^c} \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y} \right) f(x, y)$$

where, for I, J of cardinality k

$$q_{I,J}(x, y; s, t) = \sum_{0 \leq l \leq k} (-1)^l (s)_{(k-l)} (t)_l \sum_{\substack{P \subset I, Q \subset J \\ \#P = \#Q = l}} \epsilon(P : I, Q : J) \Delta_{I^c \cup P, J^c \cup Q}(x) \Delta_{P^c, Q^c}(y) .$$

Proof. Apply the change of variable formula (17) to $p = \det$. \square

There is a real version of these identities and they are obtained by the same method used to prove the *real* Bernstein-Sato identities (see the proof of (12)).

Proposition 2.8. *Let $s, t \in \mathbb{C}$. For any $f \in C^\infty(V \times V)$ and $x, y \in V^\times$*

$$\left[\det \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y} \right) \right] (\det x)^{s,\epsilon} (\det y)^{t,\eta} f(x, y) = (\det x)^{s-1, -\epsilon} (\det y)^{t-1, -\eta} F_{s,t} f(x, y) . \quad (19)$$

3 Knapp-Stein intertwining operators

The definition and properties of the *Knapp-Stein intertwining operators* to be introduced later in this section are based on the study of the two (families of) distributions $(\det x)^{s,\epsilon}$. In a different terminology, there are the *local Zeta functions* on $\text{Mat}(n, \mathbb{R})$. Many authors contributed to the study of these distributions, more generally in the context of simple Jordan algebras or in the context of prehomogeneous vector spaces (see [16], [15],

[7],[14],[11],[4],[3]). For the present situation [1] turned out to be the most complete and most useful reference.

Let first consider the case where $\epsilon = +1$, and write $|\det x|^s$ instead of $(\det x)^{s,+}$. Use the notation $\mathcal{S}(V)$ (resp. $\mathcal{S}'(V)$) for the Schwartz space of smooth rapidly decreasing functions (resp. of tempered distributions) on V . Also define, for $s \in \mathbb{C}$

$$\Gamma_V(s) = \Gamma\left(\frac{s+1}{2}\right) \dots \Gamma\left(\frac{s+m}{2}\right). \quad (20)$$

Proposition 3.1.

i) for any $\varphi \in \mathcal{S}(V)$ the integral $\int_V \varphi(x) |\det(x)|^s dx$ converges for $\Re(s) > -1$ and defines a tempered distribution $T_{s,+}$ on $\mathcal{S}(V)$.

ii) the $\mathcal{S}'(V)$ -valued function $s \mapsto T_{s,+}$ defined for $\Re(s) > -1$ can be analytically continued as a meromorphic function on \mathbb{C} .

iii) the function $s \mapsto \frac{1}{\Gamma_V(s)} T_{s,+}$ extends as an entire function of s (denoted by $\tilde{T}_{s,+}$) with values in the space of tempered distributions.

Proof. See [1] and specially Theorem 5.12. A careful examination of the Γ factors in the normalizing factor $\Gamma_V(s)$ shows that the poles are at $s = -1, -2, \dots$ if $m > 1$ and at $s = -1, -3, \dots$ if $m = 1$. \square

For $f \in \mathcal{S}(V)$, define the *Euclidean Fourier transform* $\mathcal{F}f$ by

$$\mathcal{F}f(x) = \int_V e^{-2i\pi\langle x,y \rangle} f(y) dy.$$

The Fourier transform is extended to various functional spaces, and in particular to the space of tempered distributions $\mathcal{S}'(V)$. Recall the elementary formulæ, for $p \in \mathcal{P}(V)$

$$\mathcal{F}\left(p\left(\frac{\partial}{\partial x}\right)f\right) = p(2i\pi \cdot)\mathcal{F}f, \quad \mathcal{F}(pf) = p\left(-\frac{1}{2i\pi}\frac{\partial}{\partial x}\right)(\mathcal{F}f) \quad (21)$$

Proposition 3.2. *The Fourier transform of the tempered distribution $\tilde{T}_{s,+}$ is given by*

$$\mathcal{F}(\tilde{T}_{s,+}) = \pi^{-\frac{m^2}{2}-ms} \tilde{T}_{-m-s,+} \quad (22)$$

or equivalently

$$\mathcal{F}\left(\frac{1}{\Gamma_V(s)} |\det(\cdot)|^s\right) = \frac{\pi^{-\frac{m^2}{2}-ms}}{\Gamma_V(-s-m)} |\det(\cdot)|^{-m-s}. \quad (23)$$

Proof. See [1] Theorem 4.4 and Theorem 5.12. \square

Now let $\epsilon = -1$. The corresponding results do not seem to have been written, although they could be deduced from [4]. In our approach, the results for $(\det x)^{s,+}$ are used to prove those for $(\det x)^{s,-}$.

Proposition 3.3.

i) for any $\varphi \in \mathcal{S}(V)$ the integral $\int_V \varphi(x)(\det x)^{s,-} dx$ converges for $\Re(s) > -1$ and defines a tempered distribution $T_{s,-}$ on $\mathcal{S}(V)$.

ii) the $\mathcal{S}'(V)$ -valued function $s \mapsto T_{s,-}$ defined for $\Re(s) > -1$ can be analytically continued as a meromorphic function on \mathbb{C} .

iii) the function $s \mapsto \frac{1}{s \Gamma_V(s-1)} T_{s,-}$ extends as an entire function of s (denoted by $\tilde{T}_{s,-}$) with values in $\mathcal{S}'(V)$.

Proof. As a special case of (12), the following identity holds on V^\times

$$\det \left(\frac{\partial}{\partial x} \right) (\det x)^{s+1,+} = (s+1)_m (\det x)^{s,-} . \quad (24)$$

Next

$$\frac{\Gamma_V(s+1)}{\Gamma_V(s-1)} = \frac{\Gamma(\frac{s}{2}+1) \dots \Gamma(\frac{s+m-1}{2}+1)}{\Gamma(\frac{s}{2}) \dots \Gamma(\frac{s+m-1}{2})} = 2^{-m} (s)_m = 2^{-m} \frac{s}{s+m} (s+1)_m .$$

Rewrite (24) as

$$\frac{1}{s \Gamma_V(s-1)} (\det x)^{s,-} = 2^{-m} \frac{1}{s+m} \det \left(\frac{\partial}{\partial x} \right) \left(\frac{1}{\Gamma_V(s+1)} (\det x)^{s+1,+} \right) .$$

For $\Re s$ large enough, both sides extend as continuous functions on V and hence coincide as distributions. Viewed now as a distribution-valued function of s , the right hand side extends holomorphically to all of \mathbb{C} except perhaps at $s = -m$. To get the statements of Proposition 3.3, it suffices to prove that at $s = -m$ the right hand side can be continued as a holomorphic function. In turn this is a consequence of the following lemma.

Lemma 3.1.

$$\det \left(\frac{\partial}{\partial x} \right) (\tilde{T}_{-m+1,+}) = 0 . \quad (25)$$

Proof. The Fourier transform of the distribution $\tilde{T}_{-m+1,+}$ is equal (up to a non vanishing constant) to $\tilde{T}_{-1,+}$ (see (22)). Hence the statement of the lemma is equivalent to

$$(\det x) \tilde{T}_{-1,+} = 0 . \quad (26)$$

But $\tilde{T}_{-1,+}$ (the "first" residue of the meromorphic function $s \mapsto T_{s,+}$) is equal (up to a non vanishing constant) to the quasi-invariant measure on the L -orbit $\mathcal{O}_1 = \{x \in V, \text{rank}(x) = m-1\}$ (see [1] Theorem 5.12). As $\mathcal{O}_1 \subset \{x \in V, \det x = 0\}$, equation (26) follows. \square

This finishes the proof of Proposition 3.3. A careful analysis of the normalization factor $s\Gamma_V(s-1)$ shows that $T_{s,-}$ has poles at $s = -1, -2, -3, \dots$ if $m > 1$, and at $s = -2, -4, \dots$ if $m = 1$. \square

Proposition 3.4.

$$\mathcal{F}(\tilde{T}_{s,-}) = -i^m \pi^{-\frac{m^2}{2} - ms} \tilde{T}_{-m-s,-} . \quad (27)$$

Proof. During the proof of Proposition 3.3, it was established that

$$\tilde{T}_{s,-} = 2^{-m} \frac{1}{s+m} \det \left(\frac{\partial}{\partial x} \right) T_{s+1,+} .$$

Hence, using (23)

$$\mathcal{F}(\tilde{T}_{s,-}) = 2^{-m} \frac{1}{s+m} \pi^{-\frac{m^2}{2} - m(s+1)} (2i\pi)^m (\det x) \tilde{T}_{-s-m-1,+}$$

which, for generic s can be rewritten as

$$i^m \pi^{-\frac{m^2}{2} - ms} \frac{1}{s+m} \frac{1}{\Gamma_V(-s-m-1)} (\det x) T_{-s-m-1,+} .$$

Next, for $\Re(s)$ large enough, $(\det x) T_{s,+} = T_{s+1,-}$, and by analytic continuation this holds for any s where both sides are defined. Use this result to obtain (27) for generic s , and by continuity for all s . \square

For $(s, \epsilon) \in \mathbb{C} \times \{\pm\}$, let

$$\gamma(s, \epsilon) = \begin{cases} \frac{1}{\Gamma_V(s)} & \text{if } \epsilon = 1 \\ \frac{1}{s\Gamma_V(s-1)} & \text{if } \epsilon = -1 \end{cases}$$

so that

$$\tilde{T}_{s,\epsilon} = \gamma(s, \epsilon) T_{s,\epsilon} . \quad (28)$$

Let

$$\rho(s, \epsilon) = \begin{cases} \pi^{-\frac{m^2}{2}-ms} & \text{if } \epsilon = +1 \\ -im\pi^{-\frac{m^2}{2}-ms} & \text{if } \epsilon = -1 \end{cases}$$

so that

$$\mathcal{F}(\tilde{T}_{s,\epsilon}) = \rho(s, \epsilon) \tilde{T}_{-s-m,\epsilon} . \quad (29)$$

The Knapp-Stein intertwining operators play a central role in semi-simple harmonic analysis (see [8] for general results). The present approach takes advantage of the specific situation to give more explicit results.

For $(\lambda, \epsilon) \in \mathbb{C} \times \{\pm 1\}$ consider the following operator (*Knapp-Stein intertwining operator*) (formally) defined by

$$J_{\lambda,\epsilon}f(x) = \int_V \det(x-y)^{-2m+\lambda,\epsilon} f(y) dy . \quad (30)$$

The operator $J_{\lambda,\epsilon}$ verifies the following (formal) intertwining property.

Proposition 3.5. *For any $g \in G$,*

$$J_{\lambda,\epsilon} \circ \pi_{\lambda,\epsilon}(g) = \pi_{2m-\lambda,\epsilon}(g) \circ J_{\lambda,\epsilon} .$$

Proof.

$$J_{\lambda,\epsilon}(\pi_{\lambda,\epsilon}(g)f)(x) = \int_V (\det(x-y))^{-2m+\lambda,\epsilon} \alpha(g^{-1}, y)^{-\lambda,\epsilon} f(g^{-1}(y)) dy$$

which, by using (9) and the cocycle property of α can be rewritten as

$$\alpha(g^{-1}, x)^{-2m+\lambda,\epsilon} \int_V \det(g^{-1}(x) - g^{-1}(y))^{-2m+\lambda,\epsilon} \alpha(g^{-1}, y)^{-2m-\lambda+\lambda,\epsilon^2} dy$$

and use the change of variable $z = g^{-1}(y)$, $dz = |\alpha(g^{-1}, y)|^{-2m} dy$ to get

$$\begin{aligned} J_{\lambda,\epsilon}(\pi_{\lambda,\epsilon}(g)f)(x) &= \alpha(g^{-1}, x)^{-(2m-\lambda),\epsilon} \int_V \det(g^{-1}(x) - z)^{-2m+\lambda,\epsilon} f(z) dz \\ &= \pi_{2m-\lambda,\epsilon}(g)(J_{\lambda,\epsilon}f)(x) . \end{aligned}$$

□

To pass from a formal operator to an actual operator, notice that the Knapp-Stein operator is a convolution operator and hence (30) can be rewritten as

$$J_{\lambda,\epsilon}f = T_{-2m+\lambda,\epsilon} \star f$$

The study of the distributions $T_{s,\pm}$ strongly suggests to define the *normalized* intertwining operator $\tilde{J}_{\lambda,\epsilon}$ by

$$\tilde{J}_{\lambda,\epsilon}f = \tilde{T}_{-2m+\lambda,\epsilon} \star f \quad (31)$$

for $f \in \mathcal{S}(V)$, or more explicitly

$$\begin{aligned} \tilde{J}_{\lambda,+}f(x) &= \frac{1}{\Gamma_V(-2m+\lambda)} \int_V |\det(x-y)|^{-2m+\lambda} f(y) dy, \\ \tilde{J}_{\lambda,-}f(x) &= \frac{1}{(-2m+\lambda)\Gamma_V(-2m+\lambda-1)} \int_V (\det(x-y))^{-2m+\lambda,-} f(y) dy. \end{aligned}$$

The representation $\pi_{\lambda,\epsilon}$ is not properly defined on $\mathcal{S}(V)$, but its infinitesimal version is. In fact, let $\varphi \in C_c^\infty(V)$. For $g \in G$ sufficiently close to the identity, g is defined on the compact $\text{Supp}(\varphi)$, so that the following definition makes sense : for $X \in \mathfrak{g}$ let

$$d\pi_{\lambda,\epsilon}(X)\varphi = \left(\frac{d}{dt} \right)_{t=0} \pi_{\lambda,\epsilon}(\exp tX)\varphi.$$

Moreover, it is well known that the resulting operator $d\pi_{\lambda,\epsilon}(X)$ is a differential operator of order 1 on V with polynomial coefficients, hence can be extended as a continuous operator on the Schwartz space $\mathcal{S}(V)$, and by duality as an operator on $\mathcal{S}'(V)$. An operator $J : \mathcal{S}(V) \rightarrow \mathcal{S}'(V)$ is said to be an intertwining operator w.r.t. $(\pi_{\lambda,\epsilon}, \pi_{2m-\lambda,\epsilon})$ if for any $X \in \mathfrak{g}$,

$$J \circ d\pi_{\lambda,\epsilon}(X) = d\pi_{2m-\lambda,\epsilon}(X) \circ J.$$

The next statement is easily obtained by combining the results on the family of distributions $\tilde{T}_{s,\epsilon}, (s, \epsilon) \in \mathbb{C} \times \{\pm\}$ (see Propositions 3.1, 3.3), and the formal intertwining property.

Proposition 3.6.

- i) the operator $\tilde{J}_{\lambda,\epsilon}$ is a continuous operator from $\mathcal{S}(V)$ into $\mathcal{S}'(V)$.
- ii) the operator $\tilde{J}_{\lambda,\epsilon}$ intertwines the representations $\pi_{\lambda,\epsilon}$ and $\pi_{2m-\lambda,\epsilon}$
- iii) the (operator-valued) function $\lambda \mapsto \tilde{J}_{\lambda,\epsilon}$ is holomorphic.

4 Construction of the families $D_{\lambda,\mu}$ and $B_{\lambda,\mu;k}$

Recall the differential operator $F_{s,t}$ on $V \times V$, constructed in section 4. Define for $s, t \in \mathbb{C}$

$$H_{s,t} = \mathcal{F}^{-1} \circ F_{s,t} \circ \mathcal{F} \quad (32)$$

As $F_{s,t}$ is a differential operator with polynomial coefficients, $H_{s,t}$ is also a differential operator with polynomial coefficients. To be more explicit, according to (21), the passage from $F_{s,t}$ to $H_{s,t}$ consists in changing $p(\frac{\partial}{\partial x}, \frac{\partial}{\partial y})$ to multiplication by $p(-2i\pi x, -2i\pi y)$, and multiplication by $p(x, y)$ to the differential operator $p(\frac{1}{2i\pi} \frac{\partial}{\partial x}, \frac{1}{2i\pi} \frac{\partial}{\partial y})$. Observe that $q_{I,J}$ is homogeneous of degree $2m - k$ and Δ_{I^c, J^c} is homogeneous of degree $m - k$, where $k = \#I = \#J$. This leads to

$$H_{s,t} = \left(\frac{i}{2\pi}\right)^m \sum_{k=0}^m (-1)^k \sum_{\substack{I, J \subset \{1, 2, \dots, m\} \\ \#I = \#J = k}} h_{I,J} \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}; s, t\right) \left(\Delta_{I^c, J^c}(x-y)f(x, y)\right) \quad (33)$$

where the polynomial $h_{I,J}(\xi, \eta; s, t)$ is given by

$$h_{I,J}(\xi, \eta; s, t) = \sum_{0 \leq l \leq k} (s)_{(k-l)} (t)_l \sum_{\substack{P \subset I, Q \subset J \\ \#P = \#Q = l}} \epsilon(P : I, Q : J) \Delta_{I^c \cup P, J^c \cup Q}(\xi) \Delta_{P^c, Q^c}(\eta) .$$

Theorem 4.1. *The operator $H_{m-\lambda, m-\mu}$ is G -covariant with respect to $(\pi_{\lambda, \epsilon} \otimes \pi_{\mu, \eta}, \pi_{\lambda+1, -\epsilon} \otimes \pi_{\mu+1, -\eta})$.*

The (rather long) proof will be given at the end of this section. The next results are preparations for the proof.

Let M be the continuous operator on $\mathcal{S}(V \times V)$ given by

$$M\varphi(x, y) = \det(x - y)\varphi(x, y)$$

Proposition 4.1. *The operator M intertwines $\pi_{\lambda, \epsilon} \otimes \pi_{\mu, \eta}$ and $\pi_{\lambda-1, -\epsilon} \otimes \pi_{\mu-1, -\eta}$.*

Proof. Let $\varphi \in C_c^\infty(V \times V)$. Let $g \in G$, and assume that g is defined on $\text{Supp}(\varphi)$.

$$\begin{aligned} & \left(M \circ (\pi_{\lambda, \epsilon}(g) \otimes \pi_{\mu, \eta}(g)) \varphi \right)(x, y) \\ &= \det(x - y) \alpha(g^{-1}, x)^{-\lambda, \epsilon} \alpha(g^{-1}, y)^{-\mu, \eta} \varphi(g^{-1}(x), g^{-1}(y)) \end{aligned}$$

whereas

$$\begin{aligned} & \left((\pi_{\lambda-1, -\epsilon}(g) \otimes \pi_{\mu-1, -\eta}(g)) \circ M \right) \varphi(x, y) = \\ & \det(g^{-1}(x) - g^{-1}(y)) \alpha(g^{-1}, x)^{-\lambda+1, -\epsilon} \alpha(g^{-1}, y)^{-\mu+1, -\eta} \varphi(g^{-1}(x) - g^{-1}(y)) . \end{aligned}$$

Use (9) to conclude that

$$\left(M \circ (\pi_{\lambda, \epsilon}(g) \otimes \pi_{\mu, \eta}(g)) \right) \varphi = \left((\pi_{\lambda-1, -\epsilon}(g) \otimes \pi_{\mu-1, -\eta}(g)) \circ M \right) \varphi .$$

For $X \in \mathfrak{g}$, and for t small enough, $g_t = \exp tX$ is defined on $Supp(\varphi)$. Apply the previous result to g_t , differentiate w.r.t. t at $t = 0$ to get

$$M \circ (d(\pi_{\lambda, \epsilon} \otimes \pi_{\mu, \eta})(X)) \varphi = (d(\pi_{\lambda-1, -\epsilon} \otimes \pi_{\mu-1, -\eta})(X)) \circ M \varphi$$

for any $\varphi \in C_c^\infty(V \times V)$, and extend this equality to any φ in $\mathcal{S}(V \times V)$ by continuity. \square

The next proposition is the key result towards the proof.

Proposition 4.2. *For $f \in \mathcal{S}(V \times V)$*

$$M \circ (\tilde{J}_{\lambda, \epsilon} \otimes \tilde{J}_{\mu, \eta}) f = d((\lambda, \epsilon), (\mu, \eta)) ((\tilde{J}_{\lambda+1, -\epsilon} \otimes \tilde{J}_{\mu+1, -\eta}) \circ H_{-m+2\lambda, -m+2\mu}) f,$$

where $d((\lambda, \epsilon), (\mu, \eta))$ is equal to

$$\begin{array}{ll} \frac{\pi^{4m^2}}{(\lambda - m) \dots (\lambda - 2m + 2)(\mu - m) \dots (\mu - 2m + 2)} & \epsilon = +1, \eta = +1 \\ \frac{2^{-m} \pi^{4m^2}}{(\lambda - m) \dots (\lambda - 2m + 2)(\mu - m)} & \epsilon = +1, \eta = -1 \\ \frac{2^{-m} \pi^{4m^2}}{(\lambda - m)(\mu - m) \dots (\mu - 2m + 2)} & \epsilon = -1, \eta = +1 \\ \frac{2^{-2m} \pi^{4m^2}}{(\lambda - m)(\mu - m)} & \epsilon = -1, \eta = -1 . \end{array}$$

Proof. As the operators $\tilde{J}_{\lambda, \epsilon}$ and $\tilde{J}_{\mu, \eta}$ are convolution operators by a tempered distribution, the left hand side is well defined as a tempered distribution on $V \times V$, and so is its Fourier transform.

In order to alleviate the proof, c_1, \dots, c_4 are used during the proof to mean complex numbers depending on $\lambda, \epsilon, \mu, \eta$ but neither on f nor on $(x, y) \in V \times V$. Their actual values are listed at the end of the computation. By (23),

$$\begin{aligned} \mathcal{F}((\tilde{J}_{\lambda, \epsilon} \otimes \tilde{J}_{\mu, \eta}) f)(x, y) &= \mathcal{F}(\tilde{T}_{-2m+\lambda, \epsilon})(x) \mathcal{F}(\tilde{T}_{-2m+\mu, \eta})(y) \mathcal{F}f(x, y) \quad (34) \\ &= c_1 \tilde{T}_{m-\lambda, \epsilon}(x) \tilde{T}_{m-\mu, \eta}(x) \mathcal{F}f(x, y) . \end{aligned}$$

Next, for p a polynomial on $V \times V$, and $\Phi \in \mathcal{S}'(V)$,

$$\mathcal{F}(p\Phi)(x, y) = p\left((-2i\pi)^{-1} \frac{\partial}{\partial x}, (-2i\pi)^{-1} \frac{\partial}{\partial y}\right)(\mathcal{F}\Phi)(x, y) .$$

Hence

$$\begin{aligned} & \mathcal{F}(M \circ (\tilde{J}_{\lambda, \epsilon} \otimes \tilde{J}_{\mu, \eta})f)(x, y) \\ &= c_1 c_2 \det \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y} \right) \left((\det x)^{m-\lambda, \epsilon} (\det y)^{m-\mu, \eta} \mathcal{F}f(x, y) \right) . \end{aligned} \quad (35)$$

Assume temporarily that $\Re \lambda, \Re \mu < 0$ so that $(\det x)^{m-\lambda, \epsilon} (\det y)^{m-\mu, \eta}$ is a sufficiently many times differentiable function on $V \times V$. Then, use Proposition 2.8 to get

$$\begin{aligned} & \mathcal{F}(M \circ (\tilde{J}_{\lambda, \epsilon} \otimes \tilde{J}_{\mu, \eta})f)(x, y) \\ &= c_1 c_2 (\det x)^{m-(\lambda+1), -\epsilon} (\det y)^{m-(\mu+1), -\eta} F_{m-\lambda, m-\mu}(\mathcal{F}f)(x, y) , \end{aligned} \quad (36)$$

the equality being valid *a priori* on $V^\times \times V^\times$, but thanks to the assumption on λ and μ it extends to all of $V \times V$. Next, by the definition of the operator $H_{s, t}$,

$$\begin{aligned} & \mathcal{F}(M \circ (\tilde{J}_{\lambda, \epsilon} \otimes \tilde{J}_{\mu, \eta})f)(x, y) \\ &= c_1 c_2 (\det x)^{m-\lambda-1, -\epsilon} (\det y)^{m-\mu-1, -\eta} \mathcal{F}(H_{-m+2\lambda, -m+2\mu}f)(x, y) \\ &= c_1 c_2 c_3 \tilde{T}_{m-\lambda-1, -\epsilon}(x) \tilde{T}_{m-\mu-1, -\eta}(y) \mathcal{F}(H_{m-\lambda, m-\mu}f)(x, y) . \end{aligned} \quad (37)$$

Use inverse Fourier transform and (29) to conclude that

$$M \circ (\tilde{J}_\lambda \otimes \tilde{J}_\mu)f = c_1 c_2 c_3 c_4 \left((\tilde{J}_{\lambda+1, -\epsilon} \otimes \tilde{J}_{\mu+1, -\eta}) \circ H_{m-\lambda, m-\mu} \right) f . \quad (38)$$

The values of the constants $c_1, 2, c_3$ and c_4 are given by

$$\begin{aligned} c_1 &= \rho(-2m + \lambda, \epsilon) \rho(-2m + \mu, \eta) \\ c_2 &= (-1)^m (2\pi)^{-2m} \gamma(m - \lambda, \epsilon) \gamma(m - \mu, \eta) \\ c_3 &= \frac{1}{\gamma(m - \lambda - 1, -\epsilon) \gamma(m - \mu - 1, -\eta)} \\ c_4 &= \frac{1}{\gamma(\lambda + 1, -\epsilon) \gamma(\mu + 1, -\eta)} \end{aligned}$$

so that $c_1 c_2 c_3 c_4$ is equal to

$$\begin{aligned}
& \frac{\pi^{4m^2}}{(\lambda - m) \dots (\lambda - 2m + 2)(\mu - m) \dots (\mu - 2m + 2)} & \epsilon = +1, \eta = +1 \\
& \frac{2^{-m} \pi^{4m^2}}{(\lambda - m) \dots (\lambda - 2m + 2)(\mu - m)} & \epsilon = +1, \eta = -1 \\
& \frac{2^{-m} \pi^{4m^2}}{(\lambda - m)(\mu - m) \dots (\mu - 2m + 2)} & \epsilon = -1, \eta = +1 \\
& \frac{2^{-2m} \pi^{4m^2}}{(\lambda - m)(\mu - m)} & \epsilon = -1, \eta = -1
\end{aligned}$$

By analytic continuation, (4.2) holds for all λ, μ , thus proving Proposition 4.2. Incidentally, notice that the last step implies the vanishing of $((\tilde{J}_{\lambda+1, -\epsilon} \otimes \tilde{J}_{\mu+1, -\eta}) \circ H_{-m+2\lambda, -m+2\mu})$ at the poles of $d((\lambda, \epsilon), (\mu, \eta))$. \square

To finish the proof of Theorem 4.1, note that, by Lemma 4.1 and Proposition 3.6 the operator $M \circ (\tilde{J}_{\lambda, \epsilon} \otimes \tilde{J}_{\mu, \eta})$ is covariant with respect to $(\pi_{\lambda, \epsilon} \otimes \pi_{\mu, \eta}), (\pi_{2m-\lambda-1, -\epsilon} \otimes \pi_{2m-\mu-1, -\eta})$. Using Proposition 4.2, this implies, generically in (λ, μ) that for any $f \in C_c^\infty(V \times V)$ and any $g \in G$ which is defined on $Supp(f)$,

$$\begin{aligned}
& ((\tilde{J}_{\lambda+1, -\epsilon} \otimes \tilde{J}_{\mu+1, -\eta}) \circ (\pi_{\lambda+1, -\epsilon}(g) \otimes \pi_{\mu+1, -\eta}(g))) \circ H_{-m+2\lambda, -m+2\mu}) f \\
& = ((\tilde{J}_{\lambda+1, -\epsilon} \otimes \tilde{J}_{\mu+1, -\epsilon}) \circ H_{m-\lambda, m-\mu} \circ (\pi_{\lambda, \epsilon}(g) \otimes \pi_{\mu, \eta}(g))) f
\end{aligned}$$

Generically in (λ, μ) , the convolution operator $\tilde{J}_{\lambda+1, -\epsilon} \otimes \tilde{J}_{\mu+1, -\eta}$ is injective on $C_c^\infty(V)$ as can be seen after performing a Fourier transform, so that

$$\begin{aligned}
& \left((\pi_{\lambda+1, -\epsilon}(g) \otimes \pi_{\mu+1, -\eta}(g)) \circ H_{m-\lambda, m-\mu} \right) f \\
& = \left(H_{m-\lambda, m-\mu} \circ (\pi_{\lambda, \epsilon}(g) \otimes \pi_{\mu, \eta}(g)) \right) f .
\end{aligned}$$

The covariance of $H_{m-\lambda, m-\mu}$ follows, at least generically in λ, μ and hence everywhere by analytic continuation. This completes the proof of Theorem 4.1.

For convenience in the sequel, let shift the parameters in the notation by setting

$$D_{\lambda, \mu} = H_{m-\lambda, m-\mu} .$$

Perhaps is it enlightening to state a version of Theorem 4.1 in the compact picture. Going back to the notation of the Introduction, the tensor product

$\mathcal{E}_{\lambda,\epsilon} \boxtimes \mathcal{E}_{\mu,\eta}$ can be completed to a space $\mathcal{E}_{(\lambda,\epsilon),(\mu,\eta)}$ of smooth sections of the line bundle $E_{\lambda,\mu} \boxtimes E_{\mu,\eta}$ over $X \times X$. The operator M can also be transferred as a continuous operator from $\mathcal{E}_{(\lambda,\epsilon),(\mu,\eta)}$ into $\mathcal{E}_{(\lambda-1,-\epsilon),(\mu-1,-\eta)}$. Denote by $\tilde{I}_{\lambda,\epsilon} : \mathcal{E}_{\lambda,\epsilon}$ into $\mathcal{E}_{2m-\lambda,\epsilon}$ the normalized Knapp-Stein operator, which corresponds to $\tilde{J}_{\lambda,\epsilon}$ in the principal chart. The formulation to be given below is a consequence of Theorem 4.1, using the well-known fact that the Knapp-Stein intertwining operators are invertible, at least generically in λ , the inverse of $\tilde{I}_{\lambda,\epsilon}$ being equal (up to a scalar) to $\tilde{I}_{2m-\lambda,\epsilon}$.

Theorem 4.2. *The operator $D_{(\lambda,\epsilon),(\mu,\eta)}$ defined as*

$$D_{(\lambda,\epsilon),(\mu,\eta)} = \left(\tilde{I}_{2m-\lambda-1,-\epsilon} \otimes \tilde{I}_{2m-\mu-1,-\eta} \right) \circ M \circ \left(\tilde{I}_{\lambda,\epsilon} \otimes \tilde{I}_{\mu,\eta} \right)$$

which, by construction intertwines $\pi_{\lambda,\epsilon} \otimes \pi_{\mu,\eta}$ and $\pi_{\lambda+1,-\epsilon} \otimes \pi_{\mu+1,-\eta}$ (as representations of G) is a differential operator on $X \times X$.

Let $\text{res} : C^\infty(V \times V) \longrightarrow C^\infty(V)$ be the *restriction map* defined by

$$\text{res}(\varphi)(x) = \varphi(x, x) .$$

For any λ, ϵ and μ, η in $\mathbb{C} \times \{\pm\}$, the restriction map intertwines the representations $\pi_{\lambda,\epsilon} \otimes \pi_{\mu,\eta}$ and $\pi_{\lambda+\mu,\epsilon\eta}$.

Let $\lambda, \mu \in \mathbb{C}$, and $k \in \mathbb{N}$. Let $B_{\lambda,\mu,k} : C^\infty(V \times V) \longrightarrow C^\infty(V)$ be the bi-differential operator defined by

$$B_{\lambda,\mu,k} = \text{res} \circ D_{\lambda+k-1,\mu+k-1} \circ \cdots \circ D_{\lambda,\mu} .$$

The covariance property of the operators $D_{\lambda,\mu}$ and of res imply the following result.

Theorem 4.3. *Let $(\lambda, \epsilon), (\mu, \eta)$ be in $\mathbb{C} \times \{\pm\}$. The operator $B_{\lambda,\mu,k}$ is covariant w.r.t. $(\pi_{\lambda,\epsilon} \otimes \pi_{\mu,\eta}, \pi_{\lambda+\mu+2k,\epsilon\eta})$.*

A remarkable fact is that whereas the operator $H_{\lambda,\mu}$ has polynomial functions as coefficients, the operator $B_{\lambda,\mu,k}$ has *constant* coefficients, i.e. is of the form

$$\varphi \longmapsto \sum_{\alpha,\beta} a_{\alpha,\beta} \left(\frac{\partial^{|\alpha|+|\beta|}}{\partial y^\alpha \partial z^\beta} \varphi \right) (x, x)$$

where $a_{\alpha,\beta}$ are complex numbers. In fact, this is merely a consequence of the invariance of the $B_{\lambda,\mu,k}$ under the action of the translations (action of \overline{N}). More concretely, this is due to the vanishing on the diagonal $\text{diag}(V)$ of many of the coefficients of the operators $H_{\lambda,\mu}$. It seems however difficult to find a closed formula for the coefficients of $B_{\lambda,\mu,k}$ except if $m = 1$.

5 The case $m = 1$ and the Ω -process

For $m = 1$, a simple calculation yields

$$F_{s,t}f = (-tx + sy)f + xy \left(\frac{\partial^2}{\partial x \partial y} \right) f \quad (39)$$

$$H_{s,t}f = \frac{1}{2i\pi} \left((- (t-1) \frac{\partial}{\partial x} f + (s-1) \frac{\partial}{\partial y} f) - (x-y) \frac{\partial^2 f}{\partial x \partial y} \right) . \quad (40)$$

$$D_{\lambda,\mu} = \frac{1}{2i\pi} \left(\mu \frac{\partial}{\partial x} - \lambda \frac{\partial}{\partial y} - (x-y) \frac{\partial^2}{\partial x \partial y} \right) \quad (41)$$

There is a relation with the Ω -process, which we now recall following the classical spirit (see e.g.[12]), but in terms adapted to our situation.

Let $(\lambda, \epsilon) \in \mathbb{C} \times \{\pm\}$ and let $\mathcal{F}_{\lambda,\epsilon}$ be the space of smooth functions defined on $\mathbb{R}^2 \setminus \{0\}$ which satisfy

$$\forall t \in \mathbb{R}^* \quad F(tx_1, tx_2) = t^{-\lambda, \epsilon} F(x_1, x_2) .$$

To $F \in \mathcal{F}_{\lambda,\epsilon}$ associate the function f given by $f(x) = F(x, 1)$. Then f is a smooth function on \mathbb{R} , and F can be recovered from f by

$$F(x_1, x_2) = x_2^{-\lambda, \epsilon} f\left(\frac{x_1}{x_2}\right) ,$$

at least for $x_2 \neq 0$ and then extended by continuity.

Let $g \in SL_2(\mathbb{R})$ and let $g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The function $F \circ g^{-1}$ also belongs to $\mathcal{F}_{\lambda,\epsilon}$, and is explicitly given by

$$F \circ g^{-1}(x_1, x_2) = F(ax_1 + bx_2, cx_1 + dx_2)$$

Its associated function on \mathbb{R} is given by

$$(F \circ g^{-1})(x, 1) = F(ax + b, cx + d) = (cx + d)^{-\lambda, \epsilon} f\left(\frac{ax + b}{cx + d}\right) ,$$

so that the natural action of $G = SL(2, \mathbb{R})$ on $\mathcal{F}_{\lambda,\epsilon}$ is but another realization of the representation $\pi_{\lambda,\epsilon}$.

Now let $(\lambda, \epsilon), (\mu, \eta) \in \mathbb{C} \times \{\pm\}$ and consider the space $\mathcal{F}_{(\lambda,\epsilon),(\mu,\eta)}$ of smooth functions F on $\mathbb{R}^2 \setminus \{0\} \times \mathbb{R}^2 \setminus \{0\}$ which satisfy

$$\forall t, s \in \mathbb{R}^*, \quad F(t(x_1, x_2), s(y_1, y_2)) = t^{-\lambda, \epsilon} s^{-\mu, \eta} F((x_1, x_2), (y_1, y_2)) .$$

The group $SL_2(\mathbb{R})$ acts naturally (diagonally) on $\mathcal{F}_{(\lambda,\epsilon),(\mu,\eta)}$, and this action yields a realization of $\pi_{\lambda,\epsilon} \otimes \pi_{\mu,\eta}$. More explicitly, let $f(x, y) = F((x, 1), (y, 1))$.

Then for $g \in SL_2(\mathbb{R})$ such that $g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$F \circ g^{-1}((x, 1), (y, 1)) = (cx + d)^{-\lambda, \epsilon} (cy + d)^{-\mu, \eta} f\left(\frac{ax + b}{cx + d}, \frac{ay + b}{cy + d}\right).$$

The polynomial $\det \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix}$ is invariant by the action of $SL_2(\mathbb{R})$ and so is the differential operator

$$\Omega = \frac{\partial^2}{\partial x_1 \partial y_2} - \frac{\partial^2}{\partial x_2 \partial y_1}.$$

The operator Ω maps $\mathcal{F}_{(\lambda,\epsilon),(\mu,\eta)}$ to $\mathcal{F}_{(\lambda+1,-\epsilon),(\mu+1,-\eta)}$ and yields a covariant differential w.r.t. $(\pi_{\lambda,\epsilon} \otimes \pi_{\mu,\eta}, \pi_{\lambda+1,-\epsilon} \otimes \pi_{\mu+1,-\eta})$.

Let $F \in \mathcal{F}_{(\lambda,\epsilon),(\mu,\eta)}$. As above, let f be the function on $\mathbb{R} \times \mathbb{R}$ obtained by deshomogenization of F i.e. $f(x, y) = F((x, 1), (y, 1))$. The corresponding differential operator on $\mathbb{R} \times \mathbb{R}$ is given by

$$\omega_{\lambda,\mu} f(x, y) = (\Omega F)((x, 1), (y, 1)) = -\mu \frac{\partial f}{\partial x} + \lambda \frac{\partial f}{\partial y} + (x - y) \frac{\partial^2 f}{\partial x \partial y},$$

independently of ϵ and η , so that $D_{\lambda,\mu} = -2i\pi\omega_{\lambda,\mu}$.

For $k \in \mathbb{N}$, let $R_k : C^\infty(\mathbb{R}^2 \times \mathbb{R}^2) \mapsto C^\infty(\mathbb{R}^2)$ be the bi-differential operator given by $R_k = \text{res} \circ \Omega^k$ or more explicitly

$$x \in V, \quad R_k F(x) = \Omega^k F(x, x) \quad (42)$$

The operator R_k commutes to the action of $SL(2, \mathbb{R})$. If F belongs to $\mathcal{F}_{(\lambda,\epsilon),(\mu,\eta)}$, the function $R_k F$ is homogeneous of degree $(\lambda + \mu + 2k, \epsilon\eta)$. By deshomogenization, the corresponding operator is

$$r_{\lambda,\mu;k} = \text{res} \circ \omega_{\lambda+k-1,\mu+k-1} \circ \cdots \circ \omega_{\lambda,\mu}$$

so that $B_{\lambda,\mu;k} = (-2i\pi)^k r_{\lambda,\mu;k}$.

A classical computation in the theory of the Ω -process yields an explicit expression for $r_{\lambda,\mu,k}$

$$r_{\lambda,\mu;k} = \text{res} \circ \left(k! \sum_{i+j=k} (-1)^j \binom{-\lambda-i}{j} \binom{-\mu-j}{i} \frac{\partial^k}{\partial x^i \partial y^j} \right). \quad (43)$$

The computation can be found in [13], where the indices λ and μ are supposed to be negative integers, but the computation goes through without this assumption.

Two special cases are worth being reported, both corresponding to cases where the representations $\pi_{\lambda,\epsilon}, \pi_{\mu,\eta}$ are *reducible*.

Suppose that $\lambda = k \in \mathbb{Z}$. Choose $\epsilon = (-1)^k$, so that for any $t \in \mathbb{R}^*$, $t^{\lambda,\epsilon} = t^k$. Then for $g \in G$ such that $g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$\pi_{k,(-1)^k}(g)f(x) = (cx + d)^{-k} f\left(\frac{ax + b}{cx + d}\right).$$

Let first consider the case where $\lambda \in -\mathbb{N}$, say $\lambda = -l, l \in \mathbb{N}$. Then the space \mathcal{P}_l of polynomials of degree less than l is preserved by the representation $\pi_{-l,(-1)^l}$. Similarly, let $\mu = -m$ for some $m \in \mathbb{N}$. Let $p \in \mathcal{P}_l, q \in \mathcal{P}_m$. Let P (resp. Q) be the homogeneous polynomial on \mathbb{R}^2 obtained by homogenization of p (resp. q). For $k \leq \inf(l, m)$, the function $R_k(P \otimes Q)$ is a polynomial which is homogeneous of degree $l + m - 2k$ and which in the classical theory of invariants is called the k^{th} *transvectant* of P and Q usually denoted by $[P, Q]_k$. So $B_{-l,-m;k}$ just expresses the k -th transvectant at the level of inhomogeneous polynomials.

Now suppose that $\lambda = l, l \in \mathbb{N}$. Then restrictions of holomorphic functions to \mathbb{R} are preserved by the representation $\pi_{l,(-1)^l}$. Suppose also $\mu = m \in \mathbb{N}$. Then the operators $D_{l,m}$ and $B_{l,m,k}$, extended as holomorphic differential operators are still covariant under the action of G . If f is an automorphic form of degree l and g of degree m , then the covariance property of $B_{l,m;k}$ implies that $B_{l,m,k}(f \otimes g)$ is an automorphic form of degree $l + m + 2k$. The operators $B_{l,m;k}$ essentially coincide with the *Rankin-Cohen brackets*, as easily deduced from formula (43).

6 The general case and some open problems

When $m \geq 2$, the Ω -process can be extended along the same lines (see [13]). Let $\mathcal{F}_{\lambda,\epsilon}$ be the space of functions $F : V \times V$ which are *determinantially homogeneous of weight* (λ, ϵ) , i.e. satisfying

$$\forall \gamma \in GL(V) \quad F(x\gamma, y\gamma) = (\det \gamma)^{-\lambda,\epsilon} F(x, y).$$

To such a function F , associate the function f on V defined by $f(x) = F(x, \mathbf{1}_m)$. Then F can be recovered from f by

$$F(x, y) = (\det y)^{-\lambda,\epsilon} f(xy^{-1}), \quad (44)$$

at least when $y \in V^\times$ and everywhere by continuity.

The group $G = SL(2m, \mathbb{R})$ acts on $V \times V$ by left multiplication, i.e. if

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$(g, (x, y)) \longmapsto g \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}.$$

The determinantal homogeneity of functions is preserved by this action, and hence the representation of G on $\mathcal{F}_{\lambda, \epsilon}$ is but another realization of $\pi_{\lambda, \epsilon}$ as can be seen by transferring the action through the correspondence $F \mapsto f$ given by (44). Using this time the polynomial $\det_{2m} \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix}$, an operator Ω can be defined along the same line as in the case $m = 1$. As the action of G commutes to the action (on the right) of $GL(V)$, Ω^k maps $\mathcal{F}_{\lambda, \epsilon} \otimes F_{\mu, \eta}$ into $\mathcal{F}_{\lambda+1, -\epsilon} \otimes F_{\mu+1, -\eta}$ and is covariant for the action of G . Again, using the correspondence $F \mapsto f$, Ω lifts to a differential operator on $V \times V$ which is covariant w.r.t. $(\pi_{\lambda, \epsilon} \otimes \pi_{\mu, \eta}, \pi_{\lambda+1, -\epsilon} \otimes \pi_{\mu+1, -\eta})$ and which can be used for defining the covariant bi-differential operators.

In case $m \geq 2$, it is not clear whether the two approaches coincide, as computations get very complicated. One way of attacking this question would be to determine the dimension of the space of covariant bi-differential operators w.r.t. $(\pi_{\lambda, \epsilon} \otimes \pi_{\mu, \eta}, \pi_{\lambda+\mu+2k, \epsilon, \eta})$, and/or the dimension of the space of covariant differential operators on $V \times V$ w.r.t. $(\pi_{\lambda, \epsilon} \otimes \pi_{\mu, \eta}, \pi_{\lambda+1, -\epsilon} \otimes \pi_{\mu+1, -\eta})$. If the answer (to either question) were one (at least generically in (λ, μ)), then the two approaches would coincide.

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