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TIMELIKE HILBERT AND FUNK GEOMETRIES

ATHANASE PAPADOPOULOS AND SUMIO YAMADA

ABSTRACT. A timelike metric space is a Hausdorff topological space equipped with a partial order relation and a distance function satisfying a set of axioms including certain compatibility conditions between these two objects. The distance function is defined only on a certain subset (whose definition uses the partial order) of the product of the space with itself containing the diagonal. Distances between triples of points, whenever they are defined, satisfy the so-called *time inequality*, which is a reversed triangle inequality. In the 1960s, Herbert Busemann developed an axiomatic theory of timelike metric spaces and of locally timelike metric spaces. His motivation comes from the geometry underlying the theory of relativity and the classical example he gives is the n -dimensional Lorentzian spaces. Two other interesting classes of examples of timelike metric spaces introduced by Busemann are the timelike analogues of the Funk and Hilbert geometries. In this paper, we investigate these geometries. We shall find new interactions among the Euclidean, affine, projective and spherical timelike geometries. In particular, the de Sitter metric is described as a special case of a timelike spherical Hilbert metric.

Keywords. — Timelike space, timelike Hilbert geometry, timelike Funk geometry, time inequality, convexity, metric geometry, Busemann geometry, Lorentzian geometry, relativity.

AMS classification. — 53C70, 53C22, 5CC10, 53C23, 53C50, 53C45.

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1. INTRODUCTION

A timelike metric space is a Hausdorff topological space R equipped with a partial order relation $<$ and a distance function ρ which plays the role of a metric. This metric is asymmetric in the sense that the distance from x to y is not necessarily equal to the the distance from y to x . More drastically, the distance from x to y

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may be defined whereas the distance from y to x is not defined. More precisely, the distance $\rho(x, y)$ is defined only for pairs $(x, y) \in R \times R$ satisfying $x \leq y$ (that is, either $x = y$ or $x < y$). This distance function satisfies $\rho(x, x) = 0$ for every x in R , $\rho(x, y) > 0$ for every x and y such that $x < y$, and the *time inequality*, namely, the reversed triangle inequality $\rho(x, y) + \rho(y, z) \leq \rho(x, z)$ for all triples of points x, y, z satisfying $x < y < z$. The distance function ρ and the order relation $<$ satisfy a set of axioms including compatibility conditions with respect to each other. For instance, it is required that every neighborhood of a point q in R contains points x and y satisfying $x < q < y$. This axiom and others are stated precisely in the memoir [4] by Herbert Busemann.

The theories of timelike metric spaces, timelike G -spaces, locally timelike metric spaces and locally timelike G -spaces initiated by Busemann generalize the geometric theories of metric spaces and of G -spaces that he developed in his book [2] and in other papers and monographs. The motivation for the study of timelike metrics comes from the geometry underlying the theory of relativity. The classical example is the n -dimensional Lorentzian space. As other interesting examples of timelike metric spaces, Busemann introduced timelike analogues of the Funk and Hilbert geometries. In the present paper, we investigate these geometries. We establish several results concerning their geodesics, their convexity properties and their infinitesimal structure. We show in particular that they are Finsler metrics in an appropriate sense (we call them timelike Finsler metrics). We also give a description of the de Sitter space as a special case of a more general timelike Hilbert geometry in a spherical geometry setting.

2. THE TIMELIKE FUNK METRIC, ITS GEODESICS AND ITS VARIATIONAL FORMULATION

We first introduce some preliminary notions and we establish some basic facts. With few exceptions, we shall use Busemann's notation in [4], and we first recall it.

Let K be a convex hypersurface in \mathbb{R}^n , that is, the boundary of an open (possibly unbounded) convex set $I \subset \mathbb{R}^n$. For any hypersurface K which is not a plane, there is a well-defined associated open convex set I . This is the convex connected components of $\mathbb{R}^n \setminus K$. If K is a hyperplane, the connected components of $\mathbb{R}^n \setminus K$ are both convex, and in this case we make a choice of one of the two connected components, that is, a half-space bounded by the hyperplane K . We call the set I associated to K the *interior* of K . We denote the closure $K \cup I$ of I by K° .

Let \mathcal{P} be the set of supporting hyperplanes of K , that is, the hyperplanes π having nonempty intersection with K and such that the open convex set I is contained in one of the two connected components of $\mathbb{R}^n \setminus \pi$. For each $\pi \in \mathcal{P}$, we let H_π^+ be the open half-space bounded by the hyperplane π and containing I , and H_π the open half-space bounded by π and not containing I . We have:

$$I = \bigcap_{\pi \in \mathcal{P}} H_\pi^+.$$

Furthermore, if $\Omega = \mathbb{R}^n \setminus K^\circ$ is the complement of K° in \mathbb{R}^n , we have

$$\Omega = \bigcup_{\pi \in \mathcal{P}} H_\pi.$$

We now introduce a partial order relation on Ω . For a point p in Ω , we let $\tilde{C}(p) \subset \Omega$ be the union of all the rays originating at p and intersecting K . Clearly $\tilde{C}(p)$ is convex and has nontrivial interior. Depending on the shape of K and the relative position of p and K , $\tilde{C}(p) \cap \Omega$ may or may not be closed in Ω . We denote by $C(p)$ the set of interior points of $\tilde{C}(p)$. For any two distinct points p and q in Ω , we write

$$p < q \text{ if } q \in C(p),$$

and whenever this occurs, we say that q lies in the future of p . We also say that p lies in the past of q (see Figure 1). We write $p \leq q$ if either $p < q$ or $p = q$. Geometrically speaking, when $p < q$, the ray $R(p, q)$ from p through q intersects the hypersurface K transversally.

We denote by $\Omega_{<}$ (resp. Ω_{\leq}) the set of ordered pairs (p, q) in $\Omega \times \Omega$ satisfying $p < q$ (resp. $p \leq q$). Since the set $C(p)$ is open for each p , the set $\Omega_{<}$ is open in the product space $\Omega \times \Omega$. It is disjoint from the diagonal set $\{(x, x) \mid x \in \Omega\} \subset \Omega \times \Omega$.

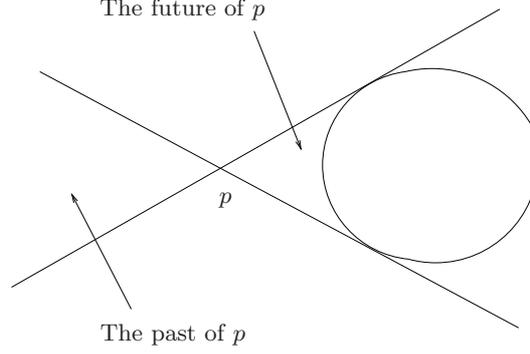


FIGURE 1.

Finally, for any a point p in \mathbb{R}^n , we denote by $\mathcal{P}(p)$ the set of supporting hyperplanes $\pi \in \mathcal{P}$ that separate K from p . In other words, the set $\mathcal{P}(p)$ is characterized by

$$(1) \quad \mathcal{P}(p) = \{\pi \in \mathcal{P} \mid p \in H_\pi\}.$$

We note that a supporting hyperplane π which contains p and is tangential to K does not belong to the set $\mathcal{P}(p)$.

We have the following:

Proposition 2.1. *For any two points p and q in Ω , we have*

$$p < q \Leftrightarrow \mathcal{P}(p) \supset \mathcal{P}(q).$$

Proof. First suppose $p < q$. We claim that every $\pi \in \mathcal{P}(q)$ is an element of $\mathcal{P}(p)$. Indeed, if this does not hold, then there exists $\pi \in \mathcal{P}(q)$ such that $p \in H_\pi^+$. Therefore q lies in H_π by definition, p lies in H_π^+ , and thus p and q lie across π from each other. In particular the ray $R(p, q)$ intersects K transversally on the H_π side, which contradicts the fact that $K \subset H_\pi^+$.

As for the converse, we argue by contraposition: if the ray $R(p, q)$ does not intersect K transversally (hence $p \not< q$) then there exists some π which lies in $\mathcal{P}(q)$ but not in $\mathcal{P}(p)$. Such a supporting hyperplane $\pi \in \mathcal{P}(q)$ intersects the interior of the segment $[p, q]$. By definition, q lies on the other side of π from K . It follows that p lies on the same side of K with respect to π . Hence the hyperplane π is an element of $\mathcal{P}(q) \setminus \mathcal{P}(p)$, namely $\mathcal{P}(p) \not\supset \mathcal{P}(q)$. □

Corollary 2.2. *Let p, q, r be three points in Ω . If $p < q$ and $q < r$, then $p < r$.*

Proof. The proof follows from Proposition 2.1, since it gives:

$$p < q \text{ and } q < r \Leftrightarrow \mathcal{P}(p) \supset \mathcal{P}(q) \supset \mathcal{P}(r).$$

□

Now we can define the timelike Funk metric $F(p, q)$ on the subset Ω_{\leq} of $\Omega \times \Omega$.

Definition 2.3 (The timelike Funk metric). The function $F(p, q)$ on pairs of distinct points p, q in Ω satisfying $p < q$ is given by the formula

$$F(p, q) = \log \frac{d(p, b(p, q))}{d(q, b(p, q))}$$

where $b(p, q)$ is the point where the ray $R(p, q)$ first hits the convex hypersurface K and $d(\cdot, \cdot)$ denotes the Euclidean distance. Note that $b(p, q) \in K$ exists since $p < q$. As the ray $R(p, q)$ intersects K transversally, we have the strict inequality $d(p, b(p, q)) > d(q, b(p, q))$, hence the value of $F(p, q)$ is strictly positive. For $p = q$, we extend the definition by setting $F(p, q) = 0$.

We remark at this point that the metric space together with the partial order relation we have defined satisfies the three axioms T_1, T'_2 and T'_3 of Busemann's [4]. (But to for the purpose of the present paper, the reader need not go through the general axiomatics of Busemann since we are only concerned here with specific examples.)

Let π_0 be a supporting hyperplane to K at $b(p, q)$, namely $\pi_0 \in \mathcal{P}(b(p, q))$. For p in \mathbb{R}^d , let $\Pi_{\pi_0}(p)$ be the foot of the point p on the hyperplane π_0 . In other words, $\Pi_{\pi_0} : \mathbb{R}^d \rightarrow \pi_0$ is the nearest point projection map. From the similarity of the triangles $\triangle(p, \Pi_{\pi_0}(p), b(p, q))$ and $\triangle(q, \Pi_{\pi_0}(q), b(p, q))$, we have

$$\log \frac{d(p, b(p, q))}{d(q, b(p, q))} = \log \frac{d(p, \pi_0)}{d(q, \pi_0)}.$$

Using the convexity of K , the quantity $F(p, q)$ can be characterized variationally as follows. For any unit vector ξ in \mathbb{R}^n and for any $\pi \in \mathcal{P}$, define $T(p, \xi, \pi)$ by $\pi \cap \{p + t\xi \mid t > 0\}$. For $p \neq q$ in \mathbb{R}^n , consider the vector $\xi = \xi_{pq} = \frac{q-p}{\|q-p\|}$ where the norm is the Euclidean one. When the hyperplane supports Ω at $b(p, q)$, we have $T(p, \xi_{pq}, \pi) = b(p, q)$ and otherwise the point $T(p, \xi_{pq}, \pi)$ lies outside Ω . When $\pi \in \mathcal{P}$ supports the convex set at some point other than p , again by the similarity of the triangles $\triangle(p, F_\pi(p), T(p, \xi_{pq}, \pi))$ and $\triangle(q, F_\pi(q), T(\xi_{pq}, \pi))$, we get

$$\frac{d(p, \pi)}{d(q, \pi)} = \frac{d(p, T(p, \xi_{pq}, \pi))}{d(q, T(p, \xi_{pq}, \pi))}.$$

Note that the furthest point from p on the ray $R(p, q) = \{p + t\xi_{pq} \mid t > 0\}$ of the form $T(p, \xi_{pq}, \pi)$ is $b(p, q)$. This in turn says that a hyperplane $\pi_{b(p, q)}$ which supports Ω at $b(p, q)$ minimizes the ratio $d(p, T(p, \xi_{pq}, \pi))/d(q, T(p, \xi_{pq}, \pi))$ among all the elements of \mathcal{P} ;

$$(2) \quad \log \frac{d(p, \pi_0)}{d(q, \pi_0)} = \inf_{\pi \in \mathcal{P}} \log \frac{d(p, \pi)}{d(q, \pi)}.$$

Notice that there is an analogous formula for the classical (non-timelike) Funk metric, where the infimum in the above formula is replaced by a supremum, see [14] Theorem 1.

As $\pi_0 \in \mathcal{P}(q) \subset \mathcal{P}(p) \subset \mathcal{P}$ (Proposition 2.1), we have the following identification:

$$(3) \quad F(p, q) = \inf_{\pi \in \mathcal{P}(q)} \log \frac{d(p, \pi)}{d(q, \pi)}.$$

Remark 2.4. The set of future points of a point p , that is, the set of points q satisfying $p < q$, has the structure of a cone $\tilde{C}(p)$ with nonempty interior $C(p)$. This reminds us of the Lorentzian space, which is the geometric setting of spacetime for the theory of relativity, where the cone of future points of p is the ambient space of the physically possible trajectories of this point. The restriction of the distance function to the cone comes from the fact that a material particle travels at a speed

which is less than the speed of light. Thus, the set of points on the rays starting at p that are on the boundary $\tilde{C}(p) \setminus C(p)$ of the solid cone $\tilde{C}(p)$ becomes an analogue of the “light cone” of spacetime (again using the language of relativity). However, in the case of Lorentzian geometry, the distance between p and a point on boundary of the light cone is zero, whereas in our setting it is undefined (recall that the distance between two points p is defined whenever we have $p < q$). In conclusion, there are analogies between Busemann’s timelike theory and the geometric setting of relativity, but one has to be careful about several details.

Remark 2.5. Equation (3) may be used as a definition of the Funk metric $F(p, q)$. One advantage would be that in the case where K is unbounded, it allows us to naturally extend the domain of definition for F so that the light cone of a point p is defined to be the set of points $q \neq p$ satisfying $F(p, q) = 0$.

We prove that the function $F(p, q)$ satisfies the reverse triangle inequality, which we call in this context, after Busemann, the *time inequality*. This inequality holds for mutually distinct points p, q and r in Ω , satisfying $p < q < r$:

Proposition 2.6 (Time inequality). *For any three points p, q and r in Ω , satisfying $p < q < r$, we have*

$$F(p, q) + F(q, r) \leq F(p, r).$$

Proof. We use the formula (3) for the timelike Funk distance. We have, from $\mathcal{P}(q) \supset \mathcal{P}(r)$ (Proposition 2.1):

$$\begin{aligned} F(p, q) + F(q, r) &= \inf_{\pi \in \mathcal{P}(q)} \log \frac{d(p, \pi)}{d(q, \pi)} + \inf_{\pi \in \mathcal{P}(r)} \log \frac{d(q, \pi)}{d(r, \pi)} \\ &\leq \inf_{\pi \in \mathcal{P}(r)} \log \frac{d(p, \pi)}{d(q, \pi)} + \inf_{\pi \in \mathcal{P}(r)} \log \frac{d(q, \pi)}{d(r, \pi)} \\ &\leq \inf_{\pi \in \mathcal{P}(r)} \left(\log \frac{d(p, \pi)}{d(q, \pi)} + \log \frac{d(q, \pi)}{d(r, \pi)} \right) \\ &= \inf_{\pi \in \mathcal{P}(r)} \log \frac{d(p, \pi)}{d(r, \pi)} \\ &= F(p, r). \end{aligned}$$

□

In the rest of this section, we study geodesics and spheres in timelike Funk geometries. We shall refer to the paper [9] for the corresponding results in the classical Funk setting, and all the properties are motivated by Busemann’s work [2].

First we consider geodesics for the timelike Funk metric. We start with the definition of a geodesic. This definition is the same as in ordinary metric spaces, except that some care has to be taken so that the distances we need to deal with are always defined.

A *geodesic* is a path $\sigma : J \rightarrow \Omega$, where J may be an arbitrary interval of \mathbb{R} , such that for every pair $t_1 \leq t_2$ in J we have $\sigma(t_1) \leq \sigma(t_2)$ and for any triple $t_1 \leq t_2 \leq t_3$ in J we have

$$F(\sigma(t_1), \sigma(t_2)) + F(\sigma(t_2), \sigma(t_3)) = F(\sigma(t_1), \sigma(t_3)).$$

It follows easily from the definition that for any $p < q$ the Euclidean segment $[p, q]$ joining p to q is the image of a geodesic. This makes the distance function F “projective” in the sense of Hilbert’s Fourth Problem [8] if this problem is extended in an appropriate way to timelike spaces. In particular, the time inequality becomes an equality when p, q and r satisfying $p < q < r$ are collinear in the Euclidean sense.

Let us make an observation which concerns the non-uniqueness of geodesics and the case of equality in the time inequality. Assume that the boundary of the convex hypersurface K contains a Euclidean segment s . Take three points p, q, r in a small neighborhood \mathcal{O} of a point on s , such that the Euclidean distance of p, q, r to s taken in that order is decreasing (Figure 2). Then, using the Euclidean intercept

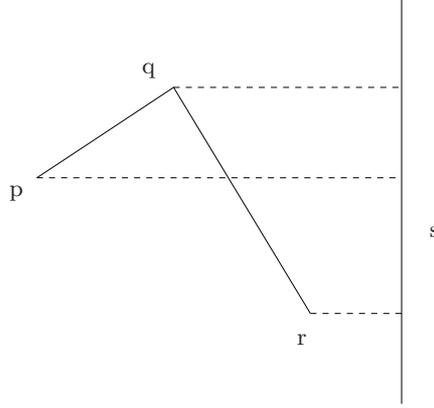


FIGURE 2. The broken segment pqr is a geodesic

theorem, we have

$$F(p, r) = F(p, q) + F(q, r).$$

Applying the same reasoning to an arbitrary ordered triple on the broken Euclidean segment $[p, q] \cup [q, r]$, we easily see that this segment is an F -geodesic. More generally, by the same argument, we see that *any* oriented arc in \mathcal{O} whose points get strictly closer to the segment s when we move along this arc, is the image of an F -geodesic.

In fact, the situation in the neighborhood \mathcal{O} is very special. It is a case where the timelike Funk and the classical Funk metrics coincide, distances being defined from a fixed point p to points in an open cone centered at this point. (Notice that for this to happen, it is necessary that the time and the triangle inequalities hold, therefore they must both be equalities.)

From the preceding remarks on Euclidean segments we can deduce the following:

Proposition 2.7. *A timelike Funk metric F defined on a set Ω_{\leq} associated to a convex hypersurface K in \mathbb{R}^n satisfies the following properties:*

- (1) *The Euclidean segments in Ω that are of the form $[p, q]$ where $p < q$ are F -geodesics.*
- (2) *Any Euclidean line $[p, b]$ from a point p in Ω to a point b in ∂K , equipped with the metric induced from the timelike Funk metric, is isometric to a Euclidean ray.*
- (3) *The Euclidean segments in (1) are the unique F -geodesic segments if and only if the convex set I is strictly convex.*

The proof is the same as that of the equivalence between (1) and (2) in Corollary 8.7 of [10], up to reversing some of the inequalities (i.e. replacing the triangle inequality by the time inequality).

After the geodesics, we consider spheres.

At each point p of Ω , given $r > 0$, a (*geometric*) *future sphere* of radius r is the set of points in Ω that are in the future of p and situated at distance r from this point.

Proposition 2.8 (Future pheres). *At each point p of Ω and for each $r > 0$, the geometric future sphere of center p and radius r is a piece of a convex hypersurface that is affinely equivalent to the portion $C(p) \cap K$, namely the part of the hypersurface K that is visible from p .*

The proof is analogous to that of Proposition 8.11 of [9], and we do not repeat it here.

This proposition implies that some affine properties of the hypersurface K are local invariants of the metric. One consequence is the following strong local rigidity theorem, which is also an analogue of a property satisfied by the classical Funk metric (cf. the concluding remarks of the paper [9]).

Corollary 2.9. *Let K and K' be two hypersurfaces in \mathbb{R}^n and Ω_{\leq} and Ω'_{\leq} the set of corresponding pairs of points for which the associated Funk metrics F and F' respectively are defined. If there exists subsets $\mathcal{O} \subset \Omega_{\leq}$ and $\mathcal{O}' \subset \Omega'_{\leq}$ and a map $\mathcal{O} \rightarrow \mathcal{O}'$ which is distance-preserving, then there is an open subset of K which is affinely equivalent to an open subset of K_2 .*

The proof follows from the fact that an isometry sends a future sphere to a future sphere.

The corollary implies for instance that if K_1 is the boundary of a polyhedron and K_2 a strictly convex hypersurface, then there is no local isometry between the associated timelike metric spaces.

The next statement is a simple observation in convex geometry, and it gives a monotonicity result for a pair of timelike Funk geometries.

Proposition 2.10. *Given as before a convex set I with associated Funk distance F , let $\hat{I} \supset I$ be another convex set and $\hat{F}(p, q)$ the associated timelike distance function defined between appropriate pairs (p, q) . Suppose that p and q are in the domain of definition of both distances F and \hat{F} , that is, $p < q$ with respect to the convex sets I and \hat{I} . Then we have*

$$\hat{F}(p, q) \geq F(p, q).$$

Proof. Using the notation of Definition 2.3, we have

$$F(p, q) = \log \frac{d(p, b(p, q))}{d(q, b(p, q))}.$$

With similar notation, we have

$$\hat{d}(p, q) = \log \frac{d(p, \hat{b}(p, q))}{d(q, \hat{b}(p, q))}.$$

Since $\hat{I} \supset I$, we have $F(p, \hat{b}(p, q)) = d(p, b(p, q)) + x$ and $d(q, \hat{b}(p, q)) = d(q, b(p, q)) + x$ for some $x \geq 0$. The result follows from the fact that the function defined for $x \geq 0$ by

$$x \mapsto \frac{a - x}{b - x},$$

where $b < a$ are two constants is increasing. \square

3. THE FINSLER STRUCTURE OF THE TIMELIKE FUNK METRIC

The timelike Funk metric associated to a convex hypersurface Ω in \mathbb{R}^n is Finsler in an appropriate sense which we describe now.

On the tangent space at each point of $\Omega = \mathbb{R}^n \setminus K^\circ$, there is a *timelike norm-function* which we also call a *(timelike) Minkowski functional*, in analogy with the Minkowski functional associated to a Finsler structure in the classical sense. We show that the timelike Funk distance $F(p, q)$ between two points p and q is obtained

by integrating this norm on tangent vectors along piecewise C^1 paths joining p to q and taking the supremum (instead of the infimum, in the classical case) of the lengths of such piecewise C^1 paths. We call such a structure a *timelike Finsler structure*. To make this more precise we start with the following definition (cf. [4] §5).

Definition 3.1 (Timelike Minkowski functional). A *timelike Minkowski functional* is a function f (which plays the role of a “norm” in this timelike setting) satisfying the following:

- (1) f is defined on an open convex cone $C \subset \mathbb{R}^n$ having the origin as apex;
- (2) $f(x) > 0$ for all x in C ;
- (3) $f(\lambda x) = \lambda f(x)$ for all x in C and $\lambda > 0$;
- (4) $f((1-t)x + ty) \geq (1-t)f(x) + tf(y)$ for all $0 < t < 1$.

The *unit sphere* B of such a norm function f is the set of vectors x in C satisfying $f(x) = 1$. In general, B is a piece of a hypersurface in \mathbb{R}^n which is concave when viewed from the origin O (see Figure 3). We allow the possibility that B is asymptotic to the boundary of the cone C . This surface is called the *indicatrix* of f .

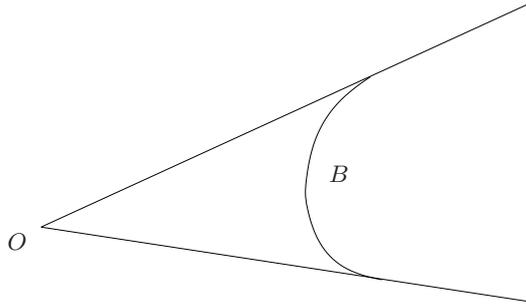


FIGURE 3. The indicatrix B is a piece of hypersurface which is concave, seen from the origin O .

The reason of the adjective *timelike* in the above definition is that in the Lorentzian setting, the Minkowski norm measures the lengths of vectors which are in the timelike cone, which is the part of spacetime where material particles move. In particular, there is a timelike Minkowski functional f for the standard Minkowski space $\mathbb{R}^{3,1}$, equipped with the Minkowski metric

$$ds^2 = -dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2.$$

It is given by

$$f(x) = \sqrt{-(-x_0^2 + x_1^2 + x_2^2 + x_3^2)}$$

and it is defined for non-zero vectors x in \mathbb{R}^4 satisfying $-x_0^2 + x_1^2 + x_2^2 + x_3^2 < 0$.

We now study the timelike Minkowski norms on tangent spaces of Funk geometries. For every point p in the timelike Funk geometry of a space Ω_{\leq} associated to a convex hypersurface K , there is a *timelike Minkowski functional* $P(p, v)$ defined on the tangent space $T_P(\Omega)$ of Ω at p consisting of the non-zero vectors v satisfying

$$p + tv \in C(p) \text{ for } t > 0$$

where we recall that $C(p)$ is the (nonempty) interior of the cone spanned by K with vertex p . We denote the set of such vectors v by $D(p) \subset T_p\Omega$. The function

$P(p, v)$ for a given point $p \in \Omega$ and for any vector $v \in D(p)$ is given by the following formula:

$$(4) \quad P(p, v) = \inf_{\pi \in \mathcal{P}(p)} \frac{\langle v, \eta_\pi \rangle}{d(p, \pi)}$$

for $v \in D(p)$ and $P(p, 0) = 0$ when $v = 0$, where $\mathcal{P}(p)$ is as in (1) above. By elementary geometric arguments (see [14] for a detailed discussion in the classical case which can be adapted to the present setting) it is shown that

$$(5) \quad P(p, v) = \frac{\|v\|}{\inf\{t \mid p + t \frac{v}{\|v\|} \in K\}}$$

for any nonzero vector $v \in D(p)$.

Note that the quantity $\inf\{t \mid p + t \frac{v}{\|v\|} \in K\}$ in the denominator is the length of the line segment from p to the point where the ray $p + tv$ hits the convex set K° for the first time. It is clear that

Proposition 3.2. *The functional $P(p, v)$ defined on the open cone $D(p)$ in $T_p\Omega \cong \mathbb{R}^n$ satisfies all the properties required by a timelike Minkowski functional.*

We say that a piecewise C^1 curve $t \mapsto \sigma(t)$ in Ω , defined on an interval J of \mathbb{R} , is *timelike* if at each time $t \in J$ the tangent vector $\sigma'(t)$ is an element of $D(\sigma(t)) \subset T_{\sigma(t)}\Omega$.

As the set $D(p)$ is a subset of an open cone bounded by a concave hypersurface $K \cap C(p)$, the sub-level set

$$s_K(p) = \{v \in D(p) \mid P(p, v) < K\}$$

is open in each tangent space $T_p\Omega$, implying that the Minkowski functional $P(p, v)$ is upper semi-continuous. Using this upper semi-continuity we define the *length* of a piecewise C^1 timelike curve $\sigma(t)$ by the Lebesgue integral

$$\text{Length}(\sigma) = \int_0^1 P(\sigma(t), \sigma'(t)) dt.$$

By standard arguments, again using the upper semi-continuity of P , it follows that the length functional is upper semi-continuous in the C^0 -topology on curves (for detailed treatments, see [13] or [6].)

We then define a “distance function” d for pairs of points (p, q) satisfying $p < q$ by setting

$$d(p, q) = \sup_{\sigma} \text{Length}(\sigma)$$

where the supremum is taken over all the timelike piecewise C^1 curves $\sigma : [0, 1] \rightarrow \Omega$ satisfying $\sigma(0) = p$ and $\sigma(1) = q$. It is easy to check that the distance function d is a timelike metric on the space Ω_{\leq} . This timelike metric is the analogue of the so-called *intrinsic metric* in the classical (non-timelike) case. We call d the *timelike intrinsic metric*. We shall prove that this distance function coincides with the Funk timelike metric. We state this as follows:

Theorem 3.3. *The value of the distance $d(p, q)$ for a pair $(p, q) \in \Omega_{\leq}$ coincides with $F(p, q)$. That is, we have*

$$F(p, q) = d(p, q).$$

In other words, we have the following

Theorem 3.4. *The timelike Funk metric is a timelike Finsler metric with timelike Finsler structure given by the Minkowski functional $P(p, v)$.*

The Minkowski functional $P(p, v)$ which underlies a Funk metric has a property which is analogous to the one noticed in [9] which makes this metric the *tautological Finsler structure* associated to the hypersurface K (or the convex body I). The term “tautological” is due to the fact that the indicatrix of the timelike Minkowski functional at $p \in \Omega$, that is the set

$$\text{Ind}(p) = \{v \in T_p\Omega \mid P(p, v) = 1\},$$

is naturally identified with the subset $K \cap C(p)$ of the hypersurface K . Recall that $C(p)$ is the open cone centered at p spanned by K .

We also note that with this identification, given a pair of points p, q with $p < q$, there always exists a distance-realizing (length-maximizing) geodesic from p to q , since the Euclidean segment $[p, q]$ is an F -geodesic. In the theory of relativity, the existence of geodesics is established by imposing the so-called global hyperbolicity ([13] or [6]) of the spacetime. The global hyperbolicity, roughly put, consists of two conditions: the non-existence of closed causal curves and the sequential compactness of geodesics in C^0 -topology. In our setting, the first condition is satisfied, while the second is not always guaranteed due to the lack of assumptions on the set $\tilde{C}(p) \setminus C(p)$.

Proof of Theorem 3.3. For a pair of points (p, q) with $p < q$, let $R(p, q)$ be the ray from p through q and let $b(p, q) \in K$ be the first intersection point of this ray with the convex set K° . Parameterize the Euclidean segment $[p, q]$ by a path $\tilde{\sigma}(t)$ having Euclidean arc-length parameter t with $\sigma(0) = p, \sigma(1) = q$. Then we have

$$\int_0^1 P(\tilde{\sigma}(t), \tilde{\sigma}'(t)) dt = \log \frac{d(p, b(p, q))}{d(q, b(p, q))} = F(p, q),$$

since

$$\frac{d}{dt} \log \frac{d(p, b(p, q))}{d(\tilde{\sigma}(t), b(p, q))} = P(\tilde{\sigma}(t), \tilde{\sigma}'(t)).$$

By taking the supremum over the set of paths from p to q , this implies the inequality

$$d(p, q) \geq F(p, q).$$

Before continuing the proof of Theorem 3.3, we show the following monotonicity for the intrinsic distance.

Lemma 3.5. *For an open convex set I bounded by K , consider the associated timelike Minkowski functional $P : T\Omega \rightarrow \mathbb{R}$ and the intrinsic distance d induced by P . Also let $\hat{I} \supset I$ be another open convex set bounded by \hat{K} , \hat{P} its associated timelike Minkowski functional, and \hat{d} the intrinsic distance. Suppose that p and q are in the domain of definition of both intrinsic distances d and \hat{d} . Then we have*

$$\hat{d}(p, q) \geq d(p, q).$$

Proof. The set of admissible paths from p to q for the convex set \hat{I} is larger than the set of paths for \hat{I} . This is due to the inclusion $\hat{C}(x) \supset C(x)$ for every x in Ω . Also between the two timelike Minkowski functionals P and \hat{P} , we have the following inequality

$$\hat{P}(x, v) \geq P(x, v)$$

whenever the two quantities are defined. This follows from the definition of the Minkowski functional:

$$(6) \quad P(p, v) = \frac{\|v\|}{\inf\{t \mid p + t\frac{v}{\|v\|} \in K\}}$$

for any nonzero vector $v \in D(p)$, as \widehat{K} is closer to p than K . Hence by integrating each functional along an admissible path and taking supremum over these paths, we have

$$\widehat{d}(p, q) \geq d(p, q)$$

□

Proof of Theorem 3.3 continued.— Suppose that we have a convex hypersurface K bounding an open convex set I , and for $(p, q) \in \Omega_<$, let $\widehat{I} := H_{\pi_{b(p,q)}}$, where $H_{\pi_{b(p,q)}}$ is the open half-space bounded by a hyperplane $\pi_{b(p,q)}$ supporting K at $b(p, q)$. The open set \widehat{I} comes with its intrinsic distance \widehat{d} . We now apply Lemma 3.5 to this setting where a convex set \widehat{I} contains I , and obtain $\widehat{d} \geq d$.

Now for the open half space $\widehat{I} = H_{\pi_{b(p,q)}}$, the values of $F(p, q)$, $\widehat{F}(p, q)$ and $d(p, q)$ all coincide, since under the hypothesis $\widehat{I} = H_{\pi_{b(p,q)}}$, the set \mathcal{P} of supporting hyperplanes consists of the single element $\pi_{b(p,q)}$, and the path $\tilde{\sigma}$ from p to q is a length-maximizing path. Also from the lemma above, we have $\widehat{d}(p, q) \geq d(p, q)$.

Hence we have

$$F(p, q) = \widehat{F}(p, q) = \widehat{d}(p, q) \geq d(p, q) \geq F(p, q)$$

and the equality $d(p, q) = F(p, q)$ follows. □

We end this section by the following convexity result for the timelike Funk distance:

Theorem 3.6. *Given a point x in Ω and $K > 0$, the set of points*

$$S_K(x) := \{p \in \Omega \mid p < x \text{ and } F(p, x) > K\}$$

is a convex set in $\Omega = \mathbb{R}^n \setminus K^\circ$.

Proof. Given p_1 and p_2 in $S_K(x)$, parameterize the Euclidean segment $[p_1, p_2]$ with an affine parameter $t \in [0, 1]$. We will show that the function

$$F(s(t), x) = \inf_{\pi \in \mathcal{P}} \log \frac{d(s(t), \pi)}{d(x, \pi)}$$

is convex in t . Fix a supporting hyperplane π in \mathcal{P} . Then

$$\frac{d}{dt} \log \frac{d(s(t), \pi)}{d(x, \pi)} = \frac{\langle -\nu_\pi(s(t)), \dot{s}(t) \rangle}{d(s(t), \pi)}$$

and

$$\frac{d^2}{dt^2} \log \frac{d(s(t), \pi)}{d(x, \pi)} = -\frac{\langle -\nu_\pi(s(t)), \dot{s}(t) \rangle^2}{d(s(t), \pi)^2} \leq 0,$$

where $\nu_\pi(x)$ is the unit vector at x perpendicular to the hypersurface π oriented toward π . In particular $-\nu_\pi$ is the gradient vector of the function $d(x, \pi)$. The sign of the second derivative says that $\log \frac{d(s(t), \pi)}{d(x, \pi)}$ is concave in t for each $\pi \in \mathcal{P}$. By taking the infimum over $\pi \in \mathcal{P}$, the resulting function $F(s(t), x)$ is concave in t .

This implies that the super-level set $S_K(x)$ of the Funk distance $F(\cdot, x)$ is convex. □

As an analogous situation in special relativity, the super-level set of the past-directed temporal distance measured from a fixed point in the Minkowski space $\mathbb{R}^{n,1}$ is convex. For example, the set below the past-directed hyperboloid: $S_1(0) = \{(x_0, x_1) \in \mathbb{R}^{1,1} \mid -x_0^2 + x_1^2 < -1, x_0 < 0\}$ is convex.

4. THE TIMELIKE HILBERT METRIC

We define the timelike Hilbert metric following the ideas of Busemann in [4], with a slight modification. For this, we consider two disjoint open, possibly unbounded, convex sets I_1 and I_2 in \mathbb{R}^n , bounded by convex hypersurfaces K_1 and K_2 respectively. Let $\Omega = \mathbb{R}^n \setminus (K_1^\circ \cup K_2^\circ)$ be the complement of the closures $K_1^\circ = K_1 \cup I_1$ and $K_2^\circ = K_2 \cup I_2$ in \mathbb{R}^n . For $i = 1, 2$, we denote by \mathcal{P}_i the set of supporting hyperplanes of I_i .

Given a point p in Ω , for $i = 1, 2$, we define $C_i(p)$ to be the set of interior points of the cone $\tilde{C}_i(p)$ spanned by K_i with apex at p .

We denote by R the set of points in Ω that are contained in the interior of Euclidean segments of the form $[a_1, a_2]$ joining a point a_1 in K_1 and a_2 in K_2 , with the open line segment (a_1, a_2) entirely contained in Ω .

We define a partial order relation $<$ on R by setting

$$p < q \Leftrightarrow q \in C_2(p) \text{ and } p \in C_1(q).$$

As usual, $p \leq q$, we mean either $p < q$ or $p = q$. We also introduce the subset $\Omega_{<}$ (resp. Ω_{\leq}) as the set of ordered pairs (p, q) in $\Omega \times \Omega$ satisfying $p < q$ (resp. $p \leq q$). Since the set $C_2(p)$ is open for each $p \in \Omega$, and the set $C_1(q)$ is open for each $q \in \Omega$, it follows that the set $\Omega_{<}$ is open in the product space $\Omega \times \Omega$. This set is disjoint from the diagonal set $\{(x, x) \mid x \in \Omega\} \subset \Omega \times \Omega$.

If $p < q$, then p and q are both in R , and there exists a pair of points $a_1 \in K_1$ and $a_2 \in K_2$, such that the open segment (a_1, a_2) is disjoint from K_1 and K_2 and the quadruple (a_1, p, q, a_2) are situated in that order on $[a_1, a_2]$ (cf. Figure 4). The converse does not hold; namely for a quadruple (a_1, p, q, a_2) aligned in that order on the segment $[a_1, a_2]$ with $a_1 \in K_1$ and $a_2 \in K_2$, when either $p \in \pi$ for some $\pi \in \mathcal{P}_2$ or $q \in \hat{\pi}$ for some $\hat{\pi} \in \mathcal{P}_1$, we have $p \not< q$.¹

Note that K_1 and K_2 do not play a symmetric role in this definition. We have $p < q$ but not $q < p$. We shall stress this fact when needed, talking for instance of an *oriented* Euclidean line joining a point in K_1 to a point in K_2 when needed, but we shall not emphasize this fact when it is understood.

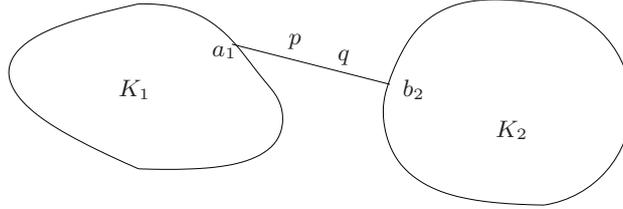


FIGURE 4.

Definition 4.1 (Timelike Hilbert metric). The timelike Hilbert metric on Ω is a distance function $H(p, q)$ defined on $\Omega_{\leq} \subset \Omega \times \Omega$ by the formula

$$H(p, q) = F_2(p, q) + F_1(q, p)$$

where for $i = 1, 2$, F_i is the timelike Funk metric associated to the convex hypersurfaces K_i .

The timelike Hilbert metric satisfies the time inequality. This follows from the definition of the timelike Hilbert metric as a sum of two timelike Funk metrics and the fact that the timelike Funk metric satisfies the time inequality. In particular,

¹In this sense, we differ from Busemann's treatment of timelike Hilbert metric in [4].

the metric space together with the partial order relation thus defined satisfies the three axioms T_1, T'_2 and T'_3 of Busemann's [4].

The timelike Hilbert metric satisfies additional properties which follow from those of the Funk metric. In particular, we have:

Proposition 4.2. (a) *In a timelike Hilbert metric H associated to two convex hypersurfaces K_1 and K_2 , the Euclidean segments of the form $[a_1, a_2]$ where $a_1 \in K_1$ and $a_2 \in K_2$, oriented from a_1 to a_2 such that $a_1 \notin \pi$ for every π in \mathcal{P}_2 and $a_2 \notin \hat{\pi}$ for every $\hat{\pi}$ in \mathcal{P}_1 , are H -geodesics and such that the open segment (a_1, a_2) is in the complement of $K_1 \cup K_2$. They are isometric to the real line.*

(b) *The oriented Euclidean segments contained in the segments of the form $[a_1, a_2]$ where $a_1 \in K_1$ and $a_2 \in K_2$ with $a_1 \notin \pi$ for every π in \mathcal{P}_2 and with $a_2 \notin \hat{\pi}$ for every $\hat{\pi}$ in \mathcal{P}_1 , and their subsegments are the unique H -geodesics if and only if there are no segments $[a_1, a_2]$ in Ω with a_1 in the interior of a segment J_1 in K_1 and a_2 in the interior of a segment J_2 in K_2 with J_1 and J_2 coplanar.*

The proof is an adaptation of that of the classical Hilbert metric (cf. [2] or [9]), and we omit it.

We show that the corresponding timelike Hilbert metric $H(p, q)$ introduced above is a Finsler metric, and we give its linear functional (the analogue of the Minkowski functional) defined on a subset of each of the tangent space.

Definition 4.3. Consider a point p in Ω so that $C_2(p) \cap -C_1(p)$ is nonempty (where $-C$ is the cone $\{-v \mid v \in C\}$.) Denote by $D(p)$ the set $C_2(p) \cap -C_1(p) \subset T_p\Omega$. Using the notation of §3 that concerns the infinitesimal Finsler metric associated to a timelike Funk metric, we define a linear functional on $D(p)$ by the formula:

$$(7) \quad P_H(p, v) = P_2(p, v) + P_1(p, -v).$$

where P_1 and P_2 are the timelike norms on tangent spaces associated to the Funk metrics associated to K_1 and K_2 respectively (see Equations (1) or (6)). Note that when v is an element of $D(p)$, v is in $C_2(p)$ and $-v$ is in $C_1(p)$.

Theorem 4.4. *The timelike Hilbert metric is a timelike Finsler metric with timelike Finsler structure given by the Minkowski functional P_H defined in (7).*

Proof. Let p and q be two points in the space $\Omega = \mathbb{R}^n \setminus (K_1^\circ \cup K_2^\circ)$ associated to the convex sets K_1 and K_2 such that (p, q) is in $\Omega_<$. In what follows, when we talk about a "Euclidean segment $[p, q]$ " joining p to q , we mean that the segment is oriented from p to q .

Recall that the Euclidean segment $[p, q]$ is an F_2 -geodesic, and the Euclidean segment $[q, p]$, is an F_1 -geodesic. Thus, we have

$$F_2(p, q) = \int_{[p, q]} P_2(x, v) dx$$

and

$$F_1(q, p) = \int_{[q, p]} P_1(x, v) dx.$$

Since the segment $[q, p]$ is the interval $[p, q]$ traversed in the opposite direction, we have

$$\int_{[q, p]} P_1(x, v) dx = \int_{[p, q]} P_1(x, -v) dx.$$

Thus, we have

$$(8) \quad H(p, q) = \int_{[p, q]} (P_2(x, v) + P_1(x, -v)) dx.$$

Furthermore, if $\gamma(p, q)$ is any path in the domain of definition of H joining p to q , then we have

$$\int_{\gamma} P_2(x, v) \leq \int_{[p, q]} P_2(x, v)$$

and

$$(9) \quad \int_{\gamma} P_1(x, -v) \leq \int_{[p, q]} P_1(x, -v).$$

Equations (8) and (9) show that H is timelike Finsler, with timelike Minkowski functional at each point x given by $P_H(p, v) = P_2(x, v) + P_1(x, -v)$. \square

The Finsler structure P_H is well-behaved in the sense that the linear functional

$$P_H(p, \cdot) : D(p) \rightarrow \mathbb{R}$$

is a timelike Minkowski functional (see §3.1) defined on the open cone $D(p) = C_2(p) \cap -C_1(p)$ in $T_p\Omega$. Since the sublevel set $s_K = \{v \in T_p\Omega : P_H(p, v) < K\}$ is open (this follows from the same argument used for the timelike Funk metric), P_H is upper semi-continuous in the tangent bundle $T\Omega$, making the length functional well-defined, and upper semi-continuous in C^0 topology of the path space.

4.1. Classical examples. We now point out some two-dimensional examples of timelike Hilbert metrics. Analogous higher-dimensional analogues hold.

Example 4.5 (The strip). Let Ω be a region contained by two parallel lines in the plane \mathbb{R}^2 , namely the complement of two half-spaces H_1, H_2 bounded by a pair of parallel hyperplanes π_1 and π_2 , which without loss of generality, are assumed to be $(-1, 1) \times \mathbb{R}$. Then any timelike curve is a geodesic for the timelike Hilbert metric. In this setting, a curve is timelike if at each point the tangent vectors are not vertical.

Consider the vertical projection $\Pi : \Omega \rightarrow (-1, 1)$. Then the Hilbert distance $H(x, y)$ for $x < y$ is equal to $H_{(-1, 1)}(\Pi(x), \Pi(y))$ where

$$H_{(-1, 1)}(a, b) = \log \frac{a-1}{b-1} \frac{b+1}{a+1}$$

is the Hilbert distance for the interval. This metric is sometimes called the ‘‘one-dimensional hyperbolic metric’’ as this is the Klein-Beltrami model of the hyperbolic space \mathbb{H}^1 . Notice that Ω is concave as well as convex in \mathbb{R}^2 .

Example 4.6 (The half-space). The half space corresponds to the limiting case of the strip discussed above, $\Omega = (-a, 1) \times \mathbb{R}$, as $a \rightarrow \infty$. Then the Hilbert metric

$$H_{(-a, 1)}(x, y) = \log \frac{\Pi(x) - 1}{\Pi(y) - 1} \frac{\Pi(y) + a}{\Pi(x) + a}$$

converges to the Funk metric

$$F(x, y) = \log \frac{\Pi(x) - 1}{\Pi(y) - 1}$$

which is the timelike Funk metric for the half-space $\mathbb{R}^2 \setminus \{x \geq 1\}$. We will come back to this example later.

5. THE NON-EUCLIDEAN TIMELIKE FUNK AND HILBERT METRICS

Leaving the setting of the Euclidean space, we now work in the n -dimensional affine space \mathbb{A}^n as underlying space, then the sphere S^n , and finally the projective space $\mathbb{R}P^n$. We first recall a few facts.

First, notice that the definition and the variational characterization of the timelike Funk metric uses only the affine structure of the Euclidean space. Namely though the Euclidean distance function appears in the definition of the timelike Funk metric, only the ratio of the distances $\frac{d(p,b(p,q))}{d(q,b(p,q))}$ among the points $p, q, b(p, q)$ on the ray $R(p, q)$ is used. The corresponding remark is well known in the setting of the classical Hilbert metric. Also in the variational characterization of the Funk distance, the ratio $\frac{d(p,\pi)}{d(q,\pi)}$ can be realized as the ratio of distances among points on the ray $R(p, q)$ transversally intersecting the supporting hyperplane π . This is an interesting fact, since it shows that perpendicularity, a concept which is Euclidean but not affine, is not needed in the variational formulation.

In the same trend, we recall that the convexity of a subset in affine space does not require a metric: A convex set in \mathbb{A}^n is a set whose intersection with any line is contractible.

5.1. Affine geometry. An affine space \mathbb{A}^n is usually thought of as a chart for the projective space $\mathbb{R}P^n$. A convex set in $\mathbb{R}P^n$ can be regarded as a convex set in an affine space \mathbb{A}^n after choosing an appropriate chart.

The notion of convexity in the Euclidean setting extends to the spherical setting where one considers only subsets which are contained in an open hemisphere. An open convex subset I of S^n can be identified with the intersection of open half spaces indexed by the set \mathcal{P} of supporting hyperplanes π , which are great hyperspheres. We have, in analogy with the Euclidean setting,

$$I = \bigcap_{\pi \in \mathcal{P}} H_{\pi}^{+}$$

where H_{π}^{+} is the open half space bounded by the hyperplane π containing I . As before, we denote the other half space by H_{π} . The complement of the closure in S^n is then

$$\Omega = \bigcup_{\pi \in \mathcal{P}} H_{\pi}.$$

Note that the set \mathcal{P} is not empty, since the convex set lies in some hemisphere. For a point $p \in \Omega$, denote the set of great circles by $\mathcal{P}(p)$ such that if $\pi \in \mathcal{P}(p)$. Then p lies on the other side of the convex set I with respect to π .

5.2. Spherical and projective geometry. We now define a timelike Hilbert metric on the sphere S^n . When talking about a convex open subset of S^n , we shall always assume that it is contained in an open hemisphere.

We consider two convex hypersurfaces K_1, K_2 with non-intersecting closures $K_1^{\circ} = K_1 \cup I_1, K_2^{\circ} = K_2 \cup I_2$. The definition of the associated timelike Hilbert metric is analogous to the Euclidean one, with some extra care. Namely, we let $\Omega = S^n \setminus (K_1^{\circ} \cup K_2^{\circ})$. We define a pre-order relation by first noticing that given a great circle which intersects K_1 and K_2 at four distinct points, we can label them so that $a_1, b_1 \in K_1, a_2, b_2 \in K_2$ and the two open arcs (a_1, a_2) and (b_1, b_2) are entirely contained in Ω , while $[a_1, a_2]$ lies in K_1° , and $[b_1, b_2]$ in K_2° . This observation says that every spherical geodesic segment in Ω connecting K_1 and K_2 is naturally oriented, where K_1 represents the past and K_2 the future. We denote by R the set of points in Ω that are contained in the interior of geodesic segments of the form $[a_1, a_2]$ joining a point a_1 in K_1 and a_2 in K_2 , with the future-oriented open line segment (a_1, a_2) entirely contained in Ω .

Given a point p in R , define $C_2(p)$ to be the set of interior points of the future-directed cone $\tilde{C}_2(p)$ spanned by K_2 with apex p . Similarly we set $C_1(p)$ to be the

set of interior points of the past-directed cone $\tilde{C}_1(p)$ spanned by K_1 with apex p . From the definition of R , $\tilde{C}_1(p)$ and $\tilde{C}_2(p)$ are non-empty.

We define a preorder relation $<$ on $R \times R \subset \Omega \times \Omega$ by setting

$$p < q \Leftrightarrow q \in C_2(p) \text{ and } p \in C_1(q).$$

We denote, as before, the set of points (p, q) in $\Omega \times \Omega$ satisfying $p < q$ by $\Omega_{<}$. We also write $p \leq q$ when $p < q$ or $p = q$. We note that in the spherical setting, the cone $C_i(p)$ is an open *lune*, with two vertices, p and its antipodal point $-p$.

Unlike the situation studied in [12], it is not possible to define a timelike Funk spherical metric, because given two distinct points in the complement of a convex subset of the sphere S^n , there is no natural way of saying that one is in the future of the other (the great circle through these points may intersect the convex set in two points).

We recall that given four points p_1, p_2, p_3, p_4 situated in that order on a great circle on the sphere, the spherical cross ratio is defined by

$$[p_1, p_2, p_3, p_4] = \frac{\sin d(p_2, p_4) \sin d(p_3, p_1)}{\sin d(p_3, p_4) \sin d(p_2, p_1)}.$$

Its values are in $\mathbb{R}_{\geq 0} \cup \{\infty\}$. The spherical cross ratio is a projectivity invariant, cf. [12].

Now, for a pair of points (p, q) in $\Omega_{<}$, let $a_1 \in K_1$ and $a_2 \in K_2$ be the intersection points between the great circle through p and q and the two hypersurfaces K_1 and K_2 , so that a_1, p, q, a_2 lie on the arc of great circle $[a_1, a_2] \subset \Omega$ in that order.

Definition 5.1 (Timelike Hilbert metric on S^n). The timelike Hilbert metric associated to the pair (K_1, K_2) is a distance function $H(p, q)$ defined on the subset Ω_{\leq} of $\Omega \times \Omega$ consisting of points p and q satisfying $p \leq q$, using the above notation for a_1, a_2 , by the formula

$$H(p, q) = \log[a_1, p, q, a_2].$$

There is a timelike Minkowski functional P_H associated to the timelike Hilbert metric H as follows. First we denote by $D(p)$ the cone in $T_p\Omega$ consisting of vectors tangent to great circles in $C_2(p) \cap -C_1(p)$. The set $D(p)$ may be empty depending on p . For a given point $p \in R$ and for any vector $v \in D(p)$, P_H is given by the following formula:

$$(10) \quad P_H(p, v) = P_2(p, v) + P_1(p, -v).$$

We also set $P_H(p, 0) = 0$ when $v = 0$, where $\mathcal{P}(p)$ is as in (1) above, and where

$$P_i(p, v) = \inf_{\pi \in \mathcal{P}(p)} \frac{\langle v, \eta_\pi \rangle}{\tan d(p, \pi)}$$

for $v \in C_i(p)$ for $i = 1, 2$.

The fact that P_H is the Finsler structure for H can be checked in the same way as for the Euclidean setting (Theorem 4.4.) Likewise, it can be shown that P_H is upper semi-continuous, that it can be used to define lengths of curves, and that the timelike Hilbert distance is the intrinsic timelike distance defined as the supremum of length of curves, as we noticed for the timelike Funk distance.

The fact that H defines a timelike distance on Ω can be proved as in the case of the Euclidean timelike Hilbert metric. The spherical geodesic segments are geodesics for this timelike spherical metric; this can be proved as in the case of the spherical (non-timelike) Hilbert metric considered in [12]. In fact, there is a relation between the Euclidean and the spherical timelike Hilbert metrics which can be seen as follows.

Assume that the sets K_1° and K_2° are both contained in an open hemisphere of S^n , which we may assume to be the northern hemisphere \mathbb{U} . (We may consider this hemisphere as an affine chart of the projective space $\mathbb{R}P^n$.) There is an isomorphism between the timelike Hilbert metric associated to the pair (K_1, K_2) and a Euclidean timelike Hilbert metric whose associated convex sets are the images \tilde{K}_i° of K_i° ($i = 1, 2$) in the Euclidean space \mathbb{R}^n seen as the hyperplane $\{x_0 = 1\}$ via the projection

$$P_S : \mathbb{U} \rightarrow \{x_0 = 1\}$$

centered at the origin of \mathbb{R}^{n+1} (see [12] for the details).

There is a notable case of a timelike spherical metric. Consider the case where the two convex hypersurfaces K_1 and K_2 are antipodal in S^n , that is, $K_2 = -K_1$ where the minus sign refers to the antipodal map $x \mapsto -x$ of S^n modeled in \mathbb{R}^{n+1} .

In particular, when K_1 is a small circle of radius $\pi/4$ in $S^n \subset \mathbb{R}^{n+1}$, then the resulting timelike Hilbert metric is isometric to the de Sitter metric restricted to the timelike vectors. Below, we will explicitly identify the de Sitter metric with the spherical timelike Hilbert metric.

Taking quotients, this case is identified with a timelike Hilbert metric on an open subset of the projective space $\mathbb{R}P^n$, in which K_1 and K_2 become a single convex hypersurface K under the antipodal quotient map $S^n \rightarrow \mathbb{R}P^n$.

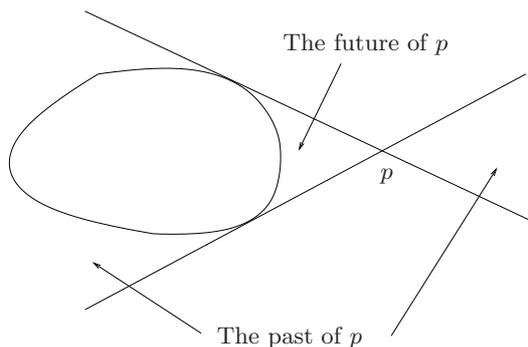


FIGURE 5.

In this setting, there is a doubling phenomenon for the rays emitted from a point p in the complement of $K^\circ = K \cup I$: if such a ray intersects K at p^+ in the future, then it also does so at a point p^- in the past (cf. Figure 5). In other words, for a pair of points p, q satisfying $p < q$, the ray $R(p, q)$ from p through q intersects K and the ray $R(q, p)$ also intersects K . In the light of the discussion about the spherical timelike Hilbert metric above, this picture corresponds to the situation where the two convex subsets of S^n are antipodally located; $K_2 = -K_1$. The line $R(p, q) \cup R(q, p) \cup \{\infty\}$ in $\mathbb{R}P^n$ is a great circle in S^n , p^+ is the intersection point between the future-directed ray and K_1 , while p^- is the intersection point between the ray and K_2 .

5.3. Light cone and null vectors. So far, we have carefully avoided the issue of null vectors in timelike geometry. We did so because there is no coherent general treatment for the timelike Funk and Hilbert metrics. However, this setting where K_1 and K_2 are antipodally located in S^n , is a particular situation worth being investigated. Let Ω be the complement of the set $K_1^\circ \cup K_2^\circ$ where $K_i^\circ = K_i \cup I_i$. Then a great circle intersecting K_1 at two points q_1, q_2 also intersect K_2 at two antipodal points $-q_1, -q_2$. Another way of saying this is that $\tilde{C}_2(p) = -\tilde{C}_1(p)$ and also $C_2(p) = -C_1(p)$. Recall that each $C_i(p)$ is a lune whose two vertices are p

and $-p$. Hence $C_1(p)$ and $C_2(p)$ are a pair of antipodal lunes. Now consider the situation when a great circle c intersect K_1 tangentially at q , which can be thought as a limiting picture for q_1 and q_2 merging on K_1 . This circle also intersects K_2 tangentially at $-p$. We consider a pair of points x, y on an arc of the great circle c in Ω , and the timelike Hilbert distance $H(x, y)$, which is the logarithm of the cross ratio of the quadruple $(-q, x, y, q)$ lying on the arc in that order. Under the antipodal quotient map $\pi : S^n \rightarrow \mathbb{RP}^n$, $\pi(q)$ and $\pi(-q)$ coincide, thus the value of the cross ratio is one, hence $H(x, y) = 0$. As the choices of x and y on the great circle c are arbitrary, we conclude that the (naturally extended) timelike Minkowski functional evaluated along the great circle tangential to K_1 and $K_2 = -K_1$ is zero. In other words, given a point p in Ω , consider the cone $\tilde{C}_2(p) \setminus C_2(p)$ consisting of great circles through p which are tangential to the convex hypersurface K_2 . These great circles are automatically tangential to $K_1 = -K_2$. The cone $D(p) = C_2(p) \cap -C_1(p)$ in $T_p\Omega$ on which the Minkowski functional $P_H(p)$ is defined is equal to $C_2(p)$. Then the tangent vectors $\overline{D}(p) \setminus D(p)$ at p constitute the future-directed light cone with respect to the timelike Minkowski functional for the spherical timelike Hilbert metric H . In this way, we have demonstrated the existence of null vectors in the de Sitter space within the framework of timelike Hilbert geometry.

5.4. The de Sitter distance equals the Hilbert distance. The quotient space of the de Sitter space is equipped with a *locally timelike* Hilbert geometry, where the quotient is taken by the \mathbb{Z}_2 antipodal symmetry of $\Omega = S^n \setminus (K_1^\circ \cup (-K_1)^\circ)$, with K_1 a small circle of radius $\pi/4$ in S^n . The timelike Hilbert metric thus defined is only *local*, as the space $\Omega = \mathbb{RP}^n \setminus K_1^\circ$ is not time-orientable. Namely consider the closed path from $p \in \Omega$ to itself, along the circle at infinity of \mathbb{RP}^n . Traversing the loop then reverses the orientation of the light cone (cf. Hawking-Ellis [6], Calabi-Marcus [5]).

We now demonstrate an explicit isometry between the locally timelike Hilbert metric and the de Sitter metric. For this, we first recall that the Sitter space is the unit sphere in the Minkowski space $\mathbb{R}^{n,1}$

$$S^{n,1} = \{(x_0, x_1, \dots, x_n) \mid -x_0^2 + \sum_{i=1}^n x_i^2 = 1\} \subset \mathbb{R}^{n,1},$$

equipped with the so-called de Sitter metric, a Lorentzian metric of type $(n, 1)$ whose first fundamental form is induced from the ambient Minkowski metric $ds^2 = -dx_0^2 + \sum dx_i^2$.

The intersection between $S^{n,1}$ and the x_0x_1 -plane is a copy of $S^{1,1} \subset \mathbb{R}^{1,1}$ which is totally geodesically embedded. By an element of $\text{SO}(n, 1)$, any pair of points (p, q) , with q lying in the future of p in $S^{n,1}$, can be isometrically transposed to a pair of points on $S^{1,1}$ so that the x_0 coordinates of the points are positive. Hence we may assume without loss of generality that p and q belong to the upper hemisphere $\{(x_0, x_1) \mid -x_0^2 + x_1^2 = 1, x_0 > 0\}$ of $S^{1,1}$ in $\mathbb{R}^{1,1}$.

We introduce a local parameter t on $S^{1,1}$ so that

$$(x_0, x_1) = (\sinh t, \cosh t).$$

Note that t is an arc-length parameter for the de Sitter metric, as the tangent vector to $\sigma(t) = (\sinh t, \cosh t)$ has norm 1. Hence for $p = \sigma(t_1)$ and $q = \sigma(t_2)$ with $t_1 < t_2$, the de Sitter distance $d(p, q)$ is equal to $t_2 - t_1$. Here the point q lies in the future of p ($p < q$.)

On the other hand, we can project a part of the hyperboloid $\{(x_0, x_1) \mid -x_0^2 + x_1^2 = 1, x_0 > 0\}$ onto the hyperplane $\{x_0 = 1\}$ along the rays from the origin of $\mathbb{R}^{1,1}$

$$(11) \quad P_{\text{dS}} : \{-x_0^2 + x_1^2 = 1\} \rightarrow \{x_0 = 1\}.$$

Let $\tilde{p} = (1, s_1)$ and $\tilde{q} = (1, s_2)$ be the images of p and q by this correspondence, where $s_1 > s_2$. The asymptotic lines $x_0 = \pm x_1$ of the hyperboloid $\{-x_0^2 + x_1^2 = 1\}$ are sent to the points $(1, 1)$ and $(1, -1)$. The cross ratio of those four points defines the Hilbert metric H for the convex set $I = \{x_0 > \pm x_1\}$ in the projective space $\mathbb{R}P^1$, and for the pair of points \tilde{p} and \tilde{q} with $\tilde{p} < \tilde{q}$, we have

$$H(\tilde{p}, \tilde{q}) = \log \frac{s_1 - 1}{s_2 - 1} \cdot \frac{s_2 + 1}{s_1 + 1}.$$

By noting the equality

$$\tilde{s}_i = \frac{\sinh t_i}{\cosh t_i},$$

the Hilbert distance $H(\tilde{p}, \tilde{q})$ is equal to $2(t_2 - t_1)$. Hence we have shown that $d(p, q) = 2H(\tilde{p}, \tilde{q})$ for $p < q$.

By post-composing the map P_{dS} with the map $P_{\text{S}}^{-1} : \{x_0 = 1\} \rightarrow \mathbb{U}$ where \mathbb{U} is the upper hemisphere $\{(x_0, x_1, \dots, x_n) | x_0^2 + \sum_{i=1}^n x_i^2 = 1, x_0 > 0\}$, the geodesic through p and q in the de Sitter space is identified with a great circle in the sphere, and the image of the map $P_{\text{S}}^{-1} \circ P_{\text{dS}}$ of the northern half of the de Sitter space is $\mathbb{U} \setminus B$ where B is the northern cap bounded by the small circle of radius $\pi/4$. This demonstrates that the timelike geometry of the de Sitter space is realized by the timelike Hilbert metric modeled on the sphere. The maps P_{dS} and P_{S} are *perspectivities*, namely they preserve the cross ratio (see [12]). We conclude that the de Sitter distance is equal to the timelike Hilbert distance.

5.5. Funk geometry in projective space. Finally we note, as Busemann mentions in his paper [4], that the Funk metric associated to a convex hypersurface K contained in an affine space \mathbb{A}^n appears as a special case of a Hilbert metric, namely, the one associated to a pair K_1, K_2 where K_1 is the hyperplane at infinity $\mathbb{R}P^{n-1}$ in the projective space $\mathbb{R}P^n = \mathbb{A}^n \cup \mathbb{R}P^{n-1}$. The set K_1 is the collection of points which are “infinite distance away” from any pair of points in $\mathbb{A}^n \setminus K_1$, in the sense that for any pair of points p, q with $p <_2 q$, we have $\frac{d(p, a_1)}{d(q, a_2)} = 1$. In that case, and using the notation of Definition 4.1, the Hilbert distance from p to q associated to the pair K_1, K_2 is just the Funk distance from p to q associated to the convex set K_2 .

6. CONCLUDING REMARKS

There is a Funk metric associated with a convex hypersurface K in the hyperbolic space \mathbb{H}^n . This was studied in [12]. In the same way, one can define a timelike Funk metric associated with convex subsets of \mathbb{H}^n . The pre-order $p < q$ is defined as in the case of the Euclidean timelike Funk metric, and the distance from p to q satisfying $p < q$ is given by the formula

$$(12) \quad F(p, q) = \log \frac{\sinh d(p, b(p, q))}{\sinh d(q, b(p, q))}$$

where $b(p, q)$ is the point where the ray $R(p, q)$ hits K for the first time, and d is the hyperbolic distance. Several properties of the hyperbolic (non-timelike) Funk metric proved in [12] hold verbatim for this timelike hyperbolic Funk metric. In particular, we have a variational formulation of the timelike hyperbolic Funk metric:

$$F(p, q) = \inf_{\pi \in \mathcal{P}(p)} \log \frac{\sinh d(p, \pi)}{\sinh d(q, \pi)}.$$

There is also a timelike hyperbolic Hilbert metric, defined in an analogous way to the timelike Hilbert metric defined in §4, replacing, in the definition, the distance by the hyperbolic sine of the distance, as we did in the definition of the timelike hyperbolic metric in (12).

The hyperbolic segments are geodesics for the timelike hyperbolic Funk and for the timelike hyperbolic Hilbert metrics. The proofs use the same as the one of the analogous result for the hyperbolic (non-timelike) Funk and Hilbert metrics considered in [12].

Finally as this is apparently a field of research revived after it was left behind by Busemann half a century ago, several problems naturally arise, and we propose the following:

- Problems 6.1.**
- (1) Study the geometric properties of specific examples (timelike geometries of individual or special classes of convex sets).
 - (2) Study the infinite-dimensional case.
 - (3) Study the reverse timelike Funk metric.
 - (4) Study the isometries of the timelike Funk metric.
 - (5) Study the horofunction boundaries of the timelike Funk and Hilbert metrics.
 - (6) Find a characterization of the timelike Hilbert geometries that can be embedded in a Lorentzian space.
 - (7) Complexify.

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