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# TWO LINNIK-TYPE PROBLEMS FOR AUTOMORPHIC $L$ -FUNCTIONS

JIANYA LIU, YAN QU & JIE WU

## ABSTRACT

Let  $m \geq 2$  be an integer, and  $\pi$  an irreducible unitary cuspidal representation for  $GL_m(\mathbb{A}_{\mathbb{Q}})$ , whose attached automorphic  $L$ -function is denoted by  $L(s, \pi)$ . Let  $\{\lambda_{\pi}(n)\}_{n=1}^{\infty}$  be the sequence of coefficients in the Dirichlet series expression of  $L(s, \pi)$  in the half-plane  $\Re s > 1$ . It is proved in this paper that, if  $\pi$  is such that the sequence  $\{\lambda_{\pi}(n)\}_{n=1}^{\infty}$  is real, then the first sign change in the sequence  $\{\lambda_{\pi}(n)\}_{n=1}^{\infty}$  occurs at some  $n \ll Q_{\pi}^{1+\varepsilon}$ , where  $Q_{\pi}$  is the conductor of  $\pi$ , and the implied constant depends only on  $m$  and  $\varepsilon$ . This improves the previous bound with the above exponent  $1 + \varepsilon$  replaced by  $m/2 + \varepsilon$ . A result of the same quality is also established for  $\{\Lambda(n)a_{\pi}(n)\}_{n=1}^{\infty}$ , the sequence of coefficients in the Dirichlet series expression of  $-\frac{L'}{L}(s, \pi)$  in the half-plane  $\Re s > 1$ .

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## 1. INTRODUCTION

Let  $\pi$  be a cuspidal automorphic representations of  $GL_m(\mathbb{A}_{\mathbb{Q}})$ . We can attach to any such  $\pi$  an automorphic  $L$ -function  $L(s, \pi)$ , which is defined by an Euler product, and for  $\sigma = \Re s > 1$ , it can be represented by an absolutely convergent Dirichlet series

$$L(s, \pi) = \sum_{n=1}^{\infty} \frac{\lambda_{\pi}(n)}{n^s}. \quad (1.1)$$

The sequence  $\{\lambda_{\pi}(n)\}_{n=1}^{\infty}$  consists of complex numbers, which we always normalize so that  $\lambda_{\pi}(1) = 1$ . It may happen that  $\lambda_{\pi}(n)$  is real for all  $n \geq 1$ ; for example, it is the case when  $\pi$  is a self-contragredient representation for  $GL_m(\mathbb{A}_{\mathbb{Q}})$  with trivial central character.

The purpose of this note is to continue the study in [4] of Linnik-type problems for automorphic  $L$ -functions  $L(s, \pi)$ . The reader is referred to [4] for general philosophy, history, and results in this direction.

The first Linnik-type problem we are going to study is, in the case  $\lambda_\pi(n)$  is real for all  $n \geq 1$ , to find the first  $n$  such that  $\lambda_\pi(n) < 0$ . Our result is as follows.

**Theorem 1.1.** *Let  $m \geq 2$  be an integer and let  $\pi$  be an irreducible unitary cuspidal representation for  $\mathrm{GL}_m(\mathbb{A}_\mathbb{Q})$ . If  $\lambda_\pi(n)$  is real for all  $n \geq 1$ , then there is some  $n$  satisfying*

$$n \ll_{m,\varepsilon} Q_\pi^{1+\varepsilon} \tag{1.2}$$

such that  $\lambda_\pi(n) < 0$ . The constant implied in (1.2) depends only on  $m$  and  $\varepsilon$ . In particular, the result is true for any self-contragredient irreducible unitary cuspidal representation  $\pi$  for  $\mathrm{GL}_m(\mathbb{A}_\mathbb{Q})$  with trivial central character.

The second Linnik-type problem considered in this note concerns sign changes in the sequence  $\{\Lambda(n)a_\pi(n)\}_{n=1}^\infty$ , which appears naturally in the Dirichlet series expression of the logarithmic derivative of  $L(s, \pi)$  in the half-plane  $\sigma > 1$ :

$$\frac{d}{ds} \log L(s, \pi) = - \sum_{n=1}^{\infty} \frac{\Lambda(n)a_\pi(n)}{n^s}. \tag{1.3}$$

Here  $\Lambda(n)$  is the von Mangoldt function, and  $\{\Lambda(n)a_\pi(n)\}_{n=1}^\infty$  is a sequence of complex numbers. The following is a Linnik-type theorem for this sequence.

**Theorem 1.2.** *Let  $m \geq 2$  be an integer and let  $\pi$  be an irreducible unitary cuspidal representation for  $\mathrm{GL}_m(\mathbb{A}_\mathbb{Q})$ . If  $\Lambda(n)a_\pi(n)$  is real for all  $n \geq 1$ , then there is some  $n$  satisfying*

$$n \ll_{m,\varepsilon} Q_\pi^{1+\varepsilon} \tag{1.4}$$

such that  $\Lambda(n)a_\pi(n) < 0$ . The constant implied in (1.4) depends only on  $m$  and  $\varepsilon$ . In particular, the result is true for any self-contragredient irreducible unitary cuspidal representation  $\pi$  for  $\mathrm{GL}_m(\mathbb{A}_\mathbb{Q})$  with trivial central character.

Theorems 1.1 and 1.2 improve significantly the corresponding results in [4], where the exponents  $1 + \varepsilon$  in (1.2) and (1.4) were replaced by the weaker exponents  $m/2 + \varepsilon$ . Note that our present exponents in (1.2) and (1.4) are independent of the dimension  $m$  of  $\mathrm{GL}_m(\mathbb{A}_\mathbb{Q})$ . New ideas leading to these improvements will be explained before Lemma 3.1 and before Lemma 3.3.

## 2. AUTOMORPHIC $L$ -FUNCTIONS

Let  $\pi$  be an irreducible unitary cuspidal representation  $\pi = \otimes \pi_p$  of  $\mathrm{GL}_m(\mathbb{A}_\mathbb{Q})$ . To every prime  $p$  at which  $\pi_p$  is unramified, there is an associated set of  $m$  nonzero complex Satake parameters

$\{\alpha_\pi(p, j)\}_{j=1}^m$ , out of which one may define local  $L$ -functions

$$L(s, \pi_p) = \prod_{j=1}^m \left(1 - \frac{\alpha_\pi(p, j)}{p^s}\right)^{-1}. \quad (2.1)$$

At  $p$  where  $\pi_p$  is ramified, the local  $L$ -function is defined in terms of the Langlands parameters of  $\pi_p$ . It is possible to write the local factors at ramified primes in the form of (2.1) with the convention that some of the  $\alpha_\pi(p, j)$ 's may be zero. Here it is appropriate to point out that the coefficients  $\{\lambda_\pi(n)\}_{n=1}^\infty$  and  $\{a_\pi(p^k)\}_{k=1}^\infty$  in (1.1) and (1.3) are actually defined, respectively, by

$$\lambda_\pi(n) = \prod_{p^\nu \parallel n} \left\{ \sum_{\nu_1 + \dots + \nu_m = \nu} \alpha_\pi(p, 1)^{\nu_1} \dots \alpha_\pi(p, m)^{\nu_m} \right\}, \quad (2.2)$$

and

$$a_\pi(p^k) = \sum_{j=1}^m \alpha_\pi(p, j)^k. \quad (2.3)$$

At the archimedean place  $\infty$ , a set of  $m$  complex Langlands parameters  $\{\mu_\pi(j)\}_{j=1}^m$  is associated to  $\pi_\infty$ . The local factor at  $\infty$  is defined to be

$$L(s, \pi_\infty) = \pi^{-\frac{ms}{2}} \prod_{j=1}^m \Gamma\left(\frac{s + \mu_\pi(j)}{2}\right). \quad (2.4)$$

For the parameters  $\{\alpha_\pi(p, j)\}_{j=1}^m$  and  $\{\mu_\pi(j)\}_{j=1}^m$ , trivial bounds state that

$$|\alpha_\pi(p, j)| \leq \sqrt{p}, \quad |\Re \mu_\pi(j)| \leq \frac{1}{2}.$$

The finite-part  $L$ -function  $L(s, \pi)$  is defined by products of local factors

$$L(s, \pi) = \prod_{p < \infty} L(s, \pi_p), \quad (2.5)$$

and the complete  $L$ -function  $\Phi(s, \pi)$  is defined by

$$\Phi(s, \pi) = L(s, \pi_\infty)L(s, \pi). \quad (2.6)$$

This complete  $L$ -function extends to an entire function on the whole complex plane via its functional equation

$$\Phi(s, \pi) = \varepsilon_\pi N_\pi^{\frac{1}{2}-s} \Phi(1-s, \tilde{\pi}) \quad (2.7)$$

where  $\tilde{\pi}$  is the contragredient of  $\pi$ ,  $\varepsilon_\pi$  a complex number of modulus 1, and  $N_\pi$  a positive integer called the arithmetic conductor of  $\pi$ . Finally  $\Phi(s, \pi)$  is of order one and bounded in the vertical strips. The reader is referred to e.g. Cogdell [1] for proofs of these properties.

The functional equation (2.7) can be re-written as

$$L(s, \pi) = \varepsilon_\pi N_\pi^{\frac{1}{2}-s} G(s) L(1-s, \tilde{\pi}), \quad (2.8)$$

where

$$G(s) = \frac{L(1-s, \tilde{\pi}_\infty)}{L(s, \pi_\infty)}. \quad (2.9)$$

The following lemma gives an estimate for  $G(s)$  on the vertical line  $\sigma = -H$ , avoiding the poles of the nominator of  $G(s)$ . Its proof is a simple application of Stirling's for the  $\Gamma$ -function, so we omit the details.

**Lemma 2.1.** *For each positive integer  $N$ , there is an  $H \in [N, N+1]$ , such that on the line  $\sigma = -H$  we have*

$$G(-H+it) \ll_{H,m} (1+|t|)^{m(\frac{1}{2}+H)} \prod_{j=1}^m (1+|\mu_\pi(j)|)^{\frac{1}{2}+H}. \quad (2.10)$$

Following Iwaniec-Sarnak [3], we define the analytic conductor of  $L(s, \pi)$  as

$$Q_\pi(t) = N_\pi \prod_{j=1}^m (1+|it + \mu_\pi(j)|). \quad (2.11)$$

Setting  $t = 0$  in the above definitions, we write

$$Q_\pi = Q_\pi(0) \quad (2.12)$$

which is called the conductor of  $\pi$ .

### 3. PREPARATIONS FOR THEOREM 1.1

The purpose of this section is to establish the preliminaries required by Theorems 1.1. Suppose that

$$\lambda_\pi(n) \geq 0 \quad \text{for all } n \leq x. \quad (3.1)$$

We start with the sum

$$S_\pi(x) = \sum_n \lambda_\pi(n) w\left(\frac{n}{x}\right), \quad (3.2)$$

where  $w(x)$  is a non-negative real valued function of  $C_c^\infty$  with compact support in  $[0, 1]$ . In this note we specify

$$w(x) := \begin{cases} \exp(-x^{-\frac{1}{4m}}) \exp\{- (1-x)^{-\frac{1}{4m}}\} & \text{if } x \in (0, 1), \\ 0 & \text{otherwise.} \end{cases} \quad (3.3)$$

We have good reasons to choose this specific weight function; the reader will see, in the discussion after the proof of Theorem 1.1 in §4, that the better-looking function

$$\begin{cases} \exp(-x^{-1}) \exp\{-(1-x)^{-1}\} & \text{if } x \in (0, 1) \\ 0 & \text{otherwise} \end{cases} \quad (3.4)$$

does not do the job.

Theorem 1.1 will follow from upper and lower bound estimates for  $S_\pi(x)$ . Our present strategy of establishing upper and lower bound estimates for  $S_\pi(x)$  is different from that in [4]. In getting our new upper bound, we apply Landau's method instead of the the direct application of convexity bound of  $L(s, \pi)$  as in [4], and this results in extra savings. Our new upper bound for  $S_\pi(x)$  is as follows.

**Lemma 3.1.** *For each positive integer  $N$ , there is an  $H \in [N, N + 1]$ , such that*

$$S_\pi(x) \ll_{H,m} x^{-H} Q_\pi^{\frac{1}{2}+H}, \quad (3.5)$$

where the implied constant depends on  $H$  and  $m$ .

*Proof.* We note that, for any positive integer  $k$ , the derivative  $w^{(k)}(x)$  has exponential decay as  $x \rightarrow 0+$  or  $1-$ . Consequently, the Mellin transform

$$W(s) = \int_0^\infty w(x) x^{s-1} dx$$

is an analytic function of  $s$ . By repeated partial integration, we have

$$|W(\sigma + it)| \ll_{A,\sigma,m} \frac{1}{(1 + |t|)^A} \quad (3.6)$$

for arbitrary positive constant  $A$ , and in particular for  $\sigma \leq 2$ . Now we apply Mellin inversion, to get

$$w(x) = \frac{1}{2\pi i} \int_{(2)} W(s) x^{-s} ds,$$

where  $(c)$  means the vertical line  $\sigma = c$ . Inserting this back to (3.2), and then using Dirichlet series expansion (1.1), we have

$$\begin{aligned} S_\pi(x) &= \frac{1}{2\pi i} \sum_n \lambda_\pi(n) \int_{(2)} W(s) \left(\frac{n}{x}\right)^{-s} ds \\ &= \frac{1}{2\pi i} \int_{(2)} x^s W(s) L(s, \pi) ds, \end{aligned}$$

where the interchange of summation and integral is guaranteed by the absolute convergence of (1.1) on the line  $\sigma = 2$ . A pre-convexity bound like

$$L(1/2 + it, \pi) \ll Q_\pi(t)^B, \quad (3.7)$$

where  $B > 0$  is some constant, can be obtained by standard analytic method via the functional equation (2.7). Actually it has been proved by Harcos [2] that any constant  $B > \frac{1}{4}$  is acceptable in (3.7). It follows that

$$L(\sigma + it, \pi) \ll_\varepsilon Q_\pi(t)^{\mu(\sigma)+\varepsilon} \quad (3.8)$$

with

$$\mu(\sigma) = \begin{cases} (1 - \sigma)/2 & \text{if } 0 \leq \sigma \leq 1, \\ 1 - \sigma/2 & \text{if } \sigma \leq 0. \end{cases}$$

Since both  $W(s)$  and  $L(s, \pi)$  are entire, we may apply (3.6) and (3.8) to shift the contour above to the vertical line  $\sigma = -H$ , getting

$$S_\pi(x) = \frac{1}{2\pi i} \int_{(-H)} x^s W(s) L(s, \pi) ds, \quad (3.9)$$

where  $H \in [N, N + 1]$  is a real number decided by Lemma 2.1.

Now we apply a classical idea of Landau to insert the functional equation (2.8) into (3.9), getting

$$\begin{aligned} S_\pi(x) &= \frac{1}{2\pi i} \int_{(-H)} x^s W(s) \varepsilon_\pi N_\pi^{\frac{1}{2}-s} G(s) L(1-s, \tilde{\pi}) ds \\ &= \frac{1}{2\pi i} \int_{(-H)} x^s W(s) \varepsilon_\pi N_\pi^{\frac{1}{2}-s} G(s) \left( \sum_{n=1}^{\infty} \frac{\lambda_{\tilde{\pi}}(n)}{n^{1-s}} \right) ds \\ &= \frac{1}{2\pi i} \sum_{n=1}^{\infty} \frac{\overline{\lambda_\pi(n)}}{n^{1+H}} \int_{(-H)} x^s W(s) \varepsilon_\pi N_\pi^{\frac{1}{2}-s} G(s) n^{s+H} ds. \end{aligned}$$

Here the interchange of summation and integral is guaranteed by the absolute convergence of the Dirichlet series as well as the rapid decay of  $W(s)$  in (3.6). Using these facts again, and

inserting (2.10) into the last integral, we get

$$\begin{aligned}
S_\pi(x) &\ll_{H,m} \int_{(-H)} \left| x^s W(s) \varepsilon_\pi N_\pi^{\frac{1}{2}-s} G(s) ds \right| \\
&\ll_{H,m} x^{-H} N_\pi^{\frac{1}{2}+H} \prod_{j=1}^m (1 + |\mu_\pi(j)|)^{\frac{1}{2}+H} \\
&= x^{-H} Q_\pi^{\frac{1}{2}+H}.
\end{aligned}$$

This proved Lemma 3.1.  $\square$

To get a suitable lower bound for  $S_\pi(x)$ , we will need the following result, which is Lemma 5.3 in [4].

**Lemma 3.2.** *Let  $m \geq 2$  be an integer and let  $\pi$  be an irreducible unitary cuspidal representation of  $GL_m(\mathbb{A}_\mathbb{Q})$ . For any prime  $p$  such that  $\pi_p$  is unramified, we have*

$$|\lambda_\pi(p^m)| + |\lambda_\pi(p^{m-1})| + \cdots + |\lambda_\pi(p)| \geq \frac{1}{m}.$$

In [4], the above lemma was applied to establish the lower bound

$$\begin{aligned}
\sum_{n \leq x} \lambda_\pi(n) &\geq \sum_{\substack{n \leq x \\ (n, N_\pi) = 1}} \lambda_\pi(n) \\
&\geq \sum_{\substack{p \leq x^{1/m} \\ p \nmid N_\pi}} \{ \lambda_\pi(p^m) + \cdots + \lambda_\pi(p) \} \\
&\gg_m \sum_{\substack{p \leq x^{1/m} \\ p \nmid N_\pi}} 1 \gg_m \frac{x^{1/m}}{\log x} - \log N_\pi,
\end{aligned} \tag{3.10}$$

which is enough to derive the main theorem there. However, (3.10) does not imply a useful lower for  $S_\pi(x)$ , since the sum  $S_\pi(x)$  actually counts the contribution essentially from  $n \in [\rho x, (1-\rho)x]$  for some small  $\rho > 0$ , which follows from the fact that our weight function  $w(x)$  has exponential decay when  $x \rightarrow 0+$  or  $1-$ . To get around this difficulty, we must find at least one integer  $n_0$ , which is close to neither 0 or 1, such that  $\lambda_\pi(n_0)$  relatively large. This is achieved in the following lemma.

**Lemma 3.3.** (i) *There is an integer  $n_0$  with canonical decomposition*

$$n_0 = p_0^{\nu_0} p_1^{\nu_1} \cdots p_r^{\nu_r}, \tag{3.11}$$



where  $p_j$ 's are distinct primes not dividing  $N_\pi$  and  $\nu_j$ 's are positive integers not exceeding  $m$  such that

$$\lambda_\pi(p_j^{\nu_j}) \geq \frac{1}{m^2} \quad \text{for all } j = 0, 1, \dots, r. \quad (3.12)$$

In addition  $p_0$  is the smallest prime not dividing  $N_\pi$ .

(ii) The above  $n_0$  falls in the interval  $[\rho x, (1 - \rho)x]$ , where

$$\rho = \frac{1}{p_0^m + 1}. \quad (3.13)$$

*Proof.* Our proof is of combinatorial nature. We begin by applying the pigeonhole principle to Lemma 3.2 to see that, for every prime  $p \nmid N_\pi$ , there exists a positive integer  $\nu \leq m$  depending on both  $p$  and  $\pi$ , such that

$$\lambda_\pi(p^\nu) \geq \frac{1}{m^2}. \quad (3.14)$$

A prime power  $p^\nu$  is called *good* with respect to  $p$ , if  $p \nmid N_\pi$  and  $\nu$  is the smallest positive integer such that (3.14) holds. An immediate consequence of the definition is that two good prime powers  $p_1^{\nu_1}$  and  $p_2^{\nu_2}$  are different if and only if  $p_1 \neq p_2$ .

Let  $0 < \rho < 1$  be a parameter, not yet specified. Take the largest good prime power  $< \rho x$ , say  $p_1^{\nu_1}$ , and form the interval

$$\left[ \frac{\rho x}{p_1^{\nu_1}}, \frac{(1 - \rho)x}{p_1^{\nu_1}} \right].$$

Then take the largest good prime power  $< \rho x / p_1^{\nu_1}$ , say  $p_2^{\nu_2}$ , and form the second interval

$$\left[ \frac{\rho x}{p_1^{\nu_1} p_2^{\nu_2}}, \frac{(1 - \rho)x}{p_1^{\nu_1} p_2^{\nu_2}} \right].$$

We repeat the process until there is an interval

$$\left[ \frac{\rho x}{p_1^{\nu_1} p_2^{\nu_2} \cdots p_r^{\nu_r}}, \frac{(1 - \rho)x}{p_1^{\nu_1} p_2^{\nu_2} \cdots p_r^{\nu_r}} \right] \quad (3.15)$$

with all these good prime powers satisfying

$$p_j^{\nu_j} < \frac{\rho x}{p_1^{\nu_1} p_2^{\nu_2} \cdots p_{j-1}^{\nu_{j-1}}} \quad \text{for all } j = 1, \dots, r, \quad (3.16)$$

such that there is no good prime power smaller than  $\rho x / (p_1^{\nu_1} p_2^{\nu_2} \cdots p_r^{\nu_r})$ . In particular, for the good prime power  $p_0^{\nu_0}$  where  $p_0$  is the smallest prime not dividing  $N_\pi$ , we must have

$$p_0^{\nu_0} \geq \frac{\rho x}{p_1^{\nu_1} p_2^{\nu_2} \cdots p_r^{\nu_r}}. \quad (3.17)$$

We want to show that  $p_0^{\nu_0}$  lies in the interval (3.15) for some suitably chosen  $\rho$ . To this end, specify  $\rho$  as in (3.13), so that  $p_0^{\nu_0} \rho \leq 1 - \rho$ . Also, (3.16) with  $j = r$  states that

$$1 < \frac{\rho x}{p_1^{\nu_1} p_2^{\nu_2} \cdots p_r^{\nu_r}}.$$

and hence

$$p_0^{\nu_0} < p_0^{\nu_0} \frac{\rho x}{p_1^{\nu_1} p_2^{\nu_2} \cdots p_r^{\nu_r}} \leq \frac{(1 - \rho)x}{p_1^{\nu_1} p_2^{\nu_2} \cdots p_r^{\nu_r}}. \quad (3.18)$$

It follows from (3.17) and (3.18) that  $p_0^{\nu_0}$  indeed lies in the interval (3.15) with  $\rho$  specified as in (3.13).

Finally we write  $n_0 = p_0^{\nu_0} p_1^{\nu_1} p_2^{\nu_2} \cdots p_r^{\nu_r}$ , and recall that all the components  $p_0^{\nu_0}, \dots, p_r^{\nu_r}$  above are good prime powers which are mutually different. It follows by construction that all the primes  $p_0, \dots, p_r$  are different and satisfy  $p_j \nmid N_\pi$ . Thus we have (3.11) and (3.12). Since  $p_0^{\nu_0}$  lies in the interval (3.15), we have  $n_0 \in [\rho x, (1 - \rho)x]$ . This proves the lemma.  $\square$

**Lemma 3.4.** *We have the lower bound*

$$S_\pi(x) \gg_m x^{-(2/\log 2) \log m} \exp(-c_0 \sqrt{\log N_\pi}), \quad (3.19)$$

where  $c_0 > 0$  is an absolute constant and the implied constant depends on  $m$ .

*Proof.* We start from the trivial observation that

$$S_\pi(x) \geq \lambda_\pi(n_0) w\left(\frac{n_0}{x}\right), \quad (3.20)$$

where  $n_0$  is decided by Lemma 3.3. The assertion of the current lemma will follow from lower bounds for  $\lambda_\pi(n_0)$  and for  $w(n_0/x)$ .

To get a lower bounds for  $\lambda_\pi(n_0)$ , we recall that

$$\lambda_\pi(n_1 n_2) = \lambda_\pi(n_1) \lambda_\pi(n_2) \quad \text{for } (n_1, n_2) = 1, (n_1 n_2, N_\pi) = 1.$$

This together with (3.11) and (3.12) implies that

$$\lambda_\pi(n_0) = \lambda_\pi(p_0^{\nu_0} p_1^{\nu_1} \cdots p_r^{\nu_r}) = \lambda_\pi(p_0^{\nu_0}) \lambda_\pi(p_1^{\nu_1}) \cdots \lambda_\pi(p_r^{\nu_r}) \geq m^{-2(r+1)}.$$

An elementary argument gives

$$r + 1 \leq \frac{\log n_0}{\log 2} \leq \frac{\log x}{\log 2},$$

and consequently

$$\lambda_\pi(n_0) \geq m^{-\frac{2\log x}{\log^2}} > x^{-3\log m}. \quad (3.21)$$

This is the desired lower bound for  $\lambda_\pi(n_0)$ .

To get a useful lower bound for  $w(n_0/x)$ , we must analyze the weight function  $w(x)$  carefully. It is easy to check that, for  $x \in (0, 1)$ ,

$$w'(x) = \frac{1}{4m}w(x)\{x^{-\frac{1}{4m}-1} - (1-x)^{-\frac{1}{4m}-1}\},$$

and consequently  $w(x)$  is increasing and decreasing, respectively, in the intervals  $[0, \frac{1}{2}]$  and  $[\frac{1}{2}, 1]$ . Lemma 3.3 asserts that  $n_0/x \in [\rho, 1 - \rho]$ , and hence

$$w\left(\frac{n_0}{x}\right) \geq w(\rho) = \exp\left(-\rho^{-\frac{1}{4m}}\right) \exp\left\{-\left(1-\rho\right)^{-\frac{1}{4m}}\right\}. \quad (3.22)$$

From (3.13) we infer  $\rho \leq \frac{1}{5}$ , and hence

$$\exp\left\{-\left(1-\rho\right)^{\frac{1}{4m}}\right\} \geq \exp\left\{-\left(4/5\right)^{\frac{1}{4m}}\right\}.$$

To evaluate the other exponential factor in (3.22), we apply (3.13) again to deduce that  $\rho \geq (2p_0)^{-m}$ , and consequently

$$\exp\left(-\rho^{-\frac{1}{4m}}\right) \geq \exp\left\{-\left(2p_0\right)^{\frac{1}{4}}\right\}.$$

Now we need to know the largest possible value of  $p_0$ , which is by definition the smallest prime not dividing  $N_\pi$ . The largest possible value of  $p_0$  occurs when  $N_\pi = q_1 \cdots q_s$  is the product of the first  $s$  primes, and in this case  $p_0 = q_{s+1}$ , the  $(s+1)$ -th prime number. Since

$$\log N_\pi = \sum_{p \leq q_s} \log p \sim \frac{q_s}{\log q_s},$$

we must have  $q_s \leq c_1(\log N_\pi) \log \log N_\pi \leq c_1(\log N_\pi)^2$  with an absolute constant  $c_1 > 0$ . It follows that

$$p_0 = q_{s+1} \leq 2q_s \leq 2c_1(\log N_\pi)^2,$$

which is the largest possible value of  $p_0$ . It follows that

$$\exp\left(-\rho^{-\frac{1}{4m}}\right) \geq \exp\left\{-\left(2p_0\right)^{\frac{1}{4}}\right\} \geq \exp\left\{-\left(4c_1\right)^{\frac{1}{4}}\sqrt{\log N_\pi}\right\},$$

which is the desired lower bound for the first exponential factor in (3.22).

Inserting everything back to (3.22), we get

$$w\left(\frac{n_0}{x}\right) \gg_m \exp\left\{-\left(4c_1\right)^{\frac{1}{4}}\sqrt{\log N_\pi}\right\},$$

which in combination with (3.20) and (3.21) gives the assertion of the lemma with  $c_0 = (4c_1)^{\frac{1}{4}}$ .  
□

#### 4. SOLUTIONS TO LINNIK-TYPE PROBLEMS

With the preparations in §3, we can now establish Theorem 1.1.

*Proof of Theorem 1.1.* It follows from Lemmas 3.1 and 3.4 that

$$x^{-3\log m} \exp(-c_0\sqrt{\log N_\pi}) \ll_m S_\pi(x) \ll_{H,m} x^{-H} Q_\pi^{\frac{1}{2}+H}.$$

One therefore has

$$x < c_2 \cdot Q_\pi^E \exp(c_0\sqrt{\log N_\pi}) \quad \text{with} \quad E = \frac{H + 1/2}{H - 3\log m},$$

where  $c_2$  is a constant depending on  $H$  and  $m$ . Taking  $H$  sufficiently large, this becomes  $x < c_3 \cdot Q_\pi^{1+\varepsilon}$ , and now the constant  $c_3$  depends on  $\varepsilon$  and  $m$ . The assertion of Theorem 1.1 is proved. □

*Remark.* Had we chosen the better-looking weight function in (3.4), we would have the same Lemma 3.1, but have Lemma 3.4 replaced by

$$S_\pi(x) \gg_m x^{-3\log m} \exp\{-c_4(\log N_\pi \log \log N_\pi)^m\}$$

for some constant  $c_4 > 0$ . This lower bound is too weak for the above proof of Theorem 1.1 to work.

*Proof of Theorem 1.2.* Theorem 1.2 follows from Theorem 1.1 in the same way as done in [4], and details are therefore omitted. □

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