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Singular conformally invariant trilinear forms, II

The higher multiplicity cases

Jean-Louis Clerc

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Abstract

Let S be the sphere of dimension $n - 1, n \geq 4$. Let $(\pi_\lambda)_{\lambda \in \mathbb{C}}$ be the scalar principal series of representations of the conformal group $SO_0(1, n)$, realized on $\mathcal{C}^\infty(S)$. For $\lambda = (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{C}^3$, let $Tri(\lambda)$ be the space of continuous trilinear forms on $\mathcal{C}^\infty(S) \times \mathcal{C}^\infty(S) \times \mathcal{C}^\infty(S)$ which are invariant under $\pi_{\lambda_1} \otimes \pi_{\lambda_2} \otimes \pi_{\lambda_3}$. For each value of λ , the dimension of $Tri(\lambda)$ is computed and a basis of $Tri(\lambda)$ is described.

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Introduction

This article continues the study of conformally invariant trilinear forms on the sphere $S = S^{n-1}, n \geq 4$ (see [4, 1]), and more specifically achieves the work begun in [3]. Notation is same as in [3] and sections of the present paper are numbered in continuity with those of [3].

For $\lambda = (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{C}^3$ let $Tri(\lambda)$ be the space of continuous trilinear forms on $\mathcal{C}^\infty(S) \times \mathcal{C}^\infty(S) \times \mathcal{C}^\infty(S)$ which are invariant under the action of the conformal group $G = SO_0(1, n)$ by (the tensor product of) three representations of the scalar principal series $(\pi_{\lambda_1}, \pi_{\lambda_2}, \pi_{\lambda_3})$. Such trilinear forms, when viewed as distributions on $S \times S \times S$ are said to be λ -invariant. In [4] was constructed (by analytic continuation) a family \mathcal{K}^λ of λ -invariant trilinear forms, depending *meromorphically* on λ . Later, in [3], a *holomorphic* family $\tilde{\mathcal{K}}^\lambda$ was introduced by renormalizing the family \mathcal{K}^λ . The zero set Z of the

function $\lambda \mapsto \tilde{\mathcal{K}}^\lambda$ was computed, and a *generic* multiplicity 1 theorem was proved, namely

$$\lambda \notin Z \iff \dim \operatorname{Tri}(\lambda) = 1 \iff \operatorname{Tri}(\lambda) = \mathbb{C}\tilde{\mathcal{K}}^\lambda .$$

The main purpose of this article is to compute $\dim \operatorname{Tri}(\lambda)$ when $\lambda \in Z$, and to give a basis of $\operatorname{Tri}(\lambda)$.

The fine structure of Z is studied in section 8. The set Z is a denumerable union of affine lines in \mathbb{C}^3 , which come in six families, each family being made of parallel lines. There is a natural partition of Z , namely

$$Z = Z_1 \cup Z_2 \cup Z_3$$

where Z_d is the subset of points of Z which belong to exactly d lines of Z (compare [9]). Notice that Z_2 and Z_3 are denumerable sets of isolated points in \mathbb{C}^3 .

The main result of the article is the following.

Theorem 1. *Let $\lambda \in Z$.*

$$\begin{aligned} \text{for } \lambda \in Z_1 & \quad \dim \operatorname{Tri}(\lambda) = 2 \\ \text{for } \lambda \in Z_2 & \quad \dim \operatorname{Tri}(\lambda) = 2 \\ \text{for } \lambda \in Z_3 & \quad \dim \operatorname{Tri}(\lambda) = 3 . \end{aligned}$$

These results may be heuristically anticipated as follows. Let $\lambda_0 \in Z$. Let $\lambda(s)$ be a holomorphic curve in \mathbb{C}^3 such that $\lambda(0) = \lambda_0$. Then $s \mapsto \tilde{\mathcal{K}}^{\lambda(s)}$ is a distribution-valued holomorphic function. The first non vanishing term of its Taylor expansion at $s = 0$ clearly is a λ_0 -invariant distribution. Variations on this idea lead to the following considerations.

When λ_0 is in Z_1 , hence belongs to a unique affine line $d_1 \subset Z$, then $\nabla_1 \tilde{\mathcal{K}}^\lambda(\lambda_0) = 0$, where ∇_1 stands for the derivative in the direction of d_1 . Let \vec{d}_2 and \vec{d}_3 be two complementary directions to \vec{d}_1 in \mathbb{C}^3 . Then $\nabla_2 \tilde{\mathcal{K}}^\lambda(\lambda_0)$ and $\nabla_3 \tilde{\mathcal{K}}^\lambda(\lambda_0)$ are “natural” candidates to generate $\operatorname{Tri}(\lambda_0)$.

When λ_0 is in Z_2 , hence belongs to two distinct lines d_1 and d_2 , let \vec{d}_3 be a complementary direction to \vec{d}_1 and \vec{d}_2 . Then $\nabla_1 \tilde{\mathcal{K}}^\lambda(\lambda_0) = \nabla_2 \tilde{\mathcal{K}}^\lambda(\lambda_0) = 0$. In this case, the second mixed derivative $\nabla_1 \nabla_2 \tilde{\mathcal{K}}^\lambda(\lambda_0)$ is easily seen to be a λ_0 -invariant distribution. Then $\nabla_3 \tilde{\mathcal{K}}^\lambda(\lambda_0)$ and $\nabla_1 \nabla_2 \tilde{\mathcal{K}}^\lambda(\lambda_0)$ are “natural” candidates to generate $\operatorname{Tri}(\lambda_0)$.

When λ_0 is in Z_3 , hence belong to three lines d_1, d_2, d_3 (it turns out that $\vec{d}_1, \vec{d}_2, \vec{d}_3$ form a basis of \mathbb{C}^3), then all partial derivatives of $\tilde{\mathcal{K}}^\lambda$ vanish at λ_0

and any second order partial derivative of $\tilde{\mathcal{K}}^\lambda(\lambda)$ at λ_0 is λ_0 -invariant. Then $\nabla_j \nabla_j \tilde{\mathcal{K}}^\lambda(\lambda_0) = 0$ for $j = 1, 2, 3$, so that the three second mixed derivatives $\nabla_i \nabla_j \tilde{\mathcal{K}}^\lambda(\lambda_0) = 0$, ($1 \leq i < j \leq 3$) are “natural” candidates to generate $Tri(\lambda_0)$.

When going to the actual construction of these invariant distributions and proof that they indeed generate $Tri(\lambda)$, a more refined (and more technical) partition of Z is needed, essentially related to the type of the poles (see details in section 8).

An important ingredient of this article is the connection with the results obtained in [1], where residues (in a generic sense) of the meromorphic function $\lambda \mapsto \mathcal{K}^\lambda$ were computed. The normalization (passage from \mathcal{K}^λ to $\tilde{\mathcal{K}}^\lambda$) involves four Γ factors. At a generic pole λ_0 (i.e. belonging to a unique plane of poles) exactly one Γ factor becomes singular at λ_0 , and $\tilde{\mathcal{K}}^{\lambda_0}$ is easily seen to be a multiple of the residue of \mathcal{K}^λ at λ_0 . For more singular poles, some variation of this relation does make sense. For each plane of poles, the residues at generic poles form a meromorphic family, which can be renormalized to get a holomorphic family. If λ_0 is a non generic pole (i.e. belongs to several planes of poles, so that several Γ factors of the renormalization vanish at λ_0), then the values at λ_0 of the renormalized families of residues corresponding to the planes of poles containing λ_0 are equal (up to constants) to partial derivatives of $\tilde{\mathcal{K}}^\lambda$ at λ_0 .

To each plane of poles, a holomorphic family of invariant distributions is constructed, which generically coincides (up to a scalar) with the residue. Recall that planes of poles come in two types called type I and type II (see [4, 3]).

For a plane of poles of type I, the family (generically denoted by $\tilde{\mathcal{T}}$) constructed in section 9 is obtained by renormalizing the meromorphic family constructed in [1]. The distributions are supported on a submanifold of codimension $n - 1$ and their expression involve *covariant differential operators*.

For a plane of poles of type II, the family (generically denoted by \mathcal{S}) is constructed in section 11. Although based on computations done in [1], the construction as done in this article use new arguments and in particular a more satisfactory determination of the (partial) Bernstein-Sato polynomial for the kernel k_α (Proposition 11.3), which has its own interest. The distributions are supported on the diagonal of $S \times S \times S$ and their expressions involve *covariant bi-differential operators*.

A last family (generically denoted by \mathcal{R}) is constructed in section 12 and has a different nature. The planes in \mathbb{C}^3 to which they are attached

are not planes of poles. I was led to introduce this family by the heuristic consideration above about derivatives of $\tilde{\mathcal{K}}^\lambda$, when dealing with poles of type II which are in Z_1 . The corresponding distributions generically have full support in $S \times S \times S$.

These holomorphic families will be almost sufficient for describing bases of $Tri(\lambda)$. However, when λ belongs to Z_2 , some exceptional distributions, denoted by $\mathcal{Q}_{l_1, m_2, m_3}$ are needed. They are obtained as second order derivatives of the function $\tilde{\mathcal{K}}^\lambda$.

Let us observe that all the distributions involved in the description of the invariant trilinear forms are in the *closure* of the meromorphic family \mathcal{K}^λ in the sense of Oshima (see [10] section 6).

Having constructed candidates for a basis of $Tri(\lambda)$, a proof of their linear independence is required. This is usually obtained by using their K -coefficients¹. In [3] was introduced the family $(p_{a_1, a_2, a_3})_{a_1, a_2, a_3 \in \mathbb{N}}$ of polynomial functions on $S \times S \times S$ which have the property that a K -invariant distribution T is equal to 0 if and only if all its K -coefficients $T(p_{a_1, a_2, a_3})$ are equal to 0. It turns out that it is possible to *explicitly* compute the K -coefficients of all the distributions we are concerned with, which allows to answer the questions of independence. The main ingredient is the evaluation of an integral (Proposition 3.2 in [3]), related to the computation of so-called *Bernstein-Reznikov integrals* (see [5] for more examples), and which is recalled for convenience in appendix A1.

On the other hand, the verification that the candidates indeed generate $Tri(\lambda)$ is more difficult and involve some delicate analysis dealing with singular distributions. This sort of problem was already encountered in establishing the multiplicity 1 theorem for $\lambda \notin Z$. We refer to the presentation of the main ideas in section 5 and 6 of [3]. The proofs are based on two techniques. The first one consists in non-extension results, obtained in [3] and presented (slightly reformulated) in appendix A2. The second technique concerns the determination of the dimension of the space $Tri(\lambda, diag)$ of λ -invariant distributions which are supported on the diagonal of $S \times S \times S$. The usual technique of Bruhat (see [2]) is not powerful enough to give a sharp bound for this dimension, due to the fact that the stabilizer of a generic point in the diagonal is a parabolic subgroup of G , hence *not reductive* in G . So it is necessary to make a specific study. Those singular distributions are in 1-1 correspondence with *covariant bi-differential operators*. In turn those have been studied in [11] and their determination is reduced to solving an explicit system of linear homogeneous equations. A systematic study of this

¹Recall that $K \simeq SO(n)$ is a maximal compact subgroup of G

system is presented in appendix A3. However a better understanding (i.e. based on geometry and/or harmonic analysis) of these results is desirable. As it is well-known, the determination of the covariant bi-differential operators is linked to problems about tensor product of two generalized Verma modules (see [12] for some insight in this question).

Let us finally mention two connected problems which are not treated in this (already long) article. First, when one (or several) of the representations $\pi_{\lambda_j}, 1 \leq j \leq 3$ are *reducible*, the question arises whether a (or which) λ -invariant trilinear distribution(s) do generate (by restriction and/or passage to quotients) non trivial invariant trilinear forms on the subspaces/quotients. Only one such situation is treated in this article, namely a case where it leads to trilinear forms on three finite dimensional representations (see 15.3). Secondly let mention that it is possible to prove factorization results for covariant bi-differential operators, similar to the results obtained by A. Juhl for conformally covariant differential operators from S^{d+1} to S^d (see [7, 8]).

Summary

8. Refined description of Z

9. The holomorphic families $\tilde{\mathcal{T}}_{\cdot, \cdot}^{(j, k)}$

10. The multiplicity 2 theorem for $\alpha \in Z_{1, I}$

10.1 The general case

10.2 The special case

11. The holomorphic families $\mathcal{S}_{\lambda_1, \lambda_2}^{(k)}$

11.1 The family $\tilde{E}_{\lambda_1, \lambda_2}$

11.2 A Bernstein-Sato identity for the kernel k_α

11.3 The families $\mathcal{S}_{\lambda_1, \lambda_2}^{(k)}$

12. The holomorphic families $\mathcal{R}_{\cdot, \cdot}^{(j, m)}$

13. The multiplicity 2 theorem for $\lambda \in Z_{1, II}$

13.1 The general case

13.2 The special case

14. The multiplicity 2 theorem for $\alpha \in Z_{2, I}$

15. The multiplicity 2 theorem for $\alpha \in Z_{2, II}$

15.1 The odd case

15.2 The even case

15.3 Invariant trilinear form for three f.d. representations

16. The multiplicity 3 theorem for $\alpha \in Z_3$

16.1 The odd case

16.1 The even case

Appendix

A1 Evaluation of the K -coefficients of \tilde{K}_α

A2 Non extension results

A3 Discussion of the system $S(\lambda_1, \lambda_2; k)$

8 Refined description of Z

For $\epsilon \in \{+, -\}$, $j \in \{1, 2, 3\}$, $l \in \mathbb{N}$ and $m \in \mathbb{Z}$, define

$$\mathcal{D}_{l,m}^{\epsilon,j} = \{ \lambda \in \mathbb{C}^3, \quad \lambda_j = -\rho - l, \quad \lambda_{j+1} + \epsilon \lambda_{j+2} = m \} \quad (1)$$

with the convention that $j+1$ and $j+2$ are interpreted modulo 3. For each quadruple (ϵ, j, l, m) , $\mathcal{D}_{l,m}^{\epsilon,j}$ is a complex affine line in \mathbb{C}^3 . A quadruple (ϵ, j, l, m) is said to be *admissible* if

$$m \equiv l \pmod{2}, \quad |m| \leq l.$$

Theorem 4.2 of [3] can be rewritten as

$$Z = \bigcup_{\substack{(\epsilon, j, l, m) \\ \text{admissible}}} \mathcal{D}_{l,m}^{\epsilon,j}. \quad (2)$$

The description of these lines can also be made in terms of the geometric² parameter α . The line $\mathcal{D}_{l,m}^{-1,j}$ admits the equations

$$\alpha_{j+1} = -(n-1) - 2k_{j+1}, \quad \alpha_{j+2} = -(n-1) - 2k_{j+2} \quad (3)$$

where $k_{j+1} = \frac{l+m}{2}$, $k_{j+2} = \frac{l-m}{2} \in \mathbb{N}$. The line $\mathcal{D}_{l,m}^{+1,j}$ admits the equations

$$\alpha_1 + \alpha_2 + \alpha_3 = -2(n-1) - 2k, \quad \alpha_j = 2p \quad (4)$$

where $k = \frac{l-m}{2}$, $p = \frac{l+m}{2} \in \mathbb{N}$. This justifies the following terminology.

²for the correspondance between the *spectral parameter* λ and the *geometric parameter* α , see [3] 2.1

Definition 8.1. A line \mathcal{D} contained in Z will be called

- i) of type I (or more precisely of type I_j) if $\mathcal{D} = \mathcal{D}_{l,m}^{-,j}$ for some admissible quadruplet $(-, j, l, m)$
- ii) of type II (or more precisely of type II_j) if $\mathcal{D} = \mathcal{D}_{l,m}^{+,j}$ for some admissible quadruplet $(+, j, l, m)$.

Proposition 8.1.

- i) if a line of type I_j and a line of type $II_{j'}$ intersect, then $j = j'$
- ii) if a line of type I_j and a (distinct) line of type $I_{j'}$ intersect, then $j \neq j'$
- iii) if a line of type II_j and a (distinct) line of type $II_{j'}$ intersect, then $j \neq j'$.

Proof. For i), let $\mathcal{D} = \mathcal{D}_{l_1, m_1}^{+,1}$ and $\mathcal{D}' = \mathcal{D}_{l_2, m_2}^{-,2}$. Then we need to show that $\mathcal{D}_{l_1, m_1}^{+,1} \cap \mathcal{D}_{l_2, m_2}^{-,2} = \emptyset$. The intersection is given by the set of equations

$$\lambda_1 = -\rho - l_1, \quad \lambda_2 + \lambda_3 = m_1, \quad \lambda_2 = -\rho - l_2, \quad \lambda_3 - \lambda_1 = m_2.$$

The equations imply

$$-\lambda_1 + \lambda_2 + \lambda_3 = \rho + l_1 + m_1 = -\rho - l_2 + m_2.$$

Hence a necessary condition for a nonvoid intersection is that

$$l_1 + m_1 = -(n-1) - l_2 + m_2.$$

As $|m_1| \leq l_1$ and $|m_2| \leq l_2$, the LHS is nonnegative, whereas the RHS is $\leq -(n-1)$. Hence i) is proved.

For ii) observe that two *distinct* lines $\mathcal{D}_{l,m}^{-,j}$ and $\mathcal{D}_{l',m'}^{-,j}$ necessarily belong to two *distinct* parallel planes, hence cannot intersect. A similar observation for $\mathcal{D}_{l,m}^{+,j}$ and $\mathcal{D}_{l',m'}^{+,j}$, thus proving iii). \square

As a consequence, to determine all possible cases where two distinct lines in Z intersect, it suffices to examine the cases where $\epsilon' = -\epsilon$ and $j = j'$, and the cases where $\epsilon = \epsilon'$ and $j \neq j'$.

Lemma 8.1. Let $(+, 1, l, m)$ and $(-, 1, l', m')$ be admissible quadruples.

- i) if $l \neq l'$, then

$$\mathcal{D}_{l,m}^{+,1} \cap \mathcal{D}_{l',m'}^{-,1} = \emptyset.$$

- ii) if $l = l'$, then

$$\mathcal{D}_{l,m}^{+,1} \cap \mathcal{D}_{l,m'}^{-,1} = \left\{ \left(-\rho - l, \frac{m+m'}{2}, \frac{m-m'}{2} \right) \right\}. \quad (5)$$

Proof. An element λ in $\mathcal{D}_{l,m}^{+,1} \cap \mathcal{D}_{l',m'}^{-,1}$ has to satisfy the equations

$$\lambda_1 = -\rho - l, \quad \lambda_2 + \lambda_3 = m, \quad \lambda_1 = -\rho - l', \quad \lambda_2 - \lambda_3 = m'$$

The intersection is clearly empty if $l \neq l'$. If $l = l'$, then the two lines intersect and the intersection point is equal to $(-\rho - l, \frac{m+m'}{2}, \frac{m-m'}{2})$. \square

Lemma 8.2. *Let $(+, 1, l_1, m_1)$ and $(+, 2, l_2, m_2)$ be admissible quadruples.*

i) if $l_1 - m_1 \neq l_2 - m_2$, then

$$\mathcal{D}_{l_1, m_1}^{+,1} \cap \mathcal{D}_{l_2, m_2}^{+,2} = \emptyset .$$

ii) if $l_1 - m_1 = l_2 - m_2$, then

$$\mathcal{D}_{l_1, m_1}^{+,1} \cap \mathcal{D}_{l_2, m_2}^{+,2} = \left\{ (-\rho - l_1, -\rho - l_2, \rho + m_1 + l_2) \right\} . \quad (6)$$

Proof. An element λ belongs to the intersection $\mathcal{D}_{l_1, m_1}^{+,1} \cap \mathcal{D}_{l_2, m_2}^{+,2}$ if and only if the following equations are satisfied

$$\lambda_1 = -\rho - l_1, \quad \lambda_2 + \lambda_3 = m_1, \quad \lambda_2 = -\rho - l_2, \quad \lambda_1 + \lambda_3 = m_2 .$$

A combination of these equations yields

$$\lambda_1 + \lambda_2 + \lambda_3 = -\rho - l_1 + m_1 = -\rho - l_2 + m_2 .$$

Hence a necessary condition for a non empty intersection is

$$l_1 - m_1 = l_2 - m_2$$

Assume that this condition is satisfied, then the intersection point of the two lines is equal to $(-\rho - l_1, -\rho - l_2, \rho + m_1 + l_2)$. For later reference, observe that the last coordinate can be written as $\rho + m_3$, with $m_3 = m_2 + l_1 = m_1 + l_2$. Then

$$m_3 \equiv l_1 \pm l_2 \pmod{2}, \quad |l_1 - l_2| \leq m_3 \leq l_1 + l_2 .$$

\square

Lemma 8.3. *Let $(-, 1, l_1, m_1)$ and $(-, 2, l_2, m_2)$ be admissible quadruples.*

i) if $l_1 - l_2 \neq m_1 + m_2$, then $\mathcal{D}_{l_1, m_1}^{-,1} \cap \mathcal{D}_{l_2, m_2}^{-,2} = \emptyset$.

ii) if $l_1 - l_2 = m_1 + m_2$, then

$$\mathcal{D}_{l_1, m_1}^{-,1} \cap \mathcal{D}_{l_2, m_2}^{-,2} = (-\rho - l_1, -\rho - l_2, -\rho - l_1 + m_2) .$$

Moreover, set $l_3 = l_1 - m_2$, $m_3 = -l_1 + l_2$. Then $m_3 \equiv l_3 \pmod{2}$, $|m_3| \leq l_3$, so that $(-, 3, l_3, m_3)$ is an admissible quadruple. The three lines $\mathcal{D}_{l_1, m_1}^{-,1}$, $\mathcal{D}_{l_2, m_2}^{-,2}$ and $\mathcal{D}_{l_3, m_3}^{-,3}$ satisfy

$$\mathcal{D}_{l_1, m_1}^{-,1} \cap \mathcal{D}_{l_2, m_2}^{-,2} \cap \mathcal{D}_{l_3, m_3}^{-,3} = \{(-\rho - l_1, -\rho - l_2, -\rho - l_3)\}.$$

Proof. An element λ belongs to the intersection $\mathcal{D}_{l_1, m_1}^{-,1} \cap \mathcal{D}_{l_2, m_2}^{-,2}$ if and only if the following equations are satisfied :

$$\lambda_1 = -\rho - l_1, \quad \lambda_2 - \lambda_3 = m_1, \quad \lambda_2 = -\rho - l_2, \quad \lambda_3 - \lambda_1 = m_2.$$

These equations imply $\lambda_2 - \lambda_1 = (\lambda_2 - \lambda_3) + (\lambda_3 - \lambda_1) = l_1 - l_2 = m_1 + m_2$, and hence the intersection is empty unless $l_1 - l_2 = m_1 + m_2$. Assume that this condition is satisfied. Let

$$l_3 = l_1 - m_2 = l_2 + m_2, \quad m_3 = -l_1 + l_2.$$

Then $l_3 + m_3 = l_2 - m_2 \equiv 0 \pmod{2}$ and hence $m_3 \equiv l_3 \pmod{2}$. Moreover,

$$m_3 = -l_1 + l_2 \leq m_1 + l_2 = l_3, \quad -m_3 = l_1 - l_2 \leq l_1 - m_2 = l_3.$$

so that $|m_3| \leq l_3$, showing that $(-, 3, l_3, m_3)$ is an admissible quadruplet. Now

$$\mathcal{D}_{l_1, m_1}^{-,1} \cap \mathcal{D}_{l_2, m_2}^{-,2} = (-\rho - l_1, -\rho - l_2, -\rho - l_1 + m_2).$$

Observe that the third coordinate can be written as $-\rho - l_3$, whereas $-\rho - l_1 - (\rho - l_2) = -l_1 + l_2 = m_3$, so that the intersection point also belongs to $\mathcal{D}_{l_3, m_3}^{-,3}$. \square

For later reference, observe that in the last situation,

$$l_1 + l_2 + l_3 \equiv 0 \pmod{2}, \quad |l_1 - l_2| \leq l_3 \leq l_1 + l_2, \quad (7)$$

and observe that these conditions are invariant under permutation of 1, 2, 3. Conversely, let l_1, l_2, l_3 satisfy (7). Then the three lines $\mathcal{D}_{l_1, -l_2 + l_3}^{-,1}$, $\mathcal{D}_{l_2, -l_3 + l_1}^{-,2}$ and $\mathcal{D}_{l_3, -l_1 + l_2}^{-,3}$ have a common point, namely $(-\rho - l_1, -\rho - l_2, -\rho - l_3)$.

Proposition 8.2.

i) The intersection of three distinct lines $\mathcal{D}, \mathcal{D}', \mathcal{D}''$ contained in Z is non empty, if and only if, up to a permutation of $\{1, 2, 3\}$ there exist $l_1, l_2, l_3 \in \mathbb{N}$ satisfying (7) such that

$$\mathcal{D} = \mathcal{D}_{l_1, -l_2 + l_3}^{-,1}, \quad \mathcal{D}' = \mathcal{D}_{l_2, -l_3 + l_1}^{-,2}, \quad \mathcal{D}'' = \mathcal{D}_{l_3, -l_1 + l_2}^{-,3}.$$

If this is the case, then

$$\mathcal{D} \cap \mathcal{D}' \cap \mathcal{D}'' = \{(-\rho - l_1, -\rho - l_2, -\rho - l_3)\}.$$

ii) The intersection of four distinct lines contained in Z is always empty.

Proof. Suppose first the lines are of type $I_j, I_{j'}$ and II_k . If they have a non empty intersection, by Proposition 8.1, $k = j$ and $k = j'$, but $j \neq j'$, a contradiction. A similar argument shows that three distinct lines of type $I_j, II_k, II_{k'}$ have an empty intersection. If the three lines are of type $II_j, II_{j'}, II_{j''}$, by Proposition 8.1 j, j', j'' have to be mutually distinct, hence up to a permutation, we may assume that $j = 1, j' = 2$ and $j'' = 3$. The equations of the lines are

$$\lambda_1 = -\rho - l_1, \quad \lambda_2 + \lambda_3 = m_1,$$

$$\lambda_2 = -\rho - l_2, \quad \lambda_3 + \lambda_1 = m_2$$

$$\lambda_3 = -\rho - l_3, \quad \lambda_1 + \lambda_2 = m_3$$

so that the coordinates of the intersection have to satisfy

$$2(\lambda_1 + \lambda_2 + \lambda_3) = -3(n-1) - 2(l_1 + l_2 + l_3) = m_1 + m_2 + m_3$$

which contradicts the conditions $|m_j| \leq l_j$ for $j = 1, 2, 3$.

Hence it remains to examine the case where the three lines are of type $I_j, I_{j'}$ and $I_{j''}$. Again by Proposition 8.1, j, j', j'' have to be mutually distinct, hence, up to a permutation of $\{1, 2, 3\}$, we may assume $j = 1, j' = 2, j'' = 3$. But then it is just the situation observed in Lemma 8.3. Hence i) follows.

Now ii) is a consequence of i) and Proposition 8.1. \square

Definition 8.2. For $d = 1, 2, 3$ let

$$Z_d = \{\lambda \in \mathbb{C}^3, \lambda \text{ belongs to exactly } d \text{ lines contained in } Z\}.$$

For $d = 1$, let

$$Z_{1,I} = \{\lambda \in Z_1, \lambda \text{ belongs to a line of type } I\}$$

$$Z_{1,II} = \{\lambda \in Z_1, \lambda \text{ belongs to a line of type } II\}$$

For $d = 2$, let

$$Z_{2,I} = \{\lambda \in Z_2, \lambda \text{ belongs to a line of type } I \text{ and to a line of type } II\}$$

$$Z_{2,II} = \{\lambda \in Z_2, \lambda \text{ belongs to two lines of type } II\}$$

To these definitions correspond the following partitions

$$Z = Z_1 \cup Z_2 \cup Z_3, \quad Z_1 = Z_{1,I} \cup Z_{1,II}, \quad Z_2 = Z_{2,I} \cup Z_{2,II}.$$

The next propositions describe (up to a permutation of the indices 1, 2, 3) the different subsets of Z just introduced, in terms of the spectral parameter and in terms of the geometric parameter. They are easily obtained from the previous study of the intersections of the lines contained in Z .

Proposition 8.3. *Let $\mathcal{D} = \mathcal{D}_{l,m}^{-,1}$ be a line of type I. Let $\lambda \in \mathcal{D}$. Then λ belongs to Z_1 if and only if*

$$\lambda_2 \notin \left\{ \frac{l-m}{2}, \frac{l-m}{2} + 1, \dots, \frac{l+m}{2} \right\} \cup \left(-\rho - \frac{l-m}{2} - \mathbb{N} \right).$$

Proposition 8.4. *Let \mathcal{D} be the line of type I defined by the equations*

$$\alpha_2 = -(n-1) - 2k_2, \quad \alpha_3 = -(n-1) - 2k_3,$$

where $k_2, k_3 \in \mathbb{N}$. Let $\alpha \in \mathcal{D}$. Then

$$\alpha \in Z_1 \iff \alpha_1 \notin \{0, 2, \dots, 2(k_2 + k_3)\} \cup \left(-(n-1) - 2\mathbb{N} \right).$$

Proposition 8.5. *Let λ belong to the line $\mathcal{D}_{l,m}^{+,1}$. Then $\lambda \in Z_1$ if and only if*

$$\lambda_2 \notin \left\{ \frac{m}{2} - \frac{l}{2}, \frac{m}{2} - \frac{l}{2} + 1, \dots, \frac{m}{2} + \frac{l}{2} \right\} \cup \left(-\rho + \frac{m}{2} - \frac{l}{2} - \mathbb{N} \right) \cup \left(\rho + \frac{m}{2} + \frac{l}{2} + \mathbb{N} \right)$$

Proposition 8.6. *Let α satisfy*

$$\alpha_1 + \alpha_2 + \alpha_3 = -2(n-1) - 2k, \quad \alpha_1 = 2p,$$

where $k, p \in \mathbb{N}$. Then $\alpha \in Z_1$ if and only if

$$\alpha_2 \notin \left\{ -(n-1), -(n-1)-2, \dots, -(n-1)-2k-2p \right\} \cup 2\mathbb{N} \cup \left(-2(n-1)-2k-2p-2\mathbb{N} \right)$$

Proposition 8.7. *Let $\lambda \in \mathbb{C}^3$.*

i) λ belongs to $Z_{2,I}$ if and only if, up to a cyclic permutation of 1, 2, 3,

$$\begin{aligned} m_2 \pm m_3 &\equiv l_1 \pmod{2}, \\ \lambda &= (-\rho - l_1, m_2, m_3), \\ |m_2 \pm m_3| &\leq l_1 \end{aligned} \tag{8}$$

ii) λ belongs to $Z_{2,II}$ if and only if, up to a permutation of $1, 2, 3$

$$\lambda = (-\rho - l_1, -\rho - l_2, \rho + m_3), \quad \begin{array}{l} m_3 \equiv l_1 \pm l_2 \pmod{2} \\ |l_1 - l_2| \leq m_3 \leq l_1 + l_2 \end{array}. \quad (9)$$

iii) λ belongs to Z_3 if and only

$$\lambda = (-\rho - l_1, \rho - l_2, -\rho - l_3), \quad \begin{array}{l} l_1 + l_2 + l_3 \equiv 0 \pmod{2} \\ |l_1 - l_2| \leq l_3 \leq l_1 + l_2 \end{array}. \quad (10)$$

Remark. The conditions (10) are symmetric in the three variables (l_1, l_2, l_3) .

Proposition 8.8. Let $\alpha \in \mathbb{C}^3$. Then

i) α belongs to $Z_{2,I}$ if and only if, up to a cyclic permutation of $\{1, 2, 3\}$, there exists $k_1, k_2, k_3 \in \mathbb{N}$ such that

$$\alpha = (2k_1, -(n-1) - 2k_2, -(n-1) - 2k_3), \quad k_1 \leq k_2 + k_3 \quad (11)$$

ii) α belongs to $Z_{2,II}$ if and only if, up to a cyclic permutation of $\{1, 2, 3\}$, there exists $k_1, k_2, k_3 \in \mathbb{N}$ such that

$$\alpha = (2k_1, 2k_2, -2(n-1) - 2k_3), \quad k_1 + k_2 \leq k_3 \quad (12)$$

iii) α belongs to Z_3 if and only if

$$\alpha = (-(n-1) - 2k_1, -(n-1) - 2k_2, -(n-1) - 2k_3) \quad (13)$$

for some $k_1, k_2, k_3 \in \mathbb{N}$.

The set Z is contained in the set of poles of the meromorphic function \mathcal{K}_α . The precise nature of the pole requires some further refinement in the description of Z .

Proposition 8.9. Let α satisfy

$$\alpha_2 = -(n-1) - 2k_2, \quad \alpha_3 = -(n-1) - 2k_3$$

and assume that $\alpha \in Z_{1,I}$. Then α belongs to exactly two planes of poles (namely $\beta_2 = -(n-1) - 2k_2$ and $\beta_3 = -(n-1) - 2k_3$), except if $\alpha_1 = -2p$ for some $p \in \{1, 2, \dots\}$, in which case α belongs moreover to the plane of poles

$$\beta_1 + \beta_2 + \beta_3 = -2(n-1) - 2(k_2 + k_3 + p).$$

Proposition 8.10. *Let α satisfy*

$$\alpha_1 + \alpha_2 + \alpha_3 = -2(n-1) - 2k, \quad \alpha_1 = 2p$$

where $k, p \in \mathbb{N}$ and assume that α belongs to $Z_{1,II}$. Then α belongs to a exactly one plane of poles (namely $\beta_1 + \beta_2 + \beta_3 = -2(n-1) - 2k$), except if α is (up to a permutation of the indices 2 and 3) of the form

$$\alpha_1 = 2p, \quad \alpha_2 = -(n-1) - 2k_2, \quad \alpha_3 = -(n-1) + 2q,$$

where $p, k_2, q \in \mathbb{N}, q \geq 1, k_2 - p - q \geq 0$, in which case α belongs moreover to the plane of poles $\beta_2 = -(n-1) - 2k_2$.

Proposition 8.11. *Let $\alpha \in Z_{2,I}$, i.e. (up to a permutation of 1, 2, 3) satisfy*

$$\alpha_1 = 2k_1, \alpha_2 = -(n-1) - 2k_2, \alpha_3 = -(n-1) - 2k_3$$

where $k_1, k_2, k_3 \in \mathbb{N}, k_2 + k_3 - k_1 \geq 0$. Then α belongs to three planes of poles, namely $\beta_1 = -(n-1) - 2k_1$, $\beta_2 = -(n-1) - 2k_2$ and $\beta_1 + \beta_2 + \beta_3 = -2(n-1) - 2(k_2 + k_3 - k_1)$.

Proposition 8.12. *Let $\alpha \in Z_{2,II}$, i.e. up to a permutation of 1, 2, 3 satisfy*

$$\alpha_1 = 2k_1, \quad \alpha_2 = 2k_2, \quad \alpha_3 = -2(n-1) - 2k_3$$

where $k_1, k_2, k_3 \in \mathbb{N}$ and $k_1 + k_2 \leq k_3$.

i) if $n-1$ is odd, then α belongs to the unique plane of poles

$$\alpha_1 + \alpha_2 + \alpha_3 = -2(n-1) - 2(-k_1 - k_2 + k_3) .$$

ii) if $n-1$ is even, then α belongs to two planes of poles, given respectively by the equations

$$\alpha_3 = -(n-1) - 2(\rho + k_3) \quad \text{and} \quad \alpha_1 + \alpha_2 + \alpha_3 = -2(n-1) - 2(-k_1 - k_2 + k_3) .$$

Proposition 8.13. *Let $\alpha \in Z_3$, i.e.*

$$\alpha_1 = -(n-1) - 2k_1, \alpha_2 = -(n-1) - 2k_2, \alpha_3 = -(n-1) - 2k_3$$

where $k_1, k_2, k_3 \in \mathbb{N}$.

i) if $n-1$ is odd, then α belongs to exactly three planes of poles.

ii) if $n-1$ is even, then α belongs to three planes of poles of type I and moreover to the plane of poles of type II given by the equation

$$\alpha_1 + \alpha_2 + \alpha_3 = -2(n-1) - 2(\rho + k_1 + k_2 + k_3) .$$

9 The holomorphic families $\widetilde{\mathcal{T}}_{\cdot, \cdot}^{(j, k)}$

The first holomorphic families to be introduced are related to the residues of the meromorphic distribution-valued function $\alpha \mapsto \mathcal{K}_\alpha$ at poles of type I (see [1] section 2). Let $k \in \mathbb{N}$ and consider the plane of poles of type (say)³ I_3 given by the equation $\alpha_3 = -(n-1) - 2k$. Use α_1, α_2 as coordinates in this plane.

Introduce the differential operator on S given by

$$\Delta_k = \prod_{j=1}^k (\Delta - (\rho + j - 1)(\rho - j)) ,$$

where Δ is the usual Laplacian on S . Recall that Δ_k is a *conformally covariant* differential operator on S , in the sense that, for any $g \in G$,

$$\Delta_k \circ \pi_{-k}(g) = \pi_k(g) \circ \Delta_k .$$

Let $\alpha_1, \alpha_2 \in \mathbb{C}$. For $f_1, f_2, f_3 \in \mathcal{C}^\infty(S)$, consider the expression defined by

$$\mathcal{T}_{\alpha_1, \alpha_2}^{(3, k)}(f_1, f_2, f_3) = \iint_{S \times S} f_2(y) f_3(z) \Delta_k[f_1(\cdot)|z - \cdot|^{\alpha_2}](y)|y - z|^{\alpha_1} dy dz \quad (14)$$

or equivalently, for $\varphi \in \mathcal{C}^\infty(S \times S \times S)$

$$\mathcal{T}_{\alpha_1, \alpha_2}^{(3, k)}(\varphi) = \iint_{S \times S} \Delta_k^{(1)}(\varphi(\cdot, y, z)|z - \cdot|^{\alpha_2})(y)|y - z|^{\alpha_1} dy dz , \quad (15)$$

where $\Delta_k^{(1)}$ is the covariant differential operator Δ_k acting on x .

Proposition 9.1.

i) for $\varphi \in \mathcal{C}^\infty(S \times S \times S)$ with $\text{Supp}(\varphi) \subset \mathcal{O}_4^c$, the integral (15) defines a λ -invariant distribution $\mathcal{T}_{\alpha_1, \alpha_2, \mathcal{O}_4^c}^{(3, k)}$ on \mathcal{O}_4^c .

ii) for $\Re \alpha_1$ and $\Re \alpha_2$ large enough, the integral (14) converges and defines a continuous trilinear form on $\mathcal{C}^\infty(S)$.

iii) the trilinear form $\mathcal{T}_{\alpha_1, \alpha_2}^{(3, k)}$ can be continued meromorphically (in the parameters α_1, α_2) to \mathbb{C}^2 , with simple poles along the complex lines

$$\alpha_1 + \alpha_2 = -(n-1) + 2k - 2l, \quad l = 0, 1, 2, \dots$$

³Needless to say, a similar construction can be made when α is a pole of type I_1 or I_2

Proof. For i), let $Supp(\varphi) \subset \mathcal{O}_4^c$. The function

$$\Phi(y, z) = \Delta_k^{(1)}(\varphi(\cdot, y, z)|z - \cdot|^{\beta_2})(y)$$

is in $\mathcal{C}^\infty(S \times S)$ and $Supp(\Phi) \cap \{y = z\} = \emptyset$. Hence the integral (15) converges and defines a distribution on \mathcal{O}_4^c . The open set \mathcal{O}_4^c is invariant by G and the distribution $\mathcal{T}_{\beta_1, \beta_2, \mathcal{O}_4^c}^{3, k_3}$ is verified to be λ -invariant as in [1] Proposition 2.1.

The convergence result ii) and the meromorphic extension iii) are established in [1] Theorem 2.2. The fact that the poles are simple (although not explicitly stated) is an easy consequence of the proof. \square

Remark. If α satisfies $\alpha_3 = -(n-1) - 2k$ and $\alpha_1 + \alpha_2 = -(n-1) + 2k - 2l$ for some $l \in \mathbb{N}$, then $\alpha_1 + \alpha_2 + \alpha_3 = -2(n-1) - 2l$, so that α is a pole of type I + II.

Now renormalize $\mathcal{T}_{\alpha_1, \alpha_2}^{(3, k)}$ by setting

$$\tilde{\mathcal{T}}_{\alpha_1, \alpha_2}^{(3, k)} = \frac{1}{\Gamma(\frac{\alpha_1 + \alpha_2}{2} + \rho - k)} T_{\alpha_1, \alpha_2}^{(3, k)}, \quad (16)$$

and use Hartog's theorem to extend $(\alpha_1, \alpha_2) \mapsto \tilde{\mathcal{T}}_{\alpha_1, \alpha_2}^{(3, k)}$ holomorphically to all of \mathbb{C}^2 .

Recall that a pole α is said to be *generic* if α belongs to a unique plane of poles. For α a generic pole (say) of type I_3 , the residue $Res(\mathcal{K}_\alpha)$ is defined as

$$Res(\mathcal{K}_\alpha) = \lim_{s \rightarrow 0} s \mathcal{K}_{\alpha_1, \alpha_2, \alpha_3 + 2s},$$

and was computed in [1].

Proposition 9.2. *Let α satisfy $\alpha_3 = -(n-1) - 2k$ and assume that α is a generic pole. The residue of \mathcal{K}_α at α is equal to*

$$Res(\mathcal{K}_\alpha) = \frac{\pi^\rho 2^{-2k}}{\Gamma(\rho + k)k!} \mathcal{T}_{\alpha_1, \alpha_2}^{(3, k)}. \quad (17)$$

Proposition 9.3. *Let $k \in \mathbb{N}$. Then*

$$\tilde{\mathcal{K}}_{\alpha_1, \alpha_2, -(n-1)-2k} = \frac{(-1)^k 2^{-2k} \pi^\rho}{\Gamma(\rho + k)\Gamma(\rho + \frac{\alpha_1}{2})\Gamma(\rho + \frac{\alpha_2}{2})} \tilde{\mathcal{T}}_{\alpha_1, \alpha_2}^{(3, k)}. \quad (18)$$

Proof. First assume that α is a generic pole. For $s \in \mathbb{C}$, let

$$\alpha(s) = (\alpha_1, \alpha_2, -(n-1-2k+2s)).$$

For $s \neq 0$, $\alpha(s)$ is not a pole, hence

$$\tilde{\mathcal{K}}_{\alpha(s)} = \frac{1}{\Gamma(\frac{\alpha_1}{2} + \rho)\Gamma(\frac{\alpha_2}{2} + \rho)\Gamma(-k + s)\Gamma(\frac{\alpha_1 + \alpha_2}{2} + \rho - k + s)} \mathcal{K}_{\alpha(s)}$$

As $s \rightarrow 0$, $\frac{1}{\Gamma(-k + s)} \sim (-1)^k k! s$, so that

$$\tilde{\mathcal{K}}_{\alpha_1, \alpha_2, -(n-1)-2k} = \frac{(-1)^k k!}{\Gamma(\frac{\alpha_1}{2} + \rho)\Gamma(\frac{\alpha_2}{2} + \rho)\Gamma(\frac{\alpha_1 + \alpha_2}{2} + \rho - k)} \text{Res}(\mathcal{K}_{\alpha})$$

and (18) follows from (16) and (17).

As both sides of (18) depend holomorphically on (α_1, α_2) , the equality is valid on all of \mathbb{C}^2 by analytic continuation. \square

The K -coefficients of $\tilde{\mathcal{T}}_{\alpha_1, \alpha_2}^{(3,k)}$ can be computed from (18) and the evaluation of the K -coefficients of $\tilde{\mathcal{K}}_{\alpha}$ (see (63) in Appendix).

Proposition 9.4. *Let $\alpha_1, \alpha_2 \in \mathbb{C}$, and let $k \in \mathbb{N}$. Then*

$$\tilde{\mathcal{T}}_{\alpha_1, \alpha_2}^{(3,k)}(p_{a_1, a_2, a_3}) = 0 \quad \text{for } a_3 > k$$

and for $a_3 \leq k$

$$\begin{aligned} \tilde{\mathcal{T}}_{\alpha_1, \alpha_2}^{(3,k)}(p_{a_1, a_2, a_3}) &= 2^{-\frac{5}{2}(n-1)-1} \pi^{n-1} (-1)^k \Gamma(\rho + k) 2^{\alpha_1 + \alpha_2} 2^{2(a_1 + a_2 + a_3)} \\ &\times (-k)_{a_3} \left(\frac{\alpha_1 + \alpha_2}{2} + \rho - k\right)_{a_1 + a_2 + a_3} \\ &\times \left(\frac{\alpha_1}{2} + \rho + a_1 - (k - a_3)\right)_{k-a_3} \left(\frac{\alpha_2}{2} + \rho + a_2 - (k - a_3)\right)_{k-a_3} \\ &\times \frac{1}{\Gamma(\frac{\alpha_1 + \alpha_2}{2} + 2\rho + a_1 + a_2)} . \end{aligned} \tag{19}$$

Proposition 9.5. *Let $\alpha = (\alpha_1, \alpha_2, -(n-1) - 2k)$.*

i) assume that α is not a pole of type II. Then

$$\text{Supp}(\tilde{\mathcal{T}}_{\alpha_1, \alpha_2}^{(3,k)}) = \overline{\mathcal{O}_3} .$$

ii) assume that α is moreover a pole of type II. Then

$$\text{Supp}(\tilde{\mathcal{T}}_{\alpha_1, \alpha_2}^{(3,k)}) \subset \mathcal{O}_4 .$$

Proof. In general, as α is a pole of type I_3 , $\text{Supp}(\tilde{\mathcal{K}}_\alpha) \subset \overline{\mathcal{O}_3}$ (see Proposition 3.3 in [3]), from which follows easily $\text{Supp}(\tilde{\mathcal{T}}_{\alpha_1, \alpha_2}^{(3, k)}) \subset \overline{\mathcal{O}_3}$. As $\text{Supp}(\tilde{\mathcal{T}}_{\alpha_1, \alpha_2}^{(3, k)})$ has to be invariant by G , either $\text{Supp}(\tilde{\mathcal{T}}_{\alpha_1, \alpha_2}^{(3, k)}) = \overline{\mathcal{O}_3}$ or $\text{Supp}(\tilde{\mathcal{T}}_{\alpha_1, \alpha_2}^{(3, k)}) \subset \mathcal{O}_4$.

For *i*), assume that α is not a pole of type II, which is equivalent to assuming that $\frac{\alpha_1 + \alpha_2}{2} + \rho - k \notin -\mathbb{N}$. Hence the factor $(\frac{\alpha_1 + \alpha_2}{2} + \rho - k)_{a_1 + a_2 + a_3}$ in (19) never vanishes. Now let $a_3 = k$ and choose a_1, a_2 large enough. Then $\tilde{\mathcal{T}}_{\alpha_1, \alpha_2}^{(3, k)}(p_{a_1, a_2, a_3}) \neq 0$. But the function p_{a_1, a_2, a_3} vanishes on \mathcal{O}_4 at an arbitrary large order for an appropriate choice of a_1 and a_2 . This is incompatible with the possibility that $\text{Supp}(\tilde{\mathcal{T}}_{\alpha_1, \alpha_2}^{(3, k)}) \subset \mathcal{O}_4$ (see similar argument in the proof of Lemma 6.5 in [3]).

For *ii*), assume that α is a pole of type II. By Proposition 3.3 in [3] $\text{Supp}(\tilde{\mathcal{K}}_\alpha) \subset \mathcal{O}_4$. If $\alpha_1, \alpha_2 \notin -(n-1) - 2\mathbb{N}$, then (18) shows that $\text{Supp}(\tilde{\mathcal{T}}_{\alpha_1, \alpha_2}^{(3, k)}) \subset \mathcal{O}_4$. The result follows by continuity. \square

10 The multiplicity 2 theorem for $\alpha \in Z_{1, I}$

Let $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ belongs to $Z_{1, I}$. Up to a permutation of $\{1, 2, 3\}$, α satisfies (see Proposition 8.4)

$$\begin{aligned} \alpha_2 &= -(n-1) - 2k_2, & \alpha_3 &= -(n-1) - 2k_3, \\ \alpha_1 &\notin \{0, 2, \dots, 2(k_2 + k_3)\} \cup (-(n-1) - 2\mathbb{N}). \end{aligned} \quad (20)$$

The next proposition is a preparation for the main results. The domain of α 's to which it applies is larger than $Z_{1, I}$.

Proposition 10.1. *Let α satisfy*

$$\alpha_2 = -(n-1) - 2k_2, \quad \alpha_3 = -(n-1) - 2k_3, \quad \alpha_1 \notin \{0, 2, \dots, 2(k_2 + k_3)\}.$$

Then $\tilde{\mathcal{T}}_{\alpha_1, -(n-1)-2k_3}^{(2, k_2)}$ and $\tilde{\mathcal{T}}_{\alpha_1, -(n-1)-2k_2}^{(3, k_3)}$ are two linearly independant λ -invariant distributions.

Proof. For $a_3 = k_3$, the value of the K -coefficient $\tilde{\mathcal{T}}_{\alpha_1, \alpha_2}^{(3, k_3)}(p_{a_1, a_2, a_3})$ given by (19) takes the particular form⁴

$$NVT \times \left(\frac{\alpha_1}{2} - k_2 - k_3\right)_{a_1 + a_2 + k_3} \Gamma\left(\frac{\alpha_1}{2} - k_2 + \rho + a_1 + a_2\right)^{-1}$$

Assume first that $\alpha_1 \notin 2k_2 + 2k_3 - 2\mathbb{N}$. Then the first factor is always $\neq 0$. The second factor, for a given value of a_2 , does not vanish for a_1 large

⁴Recall that the symbol NVT is used for a non vanishing term

enough. Hence for $a_3 = k_3$, $a_2 = k_2 + 1$ and a_1 large enough,

$$\tilde{\mathcal{T}}_{\alpha_1, -(n-1)-2k_2}^{(3, k_3)}(p_{a_1, k_2+1, k_3}) \neq 0 ,$$

whereas

$$\tilde{\mathcal{T}}_{\alpha_1, -(n-1)-2k_3}^{(2, k_2)}(p_{a_1, k_2+1, k_3}) = 0 ,$$

because $a_2 = k_2 + 1 > k_2$, and the statement follows in this case.

Next, assume that $\alpha_1 = -2l_1$ for some integer $l_1 \geq 1$. Let

$$a_3 = k_3, \quad a_2 = k_2 + 1, \quad a_1 = l_1 - 1 .$$

Then $(-l_1 - k_2 - k_3)_{a_1+a_2+a_3} = (-l_1 - k_2 - k_3)_{l_1+k_2+k_3} \neq 0$. Moreover, $\frac{\alpha_1}{2} - k_2 + \rho + a_1 + a_2 = \rho$. Hence $\tilde{\mathcal{T}}_{\alpha_1, -(n-1)-2k_2}^{(3, k_3)}(p_{a_1, a_2, a_3}) \neq 0$, whereas $\tilde{\mathcal{T}}_{\alpha_1, -(n-1)-2k_3}^{(2, k_2)}(p_{a_1, a_2, a_3}) = 0$ as $a_2 > k_2$.

Permuting the indices 2 and 3, the independence of the two trilinear forms follows. \square

Remark. Let $\alpha \in \mathbb{C}^3$ such that

$$\alpha_2 = -(n-1) - 2k_2, \quad \alpha_3 = -(n-1) - 2k_3, \quad \alpha_1 \in \{0, 2, \dots, 2k_2 + 2k_3\} .$$

Then the two trilinear forms $\tilde{\mathcal{T}}_{\alpha_1, -(n-1)-2k_2}^{(3, k_3)}$ and $\tilde{\mathcal{T}}_{\alpha_1, -(n-1)-2k_3}^{(2, k_2)}$ are proportional, as will be proved later (see Proposition 14.1).

The main result of this section can now be formulated.

Theorem 10.1. *Let $\alpha \in Z_{1,I}$ and let λ be its associated spectral parameter. Then $\dim \text{Tri}(\lambda) = 2$. More precisely, assume that α satisfies (20). Then*

$$\text{Tri}(\lambda) = \mathbb{C} \tilde{\mathcal{T}}_{\alpha_1, \alpha_3}^{(2, k_2)} \oplus \mathbb{C} \tilde{\mathcal{T}}_{\alpha_1, \alpha_2}^{(3, k_3)} .$$

Because of Proposition 10.1, it suffices to prove that $\dim \text{Tri}(\lambda) \leq 2$. The proof of this inequality strongly depends on the nature of α as a pole (see Proposition 8.9), hence will be divided in two subsections : one for the general case (α is not a pole of type II) and one for the special case (α is moreover a pole of type II).

10.1 The general case

Let α satisfy conditions (20), and in this subsection, assume moreover that α is not a pole of type II. This amounts to

$$\begin{aligned} \alpha_2 &= -(n-1) - 2k_2, & \alpha_3 &= -(n-1) - 2k_3 \\ \alpha_1 &\notin (2k_2 + 2k_3 - 2\mathbb{N}) \cup (-(n-1) - 2\mathbb{N}) \end{aligned} \tag{21}$$

for some $k_2, k_3 \in \mathbb{N}$.

In this case, $\text{Supp}(\tilde{\mathcal{T}}_{\alpha_1, \alpha_3}^{(2, k_2)}) = \overline{\mathcal{O}_2}$ and $\text{Supp}(\tilde{\mathcal{T}}_{\alpha_1, \alpha_2}^{(3, k_3)}) = \overline{\mathcal{O}_3}$ (Proposition 9.5), which gives in this case another proof of the linear independence of the two distributions (Proposition 10.1).

As suggested in the introduction, $\tilde{\mathcal{T}}_{\alpha_1, \alpha_3}^{(2, k_2)}$ and $\tilde{\mathcal{T}}_{\alpha_1, \alpha_2}^{(3, k_3)}$ are related to partial derivatives of $\beta \mapsto \tilde{\mathcal{K}}_\beta$ at α .

Lemma 10.1.

$$\frac{d}{ds} \left(\tilde{\mathcal{K}}_{\alpha_1, \alpha_2 + 2s, \alpha_3} \right)_{s=0} = \frac{(-1)^{k_3} 2^{-2k_3} \pi^\rho (-1)^{k_2} k_2!}{\Gamma(\rho + k_3) \Gamma(\rho + \frac{\alpha_1}{2})} \tilde{\mathcal{T}}_{\alpha_1, \alpha_2}^{(3, k_3)}$$

Proof. For $s \in \mathbb{C}$, let $\alpha(s) = (\alpha_1, \alpha_2 + 2s, \alpha_3)$ and use (18) to relate $\tilde{\mathcal{K}}_{\alpha(s)}$ and $\tilde{\mathcal{T}}_{\alpha_1, \alpha_2 + 2s}^{(3, k_3)}$. As $s \rightarrow 0$,

$$\frac{1}{\Gamma(\rho + \frac{\alpha_2}{2} + s)} = \frac{1}{\Gamma(-k_2 + s)} \sim (-1)^{k_2} k_2! s$$

and pass to the limit to obtain the lemma. \square

Proposition 10.2. *Let α satisfy (21). Let λ be its associated spectral parameter. The distribution $\mathcal{K}_{\alpha, \mathcal{O}_0}$ cannot be extended to a λ -invariant distribution on $S \times S \times S$.*

Proof. At α , exactly two Γ -factors in the normalization of \mathcal{K}_β become singular, namely $\Gamma(\frac{\beta_2}{2} + \rho)$ and $\Gamma(\frac{\beta_3}{2} + \rho)$. For $s \in \mathbb{C}$, let

$$\alpha(s) = (\alpha_1, \alpha_2 + 2s, \alpha_3 + 2s), \quad \lambda(s) = (\lambda_1 + 2s, \lambda_2 + s, \lambda_3 + s) .$$

Observe that $\alpha(0) = \alpha$, that for $s \neq 0$, $|s|$ small, $\alpha(s)$ is not a pole. Moreover, $\alpha(s)$ is transverse to both planes of poles $\beta_2 = -(n-1) - 2k_2$ and $\beta_3 = -(n-1) - 2k_3$. For $s \neq 0$ and $|s|$ small, consider the distribution $\mathcal{F}(s)$ defined by

$$\mathcal{F}(s) = \frac{(-1)^{k_2}}{k_2!} \frac{(-1)^{k_3}}{k_3!} \Gamma(\frac{\alpha_1}{2} + \rho) \Gamma(\frac{\alpha_1}{2} - k_2 - k_3 + 2s) \frac{1}{s} \tilde{\mathcal{K}}_{\alpha(s)} .$$

As $\alpha \in Z$, the (distribution-valued) function $s \mapsto \mathcal{F}(s)$ can be extended analytically near 0 and its Taylor expansion of order 2 reads

$$\mathcal{F}(s) = F_0 + s F_1 + O(s^2) ,$$

where F_0, F_1 are distributions on $S \times S \times S$. Notice further that $\mathcal{F}(s)$ is $\lambda(s)$ -invariant.

Lemma 10.2. *The distributions F_0 and F_1 satisfy*

- i) F_0 is λ -invariant and $\text{Supp}(F_0) = \overline{\mathcal{O}_2 \cup \mathcal{O}_3}$*
- ii) the restriction of F_1 to \mathcal{O}_0 is equal to $\mathcal{K}_{\alpha, \mathcal{O}_0}$*

Proof. For *i*),

$$F_0 = NVT \frac{d}{ds} \left(\tilde{\mathcal{K}}_{\alpha(s)} \right)_{s=0} = NVT \left(\frac{d}{ds} (\tilde{\mathcal{K}}_{\alpha_1, \alpha_2 + 2s, \alpha_3})_{s=0} + \frac{d}{ds} (\tilde{\mathcal{K}}_{\alpha_1, \alpha_2, \alpha_3 + 2s})_{s=0} \right)$$

and *i*) follows from Lemma 10.1 and Proposition 10.1.

For *ii*), let $\varphi \in \mathcal{C}_c^\infty(\mathcal{O}_0)$. Then, for $s \neq 0$, $\mathcal{K}_{\alpha(s)}(\varphi)$ is well defined and satisfies

$$\begin{aligned} \mathcal{K}_{\alpha(s)}(\varphi) &= (-1)^{k_2} k_2! \Gamma(-k_2 + s) (-1)^{k_3} k_3! \Gamma(-k_3 + s) s \mathcal{F}(s)(\varphi) \\ &= \frac{1}{s} F_0(\varphi) + F_1(\varphi) + O(s). \end{aligned}$$

As $s \rightarrow 0$, the left hand side converges to $\mathcal{K}_{\alpha, \mathcal{O}_0}(\varphi)$. Hence,

$$F_0(\varphi) = 0, \quad F_1(\varphi) = \mathcal{K}_{\alpha}(\varphi),$$

thus proving *ii*). □

Let $\mathcal{O} = \mathcal{O}_0 \cup \mathcal{O}_2$. The open subset \mathcal{O} is G -invariant, contains \mathcal{O}_2 as a (relatively) closed submanifold. Now restrict $\mathcal{F}(s)$ to \mathcal{O} , and similarly for F_0 and F_1 . Then, $\mathcal{F}(s)|_{\mathcal{O}}$ is $\lambda(s)$ -invariant, $\mathcal{F}(s)|_{\mathcal{O}} = F_0|_{\mathcal{O}} + s F_1|_{\mathcal{O}} + O(s^2)$, and $\text{Supp}(F_0|_{\mathcal{O}}) = \mathcal{O}_2$. By Proposition A1, $\mathcal{K}_{\alpha, \mathcal{O}_0}$ cannot be extended to a λ -invariant distribution on \mathcal{O} , a fortiori to $S \times S \times S$, which finishes the proof of Proposition 10.2. □

After this preparation, we are in condition to prove Theorem 10.1 in the general case. Let T be a λ -invariant distribution. The restriction of T to \mathcal{O}_0 has to be proportional to $\mathcal{K}_{\alpha, \mathcal{O}_0}$, hence has to be 0 by Proposition 10.2. Next, as $\alpha_1 \notin -(n-1) - 2\mathbb{N}$, the restriction of T to $\mathcal{O}_0 \cup \mathcal{O}_1$ has to be 0 (use Lemma 4.1 in [3]). So, T is supported on $\mathcal{O}_2 \cup \mathcal{O}_3 \cup \mathcal{O}_4$. Now we first restrict T to the G -invariant open set $\mathcal{O}_0 \cup \mathcal{O}_1 \cup \mathcal{O}_2$. The restriction $T|_{\mathcal{O}_0 \cup \mathcal{O}_1 \cup \mathcal{O}_2}$ is λ -invariant and supported on \mathcal{O}_2 . By Lemma 4.1 in [3], $T|_{\mathcal{O}_0 \cup \mathcal{O}_1 \cup \mathcal{O}_2}$ has to be proportional to (the restriction to $\mathcal{O}_0 \cup \mathcal{O}_1 \cup \mathcal{O}_2$ of) $\tilde{\mathcal{T}}_{\alpha_1, \alpha_3}^{(2, k_2)}$. A similar argument applies for the restriction of T to $\mathcal{O}_0 \cup \mathcal{O}_1 \cup \mathcal{O}_3$. Hence, there exist two constants C_2 and C_3 such that

$$T - C_2 \tilde{\mathcal{T}}_{\alpha_1, \alpha_3}^{(2, k_2)} - C_3 \tilde{\mathcal{T}}_{\alpha_1, \alpha_2}^{(3, k_3)}$$

is supported on \mathcal{O}_4 , and still λ -invariant. By Lemma 4.2 in [3], as $\alpha_1 + \alpha_2 + \alpha_3 \notin -2(n-1) - 2\mathbb{N}$, $T - C_2 \tilde{\mathcal{T}}_{\alpha_1, \alpha_3}^{(2, k_2)} - C_3 \tilde{\mathcal{T}}_{\alpha_1, \alpha_2}^{(3, k_3)} = 0$. Hence any λ -invariant distribution is a linear combination of $\tilde{\mathcal{T}}_{\alpha_1, \alpha_3}^{(2, k_2)}$ and $\tilde{\mathcal{T}}_{\alpha_1, \alpha_2}^{(3, k_3)}$. Q.E.D.

10.2 The special case

In this subsection, assume α satisfies (20) and is moreover a pole of type II, i.e. α is assumed to satisfy the conditions

$$\begin{aligned} \alpha_2 &= -(n-1) - 2k_2, & \alpha_3 &= -(n-1) - 2k_3, & \alpha_1 &= -2p_1 \\ k_2, k_3 &\in \mathbb{N}, & p_1 &\in \{1, 2, \dots\}, & p_1 &\notin \rho + \mathbb{N}. \end{aligned} \quad (22)$$

Notice in particular that $\alpha_1 + \alpha_2 + \alpha_3 = -2(n-1) - 2k$, where $k = p_1 + k_2 + k_3$ so that α is indeed a pole of type II.

Let λ be the associated spectral parameter. In this case, the two λ -invariant distributions $\tilde{\mathcal{T}}_{\alpha_1, \alpha_3}^{(2, k_2)}$ and $\tilde{\mathcal{T}}_{\alpha_1, \alpha_2}^{(3, k_3)}$ are now supported on \mathcal{O}_4 , but still linearly independent (see Proposition 9.5 and Proposition 10.1). Some preparatory results are needed.

Proposition 10.3. *Let A_2 and A_3 be two complex numbers, $(A_2, A_3) \neq (0, 0)$. The distribution on \mathcal{O}_4^c*

$$A_2 \mathcal{T}_{\alpha_1, \alpha_3, \mathcal{O}_4^c}^{(2, k_2)} + A_3 \mathcal{T}_{\alpha_1, \alpha_2, \mathcal{O}_4^c}^{(3, k_3)}$$

cannot be extended to a λ -invariant distribution on $S \times S \times S$.

Proof. For $s \in \mathbb{C}$ let

$$\alpha(s) = (\alpha_1 + 2s, \alpha_2, \alpha_3), \quad \lambda(s) = (\lambda_1, \lambda_2 + s, \lambda_3 + s)$$

and notice that $\alpha(s)$ is transverse to the plane $\beta_1 + \beta_2 + \beta_3 = -2(n-1) - 2k$ at $\alpha = \alpha(0)$. Let

$$\mathcal{F}(s) = \frac{(-1)^k}{k!} (A_2 \tilde{\mathcal{T}}_{\alpha_1 + 2s, \alpha_3}^{(2, k_2)} + A_3 \tilde{\mathcal{T}}_{\alpha_1 + 2s, \alpha_2}^{(3, k_3)}) .$$

Notice that the distribution $\mathcal{F}(s)$ is $\lambda(s)$ -invariant. Let

$$\mathcal{F}(s) = F_0 + sF_1 + O(|s|^2)$$

be the Taylor expansion of \mathcal{F} at 0.

Lemma 10.3.

i) $\text{Supp}(F_0) = \mathcal{O}_4$

ii) on \mathcal{O}_4^c

$$F_1|_{\mathcal{O}_4^c} = A_2 \mathcal{T}_{\alpha_1, \alpha_3, \mathcal{O}_4^c}^{(2, k_2)} + A_3 \mathcal{T}_{\alpha_1, \alpha_2, \mathcal{O}_4^c}^{(3, k_3)}$$

Proof. For i), note that $F_0 = \frac{(-1)^k}{k!} (A_2 \tilde{\mathcal{T}}_{\alpha_1, \alpha_3}^{(2, k_2)} + A_3 \tilde{\mathcal{T}}_{\alpha_1, \alpha_2}^{(3, k_3)})$, hence $F_0 \neq 0$ and $\text{Supp}(F_0) = \mathcal{O}_4$.

For ii), Let $\varphi \in \mathcal{C}_c^\infty(\mathcal{O}_4^c)$. Then, using (16)

$$\mathcal{F}(s)(\varphi) = \frac{(-1)^k}{k!} \frac{1}{\Gamma(-k+s)} (A_2 \mathcal{T}_{\alpha_1+2s, \alpha_3}^{(2, k_2)}(\varphi) + A_3 \mathcal{T}_{\alpha_1+2s, \alpha_2}^{(3, k_3)}(\varphi)) \quad (23)$$

Let $s \rightarrow 0$ to obtain

$$F_1(\varphi) = A_2 \mathcal{T}_{\alpha_1+2s, \alpha_3, \mathcal{O}_4^c}^{(2, k_2)}(\varphi) + A_3 \mathcal{T}_{\alpha_1+2s, \alpha_2, \mathcal{O}_4^c}^{(3, k_3)}(\varphi)$$

proving ii). \square

Now apply Proposition A2 to conclude that $A_2 \mathcal{T}_{\alpha_1, \alpha_3, \mathcal{O}_4^c}^{(2, k_2)} + A_3 \mathcal{T}_{\alpha_1, \alpha_2, \mathcal{O}_4^c}^{(3, k_3)}$ cannot be extended to $S \times S \times S$ as a λ -invariant distribution. \square

We are ready to start the proof of Theorem 10.1 for the special case. As in the general case, the following proposition holds.

Proposition 10.4. *Let α satisfy conditions (20). The distribution $\mathcal{K}_{\alpha, \mathcal{O}_0}$ cannot be extended to a λ -invariant distribution on $S \times S \times S$.*

Proof. Three Γ factors used in the renormalization of \mathcal{K}_{β} are singular at α , namely $\Gamma(\frac{\beta_2}{2} + \rho)$, $\Gamma(\frac{\beta_3}{2} + \rho)$ and $\Gamma(\frac{\beta_1 + \beta_2 + \beta_3}{2} + 2\rho)$. For $s \in \mathbb{C}$, $|s|$, let

$$\alpha(s) = (\alpha_1, \alpha_2 + 2s, \alpha_3 + 2s), \quad \lambda(s) = (\lambda_1 + 2s, \lambda_2 + s, \lambda_3 + s).$$

Notice that $\alpha(s)$ is transverse to the three planes of poles $\beta_2 = -(n-1) - 2k_2$, $\beta_3 = -(n-1) - 2k_3$ and $\beta_1 + \beta_2 + \beta_3 = -2(n-1) - 2k$. For $s \neq 0$ let

$$\mathcal{F}(s) = \frac{1}{2} \frac{(-1)^k}{k!} \frac{(-1)^{k_2}}{k_2!} \frac{(-1)^{k_3}}{k_3!} \Gamma(-p_1 + \rho) \frac{1}{s} \tilde{\mathcal{K}}_{\alpha(s)}.$$

As α is in Z , $\mathcal{F}(s)$ can be extended as a distribution-valued holomorphic function near 0. Its Taylor expansion at 0 reads

$$\mathcal{F}(s) = F_0 + sF_1 + s^2F_2 + O(s^3), \quad (24)$$

where F_0, F_1, F_2 are distributions on $S \times S \times S$.

Lemma 10.4. *The distributions F_0, F_1, F_2 satisfy*

- i) F_0 is λ -invariant and $\text{Supp}(F_0) = \mathcal{O}_4$*
- ii) $\text{Supp}(F_1) = \overline{\mathcal{O}_2 \cup \mathcal{O}_3}$ and the restriction of F_1 to \mathcal{O}_4^c is λ -invariant.*
- iii) the restriction of F_2 to \mathcal{O}_0 coincides with $\mathcal{K}_{\alpha, \mathcal{O}_0}$.*

Proof. The proof of *i)* is similar to what was done in the general case (see Lemma 10.1 and Lemma 10.2), except that the distributions $\tilde{\mathcal{T}}_{\alpha_2, \alpha_3}^{(2, k_2)}$ and $\tilde{\mathcal{T}}_{\alpha_1, \alpha_2}^{(3, k_3)}$ are now supported in \mathcal{O}_4 but still linearly independent by Proposition 10.1.

For *iii)*, let $\varphi \in \mathcal{C}_c^\infty(\mathcal{O}_0)$. Then, for $s \neq 0$,

$$\begin{aligned} (\mathcal{F}(s), \varphi) &= \\ \frac{1}{2} \frac{(-1)^k}{k!} \frac{1}{\Gamma(-k+2s)} \frac{(-1)^{k_2}}{k_2!} \frac{1}{\Gamma(-k_2+s)} \frac{(-1)^{k_3}}{k_3!} \frac{1}{\Gamma(-k_3+s)} \frac{1}{s} \mathcal{K}_{\alpha(s), \mathcal{O}_0}(\varphi) \\ &= s^2 \mathcal{K}_{\alpha, \mathcal{O}_0}(\varphi) + O(s^3) \end{aligned}$$

As $s \rightarrow 0$, $F_0(\varphi) = F_1(\varphi) = 0$ and $F_2(\varphi) = \mathcal{K}_{\alpha, \mathcal{O}_0}(\varphi)$, proving *iii)* and the fact that $\text{Supp}(F_1) \subset \overline{\mathcal{O}_1 \cup \mathcal{O}_2 \cup \mathcal{O}_3}$.

It remains to prove that $\text{Supp}(F_1) = \overline{\mathcal{O}_2 \cup \mathcal{O}_3}$. As

$$F_1|_{\mathcal{O}_4^c} = \lim_{s \rightarrow 0} \frac{1}{s} \mathcal{F}(s)|_{\mathcal{O}_4^c},$$

$F_1|_{\mathcal{O}_4^c}$ is λ -invariant. In particular $F_1|_{\mathcal{O}_0 \cup \mathcal{O}_1}$ is λ -invariant and supported on \mathcal{O}_1 . But $\alpha_1 \notin (-(n-1) - 2\mathbb{N})$, so that $F_1|_{\mathcal{O}_0 \cup \mathcal{O}_1} = 0$ (use Lemma 4.1 in [3]). Hence $\text{Supp}(F_1|_{\mathcal{O}_4^c}) \subset \mathcal{O}_2 \cup \mathcal{O}_3$. There are only three possibilities :

$$\text{Supp}(F_1|_{\mathcal{O}_4^c}) = \mathcal{O}_2, \quad \text{Supp}(F_1|_{\mathcal{O}_4^c}) = \mathcal{O}_3 \quad \text{or} \quad \text{Supp}(F_1|_{\mathcal{O}_4^c}) = \mathcal{O}_2 \cup \mathcal{O}_3.$$

Assume $\text{Supp}(F_1|_{\mathcal{O}_4^c}) = \mathcal{O}_3$. Then $\text{Supp}(F_1) = \overline{\mathcal{O}_3}$. To see a contradiction, use again the evaluation of the K -coefficients of $\tilde{\mathcal{K}}_{\alpha}(p_{a_1, a_2, a_3})$ (see (63)) and choose a_1, a_2, a_3 such that $a_2 = 0$, $a_3 > k_2 + k_3$, $a_1 + a_3 > k$. For such a choice,

$$\tilde{\mathcal{K}}_{\alpha(s)}(p_{a_1, a_2, a_3}) = NVT(s) \times (-k+s)_{a_1+a_2+a_3} (-k_3+s)_{a_3}$$

where $NVT(s)$ stands for a function which does not vanish at $s = 0$. Now both $(-k+s)_{a_1+a_2+a_3}$ and $(-k_3+s)_{a_3}$ have a simple zero for $s = 0$, so that

$$\tilde{\mathcal{K}}_{\alpha(s)}(p_{a_1, a_2, a_3}) = Cs^2 + O(s^3)$$

with $C \neq 0$. Hence $F_1(p_{a_1, a_2, a_3}) \neq 0$. But for an appropriate choice of a_3 , the function p_{a_1, a_2, a_3} vanishes on the submanifold $\overline{\mathcal{O}_3}$ at an arbitrary large order. Hence $F_1(p_{a_1, a_2, a_3}) \neq 0$ for a_3 arbitrary large is incompatible with $\text{Supp}(F_1) \subset \overline{\mathcal{O}_3}$. Exchanging the role of 2 and 3, we also get that $\text{Supp}(F_1)$ cannot be contained in $\overline{\mathcal{O}_2}$. Hence *ii*) follows. \square

Proposition 10.4 can now be proved. Restrict the situation to the G -invariant open subset $\mathcal{O} = \mathcal{O}_0 \cup \mathcal{O}_2$, in which \mathcal{O}_2 is a (relatively) closed submanifold. The distribution $\frac{1}{s} \mathcal{F}(s)|_{\mathcal{O}}$ can be extended analytically in a neighborhood of $s = 0$ and has Taylor expansion

$$\frac{1}{s} \mathcal{F}(s)|_{\mathcal{O}} = F_1|_{\mathcal{O}} + s F_2|_{\mathcal{O}} + O(s^2) .$$

and $\text{Supp}(F_1|_{\mathcal{O}}) = \mathcal{O}_2$ whereas $F_2|_{\mathcal{O}_0}$ coincides with $\mathcal{K}_{\alpha, \mathcal{O}_0}$. Now apply Proposition A1 to obtain that $\mathcal{K}_{\alpha, \mathcal{O}_0}$ cannot be extended to a λ -invariant distribution on \mathcal{O} , a fortiori to a λ -invariant distribution on $S \times S \times S$. \square

We now proceed to the proof of Theorem 10.1 for the special case. Let T be a λ -invariant distribution. As in the general case, the restriction of T to \mathcal{O}_0 has to be proportional to $\mathcal{K}_{\alpha, \mathcal{O}_0}$, so must be 0 thanks to Proposition 10.4. Hence $\text{Supp}(T) \subset \mathcal{O}_1 \cup \mathcal{O}_2 \cup \mathcal{O}_3 \cup \mathcal{O}_4$. As $\alpha_1 \notin -(n-1) - 2\mathbb{N}$, using Lemma 4.1 in [3], $\text{Supp}(T) \subset \mathcal{O}_2 \cup \mathcal{O}_3 \cup \mathcal{O}_4$. Now let $\mathcal{O} = \mathcal{O}_0 \cup \mathcal{O}_1 \cup \mathcal{O}_2$. The restriction $T|_{\mathcal{O}}$ of T to \mathcal{O} is supported in the closed submanifold \mathcal{O}_2 and is λ -invariant. Now the restriction of $\mathcal{T}_{\alpha_1, \alpha_3, \mathcal{O}_4^c}^{(2, k_2)}$ to \mathcal{O} is a λ -invariant distribution on \mathcal{O} which is supported in \mathcal{O}_2 . By the uniqueness result (see Lemma 4.1 in [3]), $T|_{\mathcal{O}}$ has to be proportional to $\mathcal{T}_{\alpha_1, \alpha_3, \mathcal{O}_4^c}^{(2, k_2)}$. Permuting the indices 2 and 3, a similar result holds on the open set $\overline{\mathcal{O}_0} \cup \mathcal{O}_1 \cup \mathcal{O}_3$. The two distributions $\mathcal{T}_{\alpha_1, \alpha_3, \mathcal{O}_4^c}^{(2, k_2)}$ and $\mathcal{T}_{\alpha_1, \alpha_2, \mathcal{O}_4^c}^{(3, k_3)}$ have disjoint supports, so that there exists two constants A_2 and A_3 such that

$$T|_{\mathcal{O}_4^c} = A_2 \mathcal{T}_{\alpha_1, \alpha_3, \mathcal{O}_4^c}^{(2, k_2)} + A_3 \mathcal{T}_{\alpha_1, \alpha_2, \mathcal{O}_4^c}^{(3, k_3)} .$$

But this contradicts Proposition 10.3, unless $A_2 = A_3 = 0$. Hence necessarily $\text{Supp}(T) \subset \mathcal{O}_4$. In other words, a λ -invariant distribution is supported in the diagonal. Now $\dim \text{Tri}(\lambda, \text{diag}) = \dim S(\lambda_1, \lambda_2; k)$ (see Appendix).

Lemma 10.5. *Let α satisfy conditions (22). Then*

$$\dim S(\lambda_1, \lambda_2; k) = 2 .$$

Proof. Conditions (22) imply

$$\lambda_1 = -\rho - k_2 - k_3, \quad \lambda_2 = -p_1 - k_3$$

Now $\lambda_1 \in \{-\rho, -\rho - 1, \dots, -\rho - (k - 1)\}$ as $k_2 + k_3 = k - p_1$ and $p_1 \geq 1$, but $\lambda_1 \notin \{-1, -2, \dots, -k\}$. This is obvious in case $n - 1$ is odd, and when $n - 1$ is even, the condition $p \notin \rho + \mathbb{N}$ implies $p < \rho$, so that $\rho + k_2 + k_3 > p + k_2 + k_3 = k$ and hence $\lambda_1 < -k$. Moreover $k_2 + k_3 + p + k_3 = k + k_3 \geq k_3$. These conditions guarantee that $\dim S(\lambda_1, \lambda_2; k) \leq 2$ (see Appendix 2). As $\dim \text{Tri}(\boldsymbol{\lambda}, \text{diag}) = \dim \text{Tri}(\boldsymbol{\lambda}) \geq 2$ by Proposition 10.1, $\dim S(\lambda_1, \lambda_2; k) \geq 2$ and the equality follows. \square

This achieves the proof of Theorem 10.1 in the exceptional case.

11 The holomorphic families $\mathcal{S}_{\lambda_1, \lambda_2}^{(k)}$

The next holomorphic family of distributions to be constructed is related to the residue of \mathcal{K}^λ at poles of type II. The determination of the residue (at least for a Zariski open set in each plane of poles of type II) was done in [1] in the non compact picture. The present exposition is its counterpart in the compact picture, which is more suited to the present purposes.

11.1 The family $\tilde{E}_{\lambda_1, \lambda_2}$

Recall the definition of the *normalized Knapp-Stein intertwining operator* \tilde{J}_λ , where $\lambda \in \mathbb{C}$

$$\tilde{J}_\lambda f(x) = \frac{1}{\Gamma(\lambda)} \int_S f(y) |x - y|^{-(n-1)+2\lambda} f(y) dy .$$

For $\Re(\lambda) > 0$, the expression makes sense for any $f \in \mathcal{C}^\infty(S)$ and defines a continuous operator on $\mathcal{C}^\infty(S)$. It is then extended by analytic continuation to a holomorphic family of operators on $\mathcal{C}^\infty(S)$. The main property of this family of operators is

$$\tilde{J}_\lambda \circ \pi_\lambda(g) = \pi_{-\lambda}(g) \circ \tilde{J}_\lambda$$

for any $g \in G$.

The operator \tilde{J}_λ is generically invertible. However, there are exceptions, precisely when π_λ is reducible.

When $\lambda = \rho + k, k \in \mathbb{N}$, then

$$\text{Im}(\tilde{J}_{\rho+k}) = \mathcal{P}_k, \quad \text{Ker}(\tilde{J}_{\rho+k}) = \mathcal{P}_k^\perp \quad (25)$$

where \mathcal{P}_k is the space of restrictions to S of polynomials on E of degree $\leq k$. Dually, when $\lambda = -\rho - k$

$$\text{Ker}(\tilde{J}_{-\rho-k}) = \mathcal{P}_k, \quad \text{Im}(\tilde{J}_{-\rho-k}) = \mathcal{P}_k^\perp. \quad (26)$$

Let $M : \mathcal{C}^\infty(S \times S) \longrightarrow \mathcal{C}^\infty(S \times S)$ be the operator given by

$$f \longmapsto Mf, \quad (Mf)(x, y) = |x - y|^2 f(x, y).$$

For any $\lambda_1, \lambda_2 \in \mathbb{C}$, the operator M intertwines the representations $\pi_{\lambda_1} \otimes \pi_{\lambda_2}$ and $\pi_{\lambda_1-1} \otimes \pi_{\lambda_2-1}$. This is a consequence of the covariance property under the conformal group of the Euclidean distance $|x - y|$ on S (equation (2) in [3]).

Consider the following diagram

$$\begin{array}{ccc} \mathcal{C}^\infty(S) \otimes \mathcal{C}^\infty(S) & \xrightarrow{E_{\lambda_1, \lambda_2}} & \mathcal{C}^\infty(S) \otimes \mathcal{C}^\infty(S) \\ \downarrow \tilde{J}_{\lambda_1} \otimes \tilde{J}_{\lambda_2} & & \uparrow \tilde{J}_{-\lambda_1-1} \otimes \tilde{J}_{-\lambda_2-1} \\ \mathcal{C}^\infty(S) \otimes \mathcal{C}^\infty(S) & \xrightarrow{M} & \mathcal{C}^\infty(S) \otimes \mathcal{C}^\infty(S) \end{array}$$

and let $E_{\lambda_1, \lambda_2} = (\tilde{J}_{-\lambda_1-1} \otimes \tilde{J}_{-\lambda_2-1}) \circ M \circ (\tilde{J}_{\lambda_1} \otimes \tilde{J}_{\lambda_2})$.

Proposition 11.1.

- i) E_{λ_1, λ_2} intertwines the representations $\pi_{\lambda_1} \otimes \pi_{\lambda_2}$ and $\pi_{\lambda_1+1} \otimes \pi_{\lambda_2+1}$
- ii) E_{λ_1, λ_2} is a differential operator on $S \times S$
- iii) $\Gamma(\lambda_1 + \rho + 1)\Gamma(-\lambda_1 + \rho)\Gamma(\lambda_2 + \rho + 1)\Gamma(-\lambda_2 + \rho) E_{\lambda_1, \lambda_2}$ can be analytically extended to \mathbb{C}^2 , yielding a homomorphic family of differential operators on $S \times S$.

Proof. i) is true by construction. ii) and iii) are obtained by passing to the non compact picture. The analogous result on \mathbb{R}^{n-1} is Proposition 3.7 in [1] (notice that the *unnormalized* Knapp-Stein operators were used in [1]). \square

However, a more direct (and enlightening) proof of iii) is available in the compact picture, based on the following lemma.

Lemma 11.1.

- i) let $\lambda_1 \in (\rho + \mathbb{N})$. Then $E_{\lambda_1, \lambda_2} = 0$
- ii) let $\lambda_1 \in (-\rho - 1 - \mathbb{N})$. Then $E_{\lambda_1, \lambda_2} = 0$.

Proof. Let $\lambda_1 = \rho + k$ for some $k \in \mathbb{N}$. Then, by (25)

$$\text{Im}(\tilde{J}_{\lambda_1} \otimes \tilde{J}_{\lambda_2}) \subset \mathcal{P}_k \otimes \mathcal{C}^\infty(S)$$

and as $|x - y|^2 = 2(1 - \langle x, y \rangle)$ on $S \times S$,

$$\text{Im}(M \circ (\tilde{J}_{\lambda_1} \otimes \tilde{J}_{\lambda_2})) \subset \mathcal{P}_{k+1} \otimes \mathcal{C}^\infty(S).$$

As $-\lambda_1 - 1 = -\rho - k - 1$, $\text{Ker}(\tilde{J}_{-\lambda_1-1}) = \mathcal{P}_{k+1}$ by (26) and hence $E_{\lambda_1, \lambda_2} = 0$.

For *ii*), let $\lambda_1 = -\rho - k - 1$, $\text{Im}(\tilde{J}_{\lambda_1}) = \mathcal{P}_{k+1}^\perp$ by (26) so that

$$\text{Im}(M \circ (\tilde{J}_{\lambda_1} \otimes \tilde{J}_{\lambda_2})) \subset \mathcal{P}_1 \mathcal{P}_{k+1}^\perp \otimes \mathcal{C}^\infty(S) \subset \mathcal{P}_k^\perp \otimes \mathcal{C}^\infty(S).$$

As $-\lambda_1 - 1 = \rho + k$, $\text{Ker}(\tilde{J}_{-\lambda_1-1}) = \mathcal{P}_k^\perp$ by (25), $E_{\lambda_1, \lambda_2} = 0$. \square

The analog statement is valid for $\lambda_2 \in \rho + \mathbb{N}$ or $\lambda_2 \in -\rho - \mathbb{N}$. From the knowledge of these zeroes of the operator-valued holomorphic function $(\lambda_1, \lambda_2 \mapsto E_{\lambda_1, \lambda_2})$, *iii*) follows from Lemma 11.1 by routine arguments, ending with the use of Hartog's theorem in \mathbb{C}^2 .

In the sequel, let⁵ $\tilde{E}_{\lambda_1, \lambda_2}$ be defined by

$$\tilde{E}_{\lambda_1, \lambda_2} = \frac{16}{\pi^{2(n-1)}} \Gamma(\lambda_1 + \rho + 1) \Gamma(-\lambda_1 + \rho) \Gamma(\lambda_2 + \rho + 1) \Gamma(-\lambda_2 + \rho) E_{\lambda_1, \lambda_2}. \quad (27)$$

Remark. In [1] Proposition 3.7, an explicit expression was given for (the analog of) the operator $\tilde{E}_{\lambda_1, \lambda_2}$ in the non compact picture. It would be of some interest to find such an expression on $S \times S$.

11.2 A Bernstein-Sato identity for the kernel k_α

Let $\alpha \in \mathbb{C}^3$, and let λ be its associated spectral parameter. Then the geometric parameter associated to $(\lambda_1 + 1, \lambda_2 + 1, \lambda_3)$ is $(\alpha_1, \alpha_2, \alpha_3 + 2)$.

Proposition 11.2. *Assume α is not a pole. Let λ be its associated spectral parameter. Then there exists a scalar $b(\alpha_1, \alpha_2, \alpha_3)$ such that, for all $f_1, f_2, f_3 \in \mathcal{C}^\infty(S)$*

$$\mathcal{K}_{\alpha_1, \alpha_2, \alpha_3+2}(\tilde{E}_{\lambda_1, \lambda_2}(f_1 \otimes f_2), f_3) = b(\alpha_1, \alpha_2, \alpha_3) \mathcal{K}_{\alpha_1, \alpha_2, \alpha_3}(f_1, f_2, f_3) \quad (28)$$

⁵The factor $\frac{16}{\pi^{2(n-1)}}$ is introduced to have a simple expression for the Bernstein-Sato polynomial, see (39).

Proof. The left hand side of (28) is λ -invariant by the covariance property of the operator $\tilde{E}_{\lambda_1, \lambda_2}$. The proposition follows from the generic uniqueness theorem for invariant trilinear forms (see [4] Theorem 3.1). \square

Equation (28) is a disguised form of a *Bernstein-Sato identity* for the kernel $k_\alpha(x, y, z) = |x - y|^{\alpha_3} |y - z|^{\alpha_1} |z - x|^{\alpha_2}$. More precisely, equation (28) implies for the distribution kernels of both sides

$$\tilde{E}_{\lambda_1, \lambda_2}^t(k_{\alpha_1, \alpha_2, \alpha_3+2}) = b(\alpha_1, \alpha_2, \alpha_3) k_{\alpha_1, \alpha_2, \alpha_3}, \quad (29)$$

where $\tilde{E}_{\lambda_1, \lambda_2}^t$ is the transpose of the differential operator $\tilde{E}_{\lambda_1, \lambda_2}$, acting on the variables x, y .

The explicit computation of the function b was done in [1] by brute force computation (with the help of a computer, thanks Ralf!), starting from (29) and the explicit expression (in the non compact picture) of (the analog of) the operator $\tilde{E}_{\lambda_1, \lambda_2}$. We propose here a different computation of the function b , based on the identity

$$\mathcal{K}_{\alpha_1, \alpha_2, \alpha_3+2}(\tilde{E}_{\lambda_1, \lambda_2}(1 \otimes 1), 1) = b(\alpha_1, \alpha_2, \alpha_3) \mathcal{K}_{\alpha_1, \alpha_2, \alpha_3}(1, 1, 1) \quad (30)$$

First recall some formulæ which are needed for the computation of the left hand side of (30).

$$\int_S |1 - x|^s dx = (2\sqrt{\pi})^{n-1} 2^s \frac{\Gamma(\frac{s}{2} + \rho)}{\Gamma(\frac{s}{2} + 2\rho)}, \quad (31)$$

which implies

$$\tilde{J}_\lambda(1) = \pi^\rho \frac{2^{2\lambda}}{\Gamma(\lambda + \rho)} 1. \quad (32)$$

Needless to say, these formulæ (and more to come) have to be interpreted as equality of meromorphic functions of s (or λ). They are usually proved via analytic continuation.

Lemma 11.2. *The following identities hold*

$$i) \quad \int_S |1 - x|^s x_1 dx = -(2\sqrt{\pi})^{n-1} 2^s \frac{s}{2} \frac{\Gamma(\frac{s}{2} + \rho)}{\Gamma(\frac{s}{2} + n)} \quad (33)$$

ii) for any $y, u \in S$,

$$\int_S |x - y|^s \langle u, x \rangle dx = -(2\sqrt{\pi})^{n-1} 2^s \frac{s}{2} \frac{\Gamma(\frac{s}{2} + \rho)}{\Gamma(\frac{s}{2} + n)} \langle u, y \rangle. \quad (34)$$

Proof. For $i)$, as $x_1 = 1 - \frac{1}{2}|\mathbf{1} - x|^2$

$$\begin{aligned} \int_S |\mathbf{1} - x|^s x_1 dx &= \int_S |\mathbf{1} - x|^s dx - \frac{1}{2} \int_S |\mathbf{1} - x|^{s+2} dx \\ &= (2\sqrt{\pi})^{n-1} \left(2^s \frac{\Gamma(\frac{s}{2} + \rho)}{\Gamma(\frac{s}{2} + 2\rho)} - \frac{1}{2} 2^{s+2} \frac{\Gamma(\frac{s}{2} + \rho + 1)}{\Gamma(\frac{s}{2} + 2\rho + 1)} \right) \\ &= (2\sqrt{\pi})^{n-1} 2^s \frac{\Gamma(\frac{s}{2} + \rho)}{\Gamma(\frac{s}{2} + 2\rho + 1)} \left(-\frac{s}{2} \right). \end{aligned}$$

For $ii)$, let $\varphi(u, y)$ be the value of the left hand side. A basic result in harmonic analysis on S is that an operator K on $\mathcal{C}^\infty(S)$ given by $Kf(y) = \int_S k(|x-y|)f(x)dx$ is scalar on any space of homogeneous harmonic polynomials, in particular on \mathcal{H}_1 , the space of homogeneous polynomial of degree 1. Hence $\varphi(y) = c\langle u, y \rangle$ for some c independent of u and y . Let $u = y = \mathbf{1}$ and compute c , using $i)$. \square

Lemma 11.3.

$i)$ for $y \in S$,

$$\int_S |x - y|^s |\mathbf{1} - x|^2 dx = (2\sqrt{\pi})^{n-1} \frac{\Gamma(\frac{s}{2} + \rho)}{\Gamma(\frac{s}{2} + n)} 2^s (sy_1 + s + 2(n-1)).$$

$ii)$ for $y, z \in S$,

$$\int_S |x - y|^s |z - x|^2 dx = (2\sqrt{\pi})^{n-1} \frac{\Gamma(\frac{s}{2} + \rho)}{\Gamma(\frac{s}{2} + n)} 2^s (s\langle z, y \rangle + s + 2(n-1)) \quad (35)$$

Proof. For $i)$ use the fact that $|\mathbf{1} - x|^2 = 2(1 - x_1)$ and combine (31) and (33). For $ii)$, let $\Psi(y, z)$ be the value of the left hand side. Then, by a change of variables, for any $k \in SO(n)$, $\Psi(ky, kz) = \Psi(y, z)$. Hence $\Psi(y, z) = \psi(\langle y, z \rangle)$. Now let $z = \mathbf{1}$ and compute ψ using $i)$. \square

Lemma 11.4. For $u, t \in S$ and r, s in \mathbb{C}^2

$$\begin{aligned} \int_S \int_S |u - x|^s |t - y|^r |x - y|^2 dx dy &= \\ (4\pi)^{n-1} \frac{\Gamma(\frac{s}{2} + \rho) \Gamma(\frac{r}{2} + \rho)}{\Gamma(\frac{s}{2} + n) \Gamma(\frac{r}{2} + n)} 2^{s+r} \left(\frac{rs}{4} |u - t|^2 + (n-1)(r+s) + 2(n-1)^2 \right). \end{aligned} \quad (36)$$

Proof. Slightly rewrite (35) as

$$\int_S |x-y|^s |z-x|^2 dx = (2\sqrt{\pi})^{n-1} \frac{\Gamma(\frac{s}{2} + \rho)}{\Gamma(\frac{s}{2} + n)} 2^s \left(-\frac{s}{2} |z-y|^2 + 2s + 2(n-1) \right).$$

Now, by Fubini's theorem

$$\begin{aligned} & \int_S \int_S |u-x|^s |t-y|^r |x-y|^2 dx dy = \int_S |t-y|^r \left(\int_S |u-x|^s |x-y|^2 dx \right) dy \\ &= (2\sqrt{\pi})^{n-1} \frac{\Gamma(\frac{s}{2} + \rho)}{\Gamma(\frac{s}{2} + n)} 2^s \left(-\frac{s}{2} \int_S |t-y|^r |u-y|^2 dy + (2s+2(n-1)) \int_S |t-y|^r dy \right) \\ &= (4\pi)^{n-1} \frac{\Gamma(\frac{s}{2} + \rho) \Gamma(\frac{r}{2} + \rho)}{\Gamma(\frac{s}{2} + n) \Gamma(\frac{r}{2} + n)} 2^{s+r} \left(\frac{rs}{4} |u-t|^2 - rs - (n-1) + (2s+2(n-1)) \left(\frac{r}{2} + (n-1) \right) \right) \end{aligned}$$

□

Lemma 11.5.

$$\begin{aligned} & \tilde{E}_{\lambda_1, \lambda_2}(1 \otimes 1)(u, t) = \\ & (\lambda_1 + \rho)(\lambda_2 + \rho) \left((\lambda_1 + \rho + 1)(\lambda_2 + \rho + 1) |u-t|^2 - 2(n-1)(\lambda_1 + \lambda_2 + 2) \right). \end{aligned} \tag{37}$$

Proof. First using (32)

$$\tilde{J}_{\lambda_1} \otimes \tilde{J}_{\lambda_2}(1 \otimes 1) = \pi^{n-1} \frac{2^{2\lambda_1+2\lambda_2}}{\Gamma(\lambda_1 + \rho) \Gamma(\lambda_2 + \rho)} 1 \otimes 1.$$

Now

$$\begin{aligned} & \tilde{J}_{-\lambda_1-1} \otimes \tilde{J}_{-\lambda_2-1}(|x-y|^2) = \frac{1}{\Gamma(-\lambda_1-1) \Gamma(-\lambda_2-1)} \times \dots \\ & \iint_{S \times S} |u-x|^{-(n-1)-2\lambda_1-2} |t-y|^{-(n-1)-2\lambda_2-2} |x-y|^2 dx dy \end{aligned}$$

and, using (36), $\iint_{S \times S} \dots$ is equal to

$$\begin{aligned} & (4\pi)^{n-1} \frac{\Gamma(-\lambda_1-1) \Gamma(-\lambda_2-1)}{\Gamma(-\lambda_1+\rho) \Gamma(-\lambda_2+\rho)} 2^{-2(n-1)-2\lambda_1-2\lambda_2-4} \times \\ & \left((\lambda_1 + \rho + 1)(\lambda_2 + \rho + 1) |u-t|^2 - 2(n-1)(\lambda_1 + \lambda_2 + 2) \right) \end{aligned}$$

Now put all things together and take into account the normalization factor for the definition of $\tilde{E}_{\lambda_1, \lambda_2}$ to get the result. □

The rest of the computation will be done using the geometric parameter, so first convert (37) to get

$$\begin{aligned} \tilde{E}_{\lambda_1, \lambda_2}(1 \otimes 1) &= \left(\frac{\alpha_2 + \alpha_3}{2} + (n-1)\right) \left(\frac{\alpha_1 + \alpha_3}{2} + (n-1)\right) \\ &\quad \left(\left(\frac{\alpha_2 + \alpha_3}{2} + n\right) \left(\frac{\alpha_1 + \alpha_3}{2} + n\right) |u-t|^2 - (n-1)(\alpha_1 + \alpha_2 + 2\alpha_3 + 2n + 2)\right) \end{aligned} \quad (38)$$

Proposition 11.3.

$$b(\alpha_1, \alpha_2, \alpha_3) = \quad (39)$$

$$(\alpha_1 + \alpha_2 + \alpha_3 + 2(n-1))(\alpha_1 + \alpha_2 + \alpha_3 + n + 1)(\alpha_3 + n - 1)(\alpha_3 + 2)$$

Proof. Recall that

$$\mathcal{K}_{\alpha_1, \alpha_2, \alpha_3+2}(\tilde{E}_{\lambda_1, \lambda_2}(1 \otimes 1), 1) = b(\alpha_1, \alpha_2, \alpha_3) \mathcal{K}_{\alpha_1, \alpha_2, \alpha_3}(1, 1, 1) .$$

Now, using (38) and the evaluation of the Bernstein-Reznikov integral (see [3] Proposition 2.1)

$$\begin{aligned} \mathcal{K}_{\alpha_1, \alpha_2, \alpha_3+2}(\tilde{E}_{\lambda_1, \lambda_2}(1 \otimes 1), 1) &= \left(\frac{\alpha_2 + \alpha_3}{2} + (n-1)\right) \left(\frac{\alpha_1 + \alpha_3}{2} + (n-1)\right) \\ &\quad \left\{ \left(\frac{\alpha_2 + \alpha_3}{2} + n\right) \left(\frac{\alpha_1 + \alpha_3}{2} + n\right) \mathcal{K}_{\alpha_1, \alpha_2, \alpha_3+4}(1, 1, 1) \right. \\ &\quad \left. - (n-1)(\alpha_1 + \alpha_2 + 2\alpha_3 + 2n + 2) \mathcal{K}_{\alpha_1, \alpha_2, \alpha_3+2}(1, 1, 1) \right\} \\ &= \left(\frac{\pi}{2}\right)^{\frac{3}{2}(n-1)} 2^{\alpha_1 + \alpha_2 + \alpha_3} 4 \frac{\Gamma(\frac{\alpha_1 + \alpha_2 + \alpha_3}{2} + 2\rho + 1) \Gamma(\frac{\alpha_1}{2} + \rho) \Gamma(\frac{\alpha_2}{2} + \rho) \Gamma(\frac{\alpha_3}{2} + \rho + 1)}{\Gamma(\frac{\alpha_1 + \alpha_2}{2} + 2\rho) \Gamma(\frac{\alpha_2 + \alpha_3}{2} + 2\rho) \Gamma(\frac{\alpha_3 + \alpha_1}{2} + 2\rho)} \times \\ &\quad \left\{ (\alpha_1 + \alpha_2 + \alpha_3 + 2n)(\alpha_3 + n + 1) - (n-1)(\alpha_1 + \alpha_2 + 2\alpha_3 + 2n + 2) \right\} . \end{aligned}$$

Now

$$\begin{aligned} &\left(\frac{\pi}{2}\right)^{\frac{3}{2}(n-1)} 2^{\alpha_1 + \alpha_2 + \alpha_3} 4 \frac{\Gamma(\frac{\alpha_1 + \alpha_2 + \alpha_3}{2} + 2\rho + 1) \Gamma(\frac{\alpha_1}{2} + \rho) \Gamma(\frac{\alpha_2}{2} + \rho) \Gamma(\frac{\alpha_3}{2} + \rho + 1)}{\Gamma(\frac{\alpha_1 + \alpha_2}{2} + 2\rho) \Gamma(\frac{\alpha_2 + \alpha_3}{2} + 2\rho) \Gamma(\frac{\alpha_3 + \alpha_1}{2} + 2\rho)} \\ &= 4 \left(\frac{\alpha_1 + \alpha_2 + \alpha_3}{2} + (n-1)\right) \left(\frac{\alpha_3}{2} + \rho\right) \mathcal{K}_{\alpha_1, \alpha_2, \alpha_3}(1, 1, 1) , \end{aligned}$$

and

$$\begin{aligned} &(\alpha_1 + \alpha_2 + \alpha_3 + 2n)(\alpha_3 + n + 1) - (n-1)(\alpha_1 + \alpha_2 + 2\alpha_3 + 2n + 2) \\ &= (\alpha_1 + \alpha_2 + \alpha_3 + n + 1)(\alpha_3 + 2) . \end{aligned}$$

□

A variant of (28) is

$$\begin{aligned} & \tilde{\mathcal{K}}_{\alpha_1, \alpha_2, \alpha_3+2}(\tilde{E}_{\lambda_1, \lambda_2}(f_1 \otimes f_2), f_3) = \\ & 4(\alpha_1 + \alpha_2 + \alpha_3 + n + 1)(\alpha_3 + n - 1) \tilde{\mathcal{K}}_{\alpha_1, \alpha_2, \alpha_3}(f_1, f_2, f_3) \end{aligned} \quad (40)$$

11.3 The family $\mathcal{S}_{\lambda_1, \lambda_2}^{(k)}$

From Proposition 11.1, it is easy to construct covariant bi-differential operators.

Denote by $res : \mathcal{C}^\infty(S \times S) \longrightarrow \mathcal{C}^\infty(S)$ the restriction map from $S \times S$ to S , defined for $f \in \mathcal{C}^\infty(S \times S)$ by

$$f \longmapsto res(f), \quad res(f)(x) = f(x, x) .$$

Lemma 11.6. *The map res intertwines $\pi_{\mu_1} \otimes \pi_{\mu_2}$ with $\pi_{\mu_1 + \mu_2 + \rho}$.*

For $\lambda_1, \lambda_2 \in \mathbb{C}$ define by induction on k the differential operators $\tilde{E}_{\lambda_1, \lambda_2}^{(k)}$ on $S \times S$ by

$$\tilde{E}_{\lambda_1, \lambda_2}^{(0)} = Id, \quad \tilde{E}_{\lambda_1, \lambda_2}^{(k)} = \tilde{E}_{\lambda_1 + k - 1, \lambda_2 + k - 1} \circ \cdots \circ \tilde{E}_{\lambda_1, \lambda_2}$$

and introduce the bi-differential operators $D_{\lambda_1, \lambda_2}^{(k)} : \mathcal{C}^\infty(S \times S) \longrightarrow \mathcal{C}^\infty(S)$ given by

$$D_{\lambda_1, \lambda_2}^{(0)} = res, \quad D_{\lambda_1, \lambda_2}^{(k)} = res \circ \tilde{E}_{\lambda_1, \lambda_2}^{(k)}$$

Proposition 11.4. *Let $\lambda_1, \lambda_2 \in \mathbb{C}$ and $k \in \mathbb{N}$. Then*

$$D_{\lambda_1, \lambda_2}^{(k)} \circ (\pi_{\lambda_1} \otimes \pi_{\lambda_2})(g) = \pi_{\lambda_1 + \lambda_2 + \rho + 2k}(g) \circ D_{\lambda_1, \lambda_2}^{(k)} .$$

This is a consequence of the covariance property of E_{μ_1, μ_2} (Proposition 11.1 i) and Lemma 11.6.

For $f_1, f_2, f_3 \in \mathcal{C}^\infty(S)$, let

$$\mathcal{S}_{\lambda_1, \lambda_2}^{(k)}(f_1, f_2, f_3) = \int_S D_{\lambda_1, \lambda_2}^{(k)}(f_1 \otimes f_2)(x) f_3(x) dx .$$

This defines a distribution on $\mathcal{C}^\infty(S \times S \times S)$, supported on the diagonal \mathcal{O}_4 .⁶

⁶To give an explicit expression of a distribution supported on \mathcal{O}_4 , a choice of a transverse submanifold (and associated transverse differential operators) at each point of \mathcal{O}_4 is needed. Our choice leads to use transverse differential operators acting on the variables (x, y) .

Proposition 11.5. *Let $k \in \mathbb{N}$ and let $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ satisfy*

$$\lambda_1 + \lambda_2 + \lambda_3 = -\rho - 2k .$$

The trilinear form $\mathcal{S}_{\lambda_1, \lambda_2}^{(k)}$ belongs to $\text{Tri}(\lambda, \text{diag})$.

Proof. As $\lambda_3 = -(\lambda_1 + \lambda_2 + \rho + 2k)$, the λ -invariance is a consequence of the covariance property of $D_{\lambda_1, \lambda_2}^{(k)}$ and the duality between π_λ and $\pi_{-\lambda}$. \square

The next statement is easily obtained from (40) by induction on k .

Proposition 11.6. *Let $\alpha \in \mathbb{C}^3$. Then*

$$\begin{aligned} & \tilde{\mathcal{K}}_{\alpha_1, \alpha_2, \alpha_3 + 2k}((E_{\lambda_1, \lambda_2}^{(k)}(f_1 \otimes f_2), f_3) = \\ & 4^k \prod_{l=1}^k (\alpha_1 + \alpha_2 + \alpha_3 + n - 1 + 2l)(\alpha_3 + 2l) \tilde{\mathcal{K}}_{\alpha_1, \alpha_2, \alpha_3}(f_1, f_2, f_3) . \end{aligned}$$

As said in the introduction, the family $\mathcal{S}_{\lambda_1, \lambda_2}^{(k)}$ was introduced in [1] because of its relation with the residues of the meromorphic family \mathcal{K}^λ at poles of type II. The presentation follows closely [1], with changes due to the choice of the compact picture instead of the non compact picture.

Lemma 11.7. *Let α satisfy $\alpha_1 + \alpha_2 + \alpha_3 = -2(n-1)$. Then, for any $f \in \mathcal{C}^\infty(S \times S \times S)$*

$$\tilde{\mathcal{K}}_\alpha(f) = \left(\frac{\pi}{16}\right)^{n-1} \frac{(n-2)!}{\Gamma(\rho)} \frac{1}{\Gamma(-\frac{\alpha_1}{2})\Gamma(-\frac{\alpha_2}{2})\Gamma(-\frac{\alpha_3}{2})} \int_S f(x, x, x) dx . \quad (41)$$

Proof. Let λ be the associated spectral parameter. As easily verified by a change of variable, the trilinear form

$$(f_1, f_2, f_3) \mapsto \int_S f_1(x) f_2(x) f_3(x) dx$$

defines a (non trivial) λ -invariant distribution on $S \times S \times S$. For $\alpha \notin Z$ (i.e. α in a dense open subset of the plane $\beta_1 + \beta_2 + \beta_3 = -2(n-1)$) the uniqueness theorem (see [3]) for λ -invariant trilinear forms applies and hence $\tilde{\mathcal{K}}_\alpha$ has to be a multiple of this distribution. To determine the value of the scalar, evaluate both sides of (41) on the function $f = 1$. By analytic continuation, the equality extends to all α 's in the plane $\alpha_1 + \alpha_2 + \alpha_3 = -2(n-1)$. \square

Proposition 11.7. *Let $\alpha \in \mathbb{C}^3$ such that $\alpha_1 + \alpha_2 + \alpha_3 = -2(n-1) - 2k$ for $k \in \mathbb{N}$. Then*

$$\tilde{\mathcal{K}}_\alpha = \left(\frac{\pi}{16}\right)^{n-1} \frac{(n-2)!}{\Gamma(\rho+k)} 16^{-k} \frac{1}{\Gamma(-\frac{\alpha_1}{2})\Gamma(-\frac{\alpha_2}{2})\Gamma(-\frac{\alpha_3}{2})} \mathcal{S}_{\lambda_1, \lambda_2}^{(k)} \quad (42)$$

Proof. Observe that $(\alpha_1, \alpha_2, \alpha_3 + 2k)$ satisfies the assumption of Lemma 11.7, so that

$$\begin{aligned} & \tilde{\mathcal{K}}_{\alpha_1, \alpha_2, \alpha_3+2k}(\tilde{E}_{\lambda_1, \lambda_2}^{(k)}(f_1 \otimes f_2), f_3) = \\ & \left(\frac{\pi}{16}\right)^{n-1} \frac{(n-2)!}{\Gamma(\rho)} \frac{1}{\Gamma(-\frac{\alpha_1}{2})\Gamma(-\frac{\alpha_2}{2})\Gamma(-\frac{\alpha_3}{2}-k)} \int_S D_{\lambda_1, \lambda_2}^{(k)}(f_1 \otimes f_2)(x) f_3(x) dx \\ & = \left(\frac{\pi}{16}\right)^{n-1} \frac{(n-2)!}{\Gamma(\rho)} \frac{1}{\Gamma(-\frac{\alpha_1}{2})\Gamma(-\frac{\alpha_2}{2})\Gamma(-\frac{\alpha_3}{2}-k)} \mathcal{S}_{\lambda_1, \lambda_2}^{(k)}(f_1, f_2, f_3). \end{aligned}$$

Now, by Proposition 11.6, the same quantity is equal to

$$4^k \prod_{l=1}^k (\alpha_1 + \alpha_2 + \alpha_3 + n - 1 + 2l) (\alpha_3 + 2l) \tilde{\mathcal{K}}_{\alpha_1, \alpha_2, \alpha_3}(f_1, f_2, f_3).$$

Observe that

$$\prod_{l=1}^k (\alpha_1 + \alpha_2 + \alpha_3 + n - 1 + 2l) = \prod_{j=0}^{k-1} (-(n-1) - 2j) = (-1)^k 2^k \prod_{j=0}^{k-1} (\rho + j) = (-1)^k 2^k (\rho)_k,$$

$$\prod_{l=1}^k (\alpha_3 + 2l) \Gamma(-\frac{\alpha_3}{2} - k) = (-1)^k 2^k \prod_{l=1}^k (-\frac{\alpha_3}{2} - l) \Gamma(-\frac{\alpha_3}{2} - k) = (-1)^k 2^k \Gamma(-\frac{\alpha_3}{2}),$$

and the result follows. \square

Proposition 11.8. *Let $k \in \mathbb{N}$ and let λ satisfying $\lambda_1 + \lambda_2 + \lambda_3 = -\rho - 2k$. Then*

if $a_1 + a_2 + a_3 > k$, $\mathcal{S}_{\lambda_1, \lambda_2}^{(k)}(p_{a_1, a_2, a_3}) = 0$, ,
if $a_1 + a_2 + a_3 \leq k$,

$$\begin{aligned} \mathcal{S}_{\lambda_1, \lambda_2}^{(k)}(p_{a_1, a_2, a_3}) &= (2\sqrt{2\pi})^{n-1} 4^k \frac{\Gamma(\rho+k)}{\Gamma(2\rho)} 2^{2(a_1+a_2+a_3)} \\ & \quad (-k)_{a_1+a_2+a_3} (-k-\lambda_1)_{a_1} (-k-\lambda_2)_{a_2} (-k-\lambda_3)_{a_3} \\ & \quad (\rho+\lambda_1+a_2+a_3)_{k-a_2-a_3} (\rho+\lambda_2+a_3+a_1)_{k-a_3-a_1} (\rho+\lambda_3+a_1+a_2)_{k-a_1-a_2} \end{aligned} \quad (43)$$

.

Proof. Assume first that λ is a generic pole of type II. Then combine (42) and the evaluation of $\tilde{\mathcal{K}}^\lambda(p_{a_1, a_2, a_3})$ (see (64)). Observe then that the two hand sides of (43) are holomorphic and conclude by the analytic continuation property. \square

12 The holomorphic families $\mathcal{R}_{\cdot, \cdot}^{(j, m)}$

In this section, define for $\alpha \in \mathbb{C}$ the operator \tilde{I}_α by

$$\tilde{I}_\alpha f(x) = \frac{1}{\Gamma(\frac{\alpha}{2} + \rho)} \int_S |x - y|^\alpha f(y) dy .$$

where $f \in \mathcal{C}^\infty(S)$. Of course, they are nothing but the Knapp-Stein intertwining operators, but with a different parametrization.

For $\alpha_1, \alpha_2 \in \mathbb{C}$, the formula

$$\mathcal{R}_{\alpha_1, \alpha_2}^{(3)}(f_1, f_2, f_3) = \int_S \tilde{I}_{\alpha_2} f_1(z) \tilde{I}_{\alpha_1} f_2(z) f_3(z) dz .$$

defines a continuous trilinear form on $\mathcal{C}^\infty(S) \times \mathcal{C}^\infty(S) \times \mathcal{C}^\infty(S)$, which as usual is regarded as a distribution on $S \times S \times S$.

Let $l \in \mathbb{N}$. Then the function $(x, y, z) \mapsto |x - y|^{2l}$ is smooth on $S \times S \times S$. So, define the distribution $\mathcal{R}_{\alpha_1, \alpha_2}^{(3, l)}$ on $S \times S \times S$ as the product of $\mathcal{R}_{\alpha_1, \alpha_2}^{(3)}$ with $|x - y|^{2l}$, or more explicitly,

$$\mathcal{R}_{\alpha_1, \alpha_2}^{(3, l)}(f) = \int_{S \times S \times S} \mathcal{R}_{\alpha_1, \alpha_2}^{(3)}(x, y, z) (|x - y|^{2l} f(x, y, z)) dx dy dz .$$

Proposition 12.1. *Let $l \in \mathbb{N}$. For $\alpha_1, \alpha_2 \in \mathbb{C}$, let λ be the spectral parameter associated to $(\alpha_1, \alpha_2, 2l)$.*

- i) $\mathcal{R}_{\alpha_1, \alpha_2}^{(3, l)}$ is λ -invariant
- ii) $\mathcal{R}_{\alpha_1, \alpha_2}^{(3, l)}$ depends holomorphically on (α_1, α_2)
- iii)

$$\tilde{\mathcal{K}}_{\alpha_1, \alpha_2, 2l} = \frac{1}{\Gamma(l + \rho)} \frac{1}{\Gamma(\frac{\alpha_1 + \alpha_2}{2} + l + 2\rho)} \mathcal{R}_{\alpha_1, \alpha_2}^{(3, l)} . \quad (44)$$

Proof. For $\Re \alpha_1, \Re \alpha_2$ large enough,

$$\mathcal{R}_{\alpha_1, \alpha_2}^{(3, l)}(f) = \frac{1}{\Gamma(\frac{\alpha_1}{2} + \rho)} \frac{1}{\Gamma(\frac{\alpha_2}{2} + \rho)} \int_{S \times S \times S} |y - z|^{\alpha_1} |z - x|^{\alpha_2} |x - y|^{2l} f(x, y, z) dx dy dz$$

$$= \frac{1}{\Gamma(\frac{\alpha_1}{2} + \rho)} \frac{1}{\Gamma(\frac{\alpha_2}{2} + \rho)} \mathcal{K}_{\alpha_1, \alpha_2, 2l}(f) .$$

Hence the distribution $\mathcal{R}_{\alpha_1, \alpha_2}^{(3, l)}$ is λ -invariant and satisfies (44). The rest follows by analytic continuation. \square

Proposition 12.2. *Let $a_1, a_2, a_3 \in \mathbb{N}$. Then*

$$\begin{aligned} \mathcal{R}_{\alpha_1, \alpha_2}^{(3, l)}(p_{a_1, a_2, a_3}) &= \left(\frac{\pi}{2}\right)^{\frac{3}{2}(n-1)} 2^{\alpha_1 + \alpha_2 + 2l} 2^{2(a_1 + a_2 + a_3)} \\ &\times \frac{\left(\frac{\alpha_1}{2} + \rho\right)_{a_1} \left(\frac{\alpha_2}{2} + \rho\right)_{a_2} \left(\frac{\alpha_1 + \alpha_2}{2} + 2\rho + a_1 + a_2\right)_{l+a_3}}{\Gamma(\frac{\alpha_1}{2} + 2\rho + l + a_1 + a_3) \Gamma(\frac{\alpha_2}{2} + 2\rho + l + a_2 + a_3)} . \end{aligned} \quad (45)$$

Use the evaluation of the K -coefficients for $\tilde{\mathcal{K}}_{\alpha_1, \alpha_2, 2l}$ (see (63) in Appendix) and the relation (44).

By a cyclic permutation of the indices $\{1, 2, 3\}$, a similar construction holds for the families for $\mathcal{R}_{\alpha_2, \alpha_3}^{(1, l)}$ and $\mathcal{R}_{\alpha_1, \alpha_3}^{(2, l)}$.

13 The multiplicity 2 theorem for λ in $Z_{1, II}$

In this section, $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ is assumed to be in $Z_{1, II}$. So, first λ belongs to some line of type II, which, up to a permutation of $\{1, 2, 3\}$ can be chosen to be $D_{l, m}^{+, 1}$ given by the equations

$$\lambda_1 = -\rho - l, \quad \lambda_2 + \lambda_3 = m$$

where $l \in \mathbb{N}, m \in \mathbb{Z}, m \equiv l \pmod{2}, |m| \leq l$.

For convenience, introduce $k = \frac{l-m}{2}$ ($k \in \mathbb{N}, k \leq l$) so that the equations become

$$\lambda_1 = -\rho - l, \quad \lambda_1 + \lambda_2 + \lambda_3 = -\rho - 2k . \quad (46)$$

Next the condition for $\lambda \in Z_{1, II}$ (that is $\lambda \notin Z_2 \cup Z_3$) reads

$$\lambda_2 \notin \{-k, -k+1, \dots, l-k\} \cup (-\rho - k - \mathbb{N}) \cup (\rho + (l-k) + \mathbb{N}) . \quad (47)$$

as a consequence of Proposition 8.5.

Proposition 13.1. *Let λ satisfy (46) and (47). Then $\mathcal{S}_{\lambda_2, \lambda_3}^{(k)} \neq 0$.*

Proof. Formula (43), applied for $a_1 = a_2 = a_3 = 0$ shows that

$$\mathcal{S}_{\lambda_2, \lambda_3}^{(k)}(1) = NVT(-l)_k (\rho + \lambda_2)_k (\rho + \lambda_3)_k .$$

Now $l \geq k$ implies $(-l)_k \neq 0$ and (47) implies $\lambda_2 + \rho \notin -k - \mathbb{N}$, so that $(\rho + \lambda_2)_k \neq 0$, and similarly for $(\lambda_3 + \rho)_k$. Hence $\mathcal{S}_{\lambda_2, \lambda_3}^{(k)}(1) \neq 0$. \square

Let α be the associated geometric parameter, so that

$$\begin{aligned} \alpha_1 + \alpha_2 + \alpha_3 &= -2(n-1) - 2k, \quad \alpha_1 = 2(l-k) \\ \alpha_2 &\notin \left\{ -(n-1), -(n-1)-2, \dots, -(n-1)-2l \right\} \cup 2\mathbb{N} \cup -2(n-1) - 2l - 2\mathbb{N} \end{aligned} \quad (48)$$

It will be convenient to let $p = l - k$. Notice that $p \in \mathbb{N}$.

13.1 The general case

In this subsection, we assume that α is a generic pole of type II, i.e. we assume moreover that $\alpha_2, \alpha_3 \notin -(n-1) - 2\mathbb{N}$.

Lemma 13.1. *Let a_1, a_2, a_3 satisfy $a_2 + a_3 > k + p$, $a_1 + a_2 > k$, $a_1 + a_3 > k$. Then $\mathcal{R}_{\alpha_2, \alpha_3}^{(1,p)}(p_{a_1, a_2, a_3}) \neq 0$.*

Proof. (45) yields

$$\begin{aligned} \mathcal{R}_{\alpha_2, \alpha_3}^{(1,p)}(p_{a_1, a_2, a_3}) &= C \left(\frac{\alpha_2}{2} + \rho \right)_{a_2} \left(\frac{\alpha_3}{2} + \rho \right)_{a_3} (-k - p + a_2 + a_3)_{p+a_1} \\ &\quad \times \frac{1}{\Gamma(\frac{\alpha_2}{2} + (n-1) + p + a_1 + a_3)} \frac{1}{\Gamma(\frac{\alpha_3}{2} + (n-1) + p + a_1 + a_2)} \end{aligned}$$

First observe that $(\frac{\alpha_2}{2} + \rho)_{a_2} (\frac{\alpha_3}{2} + \rho)_{a_3} \neq 0$ as $\alpha_2, \alpha_3 \notin -(n-1) - 2\mathbb{N}$. The assumption $a_2 + a_3 > k + p$ implies that $(-k - p + a_2 + a_3)_{p+a_1} \neq 0$. Then $\frac{\alpha_2}{2} + 2\rho + p \notin -k - \mathbb{N}$, hence $a_1 + a_3 > k$ implies $\frac{1}{\Gamma(\frac{\alpha_2}{2} + (n-1) + p + a_1 + a_3)} \neq 0$. Similarly, $a_1 + a_2 > k$ implies $\frac{1}{\Gamma(\frac{\alpha_3}{2} + (n-1) + p + a_1 + a_2)} \neq 0$. The statement follows. \square

Proposition 13.2. *Let α be a generic pole of type II which satisfies (48) and let λ be its associated spectral parameter. Then $\mathcal{S}_{\lambda_2, \lambda_3}^{(k)}$ and $\mathcal{R}_{\alpha_2, \alpha_3}^{(1,p)}$ are two linearly independent λ -invariant trilinear forms.*

Proof. By Proposition 13.1 $\mathcal{S}_{\lambda_2, \lambda_3}^{(k)} \neq 0$, and hence $\text{Supp}(\mathcal{S}_{\lambda_2, \lambda_3}^{(k)}) = \mathcal{O}_4$.

Next $\text{Supp}(\mathcal{R}_{\alpha_2, \alpha_3}^{(1,p)}) = S \times S \times S$. In fact, assume that $\text{Supp}(\mathcal{R}_{\alpha_2, \alpha_3}^{(1,p)}) \subset \mathcal{O}_1 \cup \mathcal{O}_2 \cup \mathcal{O}_3 \cup \mathcal{O}_4$. Given an arbitrary large number L , by choosing a_1, a_2, a_3 large enough, the function p_{a_1, a_2, a_3} vanishes together with its partial derivatives up to order L on $\mathcal{O}_1 \cup \mathcal{O}_2 \cup \mathcal{O}_3 \cup \mathcal{O}_4$. But by Lemma 13.1 $\mathcal{R}_{\alpha_2, \alpha_3}^{(1,p)}(p_{a_1, a_2, a_3}) \neq 0$ for arbitrary large a_1, a_2, a_3 . Hence $\text{Supp}(\mathcal{R}_{\alpha_2, \alpha_3}^{(1,p)}) \subset \mathcal{O}_1 \cup \mathcal{O}_2 \cup \mathcal{O}_3 \cup \mathcal{O}_4$ is impossible. So $\text{Supp}(\mathcal{R}_{\alpha_2, \alpha_3}^{(1,p)}) = S \times S \times S$.

Having two different supports, the two distributions are linearly independent. \square

Theorem 13.1. *Let α be a generic pole of type II and assume that $\alpha \in Z_{1,II}$. Then $\dim \text{Tri}(\lambda) = 2$. More precisely, if α satisfies (48), then*

$$\text{Tri}(\lambda) = \mathbb{C} \mathcal{R}_{\alpha_2, \alpha_3}^{(1,p)} \oplus \mathbb{C} \mathcal{S}_{\lambda_2, \lambda_3}^{(k)}.$$

Proof. Let T be a λ -invariant distribution. Consider its restriction to \mathcal{O}_0 . As $\text{Supp}(\mathcal{R}_{\alpha_2, \alpha_3}^{(1,p)}) = S \times S \times S$, it follows that $\mathcal{R}_{\alpha_2, \alpha_3}^{(1,p)}|_{\mathcal{O}_0} \neq 0$, and hence $T|_{\mathcal{O}_0} = C \mathcal{R}_{\alpha_2, \alpha_3}^{(1,p)}|_{\mathcal{O}_0}$ for some constant C . Let $U = T - C \mathcal{R}_{\alpha_2, \alpha_3}^{(1,p)}$. Then U is a λ -invariant distribution, which vanishes on \mathcal{O}_0 . Now, as none of the α_j 's belong to $-(n-1) - 2\mathbb{N}$, Lemma 4.1 in [3] shows that U is supported on \mathcal{O}_4 . Hence U and $\mathcal{S}_{\lambda_2, \lambda_3}^{1,k}$ both belong to $\text{Tri}(\lambda, \text{diag})$. In turn, the dimension of this space is equal to $\dim \text{Sol}(\lambda_1, \lambda_2; k)$. In the present situation, notice that $\lambda_1 = 2p \notin \{-1, -2, \dots, -k\} \cup \{-\rho - p - 1, \dots, -\rho - (k-1)\}$ and by Lemma A1, the dimension of the space of solutions is ≤ 1 . Hence, $\dim \text{Sol}(\lambda_1, \lambda_2; k) = 1$ and U is a multiple of $\mathcal{S}_{\lambda_1, \lambda_2}^{(k)}$, and the conclusion follows. \square

13.2 The special case

Let α be a pole of type I+II in $Z_{1,II}$. Up to a permutation of the indices 1, 2, 3, this amounts to the following assumptions

$$\begin{aligned} \alpha_1 &= 2p, & \alpha_2 &= -(n-1) - 2k_2, & \alpha_3 &= -(n-1) + 2q \\ p, k_2, q &\in \mathbb{N}, & q &\geq 1, & k_2 - p - q &\geq 0. \end{aligned} \quad (49)$$

Notice that $\alpha_1 + \alpha_2 + \alpha_3 = -2(n-1) - 2k$ where $k = k_2 - p - q$.

Lemma 13.2. *Let α satisfy (49). Then $\text{Supp}(\mathcal{R}_{\alpha_2, \alpha_3}^{1,p}) = \overline{\mathcal{O}_2}$.*

Proof. Let $f_1, f_2, f_3 \in \mathcal{C}^\infty(S)$ and assume that $\text{Supp}(f_1) \cap \text{Supp}(f_3) = \emptyset$. Then, using notation of section 13, as $\alpha_2 \in -(n-1) - 2\mathbb{N}$, \tilde{I}_{α_2} is a differential operator, and hence $\text{Supp}(\tilde{I}_{\alpha_2} f_1) \subset \text{Supp}(f_1)$. Hence $\mathcal{R}_{\alpha_2, \alpha_3}^{(1)}(f_1, f_2, f_3) = 0$. This implies $\text{Supp}(\mathcal{R}_{\alpha_2, \alpha_3}^{(1)}) \subset \overline{\mathcal{O}_2}$, and a fortiori $\text{Supp}(\mathcal{R}_{\alpha_2, \alpha_3}^{(1,p)}) \subset \overline{\mathcal{O}_2}$. Next, a careful examination of (45), for $a_1 \leq k_1$ and a_2, a_3 large yields $\mathcal{R}_{\alpha_2, \alpha_3}^{(1,p)}(p_{a_1, a_2, a_3}) \neq 0$. Hence $\text{Supp}(\mathcal{R}_{\alpha_2, \alpha_3}^{(1,p)}) \not\subset \mathcal{O}_4$. As $\mathcal{R}_{\alpha_2, \alpha_3}^{(1,p)}$ is λ -invariant, its support is invariant under G , and hence $\text{Supp}(\mathcal{R}_{\alpha_2, \alpha_3}^{(1,p)}) = \overline{\mathcal{O}_2}$. \square

Proposition 13.3. *Let α satisfy (49). Then $\mathcal{S}_{\lambda_2, \lambda_3}^{(k)}$ and $\mathcal{R}_{\alpha_2, \alpha_3}^{(1,p)}$ are two linearly independent λ -invariant trilinear forms.*

Proof. By Proposition 13.1, $\mathcal{S}_{\lambda_2, \lambda_3}^{(k)} \neq 0$ and $\text{Supp}(\mathcal{S}_{\lambda_2, \lambda_3}^{(k)}) = \mathcal{O}_4$. By Proposition 13.2, $\text{Supp}(\mathcal{R}_{\alpha_2, \alpha_3}^{1,p}) = \overline{\mathcal{O}_2}$. Hence the two distributions are linearly independent. \square

Theorem 13.2. *Let α be a pole of type I+II which belongs to $Z_{1,II}$. Then $\dim \text{Tri}(\lambda) = 2$. More precisely, assume α satisfies (49). Then*

$$\text{Tri}(\lambda) = \mathbb{C}\mathcal{S}_{\lambda_2, \lambda_3}^{(k)} \oplus \mathbb{C}\mathcal{R}_{\alpha_2, \alpha_3}^{(1,p)}.$$

To prepare for the proof, the following result is needed.

Proposition 13.4. *The distribution $\mathcal{K}_{\alpha, \mathcal{O}_0}$ cannot be extended as a λ -invariant distribution on $S \times S \times S$.*

Proof. For $s \in \mathbb{C}$, let $\alpha(s) = (\alpha_1, \alpha_2 + 2s, \alpha_3)$. For $|s|$ small, let

$$\mathcal{F}(s) = \frac{(-1)^k}{k!} \frac{(-1)^{k_2}}{k_2!} \Gamma\left(\frac{\alpha_1}{2} + \rho\right) \Gamma\left(\frac{\alpha_3}{2} + \rho\right) \frac{1}{s} \tilde{\mathcal{K}}_{\alpha(s)}.$$

This defines a holomorphic function in a neighborhood of 0, and the Taylor expansion at 0 reads

$$\mathcal{F}(s) = F_0 + sF_1 + O(s^2)$$

where F_0 and F_1 are distributions on $S \times S \times S$.

Lemma 13.3. *The distributions F_0 and F_1 satisfy*

- i) $\text{Supp}(F_0) = \overline{\mathcal{O}_2}$
- ii) *the restriction of F_1 to \mathcal{O}_0 coincides with $\mathcal{K}_{\alpha, \mathcal{O}_0}$*

Proof. i) As $\alpha_1(s) = \alpha_1 = 2p$, $\alpha(s)$ belongs to the plane $\beta_1 = 2p$, and hence, by (44)

$$\tilde{\mathcal{K}}_{2p, \alpha_2 + 2s, \alpha_3} = \frac{1}{\Gamma(p + \rho)} \frac{1}{\Gamma(-k + s)} \mathcal{R}_{\alpha_2 + 2s, \alpha_3}^{(1,p)}.$$

Differentiate at $s = 0$ to get

$$\left(\frac{d}{ds} \tilde{\mathcal{K}}_{2p, \alpha_2 + s, \alpha_3}\right)_{s=0} = \frac{1}{\Gamma(p + \rho)} (-1)^k k! \mathcal{R}_{\alpha_2, \alpha_3}^{(1,p)}.$$

Hence $\text{Supp}(F_0) = \text{Supp}(\mathcal{R}_{\alpha_2, \alpha_3}^{(1,p)}) = \overline{\mathcal{O}_2}$.

For ii), assume that $\varphi \in \mathcal{C}^\infty(S \times S \times S)$ has its support contained in \mathcal{O}_0 . Then

$$(\mathcal{F}(s), \varphi) = \frac{(-1)^k}{k!} \frac{1}{\Gamma(-k + s)} \frac{(-1)^{k_2}}{k_2!} \frac{1}{\Gamma(-k_2 + s)} (\mathcal{K}_{\alpha(s), \mathcal{O}_0}, \varphi)$$

As $s \rightarrow 0$,

$$(\mathcal{F}(s), \varphi) \sim s(\mathcal{K}_{\alpha, \mathcal{O}_0}, \varphi)$$

and hence $(F_0, \varphi) = (\mathcal{K}_{\alpha, \mathcal{O}_0}, \varphi)$, thus proving *ii*. \square

To prove Proposition 13.4, restrict the previous situation to $\mathcal{O}_0 \cup \mathcal{O}_2$, and apply Proposition A1. Hence $\mathcal{K}_{\alpha, \mathcal{O}_0}$ cannot be extended to $\mathcal{O}_0 \cup \mathcal{O}_2$ (a fortiori to $S \times S \times S$) as a λ -invariant distribution. \square

We now are in position to prove Theorem 13.2. Let T be a λ -invariant distribution. Then $T|_{\mathcal{O}_0}$ has to be 0 by Proposition 13.4. Hence T is supported in $\mathcal{O}_1 \cup \mathcal{O}_2 \cup \mathcal{O}_3 \cup \mathcal{O}_4$. But by using twice Lemma 4.1 in [3], $\text{Supp}(T)$ has to be contained in $\mathcal{O}_2 \cup \mathcal{O}_4$. Next, consider the restriction $T|_{\mathcal{O}_4^c}$ of T to the open G -invariant subset \mathcal{O}_4^c , which contains \mathcal{O}_2 as a closed submanifold. Then, again by Lemma 4.1 in [3], $T|_{\mathcal{O}_4^c}$ has to be a multiple of $\mathcal{R}_{\alpha_2, \alpha_3}^{(1, p)}|_{\mathcal{O}'}$. Otherwise stated, there exists a constant c_1 such that $T - c_1 \mathcal{R}_{\alpha_2, \alpha_3}^{(1, p)}$ is supported on \mathcal{O}_4 .

Observe that $\lambda_2 = \rho + \frac{\alpha_1 + \alpha_3}{2} = p + q \notin \{-1, -2, \dots, -k\} \cup \{-\rho, -\rho - 1, \dots, -\rho - (k - 1)\}$, and hence by Lemma A1, $\dim \text{Tri}(\lambda, \text{diag}) \leq 1$. But $\mathcal{S}_{\lambda_2, \lambda_3}^{(k)}$ is a no trivial element of $\text{Tri}(\lambda, \text{diag})$ and generates $\text{Tri}(\lambda, \text{diag})$. Hence there exists a constant c_2 such that $T - c_1 \mathcal{R}_{\alpha_2, \alpha_3}^{(1, p)} = c_2 \mathcal{S}_{\lambda_2, \lambda_3}^{(k)}$, thus finishing the proof of Theorem 13.2.

14 The multiplicity 2 result for $\alpha \in Z_{2, I}$

Let α be in $Z_{2, I}$. Up to a permutation of the indices, this amounts to

$$\alpha_1 = 2k_1, \quad \alpha_2 = -(n - 1) - 2k_2, \quad \alpha_3 = -(n - 1) - 2k_3 \quad (50)$$

where $k_1, k_2, k_3 \in \mathbb{N}$, with $k = k_2 + k_3 - k_1 \geq 0$. The conditions imply that

$$\alpha_1 + \alpha_2 + \alpha_3 = -2(n - 1) - 2k .$$

The point α belongs to two lines contained in Z , namely

$$\begin{aligned} \mathcal{D}^- & \quad \begin{cases} \alpha_2 = -(n - 1) - 2k_2 \\ \alpha_3 = -(n - 1) - 2k_3 \end{cases} , \\ \mathcal{D}^+ & \quad \begin{cases} \alpha_1 + \alpha_2 + \alpha_3 = -2(n - 1) - 2k \\ \alpha_1 = 2k_1 \end{cases} . \end{aligned}$$

The associated spectral parameter λ is given by

$$\lambda_1 = -\rho - k_2 - k_3, \quad \lambda_2 = k_1 - k_3, \quad \lambda_3 = k_1 - k_2,$$

or

$$\lambda_1 = -\rho - k_2 - k_3, \quad \lambda_2 = -k + k_2, \quad \lambda_3 = -k + k_3$$

where $k \leq k_2 + k_3$.

Change notation and set

$$\lambda_1 = -\rho - l_1, \quad \lambda_2 = m_2, \quad \lambda_3 = m_3,$$

where

$$l_1 \in \mathbb{N}, \quad m_2, m_3 \in \mathbb{Z}, \quad l_1 + m_2 + m_3 \equiv 0 \pmod{2}, \quad |m_2 \pm m_3| \leq l_1.$$

The correspondence between notations is given by

$$l_1 = k_2 + k_3, \quad m_2 = -k_3 + k_1, \quad m_3 = -k_2 + k_1.$$

Proposition 14.1. *The differential $d\tilde{\mathcal{K}}_\alpha$ of $\beta \mapsto \tilde{\mathcal{K}}_\beta$ at α is of rank 1. The distributions $\mathcal{R}_{\alpha_2, \alpha_3}^{(1, k_1)}$, $\tilde{\mathcal{T}}_{\alpha_1, \alpha_3}^{(2, k_2)}$, $\tilde{\mathcal{T}}_{\alpha_1, \alpha_2}^{(3, k_3)}$ and $\mathcal{S}_{\lambda_2, \lambda_3}^{(k)}$ are proportional to $d\tilde{\mathcal{K}}_\alpha$.*

Proof. Consider the differential of $\beta \mapsto \tilde{\mathcal{K}}_\beta$. As $\tilde{\mathcal{K}}_\beta$ vanishes on \mathcal{D}^+ and \mathcal{D}^- , the rank of its differential at $\alpha = \mathcal{D}^+ \cap \mathcal{D}^-$ is at most 1.

Consider the plane $\beta_1 = 2k_1$. For arbitrary β_2, β_3 , by (44)

$$\tilde{\mathcal{K}}_{2k_1, \beta_2, \beta_3} = \frac{1}{\Gamma(k_1 + \rho)} \frac{1}{\Gamma(\frac{\beta_2 + \beta_3}{2} + k_1 + 2\rho)} \mathcal{R}_{\beta_2, \beta_3}^{(1, k_1)}.$$

By differentiation this implies

$$\frac{d}{ds} (\tilde{\mathcal{K}}_{2k_1, \alpha_2 + s, \alpha_3 + s})_{s=0} = \frac{(-1)^k k!}{\Gamma(k_1 + \rho)} \mathcal{R}_{\alpha_2, \alpha_3}^{(1, k_1)}. \quad (51)$$

To see that $\mathcal{R}_{\alpha_2, \alpha_3}^{(1, k_1)} \neq 0$, use (45) to get

$$\begin{aligned} \mathcal{R}_{\alpha_2, \alpha_3}^{(1, k_1)}(p_{a_1, a_2, a_3}) &= NVT \times (-k_2)_{a_2} (-k_3)_{a_3} (-k_2 - k_3 + a_2 + a_3)_{k_1 + a_1} \\ &\quad \times \frac{1}{\Gamma(-k_2 + \rho + k_1 + a_2 + a_1) \Gamma(-k_3 + \rho + k_1 + a_1 + a_3)}. \end{aligned}$$

Choose $a_2 = a_3 = 0$ and $a_1 = k_2 + k_3 - k_1$ to get $\mathcal{R}_{\alpha_2, \alpha_3}^{(1, k_1)}(p_{a_1, 0, 0}) \neq 0$, and hence $\mathcal{R}_{\alpha_2, \alpha_3}^{1, k_1} \neq 0$. It follows that the rank of $d\mathcal{K}_\alpha$ is 1, and the fact that $\mathcal{R}_{\alpha_2, \alpha_3}^{(1, k_1)}$ is proportional to $d\tilde{\mathcal{K}}_\alpha$.

As a consequence, any partial derivative in a direction that does not lie in the plane containing \mathcal{D}^+ and \mathcal{D}^- of $\tilde{\mathcal{K}}_\beta$ at α does not vanish, and is a multiple of $d\tilde{\mathcal{K}}_\alpha$. So similar results can be obtained for the other invariant distributions, namely

$$\frac{d}{ds}(\tilde{\mathcal{K}}_{\alpha_1, \alpha_2, \alpha_3+2s})_{s=0} = \frac{\pi^\rho (-1)^{k_3+k_2} 4^{-k_3} k_2!}{\Gamma(\rho+k_3)\Gamma(\rho+k_1)} \tilde{\mathcal{T}}_{\alpha_1, \alpha_2}^{(3, k_3)}, \quad (52)$$

and a similar formula after exchanging the role of 2 and 3, and

$$\frac{d}{ds}(\tilde{\mathcal{K}}^{\lambda_1+2s, \lambda_2-s, \lambda_3-s})_{s=0} = \frac{(-1)^{l_1-k} (l_1-k)!}{\Gamma(\rho+k+\lambda_2)\Gamma(\rho+k+\lambda_3)} \mathcal{S}_{\lambda_2, \lambda_3}^{(k)}. \quad (53)$$

□

The plane containing \mathcal{D}^+ and \mathcal{D}^- admits the equation $\lambda_1 = -\rho - l_1$ and use (λ_2, λ_3) as coordinates in this plane. At $\mu = \lambda$, $\tilde{\mathcal{K}}^\mu$ and its partial derivatives $\frac{\partial \tilde{\mathcal{K}}^\mu}{\partial \mu_2}$ and $\frac{\partial \tilde{\mathcal{K}}^\mu}{\partial \mu_3}$ vanish, so let for $s \in \mathbb{C}$

$$\lambda(s) = (\lambda_1, \lambda_2 + 2s, \lambda_3) = (-\rho - l_1, m_2 + 2s, m_3), \quad (54)$$

$$\alpha(s) = (2k_1 + 2s, -(n-1) - 2k_2 - 2s, -(n-1) - 2k_3 + 2s). \quad (55)$$

Then the distribution

$$\mathcal{Q} = \mathcal{Q}_{l_1, m_2, m_3} = \lim_{s \rightarrow 0} \frac{1}{s^2} \tilde{\mathcal{K}}^{\lambda(s)}$$

is λ -invariant and corresponds to a mixed second order partial derivative of $\tilde{\mathcal{K}}^\lambda$ as explained in the introduction.

Lemma 14.1. *The K -coefficients of the distribution \mathcal{Q} are equal to 0 unless*

$$a_2 + a_3 > k_2 + k_3 \quad \text{and} \quad \text{either} \quad a_2 \leq k_2 \quad \text{or} \quad a_3 \leq k_3.$$

Assume $a_2 \leq k_2$ and $a_2 + a_3 > k_2 + k_3$. Then

$$\begin{aligned} \mathcal{Q}(p_{a_1, a_2, a_3}) &= \left(\frac{\sqrt{\pi}}{2}\right)^{3(n-1)} 2^{-\rho-2k+a_1+a_2+a_3} \\ &(-1)^k k! (-k+a_1+a_2+a_3-1)! (\rho+k_1)_{a_1} (-k_2)_{a_2} \\ &(-1)^{k_3} k_3! (-k_3+a_3-1)! \frac{1}{\Gamma(-k_2-k_3+a_2+a_3)} \\ &\frac{1}{\Gamma(m_2+\rho+a_3+a_1)\Gamma(m_3+\rho+a_2+a_1)}. \end{aligned} \quad (56)$$

A similar formula holds for the symmetric situation, where $a_3 \leq k_3$ and $a_2 + a_3 > k_2 + k_3$.

Proof. Use (64) in the Appendix. $\tilde{\mathcal{K}}^{\lambda(s)}(p_{a_1, a_2, a_3})$ is (essentially) a product of seven factors, which we now analyze. There is an inverse Γ factor which in the present situation does not depend on s and is equal to $\frac{1}{\Gamma(-l_1 + a_2 + a_3)}$. Hence, unless $a_2 + a_3 > l_1 = k_2 + k_3$, this factor is 0 for all s and hence $\mathcal{Q}(p_{a_1, a_2, a_3}) = 0$. Now assume that $a_2 + a_3 > l_1 = k_2 + k_3$. Another factor is equal to $(-k + s)_{a_1 + a_2 + a_3}$. In this Pochhammer's symbol the last factor is equal to $(-k + s + a_1 + a_2 + a_3 - 1)$. As $a_2 + a_3 > k_2 + k_3 \geq k$, $-k + a_1 + a_2 + a_3 - 1 \geq 0$, hence this factor has a simple zero at $s = 0$. More explicitly

$$(-k + s)_{a_1 + a_2 + a_3} = [(-1)^k k! (-k + a_1 + a_2 + a_3 - 1)!] s + O(s^2) .$$

Two other factors are equal to $(-k_2 - s)_{a_2}$ and $(-k_3 + s)_{a_3}$. If $a_2 > k_2$ and $a_3 > k_3$ each factor vanishes for $s = 0$, so that $\tilde{\mathcal{K}}^{\lambda(s)}(p_{a_1, a_2, a_3})$ has a zero of order 3 at $s = 0$ and hence $\mathcal{Q}(p_{a_1, a_2, a_3}) = 0$. Now assume $a_2 \leq k_2$ and still $a_2 + a_3 > k_2 + k_3$, which forces $k_3 > a_3$. The factor $(-k_2 - s)_{a_2}$ does not vanish for $s = 0$ and its value is $(-k_2)_{a_2}$. The factor $(-k_3 + s)_{a_3}$ has a simple zero at $s = 0$ and more precisely, $(-k_3 + s)_{a_3} = (-1)^{k_3} k_3! (-k_3 + a_3 - 1)! s + O(s^2)$. The remaining factors offer no difficulty. Collect all facts to get the statement. \square

Corollary 14.1. *The distribution \mathcal{Q} is $\neq 0$. More precisely, for a_1 large enough, $a_2 + a_3 > k_2 + k_3$ and either $a_2 \leq k_2$ or $a_3 \leq k_3$, $\mathcal{Q}(p_{a_1, a_2, a_3}) \neq 0$. Moreover,*

$$\text{Supp}(\mathcal{Q}) = \overline{\mathcal{O}_2 \cup \mathcal{O}_3} .$$

Proof. For a_1 large enough, the two inverse Γ factors on the last line of (56) do not vanish, so that the whole expression does not vanish, and hence $\mathcal{Q} \neq 0$.

Let $\varphi \in \mathcal{C}^\infty(S \times S \times S)$ and assume $\text{Supp}(\varphi) \subset \mathcal{O}_0 \cup \mathcal{O}_1$. Then

$$\inf \left(|x - y|, (x, y, z) \in \text{Supp}(\varphi) \right) > 0, \inf \left(|x - z|, (x, y, z) \in \text{Supp}(\varphi) \right) > 0 .$$

The exponent of $|y - z|$ is $\alpha_1(s) = 2k_1 + 2s$, so that, for s in a small neighborhood of 0, the integral $\mathcal{K}_{\alpha(s)}(\varphi)$ converges, and $\lim_{s \rightarrow 0} \mathcal{K}_{\alpha(s)}(\varphi) = \mathcal{K}_{\alpha}(\varphi)$. In turn,

$$\begin{aligned} \mathcal{K}_{\alpha(s)}(\varphi) &= \Gamma(k_1 + \rho + s) \Gamma(-k_2 + s) \Gamma(-k_3 + s) \Gamma(-k + s) \tilde{\mathcal{K}}_{\alpha(s)}(\varphi) \\ &= C \frac{1}{s} \mathcal{Q}(\varphi) + O(1) \end{aligned}$$

with $C \neq 0$, which forces $\mathcal{Q}(\varphi) = 0$.

As a consequence $Supp(\mathcal{Q}) \subset \overline{\mathcal{O}_2 \cup \mathcal{O}_3}$. As $Supp(\mathcal{Q})$ is invariant under the action of G , if $Supp(\mathcal{Q}) \neq \overline{\mathcal{O}_2 \cup \mathcal{O}_3}$, then $Supp(\mathcal{Q})$ has to be contained in either $\overline{\mathcal{O}_2}$ or $\overline{\mathcal{O}_3}$. Assume $Supp(\mathcal{Q}) \subset \overline{\mathcal{O}_2}$. Then $\mathcal{Q}(f) = 0$ if the smooth function f vanishes at a sufficiently large order on $\overline{\mathcal{O}_2}$. In particular, $\mathcal{Q}(p_{a_1, a_2, a_3}) = 0$ if a_2 is large enough. But this contradicts Corollary 14.1 (choose $a_3 \leq k_3$, a_2 such that $a_2 + a_3 > k_2 + k_3$ and a_1 large). Hence the only possibility is $Supp(\mathcal{Q}) = \overline{\mathcal{O}_2 \cup \mathcal{O}_3}$. \square

Proposition 14.2. *The two distributions $d\tilde{\mathcal{K}}_\alpha$ and \mathcal{Q} are linearly independent.*

Proof. The statement follows from the study of the supports of the two distributions. \square

Theorem 14.1. *Let λ be in $Z_{2,I}$. Then*

$$\dim Tri(\lambda) = 2 .$$

More precisely,

$$Tri(\lambda) = \mathbb{C}\mathcal{R}_{\alpha_2, \alpha_3}^{(1, k_1)} \oplus \mathbb{C}\mathcal{Q}_{l_1, m_2, m_3} .$$

Some preparation is needed for the proof of the theorem.

Proposition 14.3. *Let α satisfy conditions (50). The distribution $\mathcal{K}_{\alpha, \mathcal{O}_0}$ cannot be extended to $S \times S \times S$ as a λ -invariant distribution.*

Proof. Recall that three of the Γ factors involved in the normalization of \mathcal{K}_α become singular at α .

For s a complex number, let $\alpha(s)$ as in (55) and let

$$\mathcal{G}(s) = - \frac{(-1)^k}{k!} \frac{(-1)^{k_2}}{k_2!} \frac{(-1)^{k_3}}{k_3!} \Gamma(k_1 + \rho + s) \frac{1}{s^2} \tilde{\mathcal{K}}_{\alpha(s)} .$$

As $\tilde{\mathcal{K}}_\alpha(s)$ vanishes together with its first derivative at $s = 0$, $\mathcal{G}(s)$ is well defined for s in a neighborhood of 0 and its Taylor expansion at $s = 0$ reads

$$\mathcal{G}(s) = G_0 + sG_1 + O(s^2) .$$

where G_0 and G_1 are distributions on $S \times S \times S$.

Lemma 14.2.

- i) $Supp(G_0) = \overline{\mathcal{O}_2 \cup \mathcal{O}_3}$
- ii) G_1 coincides on \mathcal{O}_0 with $\mathcal{K}_{\alpha, \mathcal{O}_0}$.

Proof. For i), observe that $G_0 = -\frac{(-1)^k}{k!} \frac{(-1)^{k_2}}{k_2!} \frac{(-1)^{k_3}}{k_3!} \Gamma(k_1 + \rho) \mathcal{Q}$. Hence i) follows from Corollary 14.1.

In order to prove ii), let $\varphi \in \mathcal{C}_c^\infty(\mathcal{O}_0)$. Then for $s \neq 0$,

$$\begin{aligned} & (\mathcal{F}(s), \varphi) \\ &= \frac{(-1)^k}{k!} \frac{1}{\Gamma(-k+s)} \frac{(-1)^{k_2}}{k_2!} \frac{-1}{\Gamma(-k_2-s)} \frac{(-1)^{k_3}}{k_3!} \frac{1}{\Gamma(-k_3+s)} \frac{1}{s^2} (\mathcal{K}_{\alpha(s), \mathcal{O}_0}, \varphi). \end{aligned}$$

As $s \rightarrow 0$, this identity implies

$$(\mathcal{G}(s), \varphi) \sim s(\mathcal{K}_{\alpha, \mathcal{O}_0}, \varphi) .$$

so that ii) follows. \square

Now we are ready to prove Proposition 14.3. Let $\mathcal{O} = \mathcal{O}_0 \cup \mathcal{O}_2$. This is an open G -invariant subset which contains \mathcal{O}_2 as a closed submanifold. Restrict $\mathcal{G}(s)$, G_0 and G_1 to \mathcal{O} . Now apply Proposition A1 to conclude that G_1 cannot be extended to \mathcal{O} (a fortiori to $S \times S \times S$) as a λ -invariant distribution. \square

We still need another result before starting the proof of Theorem 14.1.

Proposition 14.4. *The distribution $\mathcal{T}_{\alpha_1, \alpha_2, \mathcal{O}_4^c}^{3, k_3}$ cannot be extended to $S \times S \times S$ as a λ -invariant distribution.*

Proof. As seen in section 9, for arbitrary $\beta_1, \beta_2 \in \mathbb{C}$, the distribution $\mathcal{T}_{\beta_1, \beta_2, \mathcal{O}_4^c}^{(3, k_3)}$ is a λ -invariant distribution on \mathcal{O}_4^c . For $s \in \mathbb{C}$, let

$$\alpha(s) = (\alpha_1, \alpha_2 + 2s, \alpha_3), \quad \lambda(s) = (\lambda_1 + s, \lambda_2, \lambda_3 + s)$$

and let

$$\mathcal{F}(s) = \frac{(-1)^k}{k!} \tilde{\mathcal{T}}_{2k_1, \alpha_2 + 2s}^{(3, k_3)} .$$

The Taylor expansion of $\mathcal{F}(s)$ at $s = 0$ reads

$$\mathcal{F}(s) = F_0 + sF_1 + O(s^2) ,$$

where F_0, F_1 are distributions on $S \times S \times S$.

Lemma 14.3.

i) $\text{Supp}(F_0) = \mathcal{O}_4$

ii) the restriction of F_1 to \mathcal{O}_4^c coincides with $\mathcal{T}_{\alpha_1, \alpha_2, \mathcal{O}_4^c}^{(3, k_3)}$.

Proof. For $i)$, $F_0 = \mathcal{F}(0) = \frac{(-1)^k}{k!} \tilde{\mathcal{T}}_{2k_1, \alpha_2}^{3, k_3}$ and hence by (52), $\text{Supp}(F_0) = \mathcal{O}_4$.

For $ii)$, let $\varphi \in \mathcal{C}^\infty(S \times S \times S)$ and assume that $\text{Supp}(\varphi) \subset \mathcal{O}_4^c$. Then, using (16)

$$(\mathcal{F}(s), \varphi) = \frac{(-1)^k}{k!} \frac{1}{\Gamma(-k+s)} (\mathcal{T}_{2k_1, \alpha_2+2s, \mathcal{O}_4^c}^{(3, k_3)}, \varphi),$$

and $ii)$ follows by letting $s \rightarrow 0$. \square

Proposition A2 implies that $\mathcal{T}_{\alpha_1, \alpha_2, \mathcal{O}_4^c}^{(3, k_3)}$ cannot be extended to $S \times S \times S$ as a λ -invariant distribution. \square

A similar statement holds for $\mathcal{T}_{\alpha_1, \alpha_3, \mathcal{O}_4^c}^{(2, k_2)}$. However, remark that the restriction of Q to \mathcal{O}_4^c is a (non trivial) linear combination of $\mathcal{T}_{\alpha_1, \alpha_3, \mathcal{O}_4^c}^{(2, k_2)}$ and $\mathcal{T}_{\alpha_1, \alpha_2, \mathcal{O}_4^c}^{(3, k_3)}$.

We can now proceed to the proof of Theorem 14.1. Let T be a λ -invariant distribution. The restriction of T to \mathcal{O}_0 has to be 0 by Proposition 14.3. Moreover, as $\alpha_1 \notin (-(n-1) - 2\mathbb{N})$, the restriction of T to $\mathcal{O}_0 \cup \mathcal{O}_1$ is equal to 0. Hence $\text{Supp}(T) \subset \mathcal{O}_2 \cup \mathcal{O}_3 \cup \mathcal{O}_4$. Consider the restriction $T_{\Omega_{12}}$ of T to the G -invariant open set $\Omega_{12} = \mathcal{O}_0 \cup \mathcal{O}_1 \cup \mathcal{O}_2$. $T_{\Omega_{12}}$ is λ -invariant and supported on the (relatively) closed submanifold \mathcal{O}_2 . The restriction of $\mathcal{Q}_{l_1, m_2, m_3}$ to Ω_{12} is λ -invariant and $\neq 0$ by Corollary 14.1. Hence by Lemma 4.1 in [3], there exists a constant c such that T and cQ coincide on Ω_{12} . Now consider $T - cQ$. It is a λ -invariant distribution which is supported on $\overline{\mathcal{O}_3}$. Its restriction to \mathcal{O}_4^c is supported on \mathcal{O}_3 and must coincide on \mathcal{O}_4^c with a multiple of $\mathcal{T}_{\alpha_1, \alpha_2, \mathcal{O}_4^c}^{(3, k_3)}$ again by Lemma 4.1 in [3]. By Proposition 14.4, this forces $T - cQ$ to be 0 on \mathcal{O}_4^c , and hence, $T - cQ$ is supported on \mathcal{O}_4 .

To finish the proof it is enough to prove that the space $\text{Tri}(\lambda, \text{diag})$ has dimension 1 (hence generated by the distribution $\mathcal{R}_{\alpha_2, \alpha_3}^{(1, k_1)}$), because this will imply that $T = cQ + d\mathcal{R}_{\alpha_2, \alpha_3}^{(1, k_1)}$ for some constant d , which is the content of Theorem 14.1. Observe that

$$\lambda_1 = -\rho - k_2 - k_3 \notin \{-\rho, -\rho - 1, \dots, -\rho - (k-1)\} \cup \{-1, -2, \dots, -k\}$$

as $k_2 + k_3 \geq k$. Hence, by Appendix 2, this implies $\dim \text{Tri}(\lambda) = 1$. Q.E.D.

15 The multiplicity 2 result for $\alpha \in Z_{2, II}$

Let $\alpha \in Z_{2, II}$. Up to a permutation of the indices, it amounts to

$$\alpha_1 = 2k_1, \quad \alpha_2 = 2k_2, \quad \alpha_3 = -2(n-1) - 2k_3 \quad (57)$$

where $k_1, k_2, k_3 \in \mathbb{N}$ and $k_1 + k_2 \leq k_3$. Observe that $\alpha_1 + \alpha_2 + \alpha_3 = -2(n-1) - 2(k_3 - k_1 - k_2) = -2(n-1) - 2k$ where $k = k_3 - k_1 - k_2$, so that α is a pole of type II. If $n-1$ is odd, α is not a pole of type I, but if $n-1$ is even, then $\alpha_3 = -(n-1) - 2(\rho + k_3)$ so that α is a pole of type I+II.

The alternative description of $Z_{2,II}$ in terms of the spectral parameter is

$$\begin{aligned} \lambda_1 &= -\rho - l_1, & \lambda_2 &= -\rho - l_2, & \lambda_3 &= \rho + m_3 \\ l_1, l_2, m_3 &\in \mathbb{N}, & l_1 + l_2 + m_3 &\equiv 0 \pmod{2}, & |l_1 - l_2| &\leq m_3 \leq l_1 + l_2. \end{aligned} \quad (58)$$

Also let $l_1 + l_2 - m_3 = 2k$. Then, l_1, l_2 satisfy $k \leq l_1, k \leq l_2$.

The point λ belongs to two lines contained in Z , namely

$$\mathcal{D}^1 = \mathcal{D}_{-l_2+m_3}^{1,+} \quad \mathcal{D}^2 = \mathcal{D}_{-l_1+m_3}^{2,+}.$$

Proposition 15.1.

- i) the differential $d\tilde{\mathcal{K}}_\alpha$ of $\beta \mapsto \tilde{\mathcal{K}}_\beta$ at α is of rank 1
- ii) the distributions $\mathcal{R}_{\alpha_2, \alpha_3}^{(1, k_1)}$ and $\mathcal{R}_{\alpha_1, \alpha_3}^{(2, k_2)}$ are $\neq 0$ and proportional to $d\tilde{\mathcal{K}}_\alpha$.

Moreover,

- If $n-1$ is odd, $\text{Supp}(\mathcal{R}_{\alpha_2, \alpha_3}^{(1, k_1)}) = S \times S \times S$.
- If $n-1$ is even, then $\text{Supp}(\mathcal{R}_{\alpha_2, \alpha_3}^{(1, k_1)}) = \overline{\mathcal{O}_3}$.

Proof. As $\tilde{\mathcal{K}}_\beta$ vanishes on \mathcal{D}^1 and \mathcal{D}^2 , the rank of $d\tilde{\mathcal{K}}_\alpha$ is at most 1.

From (49) follows for generic β_3

$$\tilde{\mathcal{K}}_{2k_1, 2k_2, \beta_3} = \frac{1}{\Gamma(\rho + k_1)} \frac{1}{\Gamma(\frac{k_1+k_2}{2} + \frac{\beta_3}{2} + 2\rho)} \mathcal{R}_{2k_2, \beta_3}^{(1, k_1)}$$

Let β_3 tend to $\alpha_3 = -2(n-1) - 2k_3$ to obtain

$$\left(\frac{\partial}{\partial \beta_3} \tilde{\mathcal{K}}_{2k_1, 2k_2, \beta_3} \right)_{\beta_3 = \alpha_3} = \frac{1}{\Gamma(\rho + k_1)} (-1)^k k! \mathcal{R}_{2k_2, \alpha_3}^{(1, k_1)} \quad (59)$$

and a similar formula for $\mathcal{R}_{2k_1, \alpha_3}^{2, k_2}$.

Next, use (50) (after permuting the indices) for $a_3 = k_3$. Then

$$\begin{aligned} \mathcal{R}_{2k_2, \alpha_3}^{(1, k_1)}(p_{a_1, a_2, a_3}) &= C (k_2 + \rho)_{a_2} (-\rho - k_3)_{k_3} (k_2 + a_2)_{k_1 + k_3} \\ &\quad \frac{1}{\Gamma(k_2 + 2\rho + k_1 + a_2 + k_3)} \frac{1}{\Gamma(k_1 + a_1)} \end{aligned}$$

which is clearly different of 0. Hence $\mathcal{R}_{2k_2, \alpha_3}^{(1, k_1)}$ is $\neq 0$, $d\tilde{\mathcal{K}}_\alpha$ is of rank 1 and proportional to $\mathcal{R}_{2k_2, \alpha_3}^{(1, k_1)}$. Hence i) holds true, and after permuting the indices 1 and 2, the first part of ii) follows.

For the second statement in *ii*), assume first that $n-1$ is odd. Then (50) implies that for $a_3 > k_3$ and a_1, a_2 arbitrary, the K -coefficient $\mathcal{R}_{2k_2, \alpha_3}^{(1, k_1)}(p_{a_1, a_2, a_3})$ does not vanish. This implies $\text{Supp}(\mathcal{R}_{\alpha_2, \alpha_3}^{(1, k_1)}) = S \times S \times S$.

Now assume that $n-1$ is even. Let $\varphi \in \mathcal{C}^\infty(S \times S \times S)$ and assume that $\text{Supp}(\varphi) \subset \mathcal{O}_0 \cup \mathcal{O}_1 \cup \mathcal{O}_2$. This implies

$$\inf\{|x-y|, (x, y, z) \in \text{Supp}(\varphi)\} > 0 .$$

Hence the integral which defines $\mathcal{K}_{2k_1, 2k_2, \beta_3}(\varphi)$ is convergent. For generic β_3 ,

$$\mathcal{K}_{2k_1, 2k_2, \beta_3}(\varphi) = \Gamma(k_2 + \rho) \Gamma\left(\frac{\beta}{2} + \rho\right) \mathcal{R}_{2k_2, \beta_3}^{(1, k_1)}(\varphi) .$$

Now as β_3 tends to $\alpha_3 \in -(n-1) - 2\mathbb{N}$, the factor $\Gamma(\frac{\beta_3}{2} + \rho)$ becomes singular, so that necessarily, $\mathcal{R}_{2k_2, \beta_3}^{(1, k_1)}(\varphi) = 0$. Hence $\text{Supp}(\mathcal{R}_{2k_2, \alpha_3}^{(1, k_1)}) \subset \overline{\mathcal{O}_3}$.

As for $a_3 = k_3$ and a_1, a_2 arbitrary, $\mathcal{R}_{2k_2, \alpha_3}^{(1, k_1)}(p_{a_1, a_2, a_3}) \neq 0$, this rules out the possibility that $\text{Supp}(\mathcal{R}_{2k_2, \alpha_3}^{(1, k_1)}) = \mathcal{O}_4$. Hence *ii*) follows. \square

Proposition 15.2. *The distribution $\mathcal{S}_{\lambda_1, \lambda_2}^{(k)}$ is $\neq 0$ and $\text{Supp}(\mathcal{S}_{\lambda_1, \lambda_2}^{(k)}) = \mathcal{O}_4$.*

Proof. It suffices to show that some K -coefficient of $\mathcal{S}_{\lambda}^{(k)}$ does not vanish. Choose $a_1 = a_2 = 0$ and $a_3 = k$. Then, from (48) follows

$$\mathcal{S}_{\lambda_1, \lambda_2}^{(k)}(p_{0,0,k}) = C(-k)_k (-k - \rho - m_3)_k (2\rho + m_3)_k \neq 0 . \quad (60)$$

\square

The relation of $\mathcal{S}_{\lambda_1, \lambda_2}^{(k)}$ to derivatives of $\tilde{\mathcal{K}}^\mu$ at λ (as suggested in the introduction) is given by the following lemma.

Lemma 15.1.

$$\frac{d^2}{ds^2} (\tilde{\mathcal{K}}^{\lambda_1-s, \lambda_2+s, \lambda_3})_{s=0} = \gamma \mathcal{S}_{\lambda_1, \lambda_2}^{(k)} .$$

with

$$\gamma = 2 \left(\frac{\pi}{16}\right)^{n-1} \frac{(n-2)!}{\Gamma(\rho+k)} 16^{-k} (-1)^{k_1} k_1! (-1)^{k_2} k_2! \frac{1}{\Gamma(n-1+k_3)}$$

Proof. Let

$$\lambda(s) = (\lambda_1 - s, \lambda_2 + s, \lambda_3), \quad \alpha(s) = (\alpha_1 + 2s, \alpha_2 - 2s, \alpha_3) .$$

Observe that $\alpha(s)$ belongs to the plane $\{\beta_1 + \beta_2 + \beta_3 = -2(n-1) - 2k\}$. In particular, by (42)

$$\tilde{\mathcal{K}}_{\alpha(s)} = \left(\frac{\pi}{16}\right)^{n-1} \frac{(n-2)!}{\Gamma(\rho+k)} 16^{-k} \frac{1}{\Gamma(-k_1+s)\Gamma(-k_2-s)\Gamma(-\frac{\alpha_3}{2})} S_{\lambda_1-s, \lambda_2+s}^{(k)} . \quad (61)$$

As

$$\frac{d^2}{ds^2} (\tilde{\mathcal{K}}^{\lambda_1-s, \lambda_2+s, \lambda_3})_{s=0} = 2 \lim_{s \rightarrow 0} \frac{\tilde{\mathcal{K}}_{\alpha(s)}}{s^2}$$

the result follows by letting s tend to 0. \square

Proposition 15.3. *The distributions $R_{k_2, \alpha_3}^{(1, k_1)}$ and $\mathcal{S}_{\lambda_1, \lambda_2}^{(k)}$ are two linearly independant λ -invariant distributions.*

Proof. The two distributions $R_{k_2, \alpha_3}^{(1, k_1)}$ and $\mathcal{S}_{\lambda_1, \lambda_2}^{(k)}$ are $\neq 0$ and have unequal supports. Hence they are linearly independant. \square

Theorem 15.1. *Let α be in $Z_{2, II}$. Then $\dim Tri(\lambda) = 2$. More precisely, assume that α satisfies (57). Then*

$$Tri(\lambda) = \mathbb{C} R_{k_2, \alpha_3}^{(1, k_1)} \oplus \mathbb{C} \mathcal{S}_{\lambda_1, \lambda_2}^{(k)} .$$

From Proposition 15.3 follows that $\dim Tri(\lambda) \geq 2$. Hence it will be sufficient to prove that $\dim(Tri(\lambda)) \leq 2$, and this is the content of the next subsections.

15.1 The odd case

Assume that α satisfy (57), and assume that $n-1$ is odd. In this case, α is a generic pole of type II.

Proposition 15.4. *Let T be a λ -invariant distribution on $S \times S \times S$. Then there exist two constants c, d such that $T = c R_{k_2, \alpha_3}^{(1, k_1)} + d \mathcal{S}_{\lambda_1, \lambda_2}^{(k)}$.*

Proof. The restriction to \mathcal{O}_0 of any λ -invariant distribution has to be proportional to $\mathcal{K}_{\alpha, \mathcal{O}_0}$. Hence there exists a constant c such that $S = T - c R_{k_2, \alpha_3}^{1, k_1}$ is supported in $\mathcal{O}_1 \cup \mathcal{O}_2 \cup \mathcal{O}_3 \cup \mathcal{O}_4$.

Now $\alpha_1, \alpha_2, \alpha_3 \notin (-(n-1) - 2\mathbb{N})$. As already seen, this implies that S is supported in \mathcal{O}_4 , or otherwise said S belongs to $Tri(\lambda, diag)$. The dimension of this space is equal to $\dim Sol(\lambda_1, \lambda_2; k)$.

Notice that $\lambda_3 = \rho + m_3 \notin \{-\rho, -\rho-1, \dots, -\rho-(k-1)\} \cup \{-1, -2, \dots, -k\}$ and hence, by Appendix 3 $\dim Tri(\lambda, diag) = 1$. So there exists a constant d such that $S = d \mathcal{S}_{\lambda}^{(k)}$. Summarizing, $T = c R_{k_2, \alpha_3}^{1, k_1} + d \mathcal{S}_{\lambda}^{(k)}$. Q.E.D. \square

15.2 The even case

Assume α satisfies (57), and assume that $n - 1$ is even. In this case, α is a pole of type I+II.

Proposition 15.5. *The distribution $\mathcal{K}_{\alpha, \mathcal{O}_0}$ cannot be extended to a λ -invariant distribution on $S \times S \times S$.*

Two of the normalizing Γ factors become singular at α , namely $\Gamma(\frac{\alpha_3}{2} + \rho)$ and $\Gamma(\frac{\alpha_1 + \alpha_1 + \alpha_2}{2} + 2\rho)$. For $s \in \mathbb{C}$, set

$$\alpha(s) = (\alpha_1, \alpha_2, \alpha_3 + 2s), \quad \lambda(s) = (\lambda_1 + s, \lambda_2 + s, \lambda_3),$$

and notice that $\alpha(s)$ is transverse at α to both planes of poles. Let

$$\mathcal{F}(s) = \Gamma(k_1 + \rho)\Gamma(k_2 + \rho)(-1)^{\rho+k_3}(\rho + k_3)!(-1)^k k! \frac{1}{s} \tilde{\mathcal{K}}_{\alpha(s)}.$$

This defines a distribution-valued holomorphic function on \mathbb{C} , which admits the following Taylor expansion at 0

$$\mathcal{F}(s) = F_0 + sF_1 + O(s^2).$$

Lemma 15.2.

- i) F_0 is λ -invariant and $\text{Supp}(F_0) = \overline{\mathcal{O}_3}$.
- ii) the restriction of F_1 to \mathcal{O}_0 coincides with $\mathcal{K}_{\alpha, \mathcal{O}_0}$.

Proof. For i), notice that F_0 is a multiple ($\neq 0$) of $\mathcal{R}_{\alpha_2, \alpha_3}^{1, k_1}$ (cf (59)), hence $\text{Supp}(F_0) = \overline{\mathcal{O}_3}$ by Proposition 15.1.

Let $s \neq 0$, and let $\varphi \in \mathcal{C}_c^\infty(\mathcal{O}_0)$. Then

$$\mathcal{F}(s)(\varphi) = \frac{(-1)^{\rho+k_3}(\rho + k_3)!}{\Gamma(-\rho - k_3 + s)} \frac{(-1)^k k!}{\Gamma(-k + s)} \frac{1}{s} \mathcal{K}_{\alpha(s)}(\varphi).$$

Let s tend to 0. The right hand side is equivalent to $s\mathcal{K}_{\alpha, \mathcal{O}_0}(\varphi)$. Hence $F_0(\varphi) = 0$ and $F_1(\varphi) = \mathcal{K}_{\alpha, \mathcal{O}_0}(\varphi)$, from which ii) follows. \square

Proposition 15.5 now follows from Proposition A1 in Appendix.

To finish the proof of Theorem 15.1, assume T is a λ -invariant distribution. By Proposition 15.5, T must be 0 on \mathcal{O}_0 . Hence $\text{Supp}(T)$ is included in $\mathcal{O}_1 \cup \mathcal{O}_2 \cup \mathcal{O}_3 \cup \mathcal{O}_4$. As $\alpha_1, \alpha_2 \notin -(n-1) - 2\mathbb{N}$, it even follows that $\text{Supp}(T) \subset \overline{\mathcal{O}_3}$. By Lemma 4.1 in [3], the restriction of T to \mathcal{O}_4^c has to be proportional to (the restriction to \mathcal{O}_4^c of) $\mathcal{R}_{2k_2, \alpha_3}^{(1, k_1)}$. In other words, there exists a constant c such that $T - c\mathcal{R}_{2k_2, \alpha_3}^{(1, k_1)}$ is λ -invariant and supported in \mathcal{O}_4 . The final conclusion is obtained as in the proof for the odd case.

15.3 Invariant trilinear form for three finite dimensional representations

The case where λ is in $Z_{2,II}$ allows to draw conclusions for finite-dimensional representations.

The following lemma is valid in full generality and in fact could be used for other situations (cf [8] for a related idea).

Lemma 15.3. *Let $\lambda = (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{C}^3$. Then, for any $f_1, f_2, f_3 \in \mathcal{C}^\infty(S)$*

$$\tilde{\mathcal{K}}^{-\lambda_1, \lambda_2, \lambda_3}(\tilde{J}_{\lambda_1} f_1, f_2, f_3) = \frac{\pi^\rho}{\Gamma(-\lambda_1 + \rho)} \tilde{\mathcal{K}}^{\lambda_1, \lambda_2, \lambda_3}(f_1, f_2, f_3)$$

Proof. Because of the intertwining relation satisfied by \tilde{J}_{λ_1} , the left hand side is a λ -invariant trilinear form. Assume that λ is not a pole. By the generic uniqueness theorem, there exists a constant c such that, for all $f_1, f_2, f_3 \in \mathcal{C}^\infty(S)$,

$$\tilde{\mathcal{K}}^{-\lambda_1, \lambda_2, \lambda_3}(\tilde{J}_{\lambda_1} f_1, f_2, f_3) = c \tilde{\mathcal{K}}^{\lambda_1, \lambda_2, \lambda_3}(f_1, f_2, f_3) .$$

Now recall that

$$\tilde{\mathcal{K}}^\lambda(1, 1, 1) = \left(\frac{\sqrt{\pi}}{2}\right)^{3(n-1)} \frac{2^{\lambda_1 + \lambda_2 + \lambda_3}}{\Gamma(\rho + \lambda_1) \Gamma(\rho + \lambda_2) \Gamma(\rho + \lambda_3)} \quad (62)$$

(see (10) in [3]). Assume moreover that for $j = 1, 2, 3$, $\lambda_j \notin -\rho - \mathbb{N}$, so that $\tilde{\mathcal{K}}^\lambda(1, 1, 1) \neq 0$. Hence the constant c can be computed from (62), using (32). The general case follows by analytic continuation. \square

Lemma 15.4. *Assume λ satisfy (58). For all $f_1, f_2, f_3 \in \mathcal{C}^\infty(S)$,*

$$\mathcal{S}_{\lambda_1, \lambda_2}^{(k)}(\tilde{J}_{\rho+l_1} f_1, \tilde{J}_{\rho+l_2} f_2, \tilde{J}_{-\rho-m_3} f_3) = 0 .$$

Proof. For $s \in \mathbb{C}$, using three times (15.3)

$$\begin{aligned} & \tilde{\mathcal{K}}^{\lambda_1-s, \lambda_2+s, \lambda_3}(\tilde{J}_{\rho+l_1+s} f_1, \tilde{J}_{\rho+l_2-s} f_2, \tilde{J}_{-\rho-m_3} f_3) = \\ & \frac{\pi^{3\rho}}{\Gamma(-l_1+s) \Gamma(-l_2-s) \Gamma(\rho+m_3)} \tilde{\mathcal{K}}^{-\lambda_1+s, -\lambda_2-s, -\lambda_3}(f_1, f_2, f_3) . \end{aligned}$$

For $s \neq 0$, multiply both sides by $\frac{1}{s^2}$ and let $s \rightarrow 0$. Then, by using Lemma 15.1

$$\mathcal{S}_{\lambda_1, \lambda_2}^{(k)}(\tilde{J}_{\rho+l_1} f_1, \tilde{J}_{\rho+l_2} f_2, \tilde{J}_{-\rho-m_3} f_3) = \gamma' \tilde{\mathcal{K}}^{-\lambda_1, -\lambda_2, -\lambda_3}(f_1, f_2, f_3)$$

where $\gamma' \neq 0$. Observe that $\lambda' = (-\lambda_1, -\lambda_2, -\lambda_3)$ belongs to Z . In fact,

$$\lambda'_1 = \rho + l_1, \lambda'_2 = \rho + l_2, \lambda'_3 = -\rho - m_3,$$

and $\lambda'_1 - \lambda'_2 = l_1 - l_2$, where $l_1 - l_2 \equiv m_3 \pmod{2}$ and $|l_1 - l_2| \leq m_3$. Hence $\tilde{\mathcal{K}}^{-\lambda_1, -\lambda_2, -\lambda_3} \equiv 0$ and the conclusion follows. \square

Now recall (cf (25) and (26)) that for $m \in \mathbb{N}$, $Im(\tilde{J}_{\rho+m}) = \mathcal{P}_m$ is an invariant subspace for the representation $\pi_{-\rho-m}$, and the restriction of $\pi_{-\rho-m}$ to \mathcal{P}_m yields a finite dimensional irreducible representation ρ_m of G . Dually, $Im(\tilde{J}_{-\rho-m}) = \mathcal{P}_m^\perp$ and $\pi_{\rho+m}$ descends to a representation ρ'_m of G on $\mathcal{P}'_m = \mathcal{C}^\infty(S)/\mathcal{P}_m^\perp$. Moreover, $Ker(\tilde{J}_{\rho+m}) = \mathcal{P}_m^\perp$, and $\tilde{J}_{\rho+m}$ passes to the quotient and gives rise to an intertwining operator for the representations ρ'_m and ρ_m . Moreover, the duality between π_λ and $\pi_{-\lambda}$ realizes ρ'_m as the contragredient representation of ρ_m .

Corollary 15.1. *Let $f_1 \in \mathcal{P}_{l_1}, f_2 \in \mathcal{P}_{l_2}, f_3 \in \mathcal{P}_{m_3}^\perp$. Then*

$$\mathcal{S}_{\lambda_1, \lambda_2}^{(k)}(f_1, f_2, f_3) = 0.$$

Proof. Use $Im(\tilde{J}_{\rho+l}) = \mathcal{P}_l$ and $Im(\tilde{J}_{-\rho-m}) = \mathcal{P}_m^\perp$. \square

Next, first restrict the trilinear form $\mathcal{S}_{\lambda_1, \lambda_2}^{(k)}$ to $\mathcal{P}_{l_1} \times \mathcal{P}_{l_2}$ in the first two variables. Then pass to the quotient $\text{mod } \mathcal{P}_{m_3}^\perp$ on the third variable to obtain a trilinear form on $\mathcal{P}_{l_1} \times \mathcal{P}_{l_2} \times \mathcal{P}'_{m_3}$. This trilinear form is not trivial. In fact, observe that $|x - y|^{2k} \in \mathcal{P}_{l_1} \otimes \mathcal{P}_{l_2}$ as $l_1, l_2 \leq k$, and $\mathcal{S}_{\lambda_1, \lambda_2}^{(k)}(|x - y|^{2k} \otimes 1) \neq 0$ by (60). This leads to the following statement, which realizes ρ_{m_3} as a component of $\rho_{l_1} \otimes \rho_{l_2}$.

Proposition 15.6. *Let $l_1, l_2, m_3 \in \mathbb{N}$ satisfy (58), and set*

$$\lambda_1 = -\rho - l_1, \quad \lambda_2 = -\rho - l_2, \quad l_1 + l_2 - m_3 = 2k$$

Then

- i) *the trilinear form $\mathcal{S}_{\lambda_1, \lambda_2}^{(k)}$ yields a non trivial continuous trilinear form on $\mathcal{P}_{l_1} \times \mathcal{P}_{l_2} \times \mathcal{P}'_{m_3}$ which is invariant w.r.t. $(\rho_{l_1}, \rho_{l_2}, \rho'_{m_3})$.*
- ii) *the bi-differential operator $D_{\lambda_1, \lambda_2}^{(k)}$ maps $\mathcal{P}_{l_1} \otimes \mathcal{P}_{l_2}$ into \mathcal{P}_{m_3} and is covariant w.r.t. $(\rho_{l_1} \otimes \rho_{l_2}, \rho_{m_3})$.*

16 Multiplicity 3 theorem for $\alpha \in Z_3$

Let α be in Z_3 . Then

$$\alpha_1 = -(n-1) - 2k_1, \quad \alpha_2 = -(n-1) - 2k_2, \quad \alpha_3 = -(n-1) - 2k_3$$

where $k_1, k_2, k_3 \in \mathbb{N}$. Correspondingly,

$$\lambda_1 = -\rho - l_1, \quad \lambda_2 = -\rho - l_2, \quad \lambda_3 = -\rho - l_3$$

where $l_1, l_2, l_3 \in \mathbb{N}$ satisfy

$$l_1 + l_2 + l_3 \equiv 0 \pmod{2}, \quad |l_1 - l_2| \leq l_3 \leq l_1 + l_2.$$

If $n-1$ is odd, then α is not a pole of type II. If $n-1$ is even (hence $\rho \in \mathbb{N}$), then $\alpha_1 + \alpha_2 + \alpha_3 = -2(n-1) - 2(\rho + k_1 + k_2 + k_3) \in -2(n-1) - 2\mathbb{N}$ and α is a pole of type II.

16.1 The odd case

Assume in this subsection that $n-1$ is odd.

Proposition 16.1. *The three distributions $\tilde{\mathcal{T}}_{\alpha_2, \alpha_3}^{(1, k_1)}, \tilde{\mathcal{T}}_{\alpha_1, \alpha_3}^{(2, k_2)}, \tilde{\mathcal{T}}_{\alpha_1, \alpha_2}^{(3, k_3)}$ are three linearly independent λ -invariant distributions.*

Proof. By Proposition 9.5 $\text{Supp}(\tilde{\mathcal{T}}_{\alpha_2, \alpha_3}^{(1, k_1)}) = \overline{\mathcal{O}_1}$ and a similar result by permuting the indices for the two other distributions. The statement follows. \square

Theorem 16.1. *The space $\text{Tri}(\lambda)$ is of dimension 3. More precisely,*

$$\text{Tri}(\lambda) = \mathbb{C}\tilde{\mathcal{T}}_{\alpha_2, \alpha_3}^{(1, k_1)} \oplus \mathbb{C}\tilde{\mathcal{T}}_{\alpha_1, \alpha_3}^{(2, k_2)} \oplus \mathbb{C}\tilde{\mathcal{T}}_{\alpha_1, \alpha_2}^{(3, k_3)}.$$

Proof. Because of Proposition 16.1, it suffices to prove that any λ -invariant distribution T is a linear combination of $\tilde{\mathcal{T}}_{\alpha_2, \alpha_3}^{(1, k_1)}, \tilde{\mathcal{T}}_{\alpha_1, \alpha_3}^{(2, k_2)}, \tilde{\mathcal{T}}_{\alpha_1, \alpha_2}^{(3, k_3)}$. The next proposition is a first step in this direction.

Proposition 16.2. *The distribution $\mathcal{K}_{\alpha, \mathcal{O}_0}$ cannot be extended to a λ -invariant distribution on $S \times S \times S$.*

Among the four Γ factors in the normalization of $\beta \mapsto \mathcal{K}_\beta$, three become singular at α , namely $\Gamma(\frac{\beta_1}{2} + \rho), \Gamma(\frac{\beta_2}{2} + \rho), \Gamma(\frac{\beta_3}{2} + \rho)$. For s in \mathbb{C} let

$$\alpha(s) = (\alpha_1 + 2s, \alpha_2 + 2s, \alpha_3 + 2s), \quad \lambda(s) = (\lambda_1 + 2s, \lambda_2 + 2s, \lambda_3 + 2s).$$

The distribution-valued function defined for $s \in \mathbb{C}$, $s \neq 0$ and $|s|$ small, by

$$\mathcal{F}(s) = (-1)^{k_1} k_1! (-1)^{k_2} k_2! (-1)^{k_3} k_3! \Gamma(-\rho - k_1 - k_2 - k_3 + 3s) \frac{1}{s^2} \tilde{\mathcal{K}}_{\alpha(s)} .$$

can be extended holomorphically at $s = 0$, because $\beta \mapsto \tilde{\mathcal{K}}_\beta$ vanishes at α as well as its first derivatives. The Taylor expansion of $\mathcal{F}(s)$ at 0 reads

$$\mathcal{F}(s) = F_0 + sF_1 + O(s^2) ,$$

where F_0 and F_1 are distributions on $S \times S \times S$.

Lemma 16.1.

- i) $Supp(F_0) = \overline{\mathcal{O}_1 \cup \mathcal{O}_2 \cup \mathcal{O}_3}$
- ii) F_1 coincides on \mathcal{O}_0 with $\mathcal{K}_{\alpha, \mathcal{O}_0}$

Proof. Let $\varphi \in \mathcal{C}^\infty(S \times S \times S)$ and assume that its support is contained in \mathcal{O}_0 . For $s \neq 0$, $\alpha(s)$ is not a pole and

$$\mathcal{F}(s)(\varphi) = \frac{(-1)^{k_1} k_1!}{\Gamma(-k_1 + s)} \frac{(-1)^{k_2} k_2!}{\Gamma(-k_2 + s)} \frac{(-1)^{k_3} k_3!}{\Gamma(-k_3 + s)} \frac{1}{s^2} \mathcal{K}_{\alpha(s), \mathcal{O}_0}(\varphi)$$

As s tends to 0, the right hand side is equal to $s \mathcal{K}_{\alpha, \mathcal{O}_0}(\varphi) + O(s^2)$ so that $(F_0, \varphi) = 0$ and $F_1(\varphi) = \mathcal{K}_{\alpha, \mathcal{O}_0}(\varphi)$, from which ii) follows.

Next,

$$\lim_{s \rightarrow 0} \frac{1}{s^2} \tilde{\mathcal{K}}_{\alpha(s)} = 2 \left(\frac{\partial^2 \tilde{\mathcal{K}}_\beta}{\partial \beta_1 \partial \beta_2}(\alpha) + \frac{\partial^2 \tilde{\mathcal{K}}_\beta}{\partial \beta_2 \partial \beta_3}(\alpha) + \frac{\partial^2 \tilde{\mathcal{K}}_\beta}{\partial \beta_3 \partial \beta_1}(\alpha) \right)$$

On the other hand, use (18) to obtain, for $s_1, s_2 \in \mathbb{C}$,

$$\tilde{\mathcal{K}}_{\alpha_1+1, \alpha_2+s_2, \alpha_3} = \frac{(-1)^{k_3} 2^{-2k_3}}{\Gamma(\rho + k_3) \Gamma(-k_1 + s_1) \Gamma(-k_2 + s_2)} \tilde{\mathcal{T}}_{\alpha_1+s_1, \alpha_2+s_2}^{(3, k_3)}$$

hence

$$\frac{\partial^2 \tilde{\mathcal{K}}_\beta}{\partial \beta_1 \partial \beta_2}(\alpha) = \frac{(-1)^{k_3} 2^{-2k_3} (-1)^{k_1} k_1! (-1)^{k_2} k_2!}{\Gamma(\rho + k_3)} \tilde{\mathcal{T}}_{\alpha_1, \alpha_2}^{(3, k_3)} ,$$

and similar formulæ by permuting the indices 1, 2, 3. So, F_0 is a linear combination (with coefficients all $\neq 0$) of three distributions with support respectively $\overline{\mathcal{O}_1}$, $\overline{\mathcal{O}_2}$ and $\overline{\mathcal{O}_3}$, so that $Supp(F_0) = \overline{\mathcal{O}_1 \cup \mathcal{O}_2 \cup \mathcal{O}_3}$. \square

Now the proof of Proposition 16.2 is obtained by using once more Proposition A1, after restricting the situation to (say) $\mathcal{O} = \mathcal{O}_0 \cup \mathcal{O}_1$.

To finish the proof of Theorem 16.1, let T be a λ -invariant distribution. Then, Proposition 16.2 shows that the restriction $T_{\mathcal{O}_0}$ has to be 0, and so $\text{Supp}(T) \subset \overline{\mathcal{O}_1 \cup \mathcal{O}_2 \cup \mathcal{O}_3}$. Let $\mathcal{O} = \mathcal{O}_0 \cup \mathcal{O}_1$. The restriction $T_{\mathcal{O}}$ of T to \mathcal{O} is supported in \mathcal{O}_1 , and hence by Lemma 4.1 in [3], there exists a constant c_1 such that T coincides on \mathcal{O} with $c_1 \tilde{\mathcal{T}}_{\alpha_2, \alpha_3}^{1, k_1}$. In other words, $T - c_1 \tilde{\mathcal{T}}_{\alpha_2, \alpha_3}^{1, k_1}$ is a λ -invariant distribution supported in $\overline{\mathcal{O}_2 \cup \mathcal{O}_3}$. By arguing in a similar way, there exists c_2 and c_3 such that $T - c_1 \tilde{\mathcal{T}}_{\alpha_2, \alpha_3}^{1, k_1} - c_2 \tilde{\mathcal{T}}_{\alpha_1, \alpha_3}^{2, k_2} - c_3 \tilde{\mathcal{T}}_{\alpha_1, \alpha_2}^{3, k_3}$ is a λ -invariant distribution supported on \mathcal{O}_4 . But as λ is *not* a pole of type II, and by Lemma 4.2 in [3], there exists no such non trivial distribution, so that $T = c_1 \tilde{\mathcal{T}}_{\alpha_2, \alpha_3}^{1, k_1} + c_2 \tilde{\mathcal{T}}_{\alpha_1, \alpha_3}^{2, k_2} + c_3 \tilde{\mathcal{T}}_{\alpha_1, \alpha_2}^{3, k_3}$. \square

16.2 The even case

Assume in this subsection that $n - 1$ is even, so that ρ is an integer. Then, as $\alpha_1 + \alpha_2 + \alpha_3 = -2(n - 1) - 2(\rho + k_1 + k_2 + k_3)$, α is a pole of type II, and we let $k = \rho + k_1 + k_2 + k_3$.

Proposition 16.3. *The three distributions $\tilde{\mathcal{T}}_{\alpha_2, \alpha_3}^{(1, k_1)}$, $\tilde{\mathcal{T}}_{\alpha_1, \alpha_3}^{(2, k_2)}$, $\tilde{\mathcal{T}}_{\alpha_1, \alpha_2}^{(3, k_3)}$ are three linearly linearly independant λ -invariant distributions.*

Proof. Although the statement is the same as in the previous case, the proof is quite different. In fact, recall that the three distributions are supported in \mathcal{O}_4 . Use once again the evaluation of (some) K -coefficients of the three distributions to prove their linear independence. The following result was obtained during the proof of Proposition 10.1.

Lemma 16.2.

- i) If $a_3 > k_3$, then $\tilde{\mathcal{T}}_{\alpha_1, \alpha_2}^{(3, k_3)}(p_{a_1, a_2, a_3}) = 0$
- ii) Let $a_1 = k_1 + \rho - 1$, $a_2 = k_2 + 1$, $a_3 = k_3$. Then $\tilde{\mathcal{T}}_{\alpha_1, \alpha_2}^{(3, k_3)}(p_{a_1, a_2, a_3}) \neq 0$.

Up to permutation of the indices, a similar result holds true for the two other distributions. As $n \geq 4$ and $n - 1$ even, $\rho \geq 2$ so that $a_1 > k_1$. Moreover, $a_2 > k_2$. As a consequence of part i) of the lemma,

$$\tilde{\mathcal{T}}_{\alpha_2, \alpha_3}^{(1, k_1)}(p_{a_1, a_2, a_3}) = 0, \quad \tilde{\mathcal{T}}_{\alpha_1, \alpha_3}^{(2, k_2)}(p_{a_1, a_2, a_3}) = 0.$$

The linear independance of the three distributions follows from this result and its variants under permutation of the indices 1, 2, 3. \square

Theorem 16.2. *The space $Tri(\boldsymbol{\lambda})$ is of dimension 3. More precisely,*

$$Tri(\boldsymbol{\lambda}) = \mathbb{C}\tilde{\mathcal{T}}_{\alpha_2, \alpha_3}^{(1, k_1)} \oplus \mathbb{C}\tilde{\mathcal{T}}_{\alpha_1, \alpha_3}^{(2, k_2)} \oplus \mathbb{C}\tilde{\mathcal{T}}_{\alpha_1, \alpha_2}^{(3, k_3)} .$$

Proof. Three facts need to be established before giving the proof, namely

- the distribution $\mathcal{K}_{\boldsymbol{\alpha}, \mathcal{O}_0}$ cannot be extended to a $\boldsymbol{\lambda}$ -invariant distribution on $S \times S \times S$.
- the distribution $\mathcal{T}_{\alpha_1, \alpha_2, \mathcal{O}_4^c}^{(3, k_3)}$ cannot be extended to a $\boldsymbol{\lambda}$ -invariant distribution on $S \times S \times S$.
- $\dim Tri(\boldsymbol{\lambda}) = 3$.

Proposition 16.4. *The distribution $\mathcal{K}_{\boldsymbol{\alpha}, \mathcal{O}_0}$ cannot be extended to a $\boldsymbol{\lambda}$ -invariant distribution on $S \times S \times S$.*

For $s \in \mathbb{C}$ define

$$\boldsymbol{\alpha}(s) = (\alpha_1 + s, \alpha_2 + s, \alpha_3 + s), \quad \boldsymbol{\lambda}(s) = (\lambda_1 + s, \lambda_2 + s, \lambda_3 + s) .$$

Observe that for $s \neq 0$ and $|s|$ small, $\boldsymbol{\alpha}(s)$ is not a pole. Next, the four Gamma factor in the normalization of $\mathcal{K}_{\boldsymbol{\alpha}}$ are singular at $\boldsymbol{\alpha}$.

By the same argument used in the odd case, the distribution-valued function defined for $s \in \mathbb{C}$, $s \neq 0$ by

$$\mathcal{F}(s) = \frac{8}{3} \frac{(-1)^{k_1}}{k_1!} \frac{(-1)^{k_2}}{k_2!} \frac{(-1)^{k_3}}{k_3!} \frac{(-1)^k}{k!} \frac{1}{s^2} \tilde{\mathcal{K}}_{\boldsymbol{\alpha}(s)}$$

can be continued to a holomorphic function in a neighborhood of $s = 0$. The Taylor expansion of \mathcal{F} at 0 reads

$$\mathcal{F}(s) = F_0 + sF_1 + s^2F_2 + O(s^3)$$

where F_0, F_1 and F_2 are distributions on $S \times S \times S$.

Proposition 16.5.

- i) F_0 is $\boldsymbol{\lambda}$ -invariant and $Supp(F_0) = \mathcal{O}_4$.
- ii) $Supp(F_1) = \overline{\mathcal{O}_1 \cup \mathcal{O}_2 \cup \mathcal{O}_3}$
- iii) the restriction of F_2 to \mathcal{O}_0 coincides with $\mathcal{K}_{\boldsymbol{\alpha}, \mathcal{O}_0}$.

Proof. The proof of i) is the same as in the odd case, except that now the three distributions $\tilde{\mathcal{T}}_{\alpha_2, \alpha_3}^{(1, k_1)}$, $\tilde{\mathcal{T}}_{\alpha_1, \alpha_3}^{(2, k_2)}$, $\tilde{\mathcal{T}}_{\alpha_1, \alpha_2}^{(3, k_3)}$ are supported in \mathcal{O}_4 .

Next, let $\varphi \in \mathcal{C}^\infty(S \times S \times S)$ and assume that $\text{Supp}(\varphi) \subset \mathcal{O}_0$. Then for $s \neq 0$ and $|s|$ small,

$$(\mathcal{F}(s), \varphi) = \frac{8}{3} \frac{(-1)^{k_1}}{k_1!} \frac{(-1)^{k_2}}{k_2!} \frac{(-1)^{k_3}}{k_3!} \frac{(-1)^k}{k!} \frac{1}{\Gamma(-k_1 + \frac{s}{2}) \Gamma(-k_2 + \frac{s}{2}) \Gamma(-k_3 + \frac{s}{2}) \Gamma(-k + \frac{3s}{2})} \frac{1}{s^2} \mathcal{K}_{\alpha(s)}(\varphi) .$$

As $s \rightarrow 0$, the right hand side is equal to $s^2 \mathcal{K}_{\alpha, \mathcal{O}_0}(\varphi) + O(s^3)$, so that $F_0(\varphi) = 0, F_1(\varphi) = 0$ and $F_2(\varphi) = \mathcal{K}_{\alpha, \mathcal{O}_0}(\varphi)$. Hence *iii*) follows, and also $\text{Supp}(F_1) \subset \overline{\mathcal{O}_1 \cup \mathcal{O}_2 \cup \mathcal{O}_3}$.

It remains to determine the supports of F_1 . Let $a_1, a_2, a_3 \in \mathbb{N}$. Then, using (11) in [3], the K -coefficient $\mathcal{F}(s)(p_{a_1, a_2, a_3})$ (up to a factor which does not vanish for $s = 0$) is equal to

$$\frac{1}{s^2} \left(-k + \frac{3}{2}s\right)_{a_1+a_2+a_3} \left(-k_1 + \frac{s}{2}\right)_{a_1} \left(-k_2 + \frac{s}{2}\right)_{a_2} \left(-k_3 + \frac{s}{2}\right)_{a_3} \cdots \frac{1}{\Gamma(-k_1 - k_2 + a_1 + a_2 + s)} \frac{1}{\Gamma(-k_2 - k_3 + a_2 + a_3 + s)} \frac{1}{\Gamma(-k_3 - k_1 + a_3 + a_1 + s)} .$$

Lemma 16.3. *Let $a_1, a_2, a_3 \in \mathbb{N}$ and assum that*

$$a_1 \leq k_1, \quad a_2 > k_1 + k_2, \quad a_3 > k_1 + k_3, \quad a_1 + a_2 + a_3 > k .$$

Then $F_1(p_{a_1, a_2, a_3}) \neq 0$.

Proof. Under the assumptions, the three Γ factors are regular at $s = 0$, the factor $(-k_1 + s)_{a_1}$ does not vanish at $s = 0$ and the three remaining factors $(-k + \frac{3}{2}s)_{a_1+a_2+a_3}, (-k_2 + s)_{a_2}, (-k_3 + s)_{a_3}$ have a simple zero at $s = 0$. Hence the conclusion holds true. \square

From the definition of F_1 and part *i*) of Proposition 16.5 follows that $F_1|_{\mathcal{O}_4}$ is λ -invariant. Hence $\text{Supp}(F_1)$ is invariant under G . Now Lemma 16.3 shows that $\text{Supp}(F_1)$ is not included in $\overline{\mathcal{O}_2 \cup \mathcal{O}_3}$. In fact, given an arbitrary integer L , choosing a_2, a_3 large enough, all partial derivatives of the function p_{a_1, a_2, a_3} of order $\leq L$ vanish on $\overline{\mathcal{O}_2 \cup \mathcal{O}_3}$. Hence if T is a distribution on $S \times S \times S$ supported on $\overline{\mathcal{O}_2 \cup \mathcal{O}_3}$, then $T(p_{a_1, a_2, a_3}) = 0$ for a_2, a_3 large enough (take for L the order of the distribution T). By permuting the indices 1, 2, 3, the only remaining possibility for $\text{Supp}(F_1)$ is $\text{Supp}(F_1) = \overline{\mathcal{O}_1 \cup \mathcal{O}_2 \cup \mathcal{O}_3}$. \square

Let us now come to the proof of Proposition 16.4. Restrict the previous situation to $\mathcal{O} = \mathcal{O}_0 \cup \mathcal{O}_1$. The distribution-valued function $\mathcal{G}(s)$ defined for $s \neq 0$ by

$$\mathcal{G}(s) = \frac{1}{s} \mathcal{F}(s)|_{\mathcal{O}} .$$

can be extended to a holomorphic function in a neighborhood of 0 in \mathbb{C} . Then, as $s \rightarrow 0$,

$$\mathcal{G}(s) = F_{1,\mathcal{O}} + sF_{2,\mathcal{O}} + O(s^2),$$

and $\text{Supp}(F_{1,\mathcal{O}}) = \mathcal{O}_1$. Now $F_{1,\mathcal{O}}$ is λ -invariant. An application of Proposition A1 implies that F_{2,\mathcal{O}_0} cannot be extended to a λ -invariant distribution on \mathcal{O} . Hence, a fortiori the distribution $\mathcal{K}_{\alpha,\mathcal{O}_0}$ cannot be extended to a λ -invariant distribution on $S \times S \times S$. \square

Proposition 16.6. *The distribution $\mathcal{T}_{\alpha_2,\alpha_3,\mathcal{O}_4^c}^{(1,k_1)}$ cannot be extended to $S \times S \times S$ as a λ -invariant distribution.*

The proof is very similar to the proof of Proposition 14.4 so that we omit it.

Proposition 16.7.

$$\dim \text{Tri}(\lambda, \text{diag}) = 3 .$$

Proof. As seen earlier, it suffices to prove that $\dim \text{Sol}(\lambda_1, \lambda_2; k) \leq 3$ (see Appendix 3). In the present case, $\lambda_1 = -\rho - k_2 - k_3$, and as $k = k_1 + k_2 + k_3 + \rho$, $\lambda_1 \in \{-\rho, -\rho - 1, \dots, -\rho - (k - 1)\} \cap \{-1, -2, \dots, -(k - 1)\}$ and similarly for λ_2 . Moreover, $-\lambda_1 - \lambda_2 = (n - 1) + k_2 + k_3 + 2k_1 \geq k$. The conclusion follows by Lemma A9 iii). \square

We now come to the proof of the main theorem in this section. Let T be a λ -invariant distribution. By Proposition 16.4, the restriction of T to \mathcal{O}_0 has to be 0. Hence $\text{Supp}(T) \subset \overline{\mathcal{O}_1 \cup \mathcal{O}_2 \cup \mathcal{O}_3}$. Let $\mathcal{O}_{01} = \mathcal{O}_0 \cup \mathcal{O}_1$. This is an open set of $S \times S \times S$ which contains \mathcal{O}_1 as a (relative) closed submanifold. The restriction $T|_{\mathcal{O}_{01}}$ of T to \mathcal{O}_{01} is supported on \mathcal{O}_1 and λ -invariant. Hence $T|_{\mathcal{O}_{01}}$ must be a multiple of $\mathcal{T}_{\alpha_2,\alpha_3,\mathcal{O}_4^c}^{1,k_1}$. By Proposition 16.6, this forces $T|_{\mathcal{O}_{01}} = 0$. By permuting the indices 1, 2, 3, T has to be supported in \mathcal{O}_4 , hence belongs to $\text{Tri}(\lambda, \text{diag})$. Hence $\text{Tri}(\lambda) = \text{Tri}(\lambda, \text{diag})$. But $\dim \text{Tri}(\lambda, \text{diag}) = \dim \text{Sol}(\lambda_1, \lambda_2; k) \leq 3$, which finishes the proof of Theorem 16.2.

Appendix

A1. Evaluation of the K -coefficients of $\tilde{\mathcal{K}}_\alpha$

For the convenience of the reader, the formalæ giving the evaluation of the K -coefficients of $\tilde{\mathcal{K}}_\alpha$ (sometimes called Bernstein-Reznikov integrals), which were obtained in [3] Proposition 3.2 are recalled. For $\alpha \in \mathbb{C}^3$, and $a_1, a_2, a_3 \in \mathbb{N}$

$$\begin{aligned} \tilde{\mathcal{K}}_\alpha(p_{a_1, a_2, a_3}) &= \left(\frac{\pi}{2}\right)^{\frac{3}{2}(n-1)} 2^{\alpha_1 + \alpha_2 + \alpha_3} 2^{2(a_1 + a_2 + a_3)} \dots \quad (63) \\ &\frac{(\frac{\alpha_1 + \alpha_2 + \alpha_3}{2} + 2\rho)_{a_1 + a_2 + a_3} (\frac{\alpha_1}{2} + \rho)_{a_1} (\frac{\alpha_2}{2} + \rho)_{a_2} (\frac{\alpha_3}{2} + \rho)_{a_3}}{\Gamma(\frac{\alpha_1 + \alpha_2}{2} + 2\rho + a_1 + a_2) \Gamma(\frac{\alpha_2 + \alpha_3}{2} + 2\rho + a_2 + a_3) \Gamma(\frac{\alpha_3 + \alpha_1}{2} + 2\rho + a_3 + a_1)} \end{aligned}$$

and its counterpart in terms of the spectral parameter λ

$$\begin{aligned} \tilde{\mathcal{K}}^\lambda(p_{a_1, a_2, a_3}) &= \left(\frac{\sqrt{\pi}}{2}\right)^{3(n-1)} 2^{\lambda_1 + \lambda_2 + \lambda_3} 2^{2(a_1 + a_2 + a_3)} \left(\frac{\lambda_1 + \lambda_2 + \lambda_3 + \rho}{2}\right)_{a_1 + a_2 + a_3} \\ &\frac{(\frac{-\lambda_1 + \lambda_2 + \lambda_3 + \rho}{2})_{a_1} (\frac{\lambda_1 - \lambda_2 + \lambda_3 + \rho}{2})_{a_2} (\frac{\lambda_1 + \lambda_2 - \lambda_3 + \rho}{2})_{a_3}}{\Gamma(\lambda_1 + \rho + a_2 + a_3) \Gamma(\lambda_2 + \rho + a_3 + a_1) \Gamma(\lambda_3 + \rho + a_1 + a_2)} . \quad (64) \end{aligned}$$

A2. Non extension results

Proposition A1. *Let \mathcal{O} be a G -invariant open subset of $S \times S \times S$, and assume that \mathcal{O}_1 is a (relatively) closed submanifold of \mathcal{O} . Let α be a pole of type I_1 , i.e. $\alpha_1 = -(n-1) - 2k_1$ for some $k_1 \in \mathbb{N}$. Let $\alpha(s)$ be a holomorphic curve defined for s in a neighborhood of 0, such that $\alpha(0) = \alpha$ and transverse to the plane $\beta_1 = -(n-1) - 2k_1$ at α . Let $\lambda(s)$ be the associated spectral parameter and let $\lambda = \lambda(0)$.*

Let $\mathcal{F}(s)$ be a family of distributions on \mathcal{O} , depending holomorphically on s in a neighborhood of 0, and such that

- i) for any s , $\mathcal{F}(s)$ is $\lambda(s)$ -invariant*
- ii) $\mathcal{F}(s) = F_0 + sF_1 + O(s^2)$ as $s \rightarrow 0$, where F_0 and F_1 are distributions on \mathcal{O}*
- iii) $\text{Supp}(F_0) = \mathcal{O}_1$.*

Then the restriction $F_1|_{\mathcal{O}'}$ of F_1 to $\mathcal{O}' = \mathcal{O} \setminus \mathcal{O}_1$ is λ -invariant, but can not be extended to \mathcal{O} as a λ -invariant distribution.

Proposition A2. *Let $\alpha \in \mathbb{C}^3$ be a pole of type II, i.e. $\alpha_1 + \alpha_2 + \alpha_3 = -2(n-1) - 2k$ for some $k \in \mathbb{N}$. Let $\alpha(s)$ be a holomorphic curve defined*

for s in a neighborhood of 0, such that $\alpha(0) = \alpha$ and transverse to the plane $\beta_1 + \beta_2 + \beta_3 = -2(n-1) - 2k$ at α . Let $\lambda(s)$ (resp. λ) be the associated spectral parameter.

Let $\mathcal{F}(s)$ be a family of distributions on $S \times S \times S$, depending holomorphically on s near 0, and such that

- i) $\mathcal{F}(s)$ is $\lambda(s)$ -invariant
- ii) $\mathcal{F}(s) = F_0 + sF_1 + O(s^2)$ as $s \rightarrow 0$, where F_0 and F_1 are distributions on $S \times S \times S$
- iii) $\text{Supp}(F_0) = \mathcal{O}_4$

Then the restriction $F_{1,\mathcal{O}'}$ of F_1 to $\mathcal{O}' = S \times S \times S \setminus \mathcal{O}_4$ is λ -invariant, but cannot be extended to $S \times S \times S$ as a λ -invariant distribution.

Proof. (sketch) The statement here is formulated in terms of holomorphic curves (in all applications only complex lines occur), to be closer to the spirit of Oshima's results (see [10] section 7).

Let $\mu = (\mu_1, \mu_2, \mu_3) = \frac{d}{ds}|_{s=0} \lambda(s)$. The Taylor expansion at $s = 0$ of the identity $\mathcal{F}(s) \circ \pi_{\lambda(s)} = \mathcal{F}(s)$ yields the functional relation, valid for any $g \in G$

$$F_1 \circ \pi_{\lambda}(g) - F_1 = C_g F_0 ,$$

where $C_g(x, y, z) = -(\mu_1 \ln \kappa(g^{-1}, x) + \mu_2 \ln \kappa(g^{-1}, y) + \mu_3 \ln \kappa(g^{-1}, z))$.

The subgroup A (a split one-parameter Cartan subgroup of G) has a fixed point in \mathcal{O}_1 (resp. \mathcal{O}_4), say \mathbf{o} . The transversality assumption guarantees that, at \mathbf{o} the character

$$\chi_{\mathbf{o}} : A \ni a \mapsto C_a(\mathbf{o})$$

is *not* trivial. The rest of the proof is as in [3], Proposition 6.1 and 6.2. \square

A3. Discussion of the system $S(\lambda_1, \lambda_2; k)$

Recall that, for $\lambda \in \mathbb{C}^3$, $\text{Tri}(\lambda, \text{diag})$ is the space of λ -invariant distributions on $S \times S \times S$ which are supported on the diagonal $\mathcal{O}_4 = \{(x, x, x), x \in S\}$.

For $(\lambda_1, \lambda_2) \in \mathbb{C}^2$ and $k \in \mathbb{N}$, $\mathcal{BD}_G(\lambda_1, \lambda_2; k)$ is the space of bi-differential operators $D : \mathcal{C}^\infty(S \times S) \rightarrow \mathcal{C}^\infty(S)$ which are covariant w.r.t. $(\pi_{\lambda_1} \otimes \pi_{\lambda_2}, \pi_{\lambda_1 + \lambda_2 + \rho + 2k})$ (see [3] section 7).

Let $k \in \mathbb{N}$ and let $(\lambda_1, \lambda_2) \in \mathbb{C}^2$. Consider the system $S(\lambda_1, \lambda_2; k)$ of homogeneous linear equations in the unknowns $(c_{r,t}), 0 \leq r, t, r+t \leq k$

given by

$$\begin{aligned} & 4(r+1)(r+1+\lambda)c_{r+1,t} \\ & +2(k-r-t)(k-r+t-1+\rho+\mu)c_{r,t} \\ & -(k-r-t+1)(k-r-t)c_{r,t-1} = 0 \end{aligned} \quad E_{r,t}^{(1)}$$

$$\begin{aligned} & 4(t+1)(t+1+\mu)c_{r,t+1} \\ & +2(k-r-t)(k+r-t-1+\rho+\lambda)c_{r,t} \\ & -(k-r-t+1)(k-r-t)c_{r-1,t} = 0 \end{aligned} \quad E_{r,t}^{(2)}$$

for $r+t \leq k-1$.

The system was introduced by Ovsienko and Redou (see [11]). Denote by $Sol(\lambda_1, \lambda_2; k)$ the space of solutions of the system $S(\lambda_1, \lambda_2; k)$.

Proposition A3. *Let $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{C}^3$ satisfy $\lambda_1 + \lambda_2 + \lambda_3 = -\rho - 2k$ for some $k \in \mathbb{N}$. Then*

$$Tri(\lambda_1, \lambda_2, \lambda_3, diag) \sim \mathcal{BD}_G(\lambda_1, \lambda_2; k) \sim Sol(\lambda_1, \lambda_2; k) .$$

Proof. The first isomorphism is proved in [1], the second is due to Ovsienko and Redou ([11]). As a consequence, the dimensions of the three spaces are equal. \square

The system $S(\lambda_1, \lambda_2; k)$ was studied by Ovsienko and Redou who proved that generically $\dim Sol(\lambda_1, \lambda_2; k) = 1$, but for our purpose, a full discussion of the system (depending on the parameters λ_1, λ_2 and k) is needed.

The next result will help lower the number of situations to be considered.

Proposition A4 (Symmetry principle).

- i) $\dim Sol(\lambda_1, \lambda_2; k) = \dim Sol(\lambda_2, \lambda_1; k)$
- ii) let $\lambda_3 = -\rho - 2k - \lambda_1 - \lambda_2$. Then

$$\dim Sol(\lambda_1, \lambda_2; k) = \dim Sol(\lambda_2, \lambda_3; k) = \dim Sol(\lambda_3, \lambda_1; k) .$$

Proof.

For i), observe (cf [11]) that if $(c_{r,t})$ is a solution of the system $S(\lambda_1, \lambda_2; k)$, then $(c_{t,r})$ is a solution of $S(\lambda_2, \lambda_1; k)$.

For ii), observe that $\dim Tri(\lambda_1, \lambda_2, \lambda_3; diag)$ is invariant by any permutation of $\{\lambda_1, \lambda_2, \lambda_3\}$. Hence Proposition 3 implies the statement. \square

The strategy to estimate $\dim \text{Sol}(\lambda_1, \lambda_2; k)$ is twofold. An algebraic study of the system will produce upper bounds for the dimension. In counterpart, the analytic study done in sections 10, 13, 14, 15 and 16 has produced linearly independent distributions inside $\text{Tri}(\lambda, \text{diag})$, giving a lower bound for the dimension. A careful inspection shows that the lower and upper estimates coincide. Of course, it could be possible to write explicitly potential solutions produced by the algebraic study and verify that they are indeed solutions of the system (as claimed in [11] for the generic case).

Lemma A1. *Let $\lambda_2 \notin \{-1, -2, \dots, -k\} \cup \{-\rho, -\rho - 1, \dots, -\rho - (k - 1)\}$. Then $\dim \text{Sol}(\lambda_1, \lambda_2; k) \leq 1$.*

Proof. Equation $E_{r,0}^{(1)}$ reads

$$4(r+1)(r+1+\lambda_1)c_{r+1,0} + 2(k-r)(k+\rho+\lambda_2-1-r)c_{r,0} = 0 .$$

As $\lambda_2 \notin \{-\rho, -\rho - 1, \dots, -\rho - (k - 1)\}$, $k + \rho + \lambda_2 - 1 - r \neq 0$ for $0 \leq r \leq k - 1$. Hence $c_{r,0}$ can be computed from $c_{r+1,0}$. Choose $c_{k,0}$ as principal unknown. Then, for $0 \leq r \leq k - 1$, $c_{r,0}$ can be computed from $c_{k,0}$.

Equation $E_{r,t}^{(2)}$ reads

$$4(t+1)(t+1+\lambda_2)c_{r,t+1} + 2(k-r)(k+r-t-1+\rho+\lambda_1)c_{r,t} - (k-r-t+1)(k-r-t)c_{r-1,t} = 0 .$$

As $\lambda_2 \notin \{-1, -2, \dots, -k\}$, the coefficient of the unknown $c_{r,t+1}$ is never 0, and hence the coefficient $c_{r,t+1}$ can be computed from $c_{r,t}$ and $c_{r-1,t}$. Hence all unknowns can be computed from $c_{k,0}$, so that $\dim \text{Sol}(\lambda_1, \lambda_2; k) \leq 1$. \square

We may now assume that $\lambda_1, \lambda_2 \in \{-\rho, -\rho - 1, \dots, -\rho - (k - 1)\} \cup \{-1, -2, \dots, -k\}$. Notice that if $n - 1$ is odd (hence $\rho \notin \mathbb{N}$), the two sets $\{-\rho, -\rho - 1, \dots, -\rho - (k - 1)\}$ and $\{-1, -2, \dots, -k\}$ are disjoint, making in this case the study somewhat easier. When $n - 1$ is even, then ρ is an integer and

$$\begin{aligned} & \{-\rho, -\rho - 1, \dots, -\rho - (k - 1)\} \cap \{-1, -2, \dots, -k\} \\ & \quad \begin{array}{ll} \emptyset & \text{if } k < \rho \\ \{-\rho, \dots, -k\} & \text{if } k \geq \rho \end{array} \end{aligned}$$

The general strategy to study the system was already used in the proof of Lemma A1. First study equations $E_{r,0}^{(1)}$ which involve only the unknowns $c_{r,0}$, then pass to the determination of the unknowns $c_{r,1}$, and so on.

Lemma A2. Let $\lambda_1 = -k_1$ for some $k_1 \in \{1, 2, \dots, k-1\}$ and $\lambda_2 = -\rho - l_2$ for some $l_2 \in \{0, 1, \dots, k-1\}$, but $\lambda_2 \notin \{-1, -2, \dots, -k\}$.

- i) if $k_1 + l_2 < k$, then $\dim \text{Sol}(\lambda_1, \lambda_2; k) \leq 1$
- ii) if $k_1 + l_2 \geq k$, then $\dim \text{Sol}(\lambda_1, \lambda_2; k) \leq 2$.

Proof. Equation $E_{k_1-1,0}^{(1)}$ reads

$$0 c_{k_1,0} + 2(k - k_1 + 1)(k - k_1 - l_2)c_{k_1-1,0} = 0 \quad (65)$$

Equation $E_{k-l_2-1,0}^{(1)}$ reads

$$4(r+1)(k-l_2-k_1)c_{k-l_2,0} + 0 c_{k-l_2-1,0} = 0. \quad (66)$$

Assume that $k_1 + l_2 < k$. (65) implies $c_{k_1-1,0} = 0$ and (66) implies $c_{k-l_2} = 0$. Repeated use of $E_{r,0}^{(1)}$ for $r \neq k_1 - 1, k - l_2 - 1$ leads to the following results

- $c_{r,0} = 0$ for $0 \leq r \leq k_1 - 1$ and for $k - l_2 \leq r \leq k$
- $c_{k_1,0}$ can be chosen as principal unknown and the remaining unknowns $c_{r,0}, k_1 < r < k - l_2$ can be expressed in terms of $c_{k_1,0}$. Next use the same method as in Lemma 1 to prove that all $c_{r,t}$ can be expressed in terms of $c_{k_1,0}$. This gives $\dim(\text{Sol}(\lambda_1, \lambda_2, k)) \leq 1$ and i) follows.

Assume now that $k_1 + l_2 \geq k$. In this case, $k - l_2 - 1 < k_1$, and choose $c_{k-l_2-1,0}$ and $c_{k_1,0}$ as principal unknowns. For $k - l_2 \leq r \leq k_1 - 1$ (this interval is empty in case $k_1 + l_2 = k$) $c_{r,0} = 0$. For $0 \leq r \leq k - l_2 - 1$, $c_{r,0}$ can be expressed in terms of $c_{k-l_2-1,0}$, whereas for $k_1 < k \leq r$, $c_{r,0}$ can be expressed in terms of $c_{k_1,0}$. Hence all $c_{r,t}$ can be expressed in terms of the two chosen principal unknowns, and so $\dim \text{Sol}(\lambda_1, \lambda_2; k) \leq 2$. \square

Lemma A3. Let $\lambda_1 = -k_1, \lambda_2 = -k_2$ where $1 \leq k_1, k_2 \leq k$.

- i) assume that $k_1 + k_2 \leq k$. Then $\dim \text{Sol}(\lambda_1, \lambda_2; k) \leq 1$.
- ii) assume that $k_1 + k_2 > k$ and assume that $\lambda_2 \notin \{-\rho, -\rho - 1, \dots, -\rho - (k-1)\}$. Then $\dim \text{Sol}(\lambda_1, \lambda_2; k) \leq 2$.

Proof.

In case i), the assumptions imply that $\lambda_3 = -\rho - 2k + k_1 + k_2$. Rewrite this as $\lambda_3 = -\rho - k - (k - k_1 - k_2)$, so that $\lambda \in -\rho - k - \mathbb{N}$ and hence $\lambda_3 \notin \{-\rho, \dots, -\rho - (k-1)\} \cup \{-1, -2, \dots, -k\}$. Use the symmetry principle and Lemma 1 to conclude.

In case of ii), assume first that $\lambda_1 \notin \{-\rho, -\rho - 1, \dots, -\rho - (k-1)\}$. The same considerations as in the beginning of the proof of Lemma 2 show that

$c_{r,0} = 0$ for $r \leq k_1 - 1$. Choose $c_{k_1,0}$ as first principal unknown, then the unknowns $c_{r,0}$ for $r \geq k_1$ can be expressed in terms of $c_{k_1,0}$. Now as long as $t \leq k_2 - 1$, it is possible to use $E_{r,t-1}^{(2)}$ to compute $c_{r,t}$ in terms of $c_{r,t-1}$ and $c_{r-1,t-1}$. It follows that, for $t \leq k_2 - 1$, $c_{r,t} = 0$ for $r \leq k_1 - 1$ and $c_{r,t}$ can be expressed in terms of $c_{k_1,0}$ for $r \geq k_1$. By symmetry, choosing $c_{k_2,0}$ as second principal unknown, it is possible for $r \leq k_1$ to express all $c_{r,t}$ in terms of c_{0,k_2} . But $r + t \leq k$ and $k > k_1 + k_2$ implies that either $r \leq k_1 - 1$ or $t \leq k_2 - 1$, hence all unknowns can be expressed in terms of the two principal unknowns $c_{k_1,0}$ and c_{0,k_2} and the statement follows.

It remains to consider the case where $\lambda_1 \in \{-\rho, -\rho-1, \dots, -\rho-(k-1)\}$. This assumption forces $n-1$ to be even, $\rho \leq k$ and hence $\lambda_1 = -k_1$, where $\rho \leq k_1 \leq k$. Moreover, $0 \leq k_2 \leq \rho - 1$. Then $\lambda_3 = -\rho - (2k - k_1 - k_2) = -\rho - l_3$ with $0 \leq l_3 = 2k - k_1 - k_2 < k$ so that $\lambda_3 \in \{-\rho, -\rho-1, \dots, -(\rho - k - 1)\}$, but writing the same equality as $\lambda_3 = -k - ((k - k_1) + (\rho - k_2))$ shows that $\lambda_3 \notin \{-1, -2, \dots, -k\}$. Moreover, $l_3 + k_1 = 2k - k_1 - k_2 + k_1 = 2k - k_2 \geq k$. So (λ_1, λ_3) satisfy the assumptions of Lemma 2 ii). By the symmetry principle, the conclusion follows also in this case. \square

Lemma A4. *Let $\lambda_1 = -\rho - l_1, \lambda_2 = -\rho - l_2$ where $0 \leq l_1, l_2 \leq k - 1$, and $\lambda_1, \lambda_2 \notin \{-1, -2, \dots, -k\}$. Then $\dim \text{Sol}(\lambda_1, \lambda_2; k) \leq 1$.*

Proof. Choose $c_{0,0}$ as principal unknown. Then, by using $E_{r-1,0}^{(1)}$ it is possible for $r \leq k - l_2 - 1$ to express $c_{r,0}$ in terms of $c_{0,0}$. Then necessarily $c_{k-l_2,0} = 0$ and for $r \geq k - l_2 + 1$ $c_{r,0} = 0$. As $\lambda_2 \notin \{-1, -2, \dots, -k\}$, it is possible, using $E_{r,t}^{(2)}$ to express $c_{r,t}$ in terms of $c_{0,0}$. \square

Lemma A5. *Let $n-1$ be even (so that $\rho \in \mathbb{N}$), and let $\lambda_1 = -k_1, \lambda_2 = -k_2$, where $\rho \leq k_1, k_2 \leq k$.*

- i) if $k_1 + k_2 \leq k$, then $\dim \text{Sol}(\lambda_1, \lambda_2; k) = 1$.*
- ii) if $k < k_1 + k_2 < k + \rho$, then $\dim \text{Sol}(\lambda_1, \lambda_2; k) = 2$.*
- iii) if $k_1 + k_2 \geq k + \rho$, then $\dim \text{Sol}(\lambda_1, \lambda_2; k) = 3$.*

Proof. If $k_1 + k_2 \leq k$, then $\lambda = -\rho - 2k + k_1 + k_2 = -\rho - k - (k - k_1 + k_2)$, so that $\lambda_3 \notin \{-\rho, -\rho-1, \dots, -\rho-(k-1)\} \cup \{-1, -2, \dots, -k\}$. Use Lemma A1 and the symmetry principle to get i).

Assume now that $k < k_1 + k_2 < k + \rho$. Then

$$\lambda_3 = -\rho - 2k + k_1 + k_2 = -\rho - k + (k_1 + k_2 - k) = -k - (k + \rho - k_1 - k_2).$$

Hence $\lambda_3 \in \{-\rho, -\rho - 1, \dots, -\rho - (k - 1)\}$ but $\lambda_3 \notin \{-1, -2, \dots, -k\}$. Use Lemma *ii*) to conclude.

Assume now that $k_1 + k_2 \geq k + \rho$.

Consider first the equation $E_{r,0}^{(1)}$, which reads

$$4(r+1)(r+1-k_1)c_{r+1,t} + 2(k-r)(k-r-1+\rho-k_2)c_{r,0}$$

The first coefficient vanishes for $r = k_1 - 1$ and the second vanishes for $r = k + \rho - 1 - k_2$. Notice that $0 < k - k_2 + \rho - 1 \leq k_1 - 1 \leq k - 1$. Choose $c_{0,0}$ as first principal unknown. Then use $E_{r-1,0}^{(1)}$ to compute $c_{r,0}$ in terms of $c_{0,0}$ up to $r = k + \rho - 1 - k_2$. The unknowns $c_{r,0}$ have to be 0 for $k + \rho - k_2 \leq r \leq k_1 - 1$ (if any). Now choose $c_{k,0}$ (second principal unknown). All $c_{r,0}$ for $r > k_1$ are determined by the equations $E_{r,0}^{(1)}$ and can be expressed in terms of $c_{k,0}$.

Next consider the equations $E_{0,t}^{(2)}$. For $t \leq k + \rho - 1 - k_1$, the unknowns $c_{0,t}$ can be expressed in terms of $c_{0,0}$, for $k + \rho - k_1 \leq t \leq k_2$ (if any) $c_{0,t}$ has to be 0. For $t > k_2$, $c_{t,0}$ can be expressed in terms of $c_{0,r}$.

Let now $1 \leq t \leq k_2 - 1$, let $r + t \leq k$, and consider the equation $E_{r,t-1}^{(2)}$. Then the coefficient of the unknown $c_{r,t}$ is equal to $2(k-r-t+1)(t-k_2) \neq 0$. Hence, if $c_{r,t-1}$ and $c_{r-1,t-1}$ are already computed, $c_{r,t}$ can be computed. As all $c_{r,0}$, $0 \leq r \leq k$ have been expressed in terms of $c_{r,0}$ and $c_{0,0}$, $c_{r,t}$ can be also expressed in terms of $c_{k,0}$ and $c_{0,0}$. The statement follows. \square

Lemmas A1 up to A5 give an upper estimate of the dimension of the space of solutions. It remains to check that in each case, this estimate matches the number of independant λ -invariant distributions supported on the diagonal which were constructed in the previous sections. Eventually, the dimension of $Sol(\lambda_1, \lambda_2; k)$ is given by the following table.

Table for $\dim \text{Sol}(\lambda_1, \lambda_2; k)$

	$\lambda_1 = -k_1$ $\lambda_1 \notin E_k^\rho$	$\lambda_1 = -\rho - l_1$ $\lambda_1 \notin E_k$	$\lambda_1 = -k_1$ $\lambda_1 \in E_k \cap E_k^\rho$
$\lambda_2 = -k_2$ $\lambda_2 \notin E_k^\rho$	$k_1 + k_2 \leq k$ 1 $k_1 + k_2 > k$ 2	$l_1 + k_2 < k$ 1 $l_1 + k_2 \geq k$ 2	$k_1 + k_2 \leq k$ 1 $k < k_1 + k_2 < k + \rho$ 2 $k_1 + k_2 \geq k + \rho$ 3
$\lambda_2 = -\rho - l_2$ $\lambda_2 \notin E_k^\rho$	$k_1 + l_2 < k$ 1 $k_1 + l_2 \geq k$ 2	1	$k_1 + l_2 < k$ 1 $k_1 + l_2 \geq k$ 2
$\lambda_2 = -k_2$ $\lambda_2 \in E_k \cap E_k^\rho$	$k_1 + k_2 \leq k$ 1 $k < k_1 + k_2 < k + \rho$ 2 $k_1 + k_2 \geq k + \rho$ 3	$k_2 + l_1 < k$ 1 $k_2 + l_1 \geq k$ 2	$k_1 + k_2 \leq k$ 1 $k < k_1 + k_2 < k + \rho$ 2 $k_1 + k_2 \geq k + \rho$ 3

where

$$E_k = \{-1, -2, \dots, -k\}, \quad E_k^\rho = \{-\rho, -\rho - 1, \dots, -\rho - (k - 1)\}.$$

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