Meromorphic quotients for some holomorphic G-actions (version 2 bis)

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Abstract. Using mainly tools from [B.13] and [B.15] we try to make a first step to obtain a "Transcendental Geometric Invariant Theory", that is to say to study conditions for the existence of "meromorphic quotients" for a holomorphic actions of a complex Lie group G on a reduced complex space X. In this article we give necessary and sufficient conditions [H.1] [H.2] and [H.3] on the $G-$orbits' configuration in X in order that a holomorphic action of a connected complex Lie group G on a reduced complex space X admits a **strongly quasi-proper meromorphic quotient**. Under these conditions a canonical (minimal) such quotient exists and it factorizes in a canonical way any $G-$invariant meromorphic map defined on X. In order to show how these conditions can be used, we apply this characterization to obtain that, when $G = K.B$ with B a closed complex subgroup of G and K a real compact subgroup of G, the existence of a strongly quasi-proper meromorphic quotient for the $B-$action implies the existence of a strongly quasi-proper meromorphic quotient for the $G-$action on X, assuming moreover that the action of B on X satisfies the condition [H.1str] on a $G-$invariant dense subset; we prove also that this last condition is automatically satisfied for G when K normalizes B and when [H.1str] [H.2] and [H.3] are satisfied for B. We also give a similar result when the connected complex Lie group has the form $G = K.A.K$ where A is a closed connected complex subgroup and K is a compact (real) subgroup assuming that the $A-$action satisfies the hypothesis [H.1str] on a $G-$invariant open set Ω_1, the hypothesis [H.2] on a $G-$invariant open set $\Omega_0 \subset \Omega_1$ and [H.3].
We prove the existence of a natural holomorphic map between the two meromorphic quotients of X for the actions of B and G (resp. of A and G) when they exist and we discuss the properness of this map.

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*Institut Elie Cartan, Géomètrie,
Université de Lorraine, CNRS UMR 7502 and Institut Universitaire de France.

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Contents

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1 Introduction

In this article we explain how the tools developed in [M.00], [B.08], [B.13] and [B.15] can be applied to produce, in suitable cases, a meromorphic quotient of a holomorphic action of a connected complex Lie group G on a reduced complex space X. This uses the notion of **strongly quasi-proper map** introduced in *loc. cit.* and our first goal is to give three hypotheses, called [H.1], [H.2], [H.3], on the $G-$orbits' configuration in X which are **equivalent** to the existence of a **strongly quasi-proper meromorphic quotient**, notion defined in the section 1.2.
The proof of this equivalence is the content of proposition 2.7.1 and theorem 2.8.1. Then we give a sufficient condition [H.1str], asking the existence of a $G-$invariant set $\Omega_1 \subset X$ which is dense, Zariski open and "good" for the action, to satisfy the condition [H.1].

Note that the conditions [H.1] [H.2] [H3] introduced in section 2.7 only depend on the $G-$orbits'configuration in X, but the condition [H.1str] depends on the action of G on X itself.
The existence theorem for a strongly quasi-proper meromorphic quotient under our three assumptions is applied to prove the following result:

Theorem 1.0.1 *Assume that we have a holomorphic action of a connected complex Lie group G on a reduced complex space X. Assume that $G = K.B$ where K is a compact (real) subgroup of G and B a connected complex closed subgroup of G. Assume that the action of B on X satisfies the condition* [H.1str] *on a $G-$invariant Zariski open dense subset Ω in X[1], and the conditions* [H.2] *and* [H.3]. *Then the $G-$action satisfies* [H.1str], [H.2] *and* [H.3]*; so it has a strongly quasi-proper meromorphic quotient.*

A first variant of this result is given by the following theorem.

Theorem 1.0.2 *Assume that we have a holomorphic action of a connected complex Lie group G on a reduced complex space X. Assume that $G = K.B$ where K is a compact (real) subgroup of G and B a connected complex closed subgroup of G. Assume that K normalizes B and that the $B-$action satisfies the conditions [H.1str], [H.2] and [H.3]. Then the $G-$action satisfies the conditions* [H.1str], [H.2] *and* [H.3] *and so has a strongly quasi-proper meromorphic quotient.*

Here is a second result obtained by a similar method.

Theorem 1.0.3 *Let G be a complex connected Lie group and assume that there exists a closed connected complex subgroup A and a compact (real) subgroup K such That $G = K.A.K$. Assume the we have a completely holomorphic action of G on an irreducible complex space X and that the action of A on X satisfies the following properties :*

i) There exists a $G-$invariant open set Ω_1 in X such that the hypothesis [H.1str] holds.

ii) There exists a dense $G-$invariant open set $\Omega_0 \subset \Omega_1$ such [H.2] holds.

iii) The hypothesis [H.3] holds.

Then [H.1str], [H.2] and [H.3] hold for the action of G on X. So there exists a SQP meromorphic quotient of X for the $G-$action.

Of course the hypothesis $G = K.A.K$ is more "general" than the case $G = K.B$. But the hypothesis of this last theorem is more restrictive for the action on X of the closed connected complex subgroup A of G : we ask also the $G-$invariance of

[1]This precisely means that there exists a $G-$invariant dense Zariski open set in X which is a "good open set" for the $B-$action (see section 2.5)

the dense open subset Ω_0 of Ω_1 (the open set Ω_0 is defined in the condition [H2]).

We conclude this article with two results relating the SQP meromorphic quotients for the actions of B and G (resp. of A and G) when they exist:

1. The existence of a holomorphic map $h : Q_B \to Q_G$ (resp. $Q_A \to Q_G$) between the corresponding quotients.

2. The existence under the hypotheses of the theorem 1.0.1 (resp. the theorem 1.0.3) of a $G-$invariant dense Zariski open set Ω disjoint from the centers of the modifications, such that the corresponding map $h_\Omega : q_B(\Omega) \to q_G(\Omega)$ (resp. $h_\Omega : q_A(\Omega) \to q_G(\Omega)$) is proper.

2 Strongly quasi-proper meromorphic quotients.

2.1 Preliminaries.

For the definition of the topology on the space $\mathcal{C}_n^f(X)$ of finite type $n-$cycles in X and its relationship with the topology of the space $\mathcal{C}_n^{loc}(X)$ we refer to [B-M] ch.IV, [B.13] and [B.15].
For the convenience of the reader we recall shortly here the definitions of a geometrically f-flat map (f-GF map) and of a strongly quasi-proper map (SQP map) between irreducible complex spaces and we give a short summary on some properties of the SQP maps. For more details on these notions see [B.13] and [B.15].

Definition 2.1.1 *Let $\pi : M \to N$ be a holomorphic map between two irreducible complex spaces and let $n := \dim M - \dim N$. We shall say that π is a* **geometrically f-flat map (a f-GF-map** *for short) if the following conditions are fulfilled:*

i) The map is quasi-proper equidimensional and surjective.

ii) There exists a holomorphic map $\varphi : N \to \mathcal{C}_n^f(M)$[2] such that for y generic[3] in N the cycle $\varphi(y)$ is reduced and equal to the set-theoretic fiber $\pi^{-1}(y)$ of π at y.

A holomorphic map $\pi : M \to N$ between two irreducible complex spaces will be **strongly quasi-proper (SQP map** *for short) if there exists a modification*[4] *$\tau : \tilde{N} \to N$ such that the strict transform*[5] *$\tilde{\pi} : \tilde{M} \to \tilde{N}$ of π by τ is a f-GF map.*
A meromorphic map $M \dashrightarrow N$ will be called **strongly quasi-proper** *when the projection on N of its graph is a SQP map.*

[2] That is to say a f-analytic family of finite type $n-$cycles in M parametrized by N.

[3] A dense subset in N of such y is enough here, thanks to the proposition 3.2.2 of [B.15].

[4] A modification is, by definition, always proper.

[5] By definition $\tilde{M}$ is the irreducible component of $M \times_N \tilde{N}$ which dominates $\tilde{N}$ and $\tilde{\pi}$ is induced by the projection.

Note that a f-GF map has, by definition, a **holomorphic fiber map** and that a SQP holomorphic (or meromorphic) map has a **meromorphic fiber map** via the composition of the holomorphic fiber map of $\tilde{\pi}$ with the (holomorphic) direct image map for finite type $n-$cycles $\tau_* : \mathcal{C}_n^f(\tilde{M}) \to \mathcal{C}_n^f(M)$. Of course, a SQP holomorphic map is quasi-proper, but the converse is not true. The notion of strongly quasi-proper map is stable by modification of the target space, property which is not true in general for a quasi-proper map having "big fibers" (see [B.15] for such an example).

Let $\pi : M \to N$ be a SQP map between irreducible complex spaces and define $n := \dim M - \dim N$. By definition of a SQP map, we can find a Zariski open dense subset N_0 in N and a holomorphic map $\varphi_0 : N_0 \to \mathcal{C}_n^f(M)$ such that

i) For each y in N_0 we have the equality of subsets $|\varphi_0(y)| = \pi^{-1}(y)$.

ii) For y generic in N_0 the cycle $\varphi_0(y)$ is reduced.

Let $\Gamma \subset N_0 \times \mathcal{C}_n^f(M)$ be the graph of φ_0. Then, thanks to the theorem 2.3.6 of [B.13], the closure $\bar{\Gamma}$ of Γ in $N \times \mathcal{C}_n^f(M)$ is proper over N. Then, using the semi-proper direct image theorem 2.3.2 of [B.15], this implies that $\tilde{N} := \bar{\Gamma}$ is an irreducible complex space (locally of finite dimension) with the structure sheaf induced by the sheaf of holomorphic functions on $N \times \mathcal{C}_n^f(M)$. Moreover the natural projection $\tau : \tilde{N} \to N$ is a (proper) modification.
Let $\tilde{M} := M \times_{N,str} \tilde{N}$ the strict transform of M by τ, that is to say the irreducible component of $M \times_N \tilde{N}$ containing the graph of the restriction π_0 of π to the open set $\pi^{-1}(N_0)$[6]. Then let $\tilde{\pi} : \tilde{M} \to \tilde{N}$ the strict transform of π by the modification τ; it is induced on $\tilde{M}$ by the natural projection of $M \times_N \tilde{N}$ onto $\tilde{N}$. The set-theoretical fiber at $\tilde{y} := (y, C) \in \tilde{N}$ of $\tilde{\pi}$ is the subset $|C| \times \{\tilde{y}\}$ in $\tilde{M}$. The map $\psi : \tilde{N} \to \mathcal{C}_n^f(\tilde{M})$ given by $(y, C) \mapsto C \times \{\tilde{y}\}$ is holomorphic and satisfies $|\psi(\tilde{y})| = \tilde{\pi}^{-1}(\tilde{y})$ for all $\tilde{y}$ in $\tilde{N}$. Moreover $\psi(\tilde{y})$ is a reduced cycle for generic $\tilde{y}$ in $\tilde{N}$. So the map $\tilde{\pi}$ is geometrically f-flat. It is the **canonical f-GF-flatning** of π.
Then we have an isomorphism induced by ψ

$$\psi : \tilde{N} \longrightarrow \mathcal{C}_n^f(\tilde{\pi})$$

where $\mathcal{C}_n^f(\tilde{\pi}) := \{C \in \mathcal{C}_n^f(\tilde{M}) \ / \ \exists \tilde{y} \in \tilde{N} \ s.t. \ |C| \subset \tilde{\pi}^{-1}(\tilde{y})\}$ is a closed analytic subset of $\mathcal{C}_n^f(\tilde{M})$ (see [B.15] proposition 2.1.7.); the inverse map is induced by the holomorphic map $\hat{\pi} : \mathcal{C}_n^f(\tilde{\pi}) \to \tilde{N}$ which associates to $\gamma \in \mathcal{C}_n^f(\tilde{\pi})$ the point in $\tilde{N}$ whose $\tilde{\pi}-$fiber contains γ (see *loc. cit.*).
The direct image of $n-$cycles by τ gives a holomorphic map $\tau_* : \mathcal{C}_n^f(\tilde{M}) \to \mathcal{C}_n^f(M)$ which sends $\tilde{N} \simeq \mathcal{C}_n^f(\tilde{\pi})$ in $\mathcal{C}_n^f(\pi)$. Let us show that it is an isomorphism of $\tilde{N}$ onto its image in $\mathcal{C}_n^f(\pi)$:
We have an obvious holomorphic map $\tilde{N} \to \mathcal{C}_n^f(\pi)$ given by $(y, C) \mapsto C$. We have also a holomorphic map $\mathcal{C}_n^f(\pi) \to N \times \mathcal{C}_n^f(M)$ given by $C \mapsto (\hat{\pi}(C), C)$ where $\hat{\pi} : \mathcal{C}_n^f(\pi) \to N$ is the map associating to $C \in \mathcal{C}_n^f(\pi)$ the point $y \in N$ such that

[6]This graph is a Zariski open set in $M \times_N \tilde{N}$ which is irreducible as M an N are irreducible.

$|C| \subset \pi^{-1}(y)$. This proves our claim.
Remark that the closed analytic subset $\mathcal{C}_n^f(\pi)$ of $\mathcal{C}_n^f(M)$ is not, in general, even locally, a complex space of finite dimension.

2.2 Action of G on $\mathcal{C}_n^f(X)$.

Let G be a Lie group. We shall say that G acts **continuously holomorphically** on the reduced complex space X when the action $f : G \times X \to X$ is a continuous map such that for each $g \in G$ fixed, the map $x \mapsto f(g, x)$ is a (biholomorphic) automorphism of X. Then there is a natural action of G induced on the set $\mathcal{C}_n^f(X)$ of finite type $n-$cycles given by $(g, C) \mapsto g_*(C)$ where we denote $g_*(C)$ the direct image of the cycle C by the automorphism of X associated to $g \in G$. When G is a complex Lie group and the map f is holomorphic we shall say that the action is **completely holomorphic**.

Proposition 2.2.1 *Let G be a Lie group acting continuously holomorphically on a reduced complex space X. Then the action of G on $\mathcal{C}_n^f(X)$ is continuously holomorphic. This means that the map*

$$G \times \mathcal{C}_n^f(X) \to \mathcal{C}_n^f(X) \quad \text{given by} \quad (g, C) \mapsto g_*(C)$$

is continuous and holomorphic for each fixed $g \in G$.
If G is complex Lie group and the action is completely holomorphic, the action of G on $\mathcal{C}_n^f(X)$ is completely holomorphic; so, for any f-analytic family of $n-$cycles $(C_s)_{s\in S}$ in X parametrized by a reduced complex space S, the family of $n-$cycles $g_*(C_s)_{(g,s)\in G\times S}$ parametrized by $G \times S$ is f-analytic.

Proof. First we prove the continuity of the action of G on $\mathcal{C}_n^{loc}(X)$. To apply the theorem IV 2.5.6 of [B-M] it is enough to see that the map $F : G \times X \to G \times X$ given by $(g, x) \mapsto (g, g.x)$ is proper. But if $L \subset G$ and $K \subset X$ are compact sets, we have $F^{-1}(L \times K) \subset L \times (L^{-1}.K)$ which is a compact set in $G \times X$.
The only point left to prove the continuity statement for the topology of $\mathcal{C}_n^f(X)$, assuming that the continuity for the topology of $\mathcal{C}_n^{loc}(X)$, is obtained as follows :
Let W be a relatively compact open set in X and $\mathcal{W}$ be the open set in $\mathcal{C}_n^f(X)$ of cycles C such any irreducible component of C meets W. Then we want to show that the set of $(g, s) \in G \times S$ such that $g_*(C_s)$ lies in $\mathcal{W}$ is an open set in $G \times S$. As the topology of $\mathcal{C}_n^f(X)$ has a countable basis[7] it is enough to show that if a sequence (g_ν, s_ν) converges to (g, s) with $g_*(C_s) \in \mathcal{W}$ then for $\nu \gg 1$ we have also $(g_\nu)_*(C_{s_\nu}) \in \mathcal{W}$. If this not the case, we can choose for infinitely many ν an irreducible component Γ_ν of $(g_\nu)_*(C_{s_\nu})$ which does not meet W. Up to pass to a sub-sequence, we may assume that the sequence (Γ_ν) converges in $\mathcal{C}_n^{loc}(X)$ to a cycle

[7] This is a corollary of the fact that this is true for $\mathcal{C}_n^{loc}(X)$ (see [B-M] ch.IV) as the topology of X has a countable basis of open sets; see the lemma 2.1.1 in [B.15] for details.

Γ which does not meet W and is contained in $g_*(C_s)$. This is a simple consequence of the continuity of the $G-$action on $\mathcal{C}_n^{loc}(X)$ and the characterization of compact subsets in $\mathcal{C}_n^{loc}(X)$ (see [B-M] ch.IV). As any irreducible component of $g_*(C_s)$ meets W this implies that Γ is the empty $n-$cycle. This means that for any compact K in X there exists an integer $\nu(K)$ such that for $\nu \geq \nu(K)$ we have $\Gamma_\nu \cap K = \emptyset$. Choose now a compact neighbourhood L of $g(K)$. For ν large enough we shall have $K \subset g_\nu^{-1}(L)$. This comes from the fact that the automorphisms g_ν^{-1} converge to g^{-1} in the compact-open topology. Then this implies that for $\nu \geq \nu(L)$ the irreducible component $g_\nu^{-1}(\Gamma_\nu)$ of C_{s_ν} does not meet K. Then, when $s_\nu \to s$ the cycles C_{s_ν} does not converge to C_s for the topology of $\mathcal{C}_n^f(X)$ because we have some "escape at infinity" in a well choosen sub-sequence. Contradiction. ■

Lemma 2.2.2 *Let $\pi : M \to N$ be a SQP map between irreducible complex spaces. Assume now that a Lie group H acts continuously holomorphically on M and N and that π is $H-$equivariant for these actions. Let $\tau : \tilde{N} \to N$ be the canonical modification giving the canonical f-GF flatning of π and let $\tilde{\pi} : \tilde{M} \to M$ be the strict transform of π by the modification τ. Then $\tilde{N}$ and $\tilde{M}$ have natural continuous holomorphic actions of H such τ and $\tilde{\pi}$ are $H-$equivariant.*
Moreover, if H is a complex Lie group and if it acts completely holomorphically on M and N, it acts also completely holomorphically on $\tilde{M}$ and $\tilde{N}$.

PROOF. Let $N_0 \subset N$ the open dense subset of points y in N such that y is normal and $\pi^{-1}(y)$ is purely $n-$dimensional. Then N_0 is $H-$stable because H acts on M and N by bi-holomorphic automorphisms. As π is quasi-proper, the theorem IV 3.4.1 of [B-M] gives a holomorphic map $\varphi_0 : N_0 \to \mathcal{C}_n^f(M)$ such that, for each $y \in N_0$, we have $|\varphi_0(y)| = \pi^{-1}(y)$, with $\varphi_0(y)$ reduced for y generic. Then the $H-$equivariance of π implies the $H-$equivariance of φ_0 for the action of H on $\mathcal{C}_n^f(M)$ defined in the proposition 2.2.1. So the graph Γ of φ_0 is stable by the $H-$action and so is its closure $\tilde{N}$ in $N \times \mathcal{C}_n^f(M)$. Then $\tau : \tilde{N} \to N$ induced by the first projection is $H-$equivariant.
Now the strict transform of $\tilde{M}$ by τ is the closure in $M \times_N \tilde{N}$ of the subset $\pi^{-1}(N_0 \times_N N_0) \simeq \pi^{-1}(N_0)$. As it is stable by the action of H, so is its closure, and the map $\tilde{\pi} : \tilde{M} \to \tilde{N}$ induced by the second projection is then $H-$equivariant. The case where H is complex and acts completely holomorphically on M and N follows from the previous proposition. ■

We shall also use the following simple tool from the cycle's space.

Proposition 2.2.3 *Let M be a reduced complex space and $(X_s)_{s\in S}$ the tautological f-continuous family of $d-$dimensional finite type cycles parametrized by a compact subset S in $\mathcal{C}_d^f(M)$. Let $(C_t)_{t\in T}$ be the tautological family of $n-$dimensional cycles in M parametrized by a compact subset $T \subset \mathcal{C}_n^{loc}(M)$. We assume the following condition:*

- *There exists a dense subset T' in T such that each $C_t, t \in T'$, is non empty and equal to the union of some X_s.* (@@)

Then the property (@@) *is satisfied for any $t \in T$ and T is in fact a compact subset of $\mathcal{C}_n^f(M)$.*

PROOF. First remark that, as S is compact in $\mathcal{C}_d^f(M)$, there exists a compact set $L \subset M$ such that any irreducible component of any X_s meets L.
Let $(t_m)_{m\in\mathbb{N}}$ be a sequence of points in T' converging to a point $t \in T$ and denote by C_m the cycle C_{t_m} for short and $C_t = C_\infty$. Now choose for each m an irreducible component Γ_m of some X_{s_m} contained in C_m. Remark that this is possible because $t_m \in T'$ implies that (@@) holds. Up to pass to a sub-sequence, we may assume that Γ_m converges in $\mathcal{C}_d^f(M)$ to a cycle Γ which is not empty (it contains at least a point in L) and is included in $|C_\infty|$. So C_∞ is not the empty cycle.
Let x be a generic point of an irreducible component D of C_∞. Then, up to pass to a sub-sequence, we may choose a sequence (x_m) of points respectively in C_m which converges to x. Choose for each m an irreducible component Γ_m of some $X_{s_m} \subset |C_m|$ which contains x_m. This is possible again because of condition (@@) holds. Now, again up to pass to a sub-sequence, we may assume that the sequence $(\Gamma_m)_{m\in\mathbb{N}}$ converges in $\mathcal{C}_d^f(M)$ to a cycle Γ containing the point x and contained in $|C_\infty|$. Note that $|\Gamma|$ contains an irreducible component of some $|X_{s_\infty}|$ containing x, as we may assume, by compactness of S, that the sequence (s_m) converges to $s_\infty \in S$. Then we have $|X_{s_\infty}| \subset |C_\infty|$. As D is the only irreducible component of C_∞ containing x, it contains at least an irreducible component of $|X_{s_\infty}|$ containing x, and so D meets L. So we have proved that C_∞ is not the the empty $n-$cycle and that any irreducible component of C_∞ meets the compact set L. This is enough to conclude thanks to the corollary 2.1.3 in [B.15]. ■

2.3 f-GF holomorphic quotient.

We shall consider a complex connected Lie group G and a completely holomorphic action of G on an irreducible complex space X.

Definition 2.3.1 *We shall say that the action of G on X has a* **quasi-proper GF holomorphic quotient, (a f-GF holomorphic quotient** *for short), when there exists a $G-$invariant quasi-proper geometrically flat holomorphic map $q : X \to Q$ onto a reduced complex space Q such that each fiber of q is set-theoretically equal to an orbit of G in X.*

REMARK. Assume that G is connected and that we have a $G-$invariant open set U and a GF surjective holomorphic map $q : U \to Q$ to a reduced complex space such that for each $x \in U$ we have the set-theoretic equality $q^{-1}(q(x)) = G.x$. Then the map q is quasi-proper: let $y := q(x)$ be any point in Q and let $V(x)$ be a relatively compact open neighbourhood of x in U. As the map q is open, $q(V(x))$ is an

open neighbourhood of y and for any $y' \in q(V(x))$ there exists $x' \in V(x)$ such that $q^{-1}(y') = G.x'$. So the fiber $q^{-1}(y')$ meets the compact set $\bar{V}(x)$ of U, proving the quasi-properness of the map q. □

Proposition 2.3.2 *In the situation of the definition above there exists a holomorphic map, where we have defined* $n := \dim X - \dim Q$,

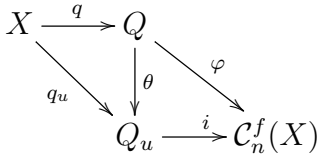
$$\Phi : X \longrightarrow \mathcal{C}_n^f(X)$$

with the following properties

i) The subset $Q_u := \Phi(X)$ *of* $\mathcal{C}_n^f(X)$ *is a closed analytic subset of finite dimension (so an irreducible complex space) with the complex structure sheaf induced by the structure sheaf of* $\mathcal{C}_n^f(X)$[8].

ii) The map $id_X \times \Phi : X \to X \times \mathcal{C}_n^f(X)$ *induces an isomorphism of* X *on the set-theoretic graph of the tautological family of finite type* $n-$*cycles in* X *parametrized by* Q_u.

iii) There is a canonical isomorphism $\theta : Q \to Q_u$ *such the diagram*



commutes, where φ *is the holomorphic map classifying the fibers of the f-GF map* q *and where* $\Phi = q_u \circ i = q \circ \varphi$.

Conversely, if there exists a holomorphic map $\Phi : X \longrightarrow \mathcal{C}_n^f(X)$ *such that for each* $x \in X$ *we have* $|\Phi(x)| = G.x$ *then* $Q_u := \Phi(X)$ *is a closed analytic subset in* $\mathcal{C}_n^f(X)$ *of finite dimension and the map* $q_u : X \to Q_u$ *induced by* Φ *is a f-GF holomorphic quotient for the* $G-$*action on* X.

Proof. The theorem 3.1.9 of [B.15] gives that the classifying map $\varphi : Q \to \mathcal{C}_n^f(X)$ for the fibers of the f-GF map q is a proper holomorphic embedding. As $\Phi := q \circ \varphi$ and q is surjective, this gives the fact that Q_u is a closed locally finite dimensional subset of $\mathcal{C}_n^f(X)$ and that the map $\theta : Q \to Q_u$ induced by φ is an isomorphism. Then the map $id_X \times \Phi$ sends X to the set-theoretic graph of the tautological family of finite type $n-$cycles in X parametrized by Q_u. The inverse map is given by the projection on X, so it is an isomorphism on this graph[9]. The point iii) is now clear. The converse is consequence of the theorem 2.3.2 of [B.15] as soon as we proved the following claim:

[8]Recall that a continuous function $g : \mathcal{U} \to \mathbb{C}$ on an open set $\mathcal{U} \subset \mathcal{C}_n^f(X)$ is holomorphic if and only if for any holomorphic map $f : S \to \mathcal{U}$ of a reduced complex space S in $\mathcal{U}$ the composed function $f \circ g$ is holomorphic on S.

[9]Note that, as the generic fiber of q_u is a reduced cycle, the cycle-graph is also reduced.

- the holomorphic map Φ is semi-proper.

Consider first the case of a non empty cycle $C_0 \in \mathcal{C}_n^f(X), C_0 \notin \Phi(X)$. There exists two points $x, y \in |C_0|$ such that $y \notin G.x$. Then choosing two adapted $n-$scales E_x, E_y to C_0 such that x and y are respectively in the domains of E_x and E_y with $\deg_{E_x}(C_0) \geq 1$ and $\deg_{E_y}(C_0) \geq 1$. Then any cycle which is near enough to C_0 in $\mathcal{C}_n^f(X)$ has the same degrees in these adapted scales. But as $\Phi(x) \neq \Phi(y)$ there exists two disjoint $G-$invariant open sets U and V in X containing respectively x and y. If we choose E_x to be a scale on U and E_y to be a scale on V we know that each cycle near enough to C_0 cannot be an orbit (set-theoretically) so is not in $\Phi(X)$.
Let now $C_0 = \Phi(x)$ and let W be an open relatively compact neighbourhood of x in X. Let $\mathcal{W}$ be the open set in $\mathcal{C}_n^f(X)$ of cycles C such any irreducible of C meets W. Then we have the equality $\Phi(X) \cap \mathcal{W} = \Phi(\bar{W}) \cap \mathcal{W}$: if $C = \Phi(y)$ is in $\mathcal{W}$ then $G.y$ has to meet W by definition of $\mathcal{W}$ and so there exists $z \in W$ such that $z \in G.y$. But then $G.z = G.y$ and we have $|\Phi(z)| = |\Phi(y)|$. But then $\Phi(y) = \Phi(z)$ as the cycles in $\Phi(X)$ are disjoint or equal; and we have find a $z \in \bar{W}$ with $\Phi(z) = \Phi(y)$, proving our claim. ■

REMARK. It is easy to see that if, for x generic in X, the cycle $\Phi(x)$ is not reduced, there exists an integer $k \geq 2$ such that, for x generic in X we have $\Phi(x) = k.[G.x]$. Then, as X is irreducible, there exists a holomorphic map $\Phi_1 : X \to \mathcal{C}_n^f(X)$ such for any $x \in X$ we have $\Phi(x) = k.\Phi_1(x)$ and we can replace Φ by Φ_1 and then, for generic x in X, the cycle $\Phi(x)$ is reduced.

The following obvious corollary of the proposition 2.3.2 will be useful.

Corollary 2.3.3 *If X admits a f-GF holomorphic quotient $q : X \to Q$ for the $G-$action, the map q is unique up to an unique isomorphism of Q.* ■

Lemma 2.3.4 *Let $(\Omega_i)_{i\in I}$ be a collection of $G-$invariant open sets in X such that for any $i \in I$ and any $x \in \Omega_i$ the orbit $G.x$ is a closed set (resp. a closed analytic subset) in Ω_i. Then for each $x \in \Omega := \cup_{i\in I}\ \Omega_i$ the orbit $G.x$ is a closed set (resp. a closed analytic subset) in Ω. Moreover, if any orbit is a closed analytic subset in Ω, the subset*

$$Z := \{(x, y) \in \Omega \times \Omega \ / \ G.x = G.y\}$$

is a locally closed analytic subset in $\Omega \times \Omega$ as soon as its intersection with $\Omega_i \times \Omega_i$ is a closed analytic subset of $\Omega_i \times \Omega_i$ for each $i \in I$.

PROOF. Let $x \in \Omega$ and $y \in \overline{G.x} \cap \Omega$. Let $j \in I$ such that y belongs to Ω_j. Choose an open neighbourhood V of y in Ω_j. Then $V \cap G.x$ is not empty. But if z is in $V \cap G.x$ we have $z \in \Omega_j$ and also $G.x = G.z$ lies in Ω_j. As $G.x$ is closed in Ω_j we conclude that y is in $G.x$ and so $G.x$ is closed in Ω. The assertion on analyticity is

obvious.
Consider now the open set

$$B := \{(x, y) \in \Omega \times \Omega \ / \ \exists i \in I \text{ such that } x \in \Omega_i \text{ and } y \in \Omega_i\}.$$

Then $Z \cap B$ is closed in B and is clearly analytic in this open set. ■

Corollary 2.3.5 *Assume that we can cover the $G-$space X by a family of by $G-$invariant open sets $(\Omega_i)_{i\in I}$ such that for each $i \in I$ we have a f-GF holomorphic quotient $q_i : \Omega_i \to Q_i$. Then if the family of closed sets $(\partial\Omega_i)_{i\in I}$ is locally finite in X, the open $G-$invariant dense set $X' := X \setminus \cup_{\imath\in I}\ \partial\Omega_i$ has a f-GF holomorphic quotient $q : X' \to Q'$.*

Proof. First remark that if X admits a holomorphic f-GF quotient $q : X \to Q$ then for any $G-$invariant open set $\Omega \subset X$ the restriction of q to Ω induces a holomorphic f-GF quotient $q_\Omega : \Omega \to Q_\Omega := q(\Omega)$ because the map q is open and $\Omega = q^{-1}(q(\Omega))$ as Ω is $G-$invariant and so $q-$saturated.
Note also that, as X is countable at infinity, we may assume that I is countable.
For any $(i, j) \in I^2$ such that the open set $\Omega_i \cap \Omega_j$ is not empty, the $G-$invariant open set $\Omega_i \cap \Omega_j$ has two holomorphic f-GF quotients :

$$q_{i|_{\Omega_i\cap\Omega_j}} : \Omega_i \cap \Omega_j \to Q_{i,j} := q_i(\Omega_i \cap \Omega_j) \quad \text{and}$$
$$q_{j|_{\Omega_i\cap\Omega_j}} : \Omega_i \cap \Omega_j \to Q_{j,i} := q_j(\Omega_i \cap \Omega_j)$$

thanks to our first remark. Then we have a canonical isomorphism $\theta_{i,j} : Q_{i,j} \to Q_{j,i}$, and again by the uniqueness assertion of the previous corollary we have

$$\theta_{i,j} \circ \theta_{j,k} \circ \theta_{k,i} = \mathrm{id} \quad \text{on} \quad \Omega_i \cap \Omega_j \cap \Omega_k \quad \forall (i, j, k) \in I^3.$$

So there exists a, may be non Hausdorff, locally reduced complex space Q obtained by identifying $Q_{i,j}$ to $Q_{j,i}$ via $\theta_{i,j}$ in the disjoint union of the $Q_i, i \in I$ and a holomorphic map $q : X \to Q$ such that its restriction to Ω_i is equal to q_i for each $i \in I$. It is then easy to see that the open subset Q' of Q obtained by deleting the image in Q of $F := \cup_{\imath\in I}\ \partial\Omega_i$ is a Hausdorff reduced complex space : to see this, it is enough to produce, for any distinct points $x \neq y$ in X' a pair of disjoint $G-$invariant neighbourhoods of x and y. If there exists $i \in I$ such both are in Ω_i this is a consequence of the separation of the quotient Q_i. If this is not the case, there exists $i \neq j \in I$ such that $x \in \Omega_i \setminus \bar{\Omega}_j$ and $y \in \Omega_j \setminus \bar{\Omega}_i$. If $V(x)$ and $V(y)$ are open neighbourhoods of x and y respectively in $\Omega_i \setminus \bar{\Omega}_j$ and $\Omega_j \setminus \bar{\Omega}_i$, then $G.V(x)$ and $G.V(y)$ answer the question.■

2.4 SQP meromorphic quotient.

Definition 2.4.1 *A* **strongly quasi-proper meromorphic quotient**, *we shall say a* **SQP-meromorphic quotient** *for short, for a completely holomorphic action $f : G \times X \to X$ of a complex Lie group G on an irreducible complex space X will be the following data:*

1. *a $G-$modification*[10] $\tau : \tilde{X} \to X$ *with center* Σ.
2. *a $G-$invariant holomorphic f-GF map $q : \tilde{X} \to Q$ where Q is a (irreducible) complex space.*
3. *an analytic $G-$invariant subset $Y \subset X$ containing Σ, with no interior point in X.*

We shall denote $\tilde{Y} := \tau^{-1}(Y)$, $\Omega := X \setminus Y, \tilde{\Omega} := \tau^{-1}(\Omega)$ and $Q' := q(\tilde{\Omega})$. Note that, as q is an open surjective map, Q' is open and dense in Q.
Now we ask that these data satisfy the following properties:

i) The restriction to Ω of the map $\tau^{-1} \circ q$ is a f-GF map onto the dense open set Q' in Q and there is an open dense subset Q'' in Q' such that the holomorphic map $\Omega_1 := (\tau^{-1} \circ q)(Q'') \to Q''$ is a f-GF holomorphic quotient.

ii) There exists an open dense $G-$invariant subset $\Omega_0 \subset \Omega_1$ such that for each $\tilde{x}$ in $\tilde{\Omega}_0 := \tau^{-1}(\Omega_0)$ the **closure** *$\overline{G.\tilde{x}}$ of $G.\tilde{x}$ in $\tilde{X}$ is exactly the reduced cycle $q^{-1}(q(\tilde{x}))$.*

Note that, in general, the $G-$invariant f-GF map $q : \tilde{X} \to Q$ is not a holomorphic f-GF quotient of $\tilde{X}$ as the generic $G-$orbits are not closed in $\tilde{X}$. Also, in general, the restriction of q to Ω is not a f-GF quotient (see the example below).

To illustrate this definition which seems a little complicate, let us look at the following very simple (algebraic) example.

Example. [from many discussions in Bochum] Let $G := \mathbb{C}^*$ and $X := \mathbb{P}_2$ the action given by $t.(x_0, x_1, x_2) := (t.x_0, t^{-1}.x_1, x_2)$. Then there exist 3 fixed points $O := (0, 0, 1), P := (1, 0, 0), Q := (0, 1, 0)$ and 3 orbits which are the punctured lines OP, OQ, PQ which are copies of $\mathbb{C}^*$. The other orbits are the conics $\{x_0.x_1 = s.x_2^2\}$ for $s \in \mathbb{C}^*$.
The SQP meromorphic quotient for this action is given by the (quasi-proper) meromorphic map

$$q : \mathbb{P}_2 \dashrightarrow \mathbb{P}_1, \quad (x_0, x_1, x_2) \mapsto (x_0.x_1, x_2^2).$$

The graph of this meromorphic map is the blow-up $\tilde{X}$ of $\mathbb{P}_2$ in the 3 points O, P, Q. Then the $G-$invariant open dense subset $\Omega_1 := \mathbb{P}_2 \setminus \{(PQ) \cup (OP)\} \simeq \mathbb{C}^* \times \mathbb{C}$ admits

[10] This means that we have a completely holomorphic $G-$action on $\tilde{X}$ and that the modification τ is $G-$equivariant.

a f-GF holomorphic quotient map $(x, y) \mapsto x.y$.
Note that we can make another choice : $\Omega_1' := \mathbb{P}_2 \setminus \{(PQ) \cup (OQ)\} \simeq \mathbb{C} \times \mathbb{C}^*$ with the same map $(x, y) \mapsto x.y$ but now x may vanishes and $y \neq 0$.
Then we can choose the $G-$invariant open set

$$\Omega_0 := \mathbb{P}_2 \setminus \{(PQ) \cup (OP) \cup (OQ)\} = \Omega_1 \cap \Omega_1'.$$

Note that for $(x, 0) \in \Omega_1$ we have

$$\tau(q^{-1}(q(x, 0)) = \overline{G.(x, 0)} \cup (OQ) \quad \text{and} \quad \tau(\overline{G.(x, 0)}) \cup \{O\} \cup \{P\}$$

so q does not induce a f-GF holomorphic quotient on $\mathbb{P}_2 \setminus \{0, P, Q\}$. □

Remark. In the situation of a connected complex Lie group acting completely holomorphically on an irreducible complex space X, the irreducibility of X gives that the subset of points in X for which the stabilizer has a dimension strictly bigger than the generic dimension is a closed analytic $G-$invariant subset Y_0 in X with no interior point. Then it is clear that any $G-$invariant open set Ω_1 for which there exists a f-GF holomorphic quotient has to be contained in $X \setminus Y_0$. In fact the $G-$invariant open dense subset $X \setminus Y_0$ is the first and best "candidate" for such an open set. But the example above shows that, even in the algebraic context, assuming moreover that each orbit in $X \setminus Y_0$ is a closed analytic subset in $X \setminus Y_0$, only some smaller open sets may have a f-GF holomorphic quotient.

Proposition 2.4.2 *Let G be a complex connected Lie group which acts completely holomorphically on an irreducible complex space X. Assume that we have a SQP meromorphic quotient for this action, given by a $G-$modification $\tau : \tilde{X} \to X$ and a $G-$invariant holomorphic f-GF map $q : \tilde{X} \to Q$.*
Let $\psi : Q \to \mathcal{C}_n^f(X)$ be the holomorphic map obtained by the composition of the fiber map of the f-GF map q and the direct image map for $n-$cycles by the modification τ. Define $Q_u := \psi(Q)$. Then we have the following properties:

1. *Q_u is a closed analytic subset in $\mathcal{C}_n^f(X)$ which is an irreducible complex space of finite dimension with the structure sheaf induced by the sheaf of holomorphic functions on $\mathcal{C}_n^f(X)$.*

2. *Let $\tilde{X}_u$ be the graph of the meromorphic map $q_u : X \dashrightarrow Q_u$ given by the holomorphic map $\psi \circ q : \tilde{X} \to Q_u$ and let $\tau_u : \tilde{X}_u \to X$ and $q_u : \tilde{X}_u \to Q_u$ be the projections on X and Q_u respectively of this graph. Then (τ_u, q_u) is also a SQP meromorphic quotient for the given $G-$action.*

3. *For any SQP meromorphic quotient (τ, q) there exists a unique holomorphic surjective map $\eta : Q \to Q_u$ such that the meromorphic maps $q : X \dashrightarrow Q$ and $q_u : X \dashrightarrow Q_u$ satisfies $\eta \circ q = q_u$.*

Definition 2.4.3 *In the situation of the previous proposition the SQP meromorphic quotient for the given $G-$action defined by (τ_u, q_u) will be called* **the minimal SQP meromorphic quotient** *of this $G-$action.*

So the proposition above says that the existence of a SQP meromorphic quotient for the given $G-$action implies the existence and uniqueness of a minimal meromorphic quotient for this $G-$action.

Proof of the proposition 2.4.2. To prove the point 1. we shall prove that the map

$$\psi \circ q : \tilde{X} \to \mathcal{C}_n^f(X)$$

is semi-proper. Let $C \neq \emptyset$ be in $\mathcal{C}_n^f(X)$ and fix a relatively compact open set W in X meeting all irreducible components of C. The subset $\mathcal{W}$ of $\mathcal{C}_n^f(X)$ of cycles C' such that any irreducible component of C' meets W is an open set containing C. Now $q(\tau^{-1}(\bar{W}))$ is a compact set in Q, as τ is proper. Take any $y \in Q$ such that $C' := \psi(y)$ is in $\mathcal{W}$. The point y is the limit in Q of points $y_\nu \in q(\tilde{\Omega}_0)$ such that the fiber of q at y is limit in $\mathcal{C}_n^f(\tilde{X})$ of the fibers $q^{-1}(y_\nu) = \overline{G.\tilde{x_\nu}}$ where, for $\nu \gg 1$, we can choose $\tilde{x}_\nu$ in $\tilde{\Omega}_0 \cap \tau^{-1}(W)$. Up to pass to a sub-sequence, we may assume that the sequence $(\tilde{x}_\nu)$ converges to a point $\tilde{x}$ in $\tau^{-1}(\bar{W})$. Then the continuity of q implies that $q(\tilde{x}) = y$ and C' is the limit of $\overline{G.\tilde{x}_\nu}$. So $|C'|$ is in the image by ψ of the compact set $q(\tau^{-1}(\bar{W}))$ and this gives the semi-properness of $\psi \circ q$.
Now the direct image theorem 2.3.2 in [B.15] shows that Q_u is an irreducible complex space (locally of finite dimension) and the point 1. is proved.
Now, using the $G-$invariance of the map $\psi \circ q$ we see that Q_u is point by point invariant by the natural action of G on $\mathcal{C}_n^f(X)$ defined in the proposition 2.2.1. Then the graph $\tilde{X}_u$ of the $G-$invariant meromorphic map $q_u : X \dashrightarrow Q_u$ is $G-$stable in $X \times Q_u$ and the $G-$action induced on it makes the projection $q_u : \tilde{X}_u \to Q_u$ $G-$invariant and the projection $\tau_u : \tilde{X}_u \to X$ $G-$equivariant.
To prove the second point we have to show that the map $q_u : \tilde{X}_u \to Q_u$ is a f-GF map. By definition $\tilde{X}_u$ is the closure in $X \times Q_u$ of the graph of the map $q_{|\Omega_0}$ where $\Omega_{0,u} := \Omega_0$ is an open dense set in X such that for any point $x \in \Omega_0$ we have $\psi(q(x)) = \overline{G.x}$ (as Ω_0 is disjoint from the center of the modification τ we identify via τ the open sets Ω_0 and $\tilde{\Omega}_0 := \tau^{-1}(\Omega_0)$). Then, by irreducibility of Q_u and X, the closed analytic subset $\tilde{X}_u \subset X \times Q_u$ is equal to the graph of the tautological family of cycles in X parametrized by $Q_u \subset \mathcal{C}_n^f(X)$. This will complete the proof of the point 2. when we shall be able to find

i) A closed $G-$invariant analytic subset $\tilde{Y}_u$ with no interieur point containing $\tau_u^{-1}(\Sigma_u)$ where Σ_u is the center of the modification τ_u; then let $Y_u := \tau(\tilde{Y}_u)$.

ii) A dense $G-$invariant open set $\Omega_{1,u} \subset X \setminus Y_u$ such that the restriction of q_u to $\Omega_{1,u}$ will give a f-GF holomorphic quotient for the action of G on $\Omega_{1,u}$.

These facts will be deduced from the point 3.
Now consider a SQP meromorphic quotient of the $G-$action given by the maps

$\tau : \tilde{X} \to X$ and $q : \tilde{X} \to Q$. Then, by the construction in the proof of the point 1, we know that $\psi \circ q(\tilde{X}) = Q_u$ and then the map $\psi \circ q$ induces a surjective holomorphic map $\eta : Q \to Q_u$. Now $\tilde{X}$ is a $G-$equivariant modification of the graph $\tilde{X}_0$ of the $G-$invariant meromorphic map $q : X \dashrightarrow Q$. Then, as $\tilde{X}_u$ is the graph of the $G-$invariant meromorphic map $q_u : X \dashrightarrow Q_u$ the holomorphic map $\mathrm{id}_X \times \eta : X \times Q \to X \times Q_u$ sends $\tilde{X}_0$ to $\tilde{X}_u$ because this is true over the dense open set Ω_0 in X where the maps q and q_u are holomorphic and satisfy $q^{-1}(q(x)) = q_u^{-1}(q_u(x)) = \overline{G.x}$. This complete the proof of the point 3.
Now the composition of the holomorphic $G-$equivariant modifications $\tilde{X} \to \tilde{X}_0$ and $\tilde{X}_0 \to \tilde{X}_u$ allows to define $\tilde{Y}_u$ and $\Omega_{1,u}$ as the images of $\tilde{Y}$ and Ω_1 by this modification. ■

Note that the subsets $\Omega_{0,u}, \Omega_{1,u}$ and $\tilde{Y}_u, Y_u$ are not intrinsically defined in $\tilde{X}_u$.

Corollary 2.4.4 *In the situation of the previous proposition, assume that another Lie group H acts continuously holomorphically on X. Assume that the action of H normalizes the action of G, that is to say that for any $(h, x) \in H \times X$ we have $h.[G.x] = G.(h.x)$. Then the minimal SQP meromorphic quotient of X for the action of G is $H-$equivariant in the following sense:*

- *There are natural continuously holomorphic $H-$actions on $\tilde{X}_u$ and on Q_u such that the maps $\tau_u : \tilde{X}_u \to X$ and $q_u : \tilde{X}_u \to Q_u$ are $H-$equivariant.*

Moreover, if H is a complex Lie group and the action of H on X is completely holomorphic, the natural actions of H on $\tilde{X}_u$ and Q_u are completely holomorphic.

Proof. As H acts on $\mathcal{C}_n^f(X)$ via $(h, C) \mapsto h_*(C)$ the only thing to prove is the fact that $\tilde{X}_u \subset X \times \mathcal{C}_n^f(X)$ is stable by the action of H. But, by definition of the minimal SQP meromorphic quotient, $\tilde{X}_u$ is the closure in $X \times \mathcal{C}_n^f(X)$ of the set of couples $(x, \overline{G.x})$ for x in a dense $G-$invariant open set in X.
We have for such an x,

$$h.(x, \overline{G.x})) = (h.x, h_*(\overline{G.x}) = (h.x, \overline{h.[G.x]}) = (h.x, \overline{G.(h.x)})$$

as $h.[G.x] = G.(h.x)$ and the fact that H acts by complex automorphisms on X.
If H is a complex Lie group and the action of H on X is completely holomorphic, its action on $\mathcal{C}_n^f(X)$, X_u and Q_u are also completely holomorphic. ■

The next result shows that the minimal SQP quotient for a $G-$action on X factorizes any $G-$invariant meromorphic map defined on X.

Theorem 2.4.5 *Let G be a complex Lie group acting on a irreducible complex space X with a minimal SQP meromorphic quotient $\tau : \tilde{X} \to X$ and $q : \tilde{X} \to Q$. Let $\gamma : \tilde{X} \to T$ be a $G-$invariant holomorphic map. Then there exists a holomorphic map $h : Q \to T$ such $\gamma = h \circ q$.*

Moreover, for any $G-$invariant meromorphic map[11] $\gamma : X \dashrightarrow T$ *there exists a $G-$equivariant modification $\theta : Z \to X$ of X on which γ and q are holomorphic and a holomorphic map $H : Q \to T$ such that $\gamma = q \circ H$ on Z.*

The main ingredient to prove this theorem is the following lemma (see lemma 2.1.8 in [B.15] for a proof).

Lemma 2.4.6 *Let U and B be open polydiscs in $\mathbb{C}^n$ and $\mathbb{C}^p$ and let $F : U \times B \to \mathbb{C}^N$ be a holomorphic map. For any positive integer k the subset $S(F)$ of $H(\bar{U}, \mathrm{Sym}^k(B))$ of the multiform graphs contained in a fiber of F is a closed banach analytic subset of $H(\bar{U}, \mathrm{Sym}^k(B))$. Moreover, the map $\tilde{F} : S(F) \to \mathbb{C}^N$ given by the value of F on $X \in S(F)$, is a holomorphic map on $S(F)$.* ■

Proposition 2.4.7 *Let X be a reduced complex space and let $(C_s)_{s\in S}$ be a f-analytic family of $n-$cycles in X parametrized by a reduced complex space S. Let $h : X \to T$ be a holomorphic map and assume that the restriction of h to the set $|C_s|$, for s in a dense set S' in S, is constant. Then there exists a holomorphic map $H : S \to T$ such that for each $s \in S$ we have $|C_s| \subset h^{-1}(H(s))$.*

Note that this means that the classifying map $\varphi : S \to \mathcal{C}_n^f(X)$ of the f-analytic family $(C_s)_{s\in S}$ takes its values in $\mathcal{C}_n^f(h)$; the holomorphic map H is then the composition of φ with the natural holomorphic map $\tilde{h} : \mathcal{C}_n^f(h) \to T$ (see [B.15] prop. 2.1.7).

Proof. A local embedding of T in an open set in $\mathbb{C}^N$ and the consideration of finitely many adapted scales[12] to a cycle C_{s_0} of the family allow to deduce the proposition from the lemma above. ■

The following corollary of the proposition 2.4.7 is immediate, as, by definition, the fibers of a f-GF holomorphic map is given by a f-analytic family of cycles.

Corollary 2.4.8 *Let $q : X \to Q$ a f-GF holomorphic map between irreducible complex spaces. Let $n := \dim X - \dim Q$. If $h : X \to T$ is a holomorphic map such that for y in a dense set in Q the restriction of h to the fiber $q^{-1}(y)$ is constant, then there exists a holomorphic map $H : Q \to T$ such that $h = q \circ H$.* ■

Proof of the theorem 2.4.5. Let $\sigma : Z \to \tilde{X}$ be the strict transform of the modification $\theta : Y \to X$ given by the projection of the graph of γ on X by the modification $\tau : \tilde{X} \to X$. As these two modifications are $G-$equivariant, this is also the case for σ, and the maps q and γ are holomorphic on Z. The generic fiber of q is the closure of a $G-$orbit in $\tilde{X}$, so it is contained in a fiber of γ as G is connected.

[11] By definition that means that there exists a $G-$equivariant modification $\theta : Y \to X$ and a $G-$invariant holomorphic map $\tilde{\gamma} : Y \to T$ corresponding to the projections of the graph of γ.

[12] Here the quasi-properness over S of the graph of the family is used in a crucial way in order that a finite number of adapted scales are enough to determine, locally on the parameter space, the cycles of the family .

Then the corollary 2.4.8 applies and we have on Z the factorization $\gamma = q \circ H$ where $H : Q \to T$ is a holomorphic map. ■

2.5 Good points, good open sets.

Let X be a reduced complex space and G be a connected complex Lie group. Let $f : G \times X \to X$ be a completely holomorphic action of G on X. We shall often note $g.x := f(g, x)$ for $g \in G$ and $x \in X$.

Definition 2.5.1 *We shall say that a point $x \in X$ is a* **good point** *for the action f if the following condition is satisfied*

- *For each compact set K in X there exists an open neighbourhood V of x and a compact set L in G such that if $y \in V$ and $g \in G$ are such that $g.y \in K$, there exists $\gamma \in L$ with $\gamma.y = g.y$*

We shall say that the action of G on X is **good** *when each point in X is a good point. If Ω is a $G-$invariant open set in X, we shall say that Ω is a* **a good open set** *for the action f when all points in Ω are good points for the $G-$action given by f* **restricted to** *Ω.*

Remarks.

1. If $x \in X$ is a good point, then, for any $g_0 \in G$, $g_0.x$ is also a good point: for K given, choose $g_0.V$ as neighbourhood of $g_0.x$ and the compact set $L.g_0^{-1} \subset G$ to satisfy the needed conditions.

2. Consider $\Omega' \subset \Omega$ two $G-$invariant open sets in X and assume that $x \in \Omega'$ is a good point in Ω. Then x is a good point in Ω'.
   So, if Ω is a good open set, Ω' is also a good open set.

3. But conversely, if $x \in \Omega' \subset \Omega$ is a good point in Ω', it is not true, in general, that x is a good point in Ω. So if Ω is a good open set, points in Ω are not in general good points for the action on X.

4. If M is a compact set of good points in X for any compact set K in X we can find a neighbourhood V of M in X and a compact set L in G such that for any point $y \in V$ and any $g \in G$ such that $g.y \in K$ there exists $\gamma \in L$ with $\gamma.y = g.y$. This is easily obtained by a standard compactness argument. We shall say that a compact set of good points is **uniformely good**.

Lemma 2.5.2 *Let x be a point in X. Then x is a good point for the $G-$action on X if and only if the map $F_X : G \times X \to X \times X$ given by $(g, x) \mapsto (x, g.x)$ is*

semi-proper at each point of $\{x\} \times X$. *As a consequence a* $G-$*invariant open set* Ω *in* X *is a good open set for the* $G-$*action if and only if the map* $F_\Omega : G \times \Omega \to \Omega \times \Omega$ *given by* $(g, x) \mapsto (x, g.x)$ *is semi-proper.*

PROOF. Let $x \in X$ be a good point and fix any $z \in X$. To prove that the map F_X is semi-proper at (x, z) choose compact neighbourhoods V_0 and K of x and z in X and apply the definition of a good point to the compact set K. So we can find a neighbourhood V of x, that we may assume to be contained in V_0, and a compact set L in G such that for any $y \in V$ such that $g.y \in K$ we have a $\gamma \in L$ with $g.y = \gamma.y$. Then we have $F_X(G \times X) \cap (V \times K) = F_X(L \times V_0) \cap (V \times K)$ and $L \times V_0$ is a compact set in $G \times X$.
Conversely, assume that the map F_X is semi-proper at each point of $\{x\} \times X$. Take a compact set K in X and apply the semi-properness to each point (x, z) where z is in K. For each $z \in K$ we obtain open neighbourhoods V_z and W_z of x and z in X and a compact set $L_z \times M_z$ in $G \times X$ such that

$$F_X(G \times X) \cap (V_z \times W_z) = F_X(L_z \times M_z) \cap (V_z \times W_z).$$

Extract a finite sub-cover $W_1, \ldots, W_N$ of K by the open sets W_z and define the compact set $L := \cup_{i \in [1,N]} L_{z_i}$ and the neighbourhood $V := \cap_{i \in [1,N]} V_{z_i}$ of x. Then if y is in V and $g.y$ is in K there exists $i \in [1, N]$ such that $g.y \in W_i$. As y is in V_i we can find a $\gamma \in L_{z_i} \subset L$ with $F_X(g, y) = F_X(\gamma, y)$ and this implies that x is a good point. The second assertion is an easy consequence of the first one. ■

Lemma 2.5.3 *Let* G *be a connected complex Lie group. Let* $f : G \times X \to X$ *be a completely holomorphic action of* G *on a reduced complex space* X. *Consider a countable family* $(\Omega_i)_{i \in I}$ *of good open sets for* f *and the family* $(F_i)_{i \in I}$ *of closed sets in* $\Omega := \cup_{i \in I} \Omega_i$ *defined by* $F_i := \partial\Omega_i \cap \Omega$. *Let* $F := \cup_{i \in I} F_i$. *Then any point in the dense set* $\Omega \setminus F$ *of* Ω *is a good point in* Ω.

PROOF. Of course Ω is a $G-$invariant open set in X as a good open set is $G-$invariant by definition. Then $F \cap \Omega$ is a $G-$invariant set in Ω which is a countable union of nowhere dense closed sets in Ω. So $\Omega \setminus F$ is a dense G_δ in Ω. Now let $x \in \Omega \setminus F$ and K be a compact subset in Ω. Choose a sub-covering $(\Omega)_i, i \in [1, N]$ of K by some open sets $\Omega_i, i \in [1, N]$ and let $i \in [1, p], p \in [0, N]$ the subset of $i \in [1, N]$ such that x is in Ω_i. Choose now a compact neighbourhood V of x such that V is contained in $\cap_{i=1}^{p} \Omega_i$ and such that $V \cap \Omega_j = \emptyset$ for each $j \in [p+1, N]$. This is possible because x is not in $\partial\Omega_j$ for $j \in [p+1, N]$. Remark that if y is in V and $g.y$ is in K, by the $G-$invariance of Ω_j we have $g.y \notin \Omega_j$ for each $j \in [p+1, N]$. So let $\tilde{K} := K \setminus \cup_{j=p+1}^{N} \Omega_j$. This is a compact set in $\cup_{i=0}^{N} \Omega_i$. Choose now compact sets $K_1, \ldots, K_p$ such that $K_i \subset \Omega_i$ and such that $\tilde{K} \subset \cup_{i=1}^{p} K_i$. For each $i \in [1, p]$, as x is in Ω_i and as K_i is a compact set in Ω_i which is a good open set, there exist an open neighbourhood $W_i \subset V$ of x and a compact set L_i in G such that for any

$y \in W_i$ and any $g \in G$ such that $g.y$ is in K_i there exists a $\gamma \in L_i$ with $\gamma.y = g.y$. Let $W := \cap_{i=1}^p W_i$ and $L := \cup_{i=1}^p L_i$. If now $y \in W$ and $g \in G$ are such that $g.y$ is in K, then first we have $g.y$ which is in $\tilde{K}$. If $g.y$ is in K_{i_0} as y is in W_{i_0} there exists $\gamma \in L_{i_0} \subset L$ such that $g.y = \gamma.y$. This shows that any x in $\Omega \setminus F$ is a good point in the $G-$invariant open set Ω. ■

Remarks.

1. If the family $(F_i)_{i\in I}$ is locally finite in Ω then $\Omega \setminus F$ is a dense good open set in Ω.

2. As Ω is countable at infinity (it is an open set in a complex space), if the family $(\Omega_i)_{i\in I}$ is not countable, it is always possible to find a countable sub-family $(\Omega_i)_{i\in I'}$ such that $\Omega = \cup_{i\in I'} \Omega_i$.

Proposition 2.5.4 *Let G be a connected complex Lie group. Let $f : G \times X \to X$ be a completely holomorphic action of G on a reduced complex space X. Then we have the following properties :*

i) If x is a good point for f the orbit $G.x$ is a closed analytic subset of X.

ii) If x is a good point for f and if $(G.x) \cap K = \emptyset$ where $K \subset X$ is a compact set, there exists a neighbourhood V of x in X such that $(\overline{G.x'}) \cap K = \emptyset$ for any x' in V (x' is not assumed here to be a good point).

iii) If x is a good point for f there exists a neighbourhood V of x such that any good point $x' \in V$ has an orbit which is a closed analytic subset of the same dimension than $G.x$.

iv) Let Ω be a good connected open set in X which is normal and let n be the dimension of $G.x$ for $x \in \Omega$. Then there exists a holomorphic map[13] *$\varphi : \Omega \to \mathcal{C}_n^f(\Omega)$ given generically by $\varphi(x) := G.x$ as a reduced $n-$cycle in Ω.*

v) When we have a good open set Ω in X which is normal, there exists a quasi-proper GF holomorphic quotient of Ω for the action restricted to Ω.

Proof. We already proved that x is a good point if the map $G \to X$ given by $g \mapsto g.x$ s semi-proper in lemma 1.3.2. Now Kuhlmann's theorem [K.64], [K.66] gives that $f_x(G) = G.x$ is a closed analytic subset of X. This proved i)

Assume ii) is not true ; then we have a compact set K such that $(G.x) \cap K = \emptyset$ and a sequence $(x_\nu)_{\nu\in\mathbb{N}}$ converging to x and such that $(\overline{G.x_\nu}) \cap K$ is not empty

[13] Recall that this means that we have an $f-$analytic family of $n-$cycles in Ω parametrized by Ω.

for each ν. Fix a compact neighbourhood $\tilde{K}$ of K such that $(G.x) \cap \tilde{K} = \emptyset$. This is possible thanks to i). Pick a point $y_\nu = \lim_{\alpha\to\infty} g_{\nu,\alpha}.x_\nu$ in $(\overline{G.x_\nu}) \cap K$ for each ν. Up to pass to a subsequence we may assume that sequence (y_ν) converges to $y \in K$ when $\nu \to +\infty$. So, for $\alpha \geq \alpha(\nu)$, we can assume that $g_{\nu,\alpha}.x_\nu$ is in $\tilde{K}$. But, as x is a good point, for the given compact set $\tilde{K}$ there exists a neighbourhood V of x and a compact set $L \subset G$ as in the definition. We may assume that x_ν is in V for $\nu \geq \nu_0$ and so we may find, for $\nu \geq \nu_0, \alpha \geq \alpha(\nu)$, elements $\gamma_{\nu,\alpha} \in L$ such that $\lim_{\alpha\to\infty} \gamma_{\nu,\alpha}.x_\nu = y_\nu \quad \forall \nu \geq \nu_0$.
Up to pass to a sub-sequence for each given $\nu \geq \nu_0$, we may assume that the sequence $(\gamma_{\nu,\alpha})_\alpha$ converges to some $\gamma_\nu \in L$. And again, that the sequence (γ_ν) converges to some $\gamma \in L$. So the continuity of f gives $y_\nu = \gamma_\nu.x_\nu \to \gamma.x = y \in \tilde{K}$ giving a contradiction because we assume $(G.x) \cap \tilde{K} = \emptyset$. This proves ii).

Let $E := (U, B, j)$ be a $n-$scale on Ω adapted to the $n-$cycle $G.x$. Then the compact set $K := j^{-1}(\bar{U} \times \partial B)$ does not meet $G.x$, by definition of an adapted scale. Using ii), there exists a neighbourhood V of x such that for any $x' \in V$ we have $(\overline{G.x'}) \cap K = \emptyset$. As for a good point $x' \in V$ we know that $G.x'$ is a closed analytic subset, the $n-$scale is then adapted to $G.x'$. This implies that the dimension of $G.x'$ is at most equal to n. But the semi-continuity of the dimension of the stabilizers implies that the dimension of $G.x' \simeq G/St(x')$ is at least equal to $n = \dim(G/St(x))$. This proves iii).

Remark that for any $x' \in V$ such that $\overline{G.x'} \cap \Omega$ is a (closed) analytic subset in Ω, the previous proof shows also that $\overline{G.x'} \cap \Omega$ is of dimension n.

To prove iv) fix a good connected open set Ω and define

$$Z := \{(g, x, y) \in G \times \Omega \times \Omega \ / \ y = g.x.\}$$

This is a closed analytic subset in $G \times \Omega \times \Omega$. It is isomorphic to $G \times \Omega$ by the projection $(g, x, y) \mapsto (g, x)$ and so the projection $p : Z \to \Omega \times \Omega$ is semi-proper, thanks to the lemma 2.5.2. Its image $p(Z)$ is then a closed analytic subset of $\Omega \times \Omega$ by Kuhlmann's theorem [K.64], [K.66], which is a generalization of Remmert's theorem [R.57] to the semi-proper case. But now the projection

$$\pi : p(Z) \to \Omega$$

is $n-$equidimensional, thanks to iii), and has irreducible generic fibers on a normal basis Ω. So its fibers (with generic multiplicity equal to 1) define an analytic family of $n-$cycles of X parametrized by Ω. It is clearly $f-$analytic because each fiber is irreducible[14] and we have an holomorphic section because each x lies in $G.x$.
To prove v) let us prove that the holomorphic map $\varphi : \Omega \to \mathcal{C}_n^f(\Omega)$ classifying the fibers of $p(Z)$ is semi-proper. Fix a non empty cycle $C \in \mathcal{C}_n^f(\Omega)$ and choose a point $x_i, i \in [1, k]$, in each irreducible component of $|C|$. Let W a relatively compact open

[14]But some multiplicities may occur.

neighbourhood of $\{x_1, \ldots, x_k\}$ in Ω and let $\mathcal{W}$ be the open set in $\mathcal{C}_n^f(\Omega)$ of cycles such that each irreducible component meets W. Let C' be in $\mathcal{W} \cap \varphi(\Omega)$; we know that if $C' = \varphi(z)$ we have $|C'| = G.z$. So $G.z$ has to meet W and we can choose y in the compact set $\bar{W}$ such that $|C'| = |\varphi(y)|$; but the equality of supports implies equality of cycles in this family. So $\varphi(z) = \varphi(y)$. This gives the semi-properness of φ. Now the semi-proper direct image theorem 2.3.2 of [B.15] implies that the image Q of φ is a locally finite dimensional reduced complex space. Moreover, it parametrizes a f-analytic family of $n-$cycles in Ω which coïncides generically with the reduced $G-$orbits. Then the holomorphic map $q : \Omega \to Q$ is a quasi-proper GF holomorphic quotient for the action f on Ω as **each** fiber of q is set-theoretically a $G-$orbit.■

2.6 Nice points.

We consider a connected complex Lie group G and a completely holomorphic action of G on a reduced complex space X given by a holomorphic map

$$f : G \times X \to X.$$

Define the closed analytic subset

$$Z := \{(g, x, y) \in G \times X \times X \ / \ y = g.x\}$$

and let $p : Z \to X \times X$ the natural projection. Remark that Z is isomorphic to $G \times X$ and that the projection $p : Z \to X \times X$ is equivalent, via this isomorphism, to the map $f \times \mathrm{id} : G \times X \to X \times X$ given by $(g, x) \mapsto (g.x, x)$.

Definition 2.6.1 *We shall say that a couple $(x, y) \in X \times X$ is a* **good couple** *when there exists open neighbourhoods $V(x)$ and $V(y)$ respectively of x and y in X and a compact subset L in G such that for any $x' \in V(x)$, any $y' \in V(y)$ and any $g \in G$ such that $y' = g.x'$ there exists $\gamma \in L$ with $y' = \gamma.x'$.*

Remarks.

1. A couple (x, y) is good if and only if the map $p : Z \to X \times X$ is semi-proper at the point (x, y), and then it is semi-proper at any point in $V(x) \times V(y)$. So the set of good couples in $X \times X$ is an open subset.

2. A couple (x, y) is good if and only the couple (y, x) is good.

3. Assume that the couple (x, y) is good ; then, with the notation of the previous definition, for any $x' \in V(x)$ the subset $G.x' \cap V(y)$ is a closed subset in $V(y)$. Also for any $y' \in V(y), G.y' \cap V(x)$ is closed in $V(x)$.

4. If a couple (x, y) is good, then for any $(g_1, g_2) \in G \times G$, the couple $(g_1.x, g_2.y)$ is a good couple. So the subset of good couple in $X \times X$ is $(G \times G)-$invariant.

5. For any point $x \in X$ the subset $\Omega(x) := \{y \in X \ / \ (x,y) \text{ is a good couple}\}$ is an open $G-$invariant subset in X. Moreover, the open set $\Omega(x)$ only depends on the orbit $G.x$ of x.

In order to obtain a canonical $G-$invariant open set in X on which there exists (at least locally) a f-GF holomorphic quotient we shall introduce the following notion.

Definition 2.6.2 *We shall say that a point x in X is a* **nice point** *when the couple (x,x) is a good couple.*

Note that nice points in X are points corresponding via the diagonal embedding $\delta : X \to X \times X$ to the intersection of the set of good couples with the diagonal. So the subset of nice points is a $G-$invariant open set in X (may be empty !). We have the following characterization of nice points in X.

Lemma 2.6.3 *A point $x \in X$ is a nice point if and only there exists a $G-$invariant open set U containing x such that x is a good point in U.*

Proof. For any $x \in X$ the subset $\Omega(x)$ is a $G-$invariant open set, and x is a nice point in X if and only if x is in $\Omega(x)$ by definition. Let us show that in this case x is a good point in $\Omega(x)$. Then take any compact set K in $\Omega(x)$. For any $y \in K$ the couple (x,y) is a good couple, so there exist $V_y(x), V(y)$, respectively open neighbourhoods of x and y in $\Omega(x)$ and a compact set L_y in G such that for any $x' \in V_y(x), y' \in V(y)$ such that $y' = g.x'$ for some $g \in G$, there exists $\gamma \in L_y$ with $\gamma.x' = y'$. Extract a finite sub-cover of the open cover of the compact K by the $V(y), y \in K$ corresponding to the points $y_1, \ldots, y_N$. Let $W(x) := \cap_{i=1}^N V_{y_i}(x)$ and $L := \cup_{i=1}^N L_{y_i}$. Then for any $x' \in W(x)$ and any $g \in G$ such that $g.x' = z$ is in K, as there exists $i \in [1,N]$ such that $z \in V(y_i)$ and as x' is in $V_{y_i}(x)$ we can find $\gamma \in L_{y_i} \subset L$ with $\gamma.x' = z$. So x is a good point in $\Omega(x)$.
Conversely, if x is a good point in the $G-$invariant open set U in X, let V be a relatively compact open neighbourhood of x in U. Let $K := \bar{V}$. As x is a good point in U, for the compact set K in U we may find $V(x)$ an open neighbourhood of x in U and a compact set L in G such that for any $x' \in V(x)$ and any $g \in G$ such that $g.x'$ lies in K we can find $\gamma \in L$ such that $\gamma.x' = g.x'$. As we may assume that $V(x)$ is contained in V we see that the couple (x,x) is a good couple thanks to the choices $V(x), V(x)$ and L. ■

Corollary 2.6.4 *A point $x \in X$ is a nice point if and only it is contained in a good open set in X.*

Proof. As the set of nice points in X is open, we can find a compact neighbourhood V of x such that any couple $(x',y') \in V \times V$ is a good couple. Now define

$$\Omega(V) := \{y \in X \ / \ \forall x' \in V \text{ the couple } (x',y) \text{ is good}\}.$$

Then $\Omega(V)$ is a $G-$invariant open set in X and the interior V_0 of V is an open set of good points in $\Omega(V)$. Then the $G-$invariant open set $G.V_0$ is a good open set containing x. The converse is obvious thanks to the previous lemma. ■

The next proposition summarizes the relations between the several definitions given above.

Proposition 2.6.5 *Let Ω be a $G-$invariant open set in X. Then the following properties holds:*

i) A point $x \in \Omega$ is a good point in Ω if and only if $\Omega \subset \Omega(x)$.

ii) Ω is a good open set if and only if $\Omega \times \Omega$ is contained in the open set of good couples in $X \times X$. This implies that each point of Ω is a nice point but also (see i)) that $\Omega \subset \cap_{x\in\Omega}\ \Omega(x)$.

iii) A $G-$invariant open set Ω in X is a good open set if and only if the restriction map $(f \times \mathrm{id})_{|\Omega} : G \times \Omega \to \Omega \times \Omega$ is semi-proper.

iv) The set of nice points in X, denoted Ω_{nice}, is the union of all good open sets in X. But, in general, it is not true that Ω_{nice} is itself a good open set (see the lemma 2.5.3).

v) On the $G-$invariant open set of normal points of Ω_{nice} we have **locally** *a f-GF holomorphic quotient for the $G-$action.*

Proof. As Ω is locally compact, it is clear that for any good point x in Ω and any point $y \in \Omega$ the couple (x, y) is a good couple. The converse is easy (see the begining of the proof of the lemma 2.6.3). If Ω is a good open set it is again easy to see that any couple $(x, y) \in \Omega \times \Omega$ is a good couple. The converse is also analoguous to the begining of the proof of the lemma 2.6.3.
The assertion iii) is already in the lemma 2.5.2. The inclusion of good open set in Ω_{nice} is obtained in lemma 2.6.3 and the corollary 2.6.4 implies the equality. The assertion v) is consequence of iv) and of the property v) in the proposition 2.5.4. ■

2.7 The conditions [H.1], [H.1str], [H.2] and [H.3].

Now we shall consider the following conditions on the action f.

- **There exists a $G-$invariant dense open set Ω_1 in X which admits a quasi-proper GF holomorphic quotient**. [**H.1**]

Recall that this means that there exists a $G-$invariant geometrically f-flat holomorphic map $q : \Omega_1 \to Q_1$ onto a reduced complex space Q_1 such that each fiber of q

over a point in Q_1 is set-theoretically an orbit in Ω_1. So the hypothesis [H.1] implies that all $G-$orbits in Ω_1 are closed analytic subsets in Ω_1 of the same dimension.

The following stronger form will be useful.

- **There exists a $G-$invariant good open set Ω_1 which is Zariski open and dense in X.** [**H.1str**]

This condition implies [H.1] thanks to the proposition 2.5.4. But it is not true in general that [H.1] implies the existence of a dense good open set in X.

Remarks.

1. The existence of a dense good open set in X is a natural hypothesis on the $G-$action to obtain [H.1], thanks to the proposition 2.5.4. We add here the condition "Zariski open" for this dense good open set because this assumption is crucial in our application in order to use the "sub-analytic lemma".

2. A good open set is always contained in Ω_{nice} which is a $G-$invariant open set canonically defined by the action. So a necessary condition for the existence of a dense good open set in X is the density of Ω_{nice} in X. Also to have a dense Zariski good open set in X it is necessary that the complement of Ω_{nice} is contained in a closed nowhere dense analytic subset in X.

3. The hypotheses [H.1] concerns only the structure of the orbits in a $G-$invariant dense open set Ω. The existence of a dense good open set in X involves the defining map $f : G \times X \to X$ for the action of G on X. So these hypotheses are at different levels.

Now assume [H.1] and define $\mathcal{R} := \{(x, y) \in \Omega_1 \times \Omega_1 \ / \ y \in G.x\}$. It is a closed analytic set in $\Omega_1 \times \Omega_1$: on the $G-$invariant open set Ω_1 on which there exists a f-GF holomorphic quotient $q_1 : \Omega_1 \to Q_1$, so the equality $G.x = G.y$ is equivalent to $q_1(x) = q_1(y)$.
Our second assumption will be :

- **The closure $\tilde{\mathcal{R}}$ of $\mathcal{R}$ in $X \times X$ is an analytic subset and there exists a $G-$invariant open dense subset $\Omega_0 \subset \Omega_1$ such that for each $x \in \Omega_0$**

$$\overline{\mathbf{G.x}} = \tilde{\mathcal{R}} \cap (\{\mathbf{x}\} \times \mathbf{X}). \qquad [\mathbf{H.2}]$$

Remark. For $x \in \Omega_0$ the orbit $G.x$ is a closed analytic subset in Ω_1, so, as we have $\tilde{\mathcal{R}} \cap (\{x\} \times \Omega_1) = (\{x\} \times G.x)$, $G.x$ is open in $\overline{G.x}$. Then $\overline{G.x}$ is irreducible of dimension n and when Ω_1 is Zariski open in X, $\overline{G.x} \setminus G.x$ is a closed analytic subset and has dimension at most $n-1$. As this analytic set is $G-$invariant, it is contained in $Y_0 \subset X$, the closed analytic subset in X where the stabilizer has a

bigger dimension than the generic one.

Now the first projection $p_1 : \tilde{\mathcal{R}} \cap (\Omega_0 \times X) \to \Omega_0$ is quasi-proper because we have a holomorphic section of this map (with irreducible fibers, thanks to $[H.2]$) which is given by $x \mapsto (x, x)$.
Assuming that Ω_0 contains only normal points[15] in X, the equidimensionality and quasi-properness on Ω_0 of the projection of $\tilde{\mathcal{R}}$ imply that there exists a holomorphic map

$$\bar{\varphi}_0 : \Omega_0 \to \mathcal{C}_n^f(X)$$

where the supports are given by $x \mapsto \overline{G.x}$ and where the multiplicity is generically equal to 1. Our last hypothesis is :

- **The first projection $p_1 : \tilde{\mathcal{R}} \to X$ is strongly quasi-proper .** [**H.3**]

Recall that this condition implies that there exists a modification $\tau : \tilde{X} \to X$ with center in the complement of Ω_0 such that the map $\bar{\varphi}_0$ extends holomorphically to $\tilde{X}$. Note that, thanks to [B.13] theorem 2.4.4 (see also [B.15] proposition 3.2.2) a sufficient condition for [H.3] is that the closure in $X \times \mathcal{C}_n^f(X)$ of the graph of $\bar{\varphi}_0$ is proper over X.
The following proposition shows that these conditions [H.1], [H.2] and [H.3] are necessary for the existence of a SQP-meromorphic quotient for a completely holomorphic action of G on X.

Proposition 2.7.1 *Assuming that $f : G \times X \to X$ is a completely holomorphic action of the connected complex Lie group G on the irreducible complex space X which has a SQP-meromorphic quotient, then the conditions* [H.1], [H.2] *and* [H.3] *are satisfied.*

Proof. Let $\tau : \tilde{X} \to X$ and $q : \tilde{X} \to Q$ be respectively the $G-$equivariant modification of X and the f-GF holomorphic map given by the existence of a SQP meromorphic quotient for the $G-$action on X. The conditions to be a SQP-meromorphic quotient gives an open set Ω_1 which is dense, $G-$stable and which admits a f-GF holomorphic quotient for the action of G on Ω_1. So [H.1] is clear.
Let S be the graph of the equivalence relation given by q on $\tilde{X}$. Then the proper direct image $(\tau \times \tau)(S)$ is a closed analytic subset in $X \times X$.
Let $\mathcal{R} := \{(x, y) \in \Omega_1 \times \Omega_1 \ / \ G.x = G.y\}$. Then we have $\mathcal{R} \subset (\tau \times \tau)(S) \cap (\Omega_1 \times \Omega_1)$. To see that $\mathcal{R}$ is dense in $(\tau \times \tau)(S)$, remark that on the dense open set $\Omega_0 \times \Omega_1$ we have the equality of $\mathcal{R}$ and $(\tau \times \tau)(S)$. So the closure $\tilde{\mathcal{R}}$ of $\mathcal{R}$ in $X \times X$ is analytic. Moreover, for $x \in \Omega_0$ the fiber of q at $q(x)$ is equal to $\overline{G.x}$, the closure in $\tilde{X}$ of $G.x$. So the fiber of $\tilde{\mathcal{R}}$ at x is $\tau(\overline{G.x})$ which is equal to the closure in X of $G.x$ because τ is proper and $G-$equivariant. So the condition [H.2] is satisfied.
The composition of q with the holomorphic classifying map $\varphi : Q \to \mathcal{C}_n^f(\tilde{X})$ for the

[15] This not restrictive, as we may always assume that $X \setminus \Omega_0$ contains the non normal points in X. We shall always assume that Ω_0 is normal in the sequel, without any more comment.

fibers of q gives a holomorphic map $\tilde{\psi} : \tilde{X} \to \mathcal{C}_n^f(\tilde{X})$. Composed with the direct image map, which is holomorphic (see [B.M] ch.IV ; the "quasi-proper" part of this result is easy, as τ is proper) $\tau_* : \mathcal{C}_n^f(\tilde{X}) \to \mathcal{C}_n^f(X)$, we obtain a holomorphic map

$$\Phi : \tilde{X} \to \mathcal{C}_n^f(X)$$

and the restriction of this map to $\tau^{-1}(\Omega_0) \simeq \Omega_0$ satisfies $|\Phi(\tilde{x})| = \overline{G.\tau(x)}$. So the map Φ is a holomorphic extension to $\tilde{X}$ of the map $\tilde{\varphi}_0 : \Omega_0 \to \mathcal{C}_n^f(X)$ classifying the fibers over Ω_0 of $\tilde{\mathcal{R}}$ via the first projection. This implies that the closure $\bar{\Gamma}$ in $X \times \mathcal{C}_n^f(X)$ of the graph Γ of $\tilde{\varphi}_0$ is contained in $(\tau \times \tau_*)(\Delta)$, where Δ is the graph of $\tilde{\psi}$. But Δ is proper on X via $\tau \circ p_1$ and the set $(\tau \times \tau_*)(\Delta)$ is closed in $X \times \mathcal{C}_n^f(X)$: if the sequence $(\tau(\tilde{x}_\nu), \tau_*(C_\nu))_\nu$ converges to $(x, C) \in X \times \mathcal{C}_n^f(X)$, up to pass to a subsequence, we may assume that the sequence $(\tilde{x}_\nu)_\nu$ converges to some $\tilde{x}$ in $\tau^{-1}(x)$, and then the sequence $(C_\nu)_\nu$ converges to $\psi(\tilde{x})$ so that $(x, C) = (\tau \times \tau_*)(\tilde{x}, \psi(\tilde{x}))$; this shows our claim. Then $\bar{\Gamma}$ is a closed subset in the $X-$proper set $(\tau \times \tau_*)(\Delta)$, so $\bar{\Gamma}$ is proper over X and the condition [H.3] is fulfilled. ■

2.8 Existence theorem for a SQP-meromorphic quotient

Now we shall prove that conditions [H.1], [H.2] and [H.3] on a completely holomorphic action of a connected complex Lie group G on an irreducible complex space X are sufficient for the existence of a SQP meromorphic quotient.

Theorem 2.8.1 *Under the hypothesis* [H.1], [H.2] *and* [H.3] *there exists a proper* $G-$*equivariant modification* $\tau : \tilde{X} \to X$ *with center contained in* $X \setminus \Omega_0$[16] *and a geometrically f-flat holomorphic map*

$$q : \tilde{X} \to Q$$

on a reduced complex space, which give a strongly quasi-proper meromorphic quotient for the given $G-$*action.*

Let $\Gamma \subset \Omega_0 \times \mathcal{C}_n^f(X)$ be the graph of the holomorphic map $\tilde{\varphi}_0 : \Omega_0 \to \mathcal{C}_n^f(X)$ classifying the fibers of the projection of $\tilde{\mathcal{R}}$ on Ω_0. Of course the complex space $\tilde{X}$ is the topological space $\bar{\Gamma}$ with a structure of a reduced complex space such that the projection on X is a proper modification. Then the space Q is the image of $\tilde{X}$ in $\mathcal{C}_n^f(X)$. So we need some semi-proper direct image theorem for such a map to prove this result. Such a result is the content of the theorem 2.3.2 of [B.15]

Proof. The first remark is that the hypothesis $[H.3]$ says that the projection $p : \bar{\Gamma} \to X$ is a proper topological modification of X. But to apply directly the part ii) of the theorem 2.3.6 of [B.13] to the projection $p_1 : \tilde{\mathcal{R}} \to X$ we need quasi-properness of this map. This is given by the proposition 3.2.2 of [B.15] as we

[16] The dense open subset $\Omega_0 \subset \Omega_1$ is defined in the condition [H.2].

have the condition $[H.3]$.
Then we obtain a proper (holomorphic) modification $\tau : \tilde{X} \to X$, with center $\Sigma \subset X \setminus \Omega_0$, and a $f-$analytic family of cycles in X parametrized by $\tilde{X}$ extending the family $(\overline{G.x})_{x\in\Omega_0}$, corresponding to a "holomorphic" map extending $\bar{\varphi}_0$:

$$\tilde{\varphi} : \tilde{X} \to \mathcal{C}_n^f(X).$$

Now let us prove that this map $\tilde{\varphi}$ is quasi-proper[17]. This will allow us to apply the theorem 2.3.2 of *loc. cit.* and to define the reduced complex space Q as the image $\tilde{\varphi}(\tilde{X})$. Then it will be easy to check that the map $\tilde{\varphi} : \tilde{X} \to Q$ is a strongly quasi-proper meromorphic quotient for the $G-$action we consider.
If C_0 is in $\mathcal{C}_n^f(X)$ and is not the empty cycle, choose a relatively compact open set W in X such that any irreducible component of $|C_0|$ meets W. Then let $\mathcal{W}$ be the open set in $\mathcal{C}_n^f(X)$ defined by the condition on the cycle C that any irreducible component of C meets W. Then we shall prove that there exists a compact set K in $\tilde{X}$ such that any irreducible component of the fiber of $\tilde{\varphi}$ at a point in $\mathcal{W} \cap \tilde{\varphi}(\tilde{X})$ meets K. Let $K := \tau^{-1}(\bar{W})$. If (y, C) is in $\tilde{X}$[18] with $C \in \mathcal{W}$, each irreducible component of C meets W. But the fiber of $\tilde{\varphi}$ at C is equal to $|C|$, so each irreducible component of $\tilde{\varphi}^{-1}(C)$ meets K and the quasi-properness is proved. ■

We conclude by a simple sufficient condition in order to obtain the condition [H.1] and [H.2] (assuming already [H.2]) which can be useful because the $G-$invariant open set Ω_{nice} is canonically defined by the action and so its density in X is a condition which can be tested directly.

Proposition 2.8.2 *Consider a completely holomorphic action of the complex connected Lie group G on the complex space X. Assume that it has a $G-$invariant dense open set Ω_1 which admits* **locally** *a f-GF holomorphic quotient and that it satisfies also the hypothesis [H.2]. Then it satisfies [H.1] (and [H.2]).*

Proof. The hypothesis [H.2] gives an open G-invariant dense subset $\Omega_0 \subset \Omega_1$ on which we have

$$\tilde{\mathcal{R}} \cap (\Omega_0 \times \Omega_0) = \mathcal{R} \cap (\Omega_0 \times \Omega_0)$$

showing that $\mathcal{R}_0 := \mathcal{R} \cap (\Omega_0 \times \Omega_0)$ is a closed analytic subset in $\Omega_0 \times \Omega_0$. Note also that we can assume Ω_0 normal, as the set of normal points in X is open dense and $G-$invariant.
Now remark that the projection $p_1 : \mathcal{R}_0 \to \Omega_0$ is quasi-proper and equidimensional. The quasi-properness is consequence of the existence of the holomorphic section $x \mapsto (x, x)$ of p_1 on Ω_0. So the map p_1 is a f-GF holomorphic map, and is the projection of the graph of the f-analytic family of cycles in Ω_0 given by the fibers of

[17] This makes sense as the fibers are closed analytic subsets of $\tilde{X}$.
[18] Recall that, as a topological space, $\tilde{X} = \bar{\Gamma}$.

p_1 corresponding to a holomorphic map

$$\varphi_0 : \Omega_0 \to \mathcal{C}_n^f(\Omega_0).$$

We shall prove now that φ_0 is semi-proper and then, using the semi-proper direct image theorem 2.3.6 in [B.15], we shall conclude that $\varphi_0(\Omega_0)$ is a reduced complex space and the map φ_0 induced a f-GF holomorphic quotient on Ω_0 proving [H.1].
So consider a non empty cycle $C \in \mathcal{C}_n^f(\Omega_0)$ and choose a point $x_i, i \in [1,k]$ in each irreducible component of C, where k is a positive integer. Let, for each i, V_i be an open relatively compact neighbourhood of x_i in Ω_0 and let $\mathcal{V}$ be the open set in $\mathcal{C}_n^f(\Omega_0)$ of cycles C' such that each irreducible of C' meets $\cup_{i\in[1,k]}\ V_i$. Then the compact set $\cup_{i\in[1,k]}\bar{V}_i$ in Ω_0 satisfies:

$$\varphi_0(\Omega_0) \cap \mathcal{V} = \varphi_0(\cup_{i\in[1,k]}\bar{V}_i) \cap \mathcal{V}$$

which gives the semi-properness of φ_0: if $\varphi_0(y)$ lies in $\mathcal{V}$ then let y_1 be in $\varphi_0(y) \cap V_1$. Then we have $y_1 \in G.y$ and so $\varphi_0(y_1)$ and $\varphi_0(y)$ have the same support. But a cycle in $\varphi_0(\Omega_0)$ is determined by its support. Then we have $\varphi_0(y_1) = \varphi_0(y)$. ■

The point v) of the in proposition 2.6.5 shows that the $G-$invariant open set of normal points in Ω_{nice} admits locally a f-GF holomorphic quotient, we obtain the following corollary of the theorem 2.8.1.

Corollary 2.8.3 *Let G be a complex connected Lie group actin completely holomorphically on an irreducible complex space X. Assume that the open set Ω_{nice} is dense in X and that the hypotheses [H.2] and [H.3] holds, then there exists a SQP meromorphic quotient.* ■

3 Application.

3.1 The sub-analytic lemma.

We shall use the following lemma (see [G-M-O]) in our application.

Lemma 3.1.1 *Let M be a reduced complex space and $Y \subset M$ a closed analytic subset with no interior point in M. Let R be a closed (complex) analytic subset in $M \setminus Y$ such that $\bar{R}$ is a sub-analytic set in M. Then $\bar{R}$ is a (complex) analytic subset in M.*

This important lemma is a consequence of Bishop's theorem (see [Bi.64]) and of a classical result on sub-analytic subsets (see [G-M-O] for more references).

3.2 The $G = K.B$ case: proof of the theorems 1.0.1 and 1.0.2.

Now we shall assume that G is a connected complex Lie group such that we have $G = K.B$ where B is a closed complex connected subgroup of G and K a compact real subgroup of G.

Lemma 3.2.1 *In the situation of the theorem 1.0.1, a couple $(x, y) \in X \times X$ is a good couple for the $G-$action if for any $k \in K$ the couple $(x, k.y)$ is a good couple for the $B-$action. Moreover, if K normalizes B for any good couple (x, y) for the $B-$action and any $k \in K$ the couple $(k.x, k.y)$ is again a good couple for the $B-$action. This implies that the open set $\Omega_{nice/B}$ is stable by K, when K normalizes B.*

PROOF. For each $k \in K$ there exist $V_k(x), V(k.y)$ respectively open neighbourhoods of x and $k.y$ and L_k a compact subset in B such that for any $x' \in V_k(x)$, any $y' \in V(k.y)$ with $y' = b.x'$ for some $b \in B$ there exists $\beta \in L_k$ with $y' = \beta.x'$. Now choose $k_1, \ldots, k_N$ in K such that the compact set $K.y$ is contained in the open set $U := \cup_{i=1}^N V(k_i.y)$. Then the subset $W(y) := \{y' \in X \ / \ K.y' \subset U\}$ is an open neighbourhood of y in X[19]. Define also $W(x) := \cap_{i=1}^N V_{k_i}(x)$ and $L := \cup_{i=1}^N L_{k_i}$. Then $W(x)$ is an open neighbourhood of x in X, L is a compact set in B and $\Lambda := K.L$ is a compact set in G.
Take now $x' \in W(x)$ and $y' \in W(y)$ such that $y' = g.x'$ for some $g \in G$. Write $g = k.b$ with $k \in K$ and $b \in B$. Then $k^{-1}.y$ is in $V(k_i.y)$ for some $i \in [1, N]$. As x' is in $W(x) \subset V_{k_i}(x)$, the equality $k^{-1}.y' = b.x'$ allows to find $\beta \in L$ such that $k^{-1}.y' = \beta.x'$ and then $\gamma := k.\beta$ is in $K.L$ and $y' = \gamma.x'$.
So the condition that for any $k \in K$ the couple $(x, k.y)$ is a good couple for the $B-$action implies that the couple (x, y) is a good couple for the $G-$action.
To prove the converse, it is enough to remark that any compact set Λ in G is contained in the compact set $K.L$ where L is the compact set in B defined as $L := (K.\Lambda) \cap B$.
If we assume now that K normalizes B then for any $k \in K$ the neighbourhoods $k.V(x)$ and $k.V(y)$ and the compact $k.L.k^{-1}$ of B give the fact that $(k.x, k.y)$ is good for the $B-$action:
if $k.y'$ and $k.x'$ are in $k.V(x)$ and $k.V(y)$ respectively and satisfy $k.y' = b.k.x'$ for some $b \in B$, we have $y' = b_1.x'$ with $b_1 := k^{-1}.b.k$ and so there exists $\beta_1 \in L$ such that $y' = \beta_1.x'$ and this implies $k.y' = \beta.k.x'$ with $\beta := k.\beta_1.k^{-1} \in k.L.k^{-1}$. ■

Corollary 3.2.2 *In the situation of the theorem 1.0.1, assume that we have a $G-$invariant open set Ω which is a good open set for the $B-$action, then Ω is a good open set for the $G-$action.*

PROOF. Consider a point $x \in \Omega$ and a compact set M in Ω. Then there exists a neighbourhood V of x in Ω and a compact set L in B such that $b.y \in M$ for some $y \in V$ and some $b \in B$ implies that we can find $\beta \in L$ with $b.y = \beta.y$. Now assume that M is $K-$invariant (here we use the $G-$invariance of Ω) and that $g.y$ is in M for some $g \in G$ and some $y \in V$. Write $g = k.b$ for some $k \in K$ and $b \in B$. Then $b.y$ is again in M so we can find $\beta \in L$ with $\beta.y = b.y$ and then $g.y = k.\beta.y$ with

[19] We let this exercice on compactness to the reader.

$k.\beta \in K.L$ which is a compact set in G. So x is a good point for the $G-$action on Ω. ■

The corollary of the next lemma will give the first part of [H.2] for the $G-$action assuming that we have a $G-$invariant dense, Zariski open, good open set Ω for the $B-$action with the condition [H.2] for the $B-$action.

Lemma 3.2.3 *Let Ω be an open $G-$invariant set. Define the map*

$$\chi : K\times X\times X \to X\times X \quad \text{by} \quad (k,x,y)\mapsto (k.x,y) \quad \text{and let} \quad p : K\times X\times X \to X\times X$$

be the natural projection. Then we have: $p(\chi^{-1}(\overline{\mathcal{R}_B})) = \overline{\mathcal{R}_G}$ where we define

$$\mathcal{R}_B := \{(x,y)\in \Omega\times\Omega \ / \ B.x = B.y\} \quad \text{and} \quad \mathcal{R}_G := \{(x,y)\in \Omega\times\Omega \ / \ G.x = G.y\},$$

and where the closures are taken in $X\times X$.

Proof. Remark first that

$$p(\chi^{-1}(\mathcal{R}_B)) = \{(x,y)\in \Omega\times\Omega \ / \ \exists k\in K \quad B.k.x = B.y\}.$$

So $(x,y)\in p(\chi^{-1}(\mathcal{R}_B))$ implies $y\in B.k.x\subset G.x$ and also $k.x\in B.y$; we conclude that x is in $K.B.y = G.y$. This gives the inclusion $p(\chi^{-1}(\mathcal{R}_B))\subset \mathcal{R}_G$. The opposite inclusion is easy because $G.x = G.y$ implies that $x\in K.B.y$ so there exists $k\in K$ such that $k.x\in B.y$.This gives the equality

$$p(\chi^{-1}(\mathcal{R}_B)) = \mathcal{R}_G.$$

Now the maps χ and p are continuous and proper, so we obtain the inclusion

$$\overline{\mathcal{R}_G}\subset p(\chi^{-1}(\overline{\mathcal{R}_B})).$$

Now take $(x,y)\in p(\chi^{-1}(\overline{\mathcal{R}_B}))$; there exists a sequence (k_ν,x_ν,y_ν) in $\chi^{-1}(\mathcal{R}_B)$ such that $(k_\nu.x_\nu,y_\nu)$ is a sequence in $\mathcal{R}_B$ converging to (x,y), as χ is proper and surjective. So we have $B.k_\nu.x_\nu = B.y_\nu$ and then $G.k_\nu.x_\nu = G.y_\nu$, so $(k_\nu.x_\nu,y_\nu)$ are in $\mathcal{R}_G$. We conclude that (x,y) is in $\overline{\mathcal{R}_G}$. ■

Corollary 3.2.4 *In the situation of the previous lemma, assume that $X\setminus\Omega$ is a (complex) analytic subset with no interior point in X. Assume also that $\mathcal{R}_G$ is a closed analytic subset in $\Omega\times\Omega$. Then if the subset $\overline{\mathcal{R}_B}$ is (complex) analytic in $X\times X$, the subset $\overline{\mathcal{R}_G}$ is also a (complex) analytic subset of $X\times X$.*

PROOF. Note first that the maps χ and p are real analytic, so assuming that $\overline{\mathcal{R}_B}$ is analytic implies that $p(\chi^{-1}(\overline{\mathcal{R}_B}))$ is sub-analytic. Then, as we know that $\mathcal{R}_G$ is an irreducible closed complex analytic subset, the conclusion follows from the lemma 3.1.1, as our assumption that Ω is a Zariski (dense) open set in X implies that $\Omega \times \Omega$ is Zariski open (and dense) in $X \times X$. ■

A first step to prove the quasi-properness of $\overline{\mathcal{R}_G}$ is our next result.

Lemma 3.2.5 *Let assume that the $B-$action on X satisfies* [H.1] *and* [H.2]. *Let $\Omega_0 \subset \Omega_1$ be an open set on which the fiber at any $x \in \Omega_0$ of $\overline{\mathcal{R}_B}$ is equal to $\overline{B.x}$ (with some multiplicity). Then the fiber at any $x \in \Omega_0$ of $\overline{\mathcal{R}_G}$ is equal to $\overline{G.x}$ (with some multiplicity).*

PROOF. As we know that the map $x \mapsto \overline{B.x}$, with generic multiplicity 1, is a f-analytic family of cycles of X parametrized by Ω_0, for each sequence $(x_\nu)_{\nu\in\mathbb{N}}$ of points in Ω_0 converging to a point $x \in \Omega_0$ we have (with suitable multiplicity) $\overline{B.x} = \lim_{\nu\to\infty} \overline{B.x_\nu}$ in the topology of $\mathcal{C}^f_d(X)$. We shall show that this implies, also with suitable multiplicity, the equality $\overline{G.x} = \lim_{\nu\to\infty} \overline{G.x_\nu}$ in the topology of $\mathcal{C}^f_n(X)$. As we have $G = K.B$ with K compact, for any $y \in X$ we have $\overline{G.y} = K.\overline{B.y}$. So the inclusion of $\lim_{\nu\to\infty} \overline{G.x_\nu}$ in the fiber at x of $\overline{\mathcal{R}_G}$ is clear. The point is to prove the opposite inclusion. Let y be a point in the fiber at $x \in \Omega_0$ of $\overline{\mathcal{R}_G}$. It is a limit of a sequence $y_\nu \in G.x_\nu$ where $x_\nu \in \Omega_0$ converges to x. Write $y_\nu = k_\nu.b_\nu.x_\nu$ with $k_\nu \in K$ and $b_\nu \in B$. Up to pass to a subsequence, we may assume that the sequence (k_ν) converges to a point $k \in K$. So we have $k^{-1}.y$ which is the limit of the sequence $b_\nu.x_\nu$. We obtain that $k^{-1}.y$ is in the limit of $\overline{B.x_\nu}$ which has support equal to $\overline{B.x}$. Then y is in $K.\overline{B.x} = \overline{G.x}$, concluding the proof. ■

PROOF OF THE THEOREM 1.0.1. The hypothesis gives a $G-$invariant dense, Zariski open Ω_1 which is a good open set for the $B-$action. The corollary 3.2.2 shows that it is also a good open set for the $G-$action.
The analyticity of $\overline{\mathcal{R}_G}$ in $X \times X$ is proved at corollary 3.2.4 as the complement of Ω_1 is Zariski closed. The lemma 3.2.5 gives a dense open set Ω_0 where the fiber of the projection p_1 of $\overline{\mathcal{R}_G}$ at each point $x \in \Omega_0$ is equal to $\overline{G.x}$ as a set. This implies the quasi-properness of p_1 over Ω_0, because x is in $G.x$ and G is connected; assuming (which is not restrictive) that Ω_0 is normal, we obtain a holomorphic map

$$\Phi : \Omega_0 \longrightarrow \mathcal{C}^f_n(X)$$

where the support of $\Phi(x)$ is equal to $\overline{G.x}$ for each $x \in \Omega_0$ and with generic multiplicity equal to 1. This complete the proof of [H.2] for the $G-$action.
Thanks to proposition 3.2.2 of [B.15], to prove [H.3] it is enough to show that the closure of the graph Γ_G of Φ in $X \times \mathcal{C}^f_n(X)$ is proper on X.
The projection $p_B : \overline{\mathcal{R}}_B \to X$ is strongly quasi-proper so, for any compact $\bar{V}$ in X,

the subset T in $\mathcal{C}_n^{loc}(X)$ of limits of the generic fibers of the projection $p_G : \overline{\mathcal{R}}_G \to X$ for $x \in \bar{V}$ is a compact set of $\mathcal{C}_n^{loc}(X)$ thanks to [B.13] theorem 2.3.6 i).
Now choose a relatively compact open set V in X and let $V' := V \cap \Omega_0$. Then T' the subset of T corresponding to the cycles $\Phi(x), x \in V'$ is a dense open set in T. Note that for each $x \in V'$ we have $|\Phi(x)| = \overline{G.x} = \cup_{k\in K}\, k.\overline{B.x}$. This means that for each $x \in V'$ the $n-$cycle $\Phi(x)$ is union of $d-$ cycles in the subset $S := K.q_B(\tau_B^{-1}(\bar{V}))$ where q_B is the minimal SQP meromorphic quotient of X for the $B-$action, and where K acts on $\mathcal{C}_d^f(X)$ by direct image of the cycles (note that Q_B is a closed analytic subset in $\mathcal{C}_d^f(X)$ by definition of the minimal SQP meromorphic quotient). Then S is a compact subset of $\mathcal{C}_d^f(X)$ and we may apply the proposition 2.2.3. It gives that T is a compact subset of $\mathcal{C}_n^f(X)$ and this proves [H.3]. ■

Proof of the theorem 1.0.2. We shall reduce the proof of this result to the theorem 1.0.1 using the following proposition.

Proposition 3.2.6 *In the situation of the theorem 1.0.2 there exists a $G-$invariant Zariski open set Ω_2' which is dense in X, disjoint from the center Σ_B of the modification $\tau_B : \tilde{X}_B \to X$ such that the map $q_B : \tilde{X}_B \to Q_B$ induces on the open set $\tau_B^{-1}(\Omega_2')$ a f-GF holomorphic quotient map on a open dense set Q'_B in Q_B.*

Proof. As the argument is not so simple we shall divide this proof in several steps.

Step 1. The theorem 2.8.1 gives the existence of a SQP meromorphic quotient for the $B-$action and thanks to the proposition 2.4.2 we may use the minimal SQP meromorphic quotient (see definition 2.4.3). Now, using the corollary 2.4.4 we can assume that K acts continuously and holomorphically on $\tilde{X}_B$ and Q_B and that the holomorphic maps τ_B and q_B are $K-$equivariant.

Step 2. As the center Σ_B of τ_B is K and $B-$invariant with no interior point in X, we can replace the Zariski open set Ω_1 by the Zariski open set $\Omega_1 \setminus \Sigma_B$ which is still dense and $B-$invariant and good for the $B-$action. To avoid too many change of notations , we shall simply assume now that Ω_1 is disjoint to Σ_B and also identify Ω_1 with the open set $\tau_B^{-1}(\Omega_1)$.
Now the set $\Omega_2 := K.\Omega_1$ is again Zariski open dense in X and admits locally a f-GF holomorphic quotient : it is an union (finite if we want, using the compactness of K) of good open sets $k.\Omega_1$ for the $B-$action, and it is stiil disjoint from Σ_B because Σ_B is $K-$invariant. So the subset

$$\mathcal{R}_B := \{(x, y) \in \Omega_2 \times \Omega_2 \ / \ B.x = B.y\}$$

is a **locally** closed analytic subset in $\Omega_2 \times \Omega_2$. But this will not be enough to apply the sub-analytic lemma as in the proof of the theorem 1.

STEP 3. We shall construct in Step 4 a Zariski open subset $\Omega'_2 \subset \Omega_2$ which is still dense, K and $B-$invariant (so $G-$invariant), such that for each $x \in \Omega'_2$ the cycle $\varphi_B(q_B(x))$ in X is irreducible. Let us show that this will implies that the subset $\mathcal{R}_B \cap (\Omega'_2 \times \Omega'_2)$ will be closed in $(\Omega'_2 \times \Omega'_2)$ (and is also analytic) where $\varphi_B : \tilde{X}_B \to \mathcal{C}^f_n(\tilde{X}_B)$ is the holomorphic map obtained by the composition of q_B with the classifying map of the fibers of the holomorphic f-GF map $q_B : \tilde{X}_B \to Q_B$.
We know that for $x \in \Omega_2$ we have $\bar{B}.x \subset |\varphi_B(q_B(x))|$. Assume that we have $(|\varphi_B(q_B(x))|) \cap \Omega'_2 = B.x \cup C$; then C has pure dimension n and is $B-$invariant. So it is a finite union of $B-$orbits in Ω'_2. But as we know that $\varphi_B(q_B(x))$ is irreducible, this implies that $C = \emptyset$ and so $B.x = B.y$ for $(x, y) \in \Omega'_2 \times \Omega'_2$ is then equivalent to $\varphi_B(q_B(x)) = \varphi_B(q_B(y))$ which is a closed (analytic) condition.

STEP 4. We know, by definition of a SQP meromorphic quotient, that there exists a $B-$invariant dense open set Ω_0 in $X \setminus \Sigma_B$ such that $\varphi_B(q_B(x)) = \overline{B.x}$ for each $x \in \Omega_0$. This shows that the general cycle in the family classified by the map $q_B \circ \varphi_B : \tilde{X}_B \to \mathcal{C}^f_n(\tilde{X}_B)$ is irreducible. So there exists a closed analytic subset Z in $\tilde{X}_B$, which is K and $B-$invariant, such this irreducibility holds on $\tilde{X}_B \setminus Z$. Then define $\Omega'_2 := \Omega_2 \cap (\tilde{X}_B \setminus Z)$.

The last step is to show that the family of $n-$cycles in Ω'_2 defined by $x \mapsto B.x$ is f-analytic in order to get a f-GF quotient for the $B-$action on Ω'_2. This is given by the next lemma.

Lemma 3.2.7 *Let S and X be irreducible complex spaces and let $\varphi : S \to \mathcal{C}^f_n(X)$ be an holomorphic map, so the classifying map of a f-analytic family of $n-$cycles in X. Assume that all cycles have irreducible supports and are generically reduced. Let $\Gamma \subset S \times X$ the graph of this family. Let X' be a Zariski open subset in X and assume that there exists a holomorphic map $\sigma : S \to X'$ such that $\sigma(x) \in |\varphi(x)|$. Then the family $s \mapsto \varphi(x) \cap X'$ is a f-analytic family of cycles in X'.*

PROOF. The only point to prove is that $Z' := Z \cap (S \times X')$ is quasi-proper on S as the restriction to an open set of a analytic family of cycles is always an analytic family of cycles of this open set. But the existence of a continuous map as σ is enough for that purpose under our hypothesis : for any compact set K in S the compact set $(\mathrm{id}_S \times \sigma)(K)$ in Z' meets any irreducible component of any cycle associated to some point in K. Remark that the fact that X' is Zariski open in X is used to insure that for any cycle $\varphi(s)$ the cycle $\varphi(s) \cap X'$ has at most one irreducible component. Then the existence of σ implies that it has exactly one irreducible component. ■

END OF THE PROOF OF THE THEOREM 1.0.2. The proposition 3.2.6 gives the $G-$invariant dense, Zariski open subset Ω'_2 satisfying [H.1str] for the $B-$action and we can apply the theorem 1.0.1. ■

3.3 The $G = K.A.K$ case: proof of the theorem 1.0.3.

The proof of the theorem 1.0.3 will use the next lemmata (analoguous to 3.2.2 and 3.2.3).

Lemma 3.3.1 *In the $G = K.A.K$ case, a $G-$invariant good open set for A is a good open set for G.*

PROOF. Let M be a $K-$invariant compact set in Ω and V be a $K-$invariant compact neighbourhood in Ω of a point x in Ω. Then, as V is uniformly good in Ω for the action of A, there exists a compact set L in A such that for $y \in V$ and $a \in A$ satisfying $a.y \in M$ there exists $\alpha \in L$ with $\alpha.y = a.y$. Assume now that for $g \in G$ and $y \in V$ we have $g.y \in M$. Write $g = k_1.a.k_2$. Then we have $a.k_2.y \in M$ and also $k_2.y \in V$ by the $K-$invariance of M and V. So there exists $\alpha \in L$ with $\alpha.k_2.y = a.k_2.y$ and so $g.y = k_1.\alpha.k_2.y$ where $k_1.\alpha.k_2$ is in the compact set $K.L.K$ of G. This shows that any point x in Ω is a good point for the $G-$action. ■

Lemma 3.3.2 *In the $G = K.A.K$ case, consider a $G-$invariant good open set Ω for the $A-$action. Let $\chi : K \times K \times X \times X \to X \times X$ be the map given by $\chi(k_1, k_2, x, y) = (k_1.x, k_2.y)$. Define $\mathcal{R}_A := \{(x, y) \in \Omega \times \Omega \ / \ A.x = A.y\}$ and $\mathcal{R}_G := \{(x, y) \in \Omega \times \Omega \ / \ G.x = G.y\}$. Then we have*

$$p(\chi^{-1}(\mathcal{R}_A)) = \mathcal{R}_G \quad \text{and} \quad p(\chi^{-1}(\overline{\mathcal{R}_A})) = \overline{\mathcal{R}_G}$$

where $p : K \times K \times X \times X \to X \times X$ is the projection.

PROOF. Remark first that for $(x, y) \in \Omega \times \Omega$, the condition $G.y = G.x$ is equivalent to the existence of $(k_1, k_2) \in K \times K$ such that $A.k_1.x = A.k_2.y$. So the inclusion $\mathcal{R}_G = p(\chi^{-1}(\mathcal{R}_A))$ is clear. As p is proper, this implies that $\overline{\mathcal{R}_G} \subset p(\chi^{-1}(\overline{\mathcal{R}_A}))$. Conversely, consider a sequence $(x_\nu, y_\nu) \in \mathcal{R}_A$ converging to $(x, y) \in X \times X$. As $p(\chi^{-1}(x, y)) = \{(k_1.x, k_2.y) \ / \ (k_1, k_2) \in K \times K\}$, we want to prove that for any fixed $(k_1, k_2) \in K \times K$ we have $(k_1.x, k_2.y) \in \overline{\mathcal{R}_G}$. There exists a sequence $((x_\nu, y_\nu))_\nu$ in $\mathcal{R}_A$ converging to (x, y). Then $(k_1^{-1}, k_2^{-1}, k_1.x_\nu, k_2.y_\nu)$ is in $\chi^{-1}((x_\nu, y_\nu))$. So $(k_1.x_\nu, k_2.y_\nu)$ is in $\mathcal{R}_G$ for each ν and this sequence converges to $(k_1.x, k_2.y)$ proving the inclusion $p(\chi^{-1}(\overline{\mathcal{R}_A})) \subset \overline{\mathcal{R}_G}$. ■

Lemma 3.3.3 *In the $K.A.K$ case with the hypotheses of the theorem 1.0.3, we have, for any $x \in \Omega_0$, the equality $\overline{G.x} = \cup_{(k_1,k_2)\in K\times K} \ k_1.\overline{A.k_2.x}$ in X.*

PROOF. The inclusion of $\cup_{(k_1,k_2)\in K\times K} \ k_1.\overline{A.k_2.x}$ in $\overline{G.x}$ is clear. To prove the opposite inclusion it is enough to prove that the right hand-side is a closed subset in X. But it is the image by τ_A of the subset $\cup_{(k_1,k_2)\in K\times K} \ q_A^{-1}(q_A(\overline{A.k_2.x})$ in $\tilde{X}_A$, where $\tau_A, \tilde{X}_A, q_A, Q_A$ define the minimal SQP quotient of X for the $A-$action which exists

thanks to the theorem 2.8.1 and the proposition 2.4.2. But $q_A(\overline{A.k_2.x}) = q_A(k_2.x)$ by $A-$invariance and continuity of q_A. Now the set $q_A(K.x)$ is compact in Q_A and so $q_A^{-1}(q_A(K.x))$ is closed in $\tilde{X}_A$ and equal to $\cup_{(k_1,k_2)\in K\times K}\ q_A^{-1}(q_A(\overline{A.k_2.x})$ because for each $y \in \Omega_0$ we have $q_A^{-1}(q_A(y)) = \overline{A.y}$. As τ_A is proper, it is a closed map and our right hand-side is a closed set in X. ■

Proof of the theorem 1.0.3. The lemma 3.3.1 implies that the Zariski dense good open set Ω_1 for the $A-$action is good for the $G-$action, so [H.1str] is true for G. The lemma 3.3.2 and the sub-analytic lemma 3.1.1 shows that $\overline{\mathcal{R}_G}$ is a closed analytic subset in $X \times X$ and its projection on Ω_0 is a f-GF flat map, as we may assume Ω_0 normal. The last point is to prove that the closure in $X \times \mathcal{C}_n^f(X)$ of the graph of the holomorphic map $\bar{\varphi}_0 : \Omega_0 \to \mathcal{C}_n^f(X)$ given generically by $x \mapsto \overline{G.x}$ is proper over X. We shall apply the proposition 2.2.3 using the following two facts :

1. For a compact set $\bar{V}$ in X the set $K.q_A(\tau_A^{-1}(\bar{V}))$ is compact in $\mathcal{C}_d^f(X)$ and parametrizes a f-continuous family of $d-$cycles in X.

2. If $\bar{V}$ is the compact closure of an open set V in X, then for each point x in $V' := V \cap \Omega_0$ the cycle $\bar{\varphi}_0(x)$ is an union of some $d-$cycles in the above family,

where we use the minimal SQP meromorphic quotient of X for the $A-$action. The first point uses the continuous action of K on $\mathcal{C}_d^f(X)$ by the direct image of cycles. The second point is consequence of the lemma 3.3.3. Then the proposition 2.2.3 gives the condition [H.3] for the $G-$action as in the proof of the theorem 1.0.1 and we conclude the proof using the theorem 2.8.1. ■

3.4 Relation between the two quotients.

In this section we consider a connected complex Lie group G and we assume that we have $G = K.B$ where K is a compact real subgroup and B a complex connected closed subgroup of G. We shall also indicate some analoguous results in the case $G = K.A.K$. We also assume that G acts completely holomorphically on a irreducible complex space X.

Proposition 3.4.1 *Assume that there exists a SQP meromorphic quotient for this action but also for the corresponding action of B. Then there exists a holomorphic map*

$$h : Q_B \to Q_G$$

where Q_B and Q_G are the minimal SQP quotients of X respectively for the $B-$action and the $G-$action, such that we have the equality $q_B \circ h = q_G$ on the strict transform of $\tilde{X}_G$ by the modification τ_B.
Moreover, if K normalizes B, there are natural $K-$actions on $\tilde{X}_B$ and Q_B, the map q_B is $K-$equivariant and the holomorphic map h is $K-$invariant.

PROOF. Consider the two diagrams corresponding to the two SQP meromorphic quotients of X by the actions of B and G

$$\begin{array}{ccc} \tilde{X}_B & \xrightarrow{q_B} & Q_B \\ {\scriptstyle \tau_B}\downarrow & & \\ X & & \end{array} \qquad \begin{array}{ccc} \tilde{X}_G & \xrightarrow{q_G} & Q_G \\ {\scriptstyle \tau_G}\downarrow & & \\ X & & \end{array}$$

and let $\sigma : Y \to \tilde{X}_B$ be the strict transform of the modification τ_G by the modification τ_B. Then the first assertion is a consequence of the theorem 2.4.5 as the meromorphic map $q_G : X \dashrightarrow Q_G$ is $B-$invariant.
The second assertion is consequence of the corollary 2.4.4. ■

In the $G = K.A.K$ case, assuming that the SQP meromorphic quotient exists for the actions of A and G, using the $A-$invariance of the meromorphic map $X \dashrightarrow Q_G$ and the theorem 2.4.5, we obtain that there exists also a holomorphic map between the corresponding minimal quotients $h : Q_A \to Q_G$ which satisfies the equality $q_A \circ h = q_G$ on the strict transform of $\tilde{X}_G$ by the modification τ_A.

Of course a natural question about the holomorphic map $h : Q_B \to Q_G$ defined in the previous proposition is its properness. Our next result gives a sufficient condition to obtain a partial result.

Proposition 3.4.2 *Assume that the $B-$action and the $G-$action on X admit a SQP meromorphic quotient. Assume that there exists a $G-$invariant open set Ω in X, disjoint from the centers of the modifications τ_B and τ_G associated to the minimal SQP quotients of X, on which we have a f-GF holomorphic quotient for the $G-$action, and such that for each $x \in \Omega$ we have $q_G^{-1}(q_G(x)) = \overline{G.x}$ in $\tilde{X}_G$. Then the map $h_\Omega : q_B(\Omega) \to q_G(\Omega)$, induced by the restriction of the holomorphic map $h : Q_B \to Q_G$, is proper.*

PROOF. We shall prove that if M is a compact set in Ω then we have the inclusion $h^{-1}(q_G(M)) \cap q_B(\Omega) \subset q_B(K.M)$. As $K.M$ is a compact set in Ω, this will prove the properness of the map h_Ω because the map q_G is open, so each compact set in $q_G(\Omega)$ can be cover by finitely many open sets $q_G(V_i)$ where V_i is a relatively compact open subset in Ω; then any compact set in $q_G(\Omega)$ is contained in $q_G(M)$ where M is the compact set $\cup_{i\in I}\bar{V}_i$ of Ω.
Consider a point $y \in h^{-1}(q_G(M)) \cap q_B(\Omega)$. So there exists a point $x \in M$ such that $h(y) = q_G(x)$. But from our hypothesis we know that $q_G^{-1}(q_G(x)) = \overline{G.x}$ in $\tilde{X}_G$. Also there exists also a point $x_0 \in \Omega$ such that $y = q_B(x_0)$. This implies the equality $h(q_B(x_0)) = q_G(x_0) = q_G(x)$ because $q_B \circ h = q_G$ on Ω. So x_0 is in $\overline{G.x} \cap \Omega = G.x$. We conclude that there exists $k \in K$ such that x_0 is in $B.k.x$ and then $q_B(x_0) = q_B(k.x)$ and $k.x$ is in $K.M$. ■

Remark that the existence of a SQP meromorphic quotient for the $G-$action implies

the existence of a $G-$invariant open dense subset Ω_0 satisfying all the hypotheses of the previous proposition *excepted* the fact that Ω_0 is disjoint from the center of τ_B. So under the hypothesis that the closed set $K.\Sigma_B$, where Σ_B is the center of the modification τ_B, has no interior point in X, the existence of the two SQP meromorphic quotients is enough to conclude that there exists a $G-$invariant open dense set Ω in X for which the map h_Ω is proper.
So the following corollary is immediate.

Corollary 3.4.3 *Under the hypotheses of the theorem 1.0.1 there exists a dense open set Ω in X, disjoint from the center of the modifications τ_B and τ_G, such that the map $h_\Omega : q_B(\Omega) \to q_G(\Omega)$ is proper.* ■

With analoguous argument we obtain also such a result in the $K.A.K$ case.

Corollary 3.4.4 *Under the hypotheses of the theorem 1.0.3 there exists a dense open set Ω in X, disjoint from the center of the modifications τ_A and τ_G, such that the map $h_\Omega : q_A(\Omega) \to q_G(\Omega)$ is proper.* ■

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