Stability Verification of Nearly Periodic Impulsive Linear Systems using Reachability Analysis

Mohammad Al Khatib, Antoine Girard, Thao Dang

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Mohammad Al Khatib * **Antoine Girard** * **Thao Dang** **

* *Univ. Grenoble Alpes, LJK, F-38000 Grenoble, France*
CNRS, LJK, F-38000 Grenoble, France
** *CNRS, Verimag, F-38000 Grenoble, France*

Abstract:
The paper provides stability analysis to certain classes of hybrid systems, more precisely impulsive linear systems. This analysis is conducted using the notion of reachable set. The main contribution in this work is the derivation of theoretical necessary and sufficient conditions for impulsive linear systems with nearly periodic resets subject to timing contracts. This characterization serves as the basis of a computational method for the stability verification of the considered class of systems. In addition, we show how this work handles the problem of timing contract synthesis for the considered class and we generalize our approach to verify stability of impulsive linear systems with stochastic reset instants. Applications on sampled-data control systems and comparisons with existing results are then discussed, showing the effectiveness of our approach.

Keywords: Reachability; Impulsive systems; Stability analysis.

1. INTRODUCTION

Impulsive dynamical systems form a class of hybrid systems which models processes that evolve continuously and undergo instantaneous changes at discrete time instants. Applications of impulsive dynamical systems include sampled-data control systems [Briat (2013)] or networked control systems [Donkers et al. (2011)]. In this paper, we consider the problem of verifying stability of impulsive linear systems subject to nearly periodic resets. More precisely, the duration between two consecutive resets is uncertain but constrained in some bounded interval given by a timing contract. Several approaches are developed in literature to analyze stability of such systems. On one hand, there is a discrete-time and convex embedding approach [Hetel et al. (2011, 2013)], a time delay technique [Liu et al. (2010); Seuret and Peet (2013)], a hybrid system formulation [Dai et al. (2010)], and an Input/Output stability approach [Omran et al. (2014); Fujioka (2009)]. On the other hand, [Fiacchini and Morarescu (2014)] derives an approach which mainly uses backward invariant set computations [Blanchini and Miani (2007)] to find a contracting polytopic set for the system. In the former methods, the stability criterion is given in terms of Linear Matrix Inequality (LMI) which numerically provides only sufficient conditions for stability. Whereas, the latter is less conservative but in its turn does provide only sufficiency for the stability verification problem, knowing that computational-wise necessity would come with possibly unbounded computations.

In this paper, we propose a new approach based on forward reachability analysis. Primarily, we state the necessary and sufficient theoretical conditions, based on reachable sets, for stability of NPILS. Then, we take advantage of previous work[Le Guernic and Girard (2010)], which provides an algorithmic scheme to compute the reachable sets for linear systems avoiding the wrapping effect (accumulation of over-approximation errors). We present a computational reachability based stability verification method for nearly periodic impulsive linear systems. Moreover, we handle the problem of synthesizing timing contracts using the previous method as well as monotonicity of the stability property with respect to parameters of the timing contract. The work is then extended to deal with the problem of stability verification in the stochastic case. Last, the efficiency of our work is shown by illustrative examples where our results are not only less conservative than several of those existing in literature but also show tightness of our approximation scheme. Advantages in using our method are seen to extend in dealing with further timing contracts as well as taking into consideration some performance specifications since an insight on the reachable set is given during our analysis.

The paper is organized as follows. First, some preliminary notations are defined before formulating the stability verification and the timing contract synthesis problems in Section 3. The main results are discussed in Section 4. Detailed explanations on the algorithms and over-approximation scheme utilized in our approach lies in Section 5. In Section 6 stability is studied for the stochastic case. We discuss more on the computation of the reachable set in Section 7. Applications on sampled-data control systems and comparisons with existing results are then

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discussed, before concluding our work. Due to space limitation proofs of the theorems, corollaries, and propositions are omitted.

Notations Let $\mathbb{R}$, $\mathbb{R}_0^+$, $\mathbb{R}^+$, $\mathbb{N}$, $\mathbb{N}^+$ denote the sets of reals, nonnegative reals, positive reals, nonnegative integers and positive integers, respectively. For $I \subseteq \mathbb{R}_0^+$, let $\mathbb{N}_I = \mathbb{N} \cap I$. Given a real matrix $A \in \mathbb{R}^{n\times n}$, $|A|$ is the matrix whose elements are the absolute values of the elements of A. Given $\mathcal{S} \subseteq \mathbb{R}^n$ and a real matrix $A \in \mathbb{R}^{n\times n}$, the set $A\mathcal{S} = \{x \in \mathbb{R}^n : (\exists y \in \mathcal{S} : x = Ay)\}$; for $a \in \mathbb{R}$, $a\mathcal{S} = (aI_n)\mathcal{S}$ where I_n is the $n \times n$ identity matrix. The interior of $\mathcal{S}$ is denoted by $\mathrm{int}(\mathcal{S})$. The convex hull of $\mathcal{S}$ is denoted by $\mathrm{ch}(\mathcal{S})$. The interval hull of $\mathcal{S}$ is the smallest interval containing the set $\mathcal{S}$ and is denoted by $\square(\mathcal{S})$. The symmetric interval hull of $\mathcal{S}$ is the smallest symmetric (with respect to 0) interval containing $\mathcal{S}$ and is denoted by $\boxdot(\mathcal{S})$. Given $\mathcal{S}, \mathcal{S}' \subseteq \mathbb{R}^n$, the Minkowski sum of $\mathcal{S}$ and $\mathcal{S}'$ is $\mathcal{S} \oplus \mathcal{S}' = \{x + x' : x \in \mathcal{S}, x' \in \mathcal{S}'\}$. A polytope $\mathcal{P}$ is a subset of $\mathbb{R}^n$ which can be defined as the intersection of a finite number of closed half-spaces, that is $\mathcal{P} = \{x \in \mathbb{R}^n : Hx \leq b\}$ where $H \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$ and the vector of inequalities is interpreted componentwise. Let H_i, $i \in \mathbb{N}_{[1,m]}$ denote the row vectors of H, and if $0 \in \mathrm{int}(\mathrm{ch}(\{H_1, \ldots, H_m\}))$, then $\mathcal{P}$ is compact. Given a template matrix $H \in \mathbb{R}^{m\times n}$ and a compact set $\mathcal{S} \subseteq \mathbb{R}^n$, let us define the polytope $\Gamma_H(\mathcal{S}) = \{x \in \mathbb{R}^n : Hx \leq b\}$ where $b_i = \max_{x\in\mathcal{S}} H_i x$, $i \in \mathbb{N}_{[1,m]}$. In other words, $\Gamma_H(\mathcal{S})$ is the smallest polytope whose facets directions are given by H and containing $\mathcal{S}$. We denote the set of all subsets of $\mathbb{R}^n$ by $2^{\mathbb{R}^n}$. We denote by $\mathcal{K}(\mathbb{R}^n)$ the set of compact subsets of $\mathbb{R}^n$ and by $\mathcal{K}_0(\mathbb{R}^n)$ the set of compact subsets of $\mathbb{R}^n$ containing 0 in their interior.

2. PROBLEM FORMULATION

We consider the class of impulsive linear systems given by:

$$\dot{x}(t) = A_c x(t),\ \forall t \in (t_k, t_{k+1}), k \in \mathbb{N}, \tag{1}$$

$$x(t_k^+) = A_d x(t_k),\ k \in \mathbb{N}, \tag{2}$$

where $(t_k)_{k\in\mathbb{N}}$ are the reset instants, $x(t) \in \mathbb{R}^n$ is the state of the system, and $x(t^+) = \lim_{\tau\to 0, \tau>0} x(t+\tau)$.

We assume that the sequence of reset instants (t_k) satisfies a *timing contract* given by

$$t_0 = 0,\ t_{k+1} - t_k = T + \tau_k,\ \tau_k \in [0, \delta],\ k \in \mathbb{N} \tag{3}$$

where $T \in \mathbb{R}^+$ represents a nominal reset period and $(\tau_k)_{k\in\mathbb{N}}$ is a bounded uncertain sequence in the compact set $[0, \delta]$, $\delta \in \mathbb{R}_0^+$. Hybrid dynamical systems described by the continuous dynamics (1), the discrete dynamics (2) and the timing contract (3) are called *nearly periodic impulsive linear systems* (NPILS).

Definition 1. The NPILS (1-3) is *globally uniformly exponentially stable* (GUES) if there exist $\lambda \in \mathbb{R}^+$ and $C \in \mathbb{R}^+$ such that, for all sequences $(t_k)_{k\in\mathbb{N}}$ verifying (3) the solutions of (1-2) verify

$$\|x(t)\| \leq Ce^{-\lambda t}\|x(0)\|,\ \forall t \in \mathbb{R}^+.$$

In this paper, we present algorithms for solving the following two problems:

Problem 1. (Stability verification). Given $A_c \in \mathbb{R}^{n\times n}$, $A_d \in \mathbb{R}^{n\times n}$, $T \in \mathbb{R}^+$, and $\delta \in \mathbb{R}_0^+$, verify that NPILS (1-3) is GUES.

Problem 2. (Timing contract synthesis). Given $A_c \in \mathbb{R}^{n\times n}$, $A_d \in \mathbb{R}^{n\times n}$, synthesize a set $\Pi \subseteq \mathbb{R}^+ \times \mathbb{R}_0^+$ such that for all $(T, \delta) \in \Pi$, NPILS (1-3) is GUES.

3. STABILITY CHARACTERIZATIONS

This section presents the main theoretical result of the paper in the form of a necessary and sufficient stability condition for NPILS (1-3). This condition can serve to derive a solution to Problem 1. We also prove several results which will be instrumental in solving Problem 2. Before proceeding to the main results, we need to define the notion of reachable set.

Definition 2. Given a continuous-time dynamical system

$$\dot{x}(t) = Ax(t),\ t \in \mathbb{R}_0^+,\ x(t) \in \mathbb{R}^n$$

the *reachable set* on $[t, t'] \subseteq \mathbb{R}_0^+$ from the set $\mathcal{S} \subseteq \mathbb{R}^n$ is

$$\mathcal{R}^A_{[t,t']}(\mathcal{S}) = \bigcup_{\tau\in[t,t']} e^{\tau A}\mathcal{S}.$$

Then, let us introduce the map: $\Phi : 2^{\mathbb{R}^n} \to 2^{\mathbb{R}^n}$, defined for all $\mathcal{S} \subseteq \mathbb{R}^n$ by

$$\Phi(\mathcal{S}) = \mathcal{R}^{A_c}_{[0,\delta]}(e^{TA_c}A_d\mathcal{S}). \tag{4}$$

The interpretation of $\Phi(\mathcal{S})$ is as follows. Let x be a trajectory of NPILS (1-3) such that $x(t_k) \in \mathcal{S}$, then $x(t_{k+1}) \in \Phi(\mathcal{S})$. It is easy to see that if $\mathcal{S}$ is compact then so is $\Phi(\mathcal{S})$. It is clear that for two sets $\mathcal{S}$, $\mathcal{S}' \subseteq \mathbb{R}^n$ and $a \in \mathbb{R}$, it holds $\Phi(\mathcal{S} \cup \mathcal{S}') = \Phi(\mathcal{S}) \cup \Phi(\mathcal{S}')$ and $\Phi(a\mathcal{S}) = a\Phi(\mathcal{S})$. We define the iterates of Φ as $\Phi^0(\mathcal{S}) = \mathcal{S}$ for all $\mathcal{S} \subseteq \mathbb{R}^n$, and $\Phi^{k+1} = \Phi \circ \Phi^k$ for all $k \in \mathbb{N}$. Then, it is clear that $x(0) \in \mathcal{S}$ implies that $x(t_k) \in \Phi^k(\mathcal{S})$ for all $k \in \mathbb{N}$.

The exact computation of Φ is often impossible and we use in this work an over-approximation $\overline{\Phi} : \mathcal{K}(\mathbb{R}^n) \to \mathcal{K}(\mathbb{R}^n)$ satisfying the following assumption:

Assumption 3. For all $\mathcal{S} \in \mathcal{K}(\mathbb{R}^n)$, $\Phi(\mathcal{S}) \subseteq \overline{\Phi}(\mathcal{S})$.

We will discuss an effective computation of $\overline{\Phi}(\mathcal{S})$ in Section 4. The iterates of the map $\overline{\Phi}$ are defined similarly to those of Φ.

The following result characterize the stability of NPILS (1-3) in terms of the map Φ:

Theorem 4. Let $\mathcal{S} \in \mathcal{K}_0(\mathbb{R}^n)$, the following statements are equivalent: [1]

(a) NPILS (1-3) is GUES,
(b) There exists a triplet $(k, j, \rho) \in \mathbb{N}^+ \times \mathbb{N}_{[0,k-1]} \times (0, 1)$ such that $\Phi^k(\mathcal{S}) \subseteq \rho\Phi^j(\mathcal{S})$,
(c) There exists a pair $(k, \rho) \in \mathbb{N}^+ \times (0, 1)$ such that $\Phi^k(\mathcal{S}) \subseteq \rho\bigcup_{j=0}^{k-1}\Phi^j(\mathcal{S})$.

We now derive sufficient conditions for stability based on an over-approximation of map Φ.

[1] Similar results for discrete-time switched systems were shown in [Athanasopoulos and Lazar (2014)].

Corollary 5. Under Assumption 3, if there exist a set $\mathcal{S} \in \mathcal{K}_0(\mathbb{R}^n)$ and a pair $(k, \rho) \in \mathbb{N}^+ \times (0, 1)$ such that $\overline{\Phi}^k(\mathcal{S}) \subseteq \rho \bigcup_{j=0}^{k-1} \overline{\Phi}^j(\mathcal{S})$, then NPILS (1-3) is GUES.

The previous corollary provides the background for designing a solution to Problem 1 in the next section.

We claim some simple results that will be instrumental in our approach to Problem 2. Let us introduce the new parameters $T_m \in \mathbb{R}^+$, $T_M \in \mathbb{R}^+$ with $T_m \leq T_M$ related to parameters T, δ by $T_m = T$, $T_M = T + \delta$. Then, timing contract (3) can be rewritten equivalently as

$$t_0 = 0,\ t_{k+1} - t_k \in [T_m, T_M],\ k \in \mathbb{N}. \tag{5}$$

Then, the following monotonicity result clearly holds:

Proposition 6. If NPILS (1-2), (5) is GUES for parameters $T_m \in \mathbb{R}^+$, $T_M \in \mathbb{R}^+$ with $T_m \leq T_M$ then it is GUES for all parameters $T'_m \in \mathbb{R}^+$, $T'_M \in \mathbb{R}^+$ with $T_m \leq T'_m \leq T'_M \leq T_M$.

In addition, the following proposition is used to constrain the search region for solving Problem 2:

Proposition 7. If NPILS (1-2), (5) is GUES for parameters $T_m \in \mathbb{R}^+$, $T_M \in \mathbb{R}^+$ with $T_m \leq T_M$, then for all $T \in [T_m, T_M]$, $e^{TA_c}A_d$ is a Schur matrix.

4. ALGORITHMS FOR STABILITY VERIFICATION AND CONTRACT SYNTHESIS

In this section, we describe algorithms for solving Problems 1 and 2. But first, we discuss the computation of an over-approximation $\overline{\Phi}$ based on reachability analysis.

4.1 Reachability analysis

The map Φ, to be over-approximated, is given by (4) in terms of the map $\mathcal{R}$. Moreover, the latter is efficiently over-approximated using the following result [Le Guernic (2009)]:

Theorem 8. For $\delta \in \mathbb{R}^+$, $A \in \mathbb{R}^{n\times n}$ and $\mathcal{S} \in \mathcal{K}(\mathbb{R}^n)$, let

$$\overline{\mathcal{R}}^A_{[0,\delta]}(\mathcal{S}) = \bigcup_{i=1}^{N} \overline{\mathcal{R}}^A_{[(i-1)h,ih]}(\mathcal{S}) \tag{6}$$

where $N \in \mathbb{N}^+$, $h = \delta/N$ is the time step, and $\overline{\mathcal{R}}^A_{[(i-1)h,ih]}(\mathcal{S})$ is defined by the recurrence equation:

$$\overline{\mathcal{R}}^A_{[0,h]}(\mathcal{S}) = \mathrm{ch}(\mathcal{S}, e^{hA}\,\mathcal{S}) \ \oplus\ 1/4\ \epsilon_h(\mathcal{S}),$$
$$\overline{\mathcal{R}}^A_{[ih,(i+1)h]}(\mathcal{S}) = e^{hA}\ \overline{\mathcal{R}}^A_{[(i-1)h,ih]}(\mathcal{S}),\ i \in \mathbb{N}_{[1,N-1]}$$

with

$$\begin{aligned}\epsilon_h(\mathcal{S}) = {} & \Box(|A|^{-1}(e^{h|A|} - I) \boxdot (A(I - e^{hA})\mathcal{S})) \oplus \\ & \Box(|A|^{-2}(e^{h|A|} - I - h|A|) \boxdot (A^2 e^{hA}\mathcal{S})).\end{aligned}$$

Then, $\mathcal{R}^A_{[0,\delta]}(\mathcal{S}) \subseteq \overline{\mathcal{R}}^A_{[0,\delta]}(\mathcal{S})$, and for all $i \in \mathbb{N}_{[1,N]}$, $\mathcal{R}^A_{[(i-1)h,ih]}(\mathcal{S}) \subseteq \overline{\mathcal{R}}^A_{[(i-1)h,ih]}(\mathcal{S})$

The over-approximation $\overline{\mathcal{R}}^A_{[0,T]}(\mathcal{S})$ of the reachable set is given by the union of N convex sets which may be quite impractical for subsequent manipulations. For that reason, it will be over-approximated by the smallest enclosing polytope whose facets direction are given by a matrix H. Finally, the over-approximation of Φ will be given as follows:

Corollary 9. Let the matrix $H \in \mathbb{R}^{m\times n}$, such that $0 \in \mathrm{int}(\mathrm{ch}(\{H_1, \ldots, H_m\}))$. Let $\overline{\Phi} : \mathcal{K}(\mathbb{R}^n) \to \mathcal{K}(\mathbb{R}^n)$ be given by

$$\overline{\Phi}(\mathcal{S}) = \Gamma_H \left(\overline{\mathcal{R}}^{A_c}_{[0,\delta]}(e^{TA_c}A_d\mathcal{S}) \right), \tag{7}$$

where $\overline{\mathcal{R}}^{A_c}_{[0,\delta]}(e^{TA_c}A_d\mathcal{S})$ is computed as in Theorem 8. Then, $\overline{\Phi}$ satisfies Assumption 3.

4.2 Stability verification

The stability verification algorithm consists of an initialization step and a main loop. In the initialization step, we compute an initial set which will then be propagated in the main loop using the map $\overline{\Phi}$ defined in (7) to check the stability condition given by Corollary 5.

Initial set computation The choice of the initial set is important in order to try to minimize the value of the integer k such that the stability condition given by Corollary 5 holds. One approach to compute the initial set is to define it as a common contracting symmetric polytope $\mathcal{P}_0$, to $L \in \mathbb{N}^+$ linear discrete-time-invariant systems, such that

$$\forall j \in \mathbb{N}_{[1,L]},\ e^{T_j A_c} A_d \mathcal{P}_0 \subseteq \mathrm{int}(\mathcal{P}_0), \tag{8}$$

where $T_j = T + \frac{(j-1)\delta}{L}$. Then, $\mathcal{P}_0$ can be computed using a backward iterative method in an analogous way as done in [Blanchini (1991)] and [Fiacchini and Morarescu (2014)]. The function computing $\mathcal{P}_0$ is denoted by $\mathsf{init}(A_c, A_d, T, \delta, L)$. Then, $\mathcal{P}_0 = \{x \in \mathbb{R}^n : Hx \leq b_0\}$. The matrix H defining $\mathcal{P}_0$ is used in the main loop of the algorithm in the computation of the map $\overline{\Phi}$.

Main loop In the main loop, the initial set is propagated using the map $\overline{\Phi}$ defined by (7). The stability condition given by Corollary 5 is checked after each iteration. If the condition is verified then NPILS (1-3) is GUES and the algorithm returns true. We impose a maximum number of iterations k_{max}, if that number of iterations is reached then the algorithm fails to prove stability and returns unknown. The algorithm for solving Problem 1 is then given as follows:

Algorithm 1. Stability verification

function is_GUES(A_c,A_d,T,δ)
input: A_c, $A_d \in \mathbb{R}^{n\times n}$, T, $\delta \in \mathbb{R}^+$
output: true if NPILS (1-3) is proved GUES, unknown otherwise
parameter: L, $k_{max} \in \mathbb{N}^+$

```
P_0 := init(A_c, A_d, T, δ, L);            ▷ compute initial set
for k = 1 to k_max do
    P_k := Φ̄(P_{k-1});                     ▷ set propagation
    if P_k ⊆ int(⋃_{j=0}^{k-1} P_j) then   ▷ stability check
        return true;
    end if
end for
return unknown;
```

The proposed approach above induces conservativeness due to over-approximation of the map Φ and to imposing a limited number of iteration. Consequently, it is possible that some stable NPILS cannot be verified by the algorithm. Nevertheless, by manipulating the parameters of

Algorithm 1, the efficiency of our approach is shown by several examples in Section 6.

4.3 Timing contract synthesis

Solving Problem 2 is clearly equivalent to determining a set $\Theta \subseteq \mathbb{R}^+ \times \mathbb{R}^+$ such that for every $(T_m, T_M) \in \Theta$ the NPILS (1-2), (5) is GUES.

We bound the search space to a set Ω defined by: $T_{min} \leq T_m, \leq T_M \leq T_{max}$. Next, we follow the steps in Algorithm 2 that synthesizes Θ inside Ω by taking increasing values T_m^i, $i \in \mathbb{N}_{[1,i_{max}]}$ on the T_m axis and finding, following bisection, the maximum value T_M^i on the T_M axis such that stability of NPILS (1-2), (5) is verified. Following Proposition 7, we assume that $e^{T_m^i A_c} A_d$ is a Schur matrix for all $i \in \mathbb{N}_{[1,i_{max}]}$.

Note that the algorithm uses the monotonicity property stated in Proposition 6 in order to avoid unnecessary computations, since we know that $T_M^i \geq \max(T_M^{i-1}, T_m^i)$.

Algorithm 2. Timing contract synthesis

input: A_c, $A_d \in \mathbb{R}^{n \times n}$
output: $\Theta \subseteq \mathbb{R}^+ \times \mathbb{R}^+$ such that for all $(T_m, T_M) \in \Theta$, NPILS (1-2), (5) is GUES
parameter: i_{max}, ϵ, T_{min}, T_{max}

$\Delta := \frac{T_{max} - T_{min}}{i_{max}}$;
$T_M^0 := T_{min}$;
for $i = 1$ *to* i_{max} **do**
  $T_m^i := T_{min} + (i-1)\Delta$;
  $T_{up} := T_{max}$;
  $T_{down} := max(T_M^{i-1}, T_m^i)$; ▷ use monotonicity
  while $T_{up} - T_{down} > \epsilon$ **do** ▷ bisection
    $T_{half} := \frac{T_{up} + T_{down}}{2}$;
    if is_GUES(A_c,A_d,T_m^i,$T_{half} - T_m^i$) **then**
      $T_{down} := T_{half}$;
    else
      $T_{up} := T_{half}$;
    end if
  end while
  $T_M^i := T_{down}$;
end for
$\Theta := \bigcup_{i=1}^{i_{max}} \{(T_m, T_M) : T_m^i \leq T_m \leq T_M \leq T_M^i\}$

Eventually, we can easily retrieve Π from Θ so that Problem 2 is solved by the following relation

$$\Pi = \{(T, \delta) = (T_m, T_M - T_m) : (T_m, T_M) \in \Theta\}.$$

Now we derive a stability analysis for the same system at hand (1-2) but for stochastic resets.

5. STABILITY ANALYSIS OF STOCHASTIC IMPULSIVE LINEAR SYSTEMS

In this section, we extend our approach to stochastic systems. *Stochastic impulsive linear systems* (SILS), considered in this section, take the same form as (1-2) with independent and identically distributed (i.i.d.) random durations between resets:

$$t_0 = 0,\ t_{k+1} - t_k = T + \tau_k,\ \tau_k \sim \mathcal{U}([0,\delta]), \text{i.i.d. } k \in \mathbb{N} \quad (9)$$

where $\mathcal{U}([0,\delta])$ is the uniform distribution over $[0,\delta]$. Let us remark that the method presented in this section can be easily extended to other types of probability distributions with compact support. We consider the following notion of stability for stochastic systems:

Definition 10. The SILS (1-2), (9) is *globally uniformly mean exponentially stable* (GUMES) if there exist $\lambda \in \mathbb{R}^+$ and $C \in \mathbb{R}^+$ such that for all sequences $(t_k)_{k \in \mathbb{N}}$ verifying (9) the solutions of (1-2) verify:

$$\mathbb{E}[\|x(t)\|] \leq Ce^{-\lambda t} \|x(0)\|,\ \forall t \in \mathbb{R}^+.$$

5.1 Sufficient stability condition

Let $\mathcal{S} \in \mathcal{K}^0(\mathbb{R}^n)$, in the following we provide a sufficient condition for GUMES based on a map $\rho_{\mathcal{S}} : [0,\delta] \to \mathbb{R}^+$ satisfying the following assumption:

Assumption 11. Let $\mathcal{S} \in \mathcal{K}^0(\mathbb{R}^n)$, for all $\tau \in [0,\delta]$, $e^{(T+\tau)A_c} A_d \mathcal{S} \subseteq \rho_{\mathcal{S}}(\tau)\mathcal{S}$.

Then, we can state the following stability condition:

Proposition 12. Under Assumption 11, if there exists a set $\mathcal{S} \in \mathcal{K}^0(\mathbb{R}^n)$ such that

$$\rho_{\mathcal{S}}^* = \mathbb{E}[\rho_{\mathcal{S}}(\tau)] < 1 \text{ where } \tau \sim \mathcal{U}([0,\delta]),$$

then SILS (1-2), (9) is GUMES.

5.2 Stability verification

We now present an approach based on reachability analysis for computing a function $\rho_{\mathcal{S}}$ satisfying Assumption 11.

Let us consider a polytope $\mathcal{P} = \{x \in \mathbb{R}^n : Hx \leq b\}$ where the matrix $H \in \mathbb{R}^{m \times n}$ is such that $0 \in \text{int}(\text{ch}(\{H_1, \ldots, H_m\}))$ and $b_i \geq 0$ for all $i \in \mathbb{N}_{[1,m]}$. Then, $\mathcal{P} \in \mathcal{K}_0(\mathbb{R}^n)$.

Proposition 13. Let $\rho_{\mathcal{P}} : [0,\delta] \to \mathbb{R}^+$ be given by

$$\rho_{\mathcal{P}}(\tau) = \rho_i,\ \text{if } \tau \in [(i-1)h, ih], i \in \mathbb{N}_{[1,N]}$$

where $N \in \mathbb{N}^+$, $h = \delta/N$ is the time step, and ρ_i satisfies for $i \in \mathbb{N}_{[1,N]}$

$$\Gamma_H(\overline{\mathcal{R}}^{A_c}_{[(i-1)h,ih]}(e^{A_c T} A_d \mathcal{P})) \subseteq \rho_i \mathcal{P},$$

with $\overline{\mathcal{R}}^{A_c}_{[(i-1)h,ih]}(e^{A_c T} A_d \mathcal{P})$ computed as in Theorem 8. Then, $\rho_{\mathcal{P}}$ satisfies Assumption 11.

Then, stability can be effectively verified using the following result:

Corollary 14. Let ρ_i, $i \in \mathbb{N}_{[1,N]}$ be computed as in Proposition 13, if

$$\sum_{i=1}^{N} \rho_i < N$$

then SILS (1-2), (9) is GUMES.

6. APPLICATIONS AND NUMERICAL RESULTS

We implement the algorithms presented in this paper in Matlab using the Multi-Parametric Toolbox [Herceg et al. (2013)]. All reported experiments are realized on a desktop with i5 4690 processor of frequency 3.5 GHz and a 7.8 GB RAM.

6.1 Sampled-data systems

We consider the problem of verifying stability for sampled-data control systems. These systems are given under the form:

$$\begin{aligned} \dot{z}(t) &= Az(t) + Bu(t_k),\ \ \forall t \in (t_k, t_{k+1}),\ \ k \in \mathbb{N} \\ u(t_k^+) &= Kz(t_k) \end{aligned} \quad (10)$$

where $t_{k+1} - t_k$ is a variable sampling interval bounded in $[T, T+\delta]$, $z(t) \in \mathbb{R}^n$ is the state of the system, $u(t) \in \mathbb{R}^m$ is the control input computed quasi-periodically at instants t_k, and $K \in \mathbb{R}^{m\times n}$ is the feedback gain. This problem could be rewritten in the form given by (1-3), with:

$$A_c = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}, \quad A_d = \begin{pmatrix} I_n & 0 \\ K & 0 \end{pmatrix}, \quad x(t) = \begin{pmatrix} z(t) \\ u(t_k) \end{pmatrix} \tag{11}$$

with I_n as the identity matrix of $\mathbb{R}^{n\times n}$, and with the same timing parameters T and δ.

Example 1. This sampled data system is taken from the article [Briat (2013)], that compares results of LMI-based approaches for stability analysis of NPILS. Consider the state space plant model given by (10) with

$$A = \begin{pmatrix} 0 & 1 \\ 0 & -0.1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 0.1 \end{pmatrix}, \quad K = \begin{pmatrix} -3.75 & -11.5 \end{pmatrix} \tag{12}$$

After rewriting the problem in the form of (1-3) with matrices defined as in (11), we set $T = 10^{-5}$. For this system the matrix $e^{\tau A_c} A_d$ is Schur for $\tau \in [0, 1.72941]$. Table 1 compares the results obtained by our approach with those obtained by other existing methods and reported in [Briat (2013)]. Our result is similar to the least conservative result which was obtained by the method presented in [Seuret and Peet (2013)]. More precisely, stability could be proven up to $\delta_{max} = 1.7294$ using Algorithm 1 with parameters $L = 2$ (number of subsystems chosen to find the initial set $\mathcal{P}_0$), $k_{max} = 1$ (i.e. $\mathcal{P}_1 \subseteq int(\mathcal{P}_0)$) and the number of time steps used for the over-approximation of the reachable set is $N = 1011$. The computation time was 0.1511 seconds.

Example 2. The second sampled-data control system is also taken from [Briat (2013)], with:

$$A = \begin{pmatrix} 0 & 1 \\ -2 & 0.1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad K = \begin{pmatrix} 1 & 0 \end{pmatrix}. \tag{13}$$

Results obtained by our approach and by several others are also reported in Table 1. Our approach has better results since it was able to verify GUES up to $\delta = 1.488$, instead of $\delta = 1.428$ for the method presented in [Seuret and Peet (2013)]. Note that the system becomes unstable for $\delta = 1.489$ since the matrix $\prod_{i\in\mathbb{N}_{[1,2]}}(e^{T_i A_c} A_d)$ has eigenvalues outside the unit circle for $T_1 = 0.4$ and $T_2 = 1.889$. This shows the tightness of our results. Algorithm 1 was used with parameters $L = 2$, $k_{max} = 30$ and the number of time steps used for the over-approximation of the reachable set is $N = 100$. The stability condition $\mathcal{P}_k \subseteq \mathrm{int}(\bigcup_{i=0}^{k-1} \mathcal{P}_i)$ was verified for $k = 14$. The computation time was 2.76 seconds.

We now consider the timing contract synthesis problem for the sampled-data system given by matrices (13). We used Algorithm 2 with parameters $i_{max} = 100$, $\varepsilon = 0.01$, $T_{min} = 0.2109$ and $T_{max} = 2.02$. Parameters of Algorithm 1 were $L = 2$, $k_{max} = 10$ and the number of time steps used for the over-approximation of the reachable set was $N = 100$. Figure 1 shows the regions Θ parameters in the (T_m, T_M) domains, for which GUES is guaranteed. The computation time was about 130 seconds.

6.2 Other example

Example 3. The following example is taken from the article [Hetel et al. (2013)] proposing an LMI-based approach to verify stability of a NPILS. Consider a system (1-3) with

$$A_c = \begin{pmatrix} 0 & -3 & 1 \\ 1.4 & -2.6 & 0.6 \\ 8.4 & -18.6 & 4.6 \end{pmatrix}, \quad A_d = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Table 1. Comparing the results on estimates of the maximum sampling uncertainty given by δ_{max}, for Examples 1 and 2.

	System (12)	**System** (13)	
	δ_{max}	T	δ_{max}
Briat (2013)	1.7279	0.4	1.427
Naghshtabrizi et al. (2008)	1.113	-	-
Liu et al. (2010)	1.695	-	-
Seuret and Peet (2013)	1.7294	0.4	1.428
Algorithm 1	1.7294	0.4	1.488

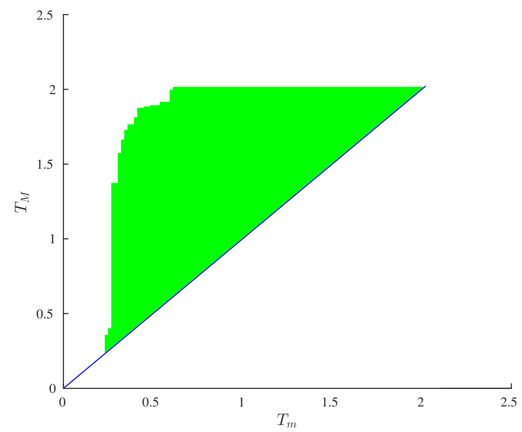

Fig. 1. Timing contract synthesis for system (12) in the (T_m, T_M) domain.

For this system the matrix $e^{TA_c} A_d$ is Schur for $T \in [0, 0.58]$. This implies that the system is stable if the reset occurs periodically, with constant reset interval $T \in [0, 0.58]$ and $\delta = 0$. Nevertheless, variations in the reset intervals may result in instability. As noted in [Hetel et al. (2013)], the matrix $\prod_{i\in\mathbb{N}_{[1,5]}}(e^{T_i A_c} A_d)$ has eigenvalues outside the unit circle for $T_1 = 0.515$ and $T_i = 0.1$, for $i \in \mathbb{N}_{[2,5]}$. As a result, for $T = 0.1$ the value $\bar{\delta} = 0.415$ is an upper bound for admissible values of δ. For $T = 0.1$, GUES could be proven up to $\delta = 0.2$ following the LMI approach in [Hetel et al. (2013)].

Table 2. Results of Algorithm 1 on Example 3 for several values of the parameters L, k_{max} and N: GUES could be proved up to $\delta = \delta_{max}$; T_{CPU} is the computation time in seconds; k is the index value for which the stability condition $\mathcal{P}_k \subseteq \mathrm{int}(\bigcup_{i=0}^{k-1} \mathcal{P}_i)$ is verified.

Parameter setup	δ_{max}	$T_{CPU}(s)$	k
A ($L = 1$, $k_{max} = 1$, $N = 100$)	0.01	0.16	1
B ($L = 1$, $k_{max} = 100$, $N = 100$)	0.4	0.8	7
C ($L = 2$, $k_{max} = 1$, $N = 100$)	0.4	2.1	1
D ($L = 2$, $k_{max} = 100$, $N = 100$)	0.414	8.5	24

Results obtained using Algorithm 1 with several parameter setups are reported in Table 2. In this example, parameter setups B, C, and D leads to less conservative results than the mentioned approach since stability is verified at least up to $\delta = 0.4$. Moreover, with parameter setup D, the verified value $\delta = 0.414$ is tight, since it is very close to the known upper-bound $\bar{\delta} = 0.415$. Figure 2 shows the projections on the first two states of $\mathcal{P}_{24}$ and $\bigcup_{i=0}^{23} \mathcal{P}_i$ computed by Algorithm 1 using parameter setup D for

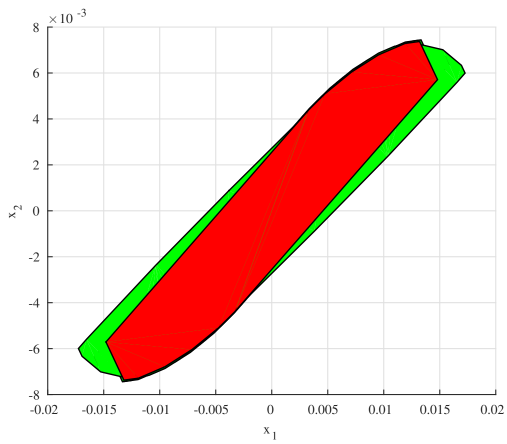

Fig. 2. $\mathcal{P}_{24}$ and $\bigcup_{i=0}^{23} \mathcal{P}_i$ computed by Algorithm 1 using parameter setup D for $\delta = 0.414$; $\mathcal{P}_{24} \subseteq \text{int}(\bigcup_{i=0}^{23} \mathcal{P}_i)$.

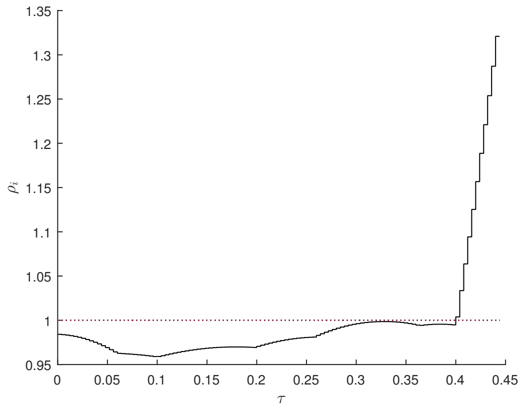

Fig. 3. $\rho_{\mathcal{P}}(\tau)$ for the polytope $\mathcal{P}$ given by the initial polytope $\mathcal{P}_0$ computed by Algorithm 1 with parameter setup C.

$\delta = 0.414$. Containment in the full space is also verified but is impractical to visualize.

Figure 3 illustrates the result on stochastic impulsive linear systems of Section 5. The figure shows the graph of the function $\rho_{\mathcal{P}}(\tau)$ defined in Proposition 13 for the polytope $\mathcal{P}$ given by the initial polytope $\mathcal{P}_0$ computed by Algorithm 1 with parameter setup C. One can check that $\rho_{\mathcal{P}}(\tau) < 1$ for $\tau \in [0, 0.4]$ which shows that the NPILS is GUES for $\delta = 0.4$. Then, we can check that the condition given by Proposition 12 for GUMES of SILS is verified for $\delta = 0.444$. The computation time was 2.16 seconds.

7. CONCLUSION

In this paper we derived a new reachability-based method to verify stability of NPILS and handle the problem of timing contract synthesis. Moreover, we have shown numerical examples where our method yields tight results and less conservativeness than several existing methods in literature. As a future work, it is interesting to verify stability for other hybrid systems under various timing contracts using reachability-based approaches.

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