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# Robust sensor localization with locally-computed, global $H_\infty$ -design

Anton Korniienko, Usman A. Khan, and Gerard Scorletti

## Abstract

This paper considers sensor localization in  $\mathbb{R}^m$ , where the problem is to find the positions of an arbitrary number of sensor nodes in the presence of at least  $m + 1$  anchor nodes, given only the inter-node distances. Assuming that the sensors lie in the convex hull of the anchors, we provide a *linear*, continuous-time update at each sensor that uses barycentric coordinates and Cayley-Menger determinants. It can be shown that the resulting update converges to the true sensor locations under certain conditions. In this paper, we consider design enhancements by adding a dynamic controller in the feed-forward loop of the location estimator at each sensor. We show that the dynamic controller has the ability to both speed-up the convergence and improve the transient behavior, while ensuring a certain disturbance rejection that could be introduced by inter-node communication. The design of the local controllers is based on the input-output approach,  $H_\infty$  design and does not require the knowledge of global parameters. Simulations are provided to illustrate the concepts described in this paper.

## Index Terms

Sensor localization, Robust control, Decentralized control, Control of networks, Sensor fusion.

## I. INTRODUCTION

This paper considers sensor localization where the problem is to find the positions of sensors placed in a Euclidean space,  $\mathbb{R}^m$ , given a certain number of *anchors* with perfectly known locations. It is well-known that the localization problem requires at least  $m + 1$  anchors, in general, given only pairwise distances. However, problems with a large number of sensors typically require a much higher number of anchors. The literature on localization is largely based on the well-known triangulation and trilateration principles, see [1] for details, or in some cases, a combination of both. Centralized localization algorithms include: maximum likelihood estimators, [2]; multi-dimensional scaling (MDS), [3], [4]; and optimization-based methods, see [5]; for additional work, see [6], [7].

On the other hand, a significant attention is also devoted to distributed localization algorithms that can be characterized into two classes: multilateration and successive refinements. In *multilateration algorithms*, [8], [9], each sensor estimates its distance from the anchors, possibly via multi-hop communication, and then calculates its location via trilateration. *Successive refinement algorithms* utilize an iterative optimization of a cost function and can be found in, e.g., [10]. Reference [11] discusses a Self-Positioning Algorithm (SPA) that provides a

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GPS-free positioning and builds a relative coordinate system. Other related work also consists of graph-theoretic approaches [12], [13], and probabilistic methods, [14], [15].

Of significant relevance to this paper are Refs. [1], [16], where a discrete-time algorithm, named *DILOC*, is described assuming that the sensors lie inside the convex hull of at least  $m+1$  anchors in  $\mathbb{R}^m$ . DILOC is completely local and distributed: each sensor finds a *triangulation set* of  $m+1$  neighboring nodes (sensors and/or anchors) such that the sensor in question lies inside the convex hull of this set. The locations estimates are subsequently updated as a linear combination of the neighboring locations (estimates) in the triangulation set. The coefficients in this linear update are the barycentric coordinates, attributed to August F. Möbius, [17]. Assuming that a triangulation set exists at each sensor, DILOC converges to the true sensor locations. It is important to note that DILOC is linear and only requires  $m+1$  anchors regardless of the number of sensors as long as the deployment (convexity) conditions are satisfied.

In [18], we provide DILOC-CT that is a continuous-time analog of DILOC and show that by using a proportional controller in the location estimators, the convergence speed can be increased arbitrarily. Since this increase may come at the price of unwanted transients especially when there is disturbance introduced by the network, we consider dynamic controllers that guarantee certain performance and disturbance rejection properties. However, the controller design in [18] is not local, i.e., it requires all of the barycentric coordinates to be known in order to compute the controller parameters. In this paper, we describe a distributed control design that is completely local and only relies on a spectral bound of the matrix of barycentric coordinates. The design in this paper, thus, can be directly implemented at the sensors and no central computation is required. Our approach is based on  $H_\infty$  design principles and uses the input-output approach, see e.g., [19]–[22].

We now describe the rest of the paper. Section II describes the problem and recaps DILOC-CT while Section III presents the interpretation of the problem under consideration in frequency domain ( $s$ -domain). Section IV discusses the main results on using the input-output approach for the decentralized controller design. Section V illustrates the concepts and Section VI concludes the paper.

**Notation:** The superscript, ‘ $T$ ’, denotes a real matrix transpose while the superscript, ‘ $*$ ’, denotes the complex-conjugate transpose. The  $N \times N$  identity matrix is denoted by  $I_N$  and the  $n \times m$  zero matrix is denoted by  $0_{n \times m}$ . The dimension of the identity or zero matrix is omitted when it is clear from the context. The diagonal aggregation of two matrices  $A$  and  $B$  is denoted by **diag**( $A, B$ ). The *Kronecker product*, denoted by  $\otimes$ , between two matrices,  $A$  and  $B$ , is defined as  $A \otimes B = [a_{ij}B]$ . We use  $T_{x \rightarrow y}(s)$  to denote the transfer function between an input,  $x(t)$ , and an output,  $y(t)$ . With matrix,  $G$ , partitioned into four blocks,  $G_{11}, G_{12}, G_{21}, G_{22}$ ,  $G \star K$  denotes the *Redheffer product*, [23], i.e.,

$$G \star K = G_{11} + G_{12}K(I - G_{22}K)^{-1}G_{21}.$$

Similarly,  $K \star G = G_{22} + G_{21}K(I - G_{11}K)^{-1}G_{12}$ . For a stable LTI system,  $G$ ,  $\|G\|_\infty$  denotes the  $H_\infty$  norm of  $G$ . For a complex matrix,  $P$ ,  $\bar{\sigma}(P)$  denotes its maximal singular value while  $\rho(P)$  denotes its spectral radius. Finally, the symbols, ‘ $\succeq$ ’ and ‘ $\succ$ ’, denote positive semi-definiteness and positive-definiteness of a matrix, respectively.

## II. PRELIMINARIES AND PROBLEM FORMULATION

Consider a network of  $M$  sensors, in the set  $\Omega$ , with unknown locations, and  $N$  anchors, in the set  $\kappa$ , with known locations, all located in  $\mathbb{R}^m$ ,  $m \geq 1$ ; let  $\Theta = \Omega \cup \kappa$ . Let  $\mathbf{x}_i^* \in \mathbb{R}^{1 \times m}$  denote the *true location* of the  $i$ th sensor,  $i \in \Omega$ ; similarly,  $\mathbf{u}_j \in \mathbb{R}^{1 \times m}$ ,  $j \in \kappa$ , for an anchor. We assume that each sensor knows its distances to the nearby nodes (sensors and/or anchors). The problem is to find the locations of the sensors in  $\Omega$ . Below, we describe DILOC-CT, see [18].

### A. DILOC-CT

DILOC-CT assumes that each sensor lies in the convex hull, denoted by  $\mathcal{C}(\kappa)$ , of the anchors. Using only the inter-node distances, each sensor, say  $i$ , finds a *triangulation set*,  $\Theta_i$ , of  $m+1$  neighbors such that  $i \in \mathcal{C}(\Theta_i)$ ,  $|\Theta_i| = m+1$ . A convex hull inclusion test is given by

$$i \in \mathcal{C}(\Theta_i), \quad \text{if } \sum_{j \in \Theta_i} A_{\Theta_i \cup \{i\} \setminus j} = A_{\Theta_i},$$

where  $A_{\Theta_i}$  denotes the  $m$ -dimensional volume, area in  $\mathbb{R}^2$  or volume in  $\mathbb{R}^3$ , of  $\mathcal{C}(\Theta_i)$ , see Fig. 1 in  $\mathbb{R}^2$ , and can be computed by Cayley-Menger determinant, [16], [24], using only the pairwise distances of the nodes in  $\{i, \Theta_i\}$ .

Given a triangulation set,  $\Theta_i$ , for every  $i \in \Omega$ , the barycentric coordinates are given by

$$p_{ij}|_{j \in \Theta_i \cap \Omega} \triangleq \frac{A_{\Theta_i \cup \{i\} \setminus j}}{A_{\Theta_i}}, \quad b_{ij}|_{j \in \Theta_i \cap \kappa} \triangleq \frac{A_{\Theta_i \cup \{i\} \setminus j}}{A_{\Theta_i}}, \quad (1)$$

where  $p_{ij}$ 's are the sensor-to-sensor coefficients and  $b_{ij}$ 's are sensor-to-anchor; note that the *barycentric coordinates* are positive and sum to 1.

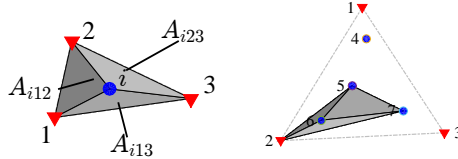


Fig. 1.  $\mathbb{R}^2$ : (Left) Sensor  $i$  (circle) lies inside the convex hull of anchors (triangles). (Right) Sensor 2, 5 and 7 form a triangulation set for sensor 6.

Let  $\mathbf{x}_i(t) \in \mathbb{R}^{1 \times m}$  denote the  $m$ -dimensional row-vector of sensor  $i$ 's location estimate, where  $t \geq 0$  is the continuous-time variable. Since the anchors' locations are known, we have  $\mathbf{u}_j(t) = \mathbf{u}_j, \forall t \geq 0, j \in \kappa$ . DILOC-CT is given by the following equation, where  $(t)$  is dropped in the sequel for convenience:

$$\dot{\mathbf{x}}_i = \alpha(-\mathbf{x}_i + \mathbf{r}_i); \quad \mathbf{r}_i \triangleq \sum_{j \in \Theta_i \cap \Omega} p_{ij} \mathbf{x}_j + \sum_{j \in \Theta_i \cap \kappa} b_{ij} \mathbf{u}_j, \quad (2)$$

and  $\Theta_i$  is the triangulation set at sensor  $i$ . Note that  $\Theta_i$  may not contain any anchor, in which case  $|\Theta_i \cap \kappa| = 0$ , and  $|\Theta_i \cap \Omega| = m+1$ . Let  $\mathbf{x}(t) \in \mathbb{R}^{M \times m}$  (and  $\mathbf{u}(t) \in \mathbb{R}^{(m+1) \times m}$ ) denote the matrix regrouping the vectors of sensor (and anchor) locations, and let  $P \triangleq \{p_{ij}\}$  and  $B \triangleq \{b_{ij}\}$ . It can be shown that the resulting dynamics,

$$\dot{\mathbf{x}} = -\alpha(I - P)\mathbf{x} + \alpha B\mathbf{u} \triangleq P_\alpha \mathbf{x} + B_\alpha \mathbf{u}, \quad (3)$$

converge to

$$\lim_{t \rightarrow \infty} \mathbf{x}(t) = (I - P)^{-1} B \mathbf{u}(0) = \mathbf{x}^*, \quad \forall \alpha > 0, \quad (4)$$

which are the true sensor locations. Successful convergence requires that each sensor is able to find a triangulation set, ensured by the convexity of sensor and anchor placements, see [1], [16], [18] for details. In particular, such deployment ensures that

$$\rho(P) < 1,$$

required for convergence, where  $\rho$  is the spectral radius of  $P$ .

### B. Problem formulation

Towards a more practical system, we consider the received signal at each sensor to incur a zero-mean additive disturbance,  $\mathbf{n}_i(t)$ , whose frequency spectrum lies in the interval,  $[\omega_z^-, \omega_z^+]$ . With this disturbance, the location estimator is given by

$$\dot{\mathbf{x}}_i = \alpha(-\mathbf{x}_i + \mathbf{r}_i + \mathbf{n}_i). \quad (5)$$

We replace the proportional gain,  $\alpha$ , with a dynamic controller,  $K(s)$ ; see Fig. 2 leading to the dynamical system,  $T_s$ , at each sensor. We note that the above localization algorithm is linear and in continuous-time, which allow us to use frequency-based design approach, which has advantages in handling multi-objective specifications and in treating the disturbance, usually present in High-Frequency (HF) range. The control design problem is to ensure global stability and global performance expressed in terms of a certain transient behavior, convergence rate, and disturbance rejection, all at the same time. In this paper, we relax the knowledge of  $P$  and  $B$ , required for controller design in [18]. In particular, we propose a controller design not for a given topology,  $P, B$ , but for a set of all topologies bounded in the sense of maximum singular value.

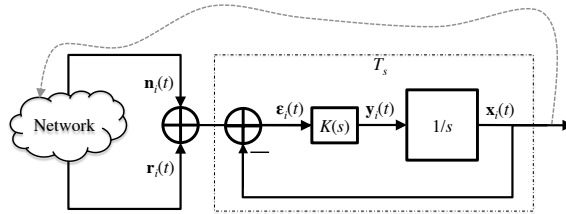


Fig. 2. DILOC-CT architecture, see [18].

### III. DILOC-CT WITH DYNAMIC CONTROLLER: $s$ -DOMAIN ANALYSIS

Since DILOC-CT is decoupled in the coordinates, each local system,  $T_s$ , see Fig. 2, can be analyzed per coordinate and the same analysis can be extended to other coordinates. Recall that the network location estimate,  $\mathbf{x}(t)$ , is an  $M \times m$  matrix where each column is associated to a location coordinate in  $\mathbb{R}^m$ . We let  $x$  to be an arbitrarily chosen column of  $\mathbf{x}$  corresponding to one chosen dimension. Similarly, we let  $n, u, r, \varepsilon^{ref}, \varepsilon$ , and  $y$  (signals in

Fig. 2) to represent  $M$ -dimensional vectors; for any of such vectors, the subscript  $i$  denotes the chosen coordinate at sensor  $i$ . Similar to [18], the global system under consideration is described by:

$$x = (I_M \otimes T_s(s)) r^n, \quad (6)$$

$$\begin{bmatrix} r^n \\ \varepsilon^{ref} \\ \varepsilon \end{bmatrix} = \underbrace{\begin{bmatrix} P & B & I_M \\ -I_M & (I_M - P)^{-1} B & 0 \\ P - I_M & B & I_M \end{bmatrix}}_{H_0} \begin{bmatrix} x \\ u \\ n \end{bmatrix}. \quad (7)$$

where  $s$  is Laplace variable, the input vector  $u \in \mathbb{R}^N$  represents the anchors position sent to the sensors,  $n \in \mathbb{R}^M$  is the input vector regrouping all sensor disturbance signals  $n_i$ ,  $x^* \in \mathbb{R}^M$  and  $x \in \mathbb{R}^M$  are the vectors regrouping the true and estimated sensor location  $x_i^*$  and  $x_i$ . The signal  $r^n \in \mathbb{R}^M$  :  $r^n \triangleq r + n = Px + Bu + n$ , is the vector regrouping all local sensor inputs  $r_i^n$ . The signal  $\varepsilon^{ref} \in \mathbb{R}^M$  :  $\varepsilon^{ref} \triangleq x^* - x = (I_M - P)^{-1} Bu - x$  is the vector regrouping all sensor localization tracking errors  $\varepsilon_i^{ref}$  and the signal  $\varepsilon \in \mathbb{R}^M$  :  $\varepsilon \triangleq r^n - x$  is the vector regrouping all local sensor errors  $\varepsilon_i$ .

The local transfer function,  $T_s$ , identical at each sensor, is

$$T_s(s) = \frac{K(s) \frac{1}{s}}{1 + K(s) \frac{1}{s}} = \frac{K(s)}{s + K(s)}. \quad (8)$$

It is complementary sensitivity function related to sensitivity transfer function  $S(s)$  by

$$S(s) = 1 - T_s(s) = \frac{s}{s + K(s)}. \quad (9)$$

Given the representation in Eqs. (6) and (7), we have

$$T_{\begin{bmatrix} u \\ n \end{bmatrix} \rightarrow \begin{bmatrix} \varepsilon^{ref} \\ \varepsilon \end{bmatrix}} = \begin{bmatrix} T_{u \rightarrow \varepsilon^{ref}} & T_{n \rightarrow \varepsilon^{ref}} \\ T_{u \rightarrow \varepsilon} & T_{n \rightarrow \varepsilon} \end{bmatrix} = (I_M \otimes T_s) \star H_0. \quad (10)$$

Let us define the following global transfer functions:

$$S^g(s) \triangleq T_{u \rightarrow \varepsilon^{ref}}(s), \quad T^g(s) \triangleq \frac{K(s)}{s} T_{n \rightarrow \varepsilon}(s), \quad (11)$$

$$KS^g(s) \triangleq K(s) T_{n \rightarrow \varepsilon}(s).$$

In [18], we proposed a method for control design,  $K(s)$ , that ensures the global stability and performance expressed in terms of frequency-dependent constraints on transfer functions,  $S^g$ ,  $T^g$  and  $KS^g$ . However, as pointed out before, the control law could be designed only if the interconnection topology, i.e.,  $P$  and  $B$ , is known in advance. In this paper, we propose an extension of the method in [18] to a set of possible interconnection topologies defined in terms of maximum singular value on interconnection matrix,  $P$ :

$$\bar{\sigma}(P) \leq \gamma < 1. \quad (12)$$

Clearly, the convergence requirement for DILOC-CT, i.e.,  $\rho(P) < 1$ , is implied by the above inequality, see Section II-A. However, the conditions for the existence of a dynamic controller that ensures the global stability

and performance for all interconnections satisfying Eq. (12) are potentially more restrictive than only for one given interconnection,  $P$ . For this reason, we propose a reduction that is justified by the following Theorem.

*Theorem 1: Consider DILOC-CT described in Fig. 2. If the local dynamic controller,  $K(s)$ :*

- ensures the global system stability,
- does not contain pure derivative actions, i.e.,  $K(0) \neq 0$ ,

*then it ensures global tracking performance, i.e., DILOC-CT converges to the true sensor locations,  $x^*$ :*

$$\lim_{t \rightarrow \infty} x(t) = (I - P)^{-1} B u = x^*. \quad (13)$$

*Proof:* Note that in Laplace domain we have:

$$x(s) = (I - T_s(s)P)^{-1} T_s(s) B u(s).$$

Since the global system is stable and  $K(0) \neq 0$ , applying the Final-Value Theorem gives:

$$\begin{aligned} \lim_{t \rightarrow \infty} x(t) &= \lim_{s \rightarrow 0} s x(s) = \\ \lim_{s \rightarrow 0} \left( s \left( I - \left( 1 - \frac{s}{s + K(s)} \right) P \right)^{-1} \left( 1 - \frac{s}{s + K(s)} \right) B \frac{1}{s} u \right) \\ &= (I - P)^{-1} B u = x^*. \end{aligned}$$

■

Based on Theorem 1, and the fact that the speed of convergence is proportional to the static gain of controller, see [18], we can reduce the global performance transfer function by eliminating the inputs and outputs related to the tracking performance:  $u$  and  $\varepsilon^{ref}$ , in Eqs. (7) and (10). We thus obtain the following reduced global system description:

$$T_{n \rightarrow \varepsilon} = (I_M \otimes T_s) \star H_1, \quad (14)$$

where ‘ $\star$ ’ is the Redheffer product, with  $T_s$  defined by Eq. (8) and  $H_1$  by

$$H_1 = \left[ \begin{array}{c|c} P & I_M \\ \hline P - I_M & I_M \end{array} \right].$$

It allows us to formulate the following control design problem:

*Problem 1 (Control problem): Given the global system in Eq. (14), for all possible sensor interconnection topology such that,  $P$  satisfies Eq. (12), find the local controller,  $K(s)$ , such that the global system is stable and satisfies the following frequency constraints:*

$$\begin{aligned} |K(j\omega)| &\geq \Omega_K(\omega), & \text{in LF range,} \\ \bar{\sigma}(T^g(j\omega)) &\leq \Omega_T(\omega), & \text{in HF range,} \\ \bar{\sigma}(K S^g(j\omega)) &\leq \Omega_{KS}(\omega), & \text{in HF range.} \end{aligned} \quad (15)$$

*Furthermore, each sensor should be able to solve the problem locally, i.e., the computation of  $K$  is to be performed at each sensor:*

We now briefly explain the frequency constraints. The first constraint,  $\Omega_K$ , ensures zero steady-state error ( $K(0) \neq 0$ ) and provides a handle on the speed of convergence for each sensor to its true location. The second and third

constraints,  $\Omega_T$  and  $\Omega_{KS}$ , impose a maximum bandwidth on  $T^g$  and a maximum gain on  $KS^g$ , which, in turn, limits the disturbance amplification by each sensor in high-frequency range. Specifics on these constraints are tabulated in Table I and are further elaborated in Section V.

TABLE I  
DILOC-CT DESIGN SPECIFICATIONS

Performance specs.	Frequency constraint	Range
Zero steady-state	Global stability and $ K(j\omega)  \geq K_0$	LF
Conv. speed		
Disturbance reduction	$-60$ dB/dec slope for $\bar{\sigma}(T^g(j\omega))$	HF
	$-40$ dB/dec slope for $\bar{\sigma}(KS^g(j\omega))$	

It is important to note, that the proposed result is general enough to cover the case of not reduced problem, i.e., imposing the frequency constraints for all transfer functions,  $S^g$ ,  $KS^g$  and  $T^g$ , as it is proposed in [18]. However, to reduce the computational load such that the proposed design method can be easily implemented at each sensor, the reduced version of the Problem 1 is solved in section V.

#### IV. MAIN RESULTS

We now describe the main results of this paper. Before we start, in subsection IV-A, we recap the input-output approach and present its main Theorem: Theorem of graph separation which can be used to solve the problem under consideration for one given interconnection topology. Then, in subsection IV-B, we extend the result to a more general case where the interconnection topology is not fixed but supposed to belong to a bounded set. Finally, in subsection IV-C, we specialize the obtained result to the case of sensor localization under consideration.

##### A. Input-output Approach

We now describe the input-output approach used to solve Problem 1. We use the concept of dissipativity taken from [19]–[22], [25], a simplified version of which is defined below.

*Definition 1 (Dissipativity):* An LTI, stable, and causal operator,  $H$ , is **strictly**  $\{X, Y, Z\}$ -**dissipative**, where  $X = X^T, Y, Z = Z^T$ , are real matrices such that

$$\begin{bmatrix} X & Y \\ Y^T & Z \end{bmatrix}$$

is full-rank; if  $\exists \varepsilon > 0$  such that for almost all  $\omega > 0$

$$\begin{bmatrix} I \\ H(j\omega) \end{bmatrix}^* \begin{bmatrix} X & Y \\ Y^T & Z \end{bmatrix} \begin{bmatrix} I \\ H(j\omega) \end{bmatrix} \leq -\varepsilon I. \quad (16)$$

If the inequality in Eq. (16) is satisfied with  $\varepsilon = 0$ , the operator is said to be  $\{X, Y, Z\}$ -**dissipative**.

Based on the dissipativity characterization of two interconnected systems,  $\mathcal{T}$  and  $\mathcal{H}$ , the following result could be obtained ensuring the stability and performance of the interconnection  $\mathcal{T} \star \mathcal{H}$ .



*Theorem 2:* Given  $\eta > 0$ , a stable LTI system,  $\mathcal{H}$ , interconnected with an LTI system,  $\mathcal{T}$ , and real matrices,  $\mathcal{X} = \mathcal{X}^T \geq 0$ ,  $\mathcal{Y}$ ,  $\mathcal{Z} = \mathcal{Z}^T \leq 0$  of appropriate dimensions, if

(i)  $\mathcal{H}$  is  $\{\mathbf{diag}(\mathcal{X}, -\eta^2 I), \mathbf{diag}(\mathcal{Y}, 0), \mathbf{diag}(\mathcal{Z}, I)\}$ -dissipative, and

(ii)  $\mathcal{T}$  is strictly  $\{-\mathcal{Z}, -\mathcal{Y}^T, -\mathcal{X}\}$ -dissipative,

then the global system,  $\mathcal{T} \star \mathcal{H}$ , is stable and

$$\|\mathcal{T} \star \mathcal{H}\|_\infty \leq \eta. \quad (17)$$

*Proof:* The proof of Theorem 2 can be found in [21], [22], and it relies on a version of the graph-separation theorem used in [19] for the stability and an S-procedure, [26], for the performance. It can also be seen as a generalization of the Kalman-Yakubovich-Popov lemma, [27]. ■

*Remark 1:* We note that for a given interconnection topology,  $P$ , Problem 1 can be solved by applying Theorem 2 with  $\mathcal{H} = H_1$ ,  $\mathcal{T} = (I \otimes T_s)$  and dissipativity matrices chosen as  $\mathcal{X} = Q \otimes X$ ,  $\mathcal{Y} = Q \otimes Y$ ,  $\mathcal{Z} = Q \otimes Z$  with a positive-definite symmetric matrix,  $Q \in \mathbb{R}^{M \times M}$ , see [18], [21] for more details. However, for this purpose, the knowledge of interconnection matrix,  $H_1$ , is necessary. In turn, the knowledge of all barycentric coordinates,  $P$ , to compute the local controller may be restrictive from an application point of view. Indeed, if the barycentric coordinates are known in advance, the position of each sensor could be easily computed at some central computer and then transmitted to the sensors. In order to avoid such knowledge, the condition (i) of Theorem 2 should be satisfied not only for one given interconnection  $\mathcal{H}$  but for a set of all possible interconnections. In this paper, we present this extension based on the idea of representing the interconnection matrix,  $P$ , as an uncertainty and exploiting the fact that it is bounded in terms of its spectral radius,  $\rho(P) \leq \gamma < 1$ , which implies Eq. (12).

### B. Global Performance for a set of Topologies

One approach to solve the Problem 1 could be to derive a similar constraint on the augmented interconnection matrix  $H_1$  based on the constraint (12), see Eq. (14):

$$\bar{\sigma}(P) \leq \gamma < 1 \Rightarrow \bar{\sigma}(H_1) \leq \tilde{\gamma} < 1 \quad (18)$$

Depending on the performance under consideration, the last condition could be very restrictive and actually never be applied in practice. For this reason, in this paper we propose to transform the system Eq. (14) in the form of an LFT (Linear Fractional Transform) interconnection of local subsystem dynamics,  $T_s$ , and one or several (repeated) interconnection matrices,  $P$ . Such a transformation is general enough to cover a large set of performance specifications. It could be easily performed in the case of sensor localization problem under consideration as shown later. In this case, it is possible to derive less restrictive global performance conditions based on the constraint in Eq. (12), instead of Eq. (18). Let us thus consider the following general description of the transformed global system:

$$p = \mathbf{diag}(I_M \otimes T_s, I_m \otimes \bar{P})(q),$$

$$\begin{bmatrix} q \\ z \end{bmatrix} = H \left( \begin{bmatrix} p \\ w \end{bmatrix} \right), \quad (19)$$

where

$$H = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix},$$

is a finite-dimensional, stable LTI system,  $w(t)$  is the input vector of dimension  $n_w$ ,  $z(t)$  is the output vector of dimension  $n_z$ , and  $q(t), p(t)$ , are internal signals of dimensions  $n_g$ . The local subsystem is  $T_s = G \star K$  with  $n_l$  inputs and outputs<sup>1</sup>. We note that  $n_g = (1 + m)n_l \times M$ .

Please also note that, to cover general result, the systems under consideration in Eq.(19) could be Multi-Input Multi-Output (MIMO) if  $n_l > 1$ . In this case, interconnection matrix  $\bar{P} = P \otimes I_{n_l}$  and its maximum singular value respect the same condition as  $P$ , i.e., Eq. (12).

The global transfer function between external input,  $w$ , and output,  $z$ , is

$$T_{w \rightarrow z} = \mathbf{diag}(I_M \otimes T_s, I_m \otimes \bar{P}) \star H,$$

and its  $H_\infty$ -norm is ensured by the local controller,  $K$ , by the following theorem which provides the main result of this paper.

*Theorem 3: Given  $\eta > 0$  and  $\gamma > 0$ , a stable LTI system,  $H$ , a local plant,  $G$ , and real matrices,  $X = X^T \geq 0$ ,  $Y$ ,  $Z = Z^T$ , if there exist*

*(i) positive-definite matrices  $Q$  and  $D$  of appropriate dimensions, such that  $H$  is  $\{\mathbf{diag}(Q \otimes X, -D \otimes I, -\eta^2 I), \mathbf{diag}(Q \otimes Y, 0, 0), \mathbf{diag}(Q \otimes Z, 0, 0)\}$ -dissipative, and*

*(ii) a local controller,  $K$ , such that  $T_s = G \star K$  is strictly  $\{-Z, -Y^T, -X\}$ -dissipative,*

*then the local controller,  $K$ , ensures that the global system,  $(I \otimes T_s, I \otimes \bar{P}) \star H$ , is stable and*

$$\| (I \otimes T_s, I \otimes \bar{P}) \star H \|_\infty \leq \eta$$

*for all interconnection matrices  $\bar{P}$  satisfying*

$$\bar{\sigma}(\bar{P}) \leq \gamma. \quad (20)$$

*Proof:* The proof is based on Theorem 2. Indeed, the first condition of Theorem 2 is ensured by the first condition of Theorem 3 with  $\mathcal{H} = H$  and

$$\begin{aligned} \mathcal{X} &= \mathbf{diag}(Q \otimes X, -D \otimes I_{n_l M}); \\ \mathcal{Y} &= \mathbf{diag}(Q \otimes Y, 0); \\ \mathcal{Z} &= \mathbf{diag}(Q \otimes Z, \gamma^2 D \otimes I_{n_l M}); \end{aligned} \quad (21)$$

<sup>1</sup>For the ease of notation and simplicity of presentation, the square local system case is presented here. It is, however, possible to consider the case of not square local systems,  $T_s$ , with different number of inputs and outputs with appropriate adjustments on the underlying system matrices.

with symmetric positive definite matrices  $Q \in \mathbb{R}^{M \times M}$  and  $D \in \mathbb{R}^{m \times m}$  which can be decomposed as follows  $Q = L_Q^T L_Q$  and  $D = L_D^T L_D$  where  $L_Q \in \mathbb{R}^{M \times M}$  and  $L_D \in \mathbb{R}^{m \times m}$  are full invertible matrices.

The constraint in Eq. (20) implies the following constraint for the operator with  $m$ -times repeated matrix,  $\bar{P}$ :

$$\begin{bmatrix} * \end{bmatrix}^T \begin{bmatrix} -I_m \otimes I_{n_l M} & 0 \\ 0 & \gamma^2 I_m \otimes I_{n_l M} \\ I_m \otimes \bar{P} \\ I_m \otimes I_{n_l M} \end{bmatrix} \geq \varepsilon I.$$

Pre- and post-multiplying the last equation by  $L_D^T \otimes I_{n_l M}$  and  $L_D \otimes I_{n_l M}$  respectively, and using the Kronecker product permutation property, [28], gives:

$$\begin{bmatrix} * \end{bmatrix}^T \begin{bmatrix} -D \otimes I_{n_l M} & 0 \\ 0 & \gamma^2 D \otimes I_{n_l M} \end{bmatrix} \begin{bmatrix} I_m \otimes \bar{P} \\ I_{m n_l M} \end{bmatrix} \geq \varepsilon I. \quad (22)$$

Similarly, for the repeated matrix,  $\bar{P}$ , condition (ii) of Theorem 3 imposes the following condition on the block-diagonal system,  $I_M \otimes T_s$ , for a positive-definite matrix,  $Q \in \mathbb{R}^{M \times M}$ :

$$\begin{bmatrix} * \end{bmatrix}^T \begin{bmatrix} Q \otimes X & Q \otimes Y \\ Q \otimes Y^T & Q \otimes Z \end{bmatrix} \begin{bmatrix} I_M \otimes T_s \\ I_{n_l M} \end{bmatrix} \geq \varepsilon I. \quad (23)$$

Combining conditions (22) and (23) together ensures that  $\tilde{T} = \mathbf{diag}(I_M \otimes T_s, I_m \otimes \bar{P})$  is strictly  $\{-\mathcal{Z}, -\mathcal{Y}^T, -\mathcal{X}\}$ -dissipative, which is exactly the second condition of Theorem 2 with  $\mathcal{T} = \tilde{T}$  and the same  $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$  as defined in Eq. (21). ■

### C. Local Control for Global Performance of DILOC-CT

In the previous subsection, we presented the Theorem that allows to solve Problem 1 for all interconnection topologies that satisfy Eq. (12) if the global system is transformed in the LFT form of Eq. (19). In this subsection, we show how this transformation is made for DILOC-CT application under consideration in order to ensure the frequency depended bounds, Eq. (15) by decentralized control,  $K$ .

Note that  $H_1$  in Eq. (14) can be written in form of an LFT of one interconnection matrix,  $P$ , since:

$$H_1 = \begin{bmatrix} 0 & I_M \\ -I_M & I_M \end{bmatrix} + \begin{bmatrix} I_M \\ I_M \end{bmatrix} P \begin{bmatrix} I_M & 0 \end{bmatrix}.$$

Therefore, Eq. (14) can be equivalently written as:

$$\begin{aligned} \begin{bmatrix} x \\ x^p \end{bmatrix} &= \mathbf{diag}(I_M \otimes T_s, P) \left( \begin{bmatrix} r^n \\ r^p \end{bmatrix} \right), \\ \begin{bmatrix} r^n \\ r^p \\ \varepsilon \end{bmatrix} &= \underbrace{\begin{bmatrix} 0 & I_M & I_M \\ I_M & 0 & 0 \\ -I_M & I_M & I_M \end{bmatrix}}_{H_2} \left( \begin{bmatrix} x \\ x^p \\ n \end{bmatrix} \right), \end{aligned} \quad (24)$$

with  $x^p = Pr^p$ ,  $r^p = x$ , and matrix,  $H_2$ , independent of uncertain interconnection matrix,  $P$ .

The Problem 1 is now solved by the following theorem according to a similar argument as in Section IV-B.

*Theorem 4 (Control Design):* Given  $\eta > 0$  and  $\gamma > 0$ , the system described in Eq. (24), and real scalars,  $X \geq 0, Y, Z \leq 0$ , if there exists positive-definite matrix,  $Q \in \mathbb{R}^{M \times M}$  and a real scalar  $D > 0$ , such that

(i)  $H_2$  is  $\{\mathbf{diag}(XQ, -D, -\eta^2 I), \mathbf{diag}(YQ, 0, 0), \mathbf{diag}(ZQ, \gamma^2 D, I)\}$ -dissipative,

and a local controller,  $K$ , such that for  $S(s)$  defined by Eq. (9):

(ii)  $T_s$  is strictly  $\{-Z, -Y, -X\}$ -dissipative with  $\|\hat{T}(s)\|_\infty < 1$ , and  $\hat{T}_s(s) = (T(s) + \frac{Y}{X}) \sqrt{\frac{X^2}{Y^2 - XZ}}$ ;

(iii)  $|S(j\omega)| \leq \omega \Omega_K^{-1}(\omega)$ , in the LF range;

(iv)  $|K(j\omega)S(j\omega)| \leq \eta^{-1} \min\{\Omega_{KS}(\omega), \omega \Omega_T(\omega)\}$ , in the HF range;

then the local controller  $K$  solves the Problem 1 for all possible interconnection matrix  $P$  satisfying

$$\bar{\sigma}(P) \leq \gamma. \quad (25)$$

*Proof:* The system given in Eq. (24) is an LFT of  $M$  systems,  $T_s$  and one interconnection matrix,  $P$ , as in Eq. (19). Hence, by the application of Theorem 3 with  $n_l = m = 1$ ,  $H = H_2$ ,  $\bar{P} = P$  two first conditions of Theorem 4 imply that the global system is stable and

$$\|\mathbf{diag}(T_s I_M, P) \star H_2\|_\infty \leq \eta, \quad (26)$$

for all interconnections,  $P$ , satisfying Eq. (25). We note that based on Eq. (9), third condition of Theorem 4 implies that in LF range we have:

$$|S(j\omega)| = \left| \frac{j\omega}{j\omega + K(j\omega)} \right| \approx \omega |K(j\omega)|^{-1} \leq \omega \Omega_K^{-1}(\omega),$$

which implies the first constraint of Problem 1 and hence, together with global stability and Theorem 1, it ensures sensor localization, see Eq. (13). Based on  $H_\infty$  norm properties, Eq. (26) implies,  $\forall \omega$ :

$$\bar{\sigma}(T_{n \rightarrow \varepsilon}(j\omega)) \leq \eta. \quad (27)$$

Note that in the HF range:

$$|KS(j\omega)| = \frac{|K(j\omega)|}{|1 - \frac{j}{\omega} K(j\omega)|} \approx |K(j\omega)|.$$

Therefore, together with Eq. (11), the condition (iv) implies the second and third conditions of Problem 1.  $\blacksquare$

*Remark 2:* Conditions (ii)-(iv) of Theorem 3 are decentralized local conditions since they are only related to local subsystem dynamics. However, due to the condition (i), these local conditions imply the appropriate global system behavior: global stability and global system performance defined in Problem 1. In the scalar case, it is possible to find  $Q$ ,  $\eta^2$  and  $X, Y, Z$  dissipativity parameters (based on quasi-convex optimization) that, for a given  $\gamma$ , satisfy condition (i), and maximally relax condition (ii), see [21] for more details. These algorithms can be implemented locally at each sensor provided the information on the number of sensors,  $M$ , is available to each sensor. In this case, the controller design reduces to a local  $H_\infty$  design problem, i.e., find a controller,  $K$ , such that it satisfies the conditions (ii)-(iv). It is a standard  $H_\infty$  design problem and can be easily solved (see [29]) and implemented by each sensor for a computed  $\eta$  and frequency constraints fixed by  $\Omega_K, \Omega_T, \Omega_{KS}$ .

## V. SIMULATIONS

We consider a network of  $M = 20$  sensors and  $N = m + 1 = 3$  anchors in  $\mathbb{R}^2$ . The sensors lie in the convex hull (triangle) formed by the anchors, see Fig. 3 for a candidate deployment where the DILOC-CT trajectories, with arbitrary initial conditions marked by ‘ $\star$ ’, are shown for four randomly chosen sensors. The sensor deployment is assumed to be free as soon as each sensor is connected to three neighbors such that it lies in their convex hull and resulting network topology  $P$  is such that it respects constraint (12).

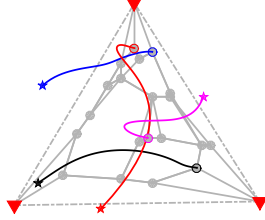


Fig. 3. DILOC-CT: One example of sensor network

First performance specification needed to be ensured by local controllers,  $K$ , is the sensor localization objective,  $\lim_{t \rightarrow \infty} x = x^*$ . Second is the rejection of the disturbance,  $n$ , each component of which is a realization of a band-limited noise with amplitude,  $A < 5$ , in frequency range,  $[\omega_z^-, \omega_z^+] = [600 \ 10^5]$  rad/sec. To achieve the specifications, we consider the frequency constraints,  $\Omega_K(\omega), \Omega_T(\omega), \Omega_{KS}(\omega)$ , see Problem 1 and Table I, shown as red dotted lines in Fig. 4.

Based on the proposed approach, see Theorem 4, each sensor is able to compute its local controller,  $K$ , that ensures the global performance specifications. The only information needed to be transmitted to the sensors is the total number of sensors,  $M = 20$ , in the network, the spectral bound for the network topology,  $\gamma = 0.99$ , and frequency constraints  $\Omega_K(\omega), \Omega_T(\omega), \Omega_{KS}(\omega)$ , see Remark 2. With this information, the quasi-convex optimization problem proposed in [21], [22] could be solved by each sensor, which for the global system description in Eq. (24), yields  $X = -3.33, Y = 1.17, Z = 1$  and the global transfer function bound,  $\eta = 9.7$ . The local controller,  $K(s)$ , is then computed using standard  $H_\infty$ -design, [29], to ensure the conditions (ii)-(iv) of Theorem 4:

$$K(s) = \frac{8.8 \cdot 10^8}{(s + 777.5)(s + 7508)}.$$

As expected, the local controller is a low-pass filter with high gain,  $K(0) \approx 150.9 \geq K_0$ , in the Low Frequency (LF) range, and the negative slope ( $-40$  dB/dec) in the HF range. According to the Theorem 4, the designed controller solves the Problem 1, i.e., it ensures the frequency constraints, Eq. (15), as can be verified in Fig. 4. It is important to note that even if the tracking performance specification was not directly imposed by the augmented global system under consideration, Eq.(24), since the global stability is ensured and the computed controller has no pure derivative actions,  $K(0) \neq 0$ , by Theorem 1, the steady-state localization error  $\varepsilon^{ref} = x^* - x \rightarrow 0$ .

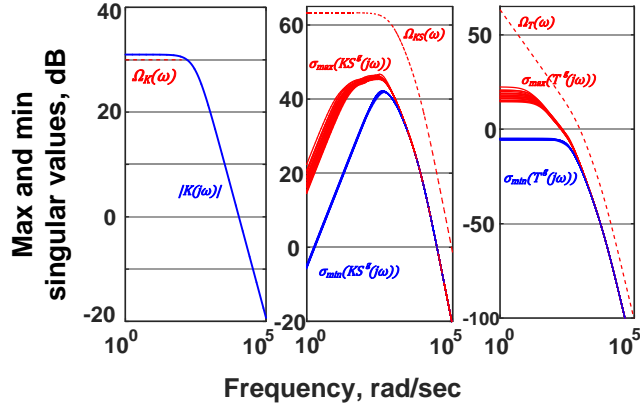


Fig. 4. Maximum (solid red line) and minimum (solid blue line) singular value of  $K$ ,  $KS^g$ , and  $T^g$  for different network topologies of 20 sensors, respecting constraint (12), together with corresponding frequency constraints (red dotted line).

This is confirmed by temporal simulations of sensor network for 10 randomly chosen interconnection topologies that respect the constraint in Eq. (12). The temporal evolution of the mean value of localization errors,  $\varepsilon_i^{ref}$ , and command signals,  $y_i$ 's, are presented in Fig. 5. To illustrate the interest of using the dynamic controller instead of a static one, Fig. 5 presents also the temporal simulations for the case of proportional gain controller  $K = \alpha = K_0$ . Even though it ensures the same tracking performance (speed of convergence) as the dynamic controller proposed in this paper, it can be seen that due to the frequency constraint imposed on global transfer functions,  $KS^g$  and  $T^g$ , the influence of the disturbance,  $n$ , at each sensor is significantly reduced.

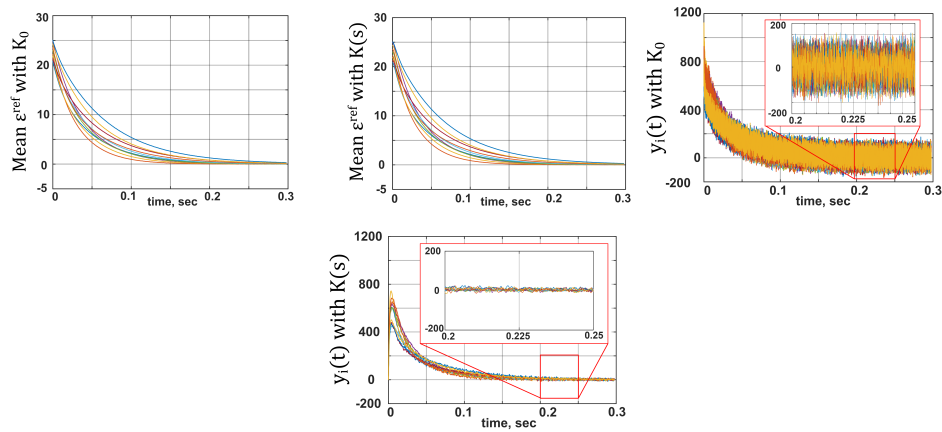


Fig. 5. Temporal simulations for 10 different topologies: (Left) Static controller,  $\alpha = K_0$ ; (Right): Dynamic controller,  $K(s)$ . Note the high values of the control command,  $y_i(t)$ 's, with the static controller, that could overexcite the local system, integrator, at each sensor.

## VI. CONCLUSIONS

In this paper, we consider a continuous-time, LTI algorithm to solve the sensor localization problem in  $\mathbb{R}^m$  with at least  $m + 1$  anchors which know their locations. Towards a more practical scenario, we consider the information exchange to incur a zero-mean additive disturbance whose frequency spectrum lies in a certain HF range. To maintain certain performance objectives while reducing the impact of the disturbance, we design a dynamic controller at each sensor with frequency-dependent performance objectives using the  $H_\infty$  theory. This controller is computed with local information and relies on a spectral bound on the interconnection matrices. We show that this dynamic controller does not only ensure global system stability but is also able to meet certain performance objectives embedded in frequency-dependent constraints.

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