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REVISITING THE CONNECTION BETWEEN THE NO-SHOW PARADOX AND MONOTONICITY*

MATÍAS NÚÑEZ^a AND M. REMZI SANVER^{a,b}

ABSTRACT. We investigate the relation between monotonicity and the no-show paradox in voting rules. Although the literature has established their logical independence, we show, by presenting logical dependency results, that the two conditions are closer than a general logical independency result would suggest. Our analysis is made both under variable and fixed-size electorates.

KEYWORDS. Monotonicity, No-Show Paradox, Participation, Reinforcement, Threshold Scoring Rules.

JEL CLASSIFICATION. D71.

1. INTRODUCTION

Among the countless contributions of Hervé Moulin to our enlightenment on the collective decision making problem, his research on the axiomatic analysis of social choice rules presents a distinguished chapter which inspired generations of scholars. We view this issue of Mathematical Social Sciences dedicated to him as a nice opportunity to revisit the connection between participation and monotonicity, two conditions of social choice theory which have been much elaborated by the fine work of Hervé Moulin.

Moulin [1988, 1991] defines participation as the vanishing of the no-show paradox introduced by Fishburn and Brams [1983]: a social choice rule exhibits the no-show paradox when the vote casted by an additional voter changes the outcome in a way which makes this new-comer worse off compared to the case he had not shown up. Thus, the paradox can be viewed as a way to manipulate social choice rules by abstaining to vote, such as Moulin [1991] who sees it as a particular case of manipulation by truncation of preferences as defined by Fishburn and Brams [1984].

Such views, however, necessitate some caution on how the new-comer/abstainer is interpreted. Here, two approaches come to the fore: One is the fixed-electorate approach where the number of voters are fixed and the abstainers are those voters who express full indifference over the set of alternatives. So a "new-comer" is an

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individual who is an incumbent member of the society who moves away from his full indifference position. The other approach necessitates a variable-electorate, as the new-comer is a voter who earlier was not a member of the society, hence "abstaining" means his altogether departing from the society to which he used to be part of.

Not only the two interpretations diverge in their meanings, but also this divergence has implications on the results obtained about the no-show paradox - a fact that we discuss in the sequel. However, we wish to note right away that the literature on the paradox has almost always adopted the variable-electorate approach, including and perhaps following the seminal paper of Moulin [1988].¹

We start, in Section 2, by considering the paradox under this standard variable-electorate interpretation and revisit its relation to a well-known monotonicity condition of social choice theory. Monotonicity, broadly speaking, requires that an "improvement" of the status of an alternative in the preferences of the electorate should result in a "raising" of the status of this alternative as the social outcome. It is clear that, different meanings can be attributed to "improvement" and "raising", each of which leading to a different definition of monotonicity. In fact, the literature exhibits a plethora of monotonicity conditions. As all of these can be connected to the (non)-manipulability of social choice rules, the logical relationship of participation to those monotonicity conditions stands out as an interesting question.

Among the various monotonicity conditions, perhaps the simplest and oldest known is the one we consider²:

MON: Raising an alternative x in voters' preferences while leaving the rankings otherwise unchanged can never result in x becoming the loser while x was initially the winner.

Although normatively appealing and simple, MON is violated by various well-known social choice rules, in particular by all point run-off systems (Smith [1973]).³ As participation (PART) is also violated by an interesting class of social choice rules, namely those which are Condorcet consistent (Moulin [1988]), the logical relationship between MON and PART turns out to be of further interest.

In fact, the general logical independence between MON and PART is already established. The question is addressed by Nurmi [1999] in p.62 who remarks that MON does not imply PART, as there exist Condorcet extensions, such as the Copeland

¹The paradox has also been considered in the framework of judgement aggregation (see Balinski and Laraki [2010]).

²For discussion on monotonicity conditions in social choice theory, one can see Fishburn [1982], Moulin [1983], Brams and Fishburn [2002] and Sanver and Zwicker [2009].

³Other interesting violations of MON are established by Fishburn [1977], Richelson [1980], Fishburn and Brams [1983].

rule, which satisfy MON but, by Moulin [1988], fail PART. Nurmi [1999] furthermore suggests the conjecture that PART implies MON which is falsified by Campbell and Kelly [2002] who give examples of social choice rules that satisfy PART but fail MON.

We present instances of logical dependencies between PART and MON. We show that in the particular case of two alternatives, PART implies MON. On the other hand, even with two alternatives, MON does not imply PART. Nevertheless, the failure of this implication merits some attention to the contextual difference regarding the definitions of the two conditions: while MON is a property that can be defined for fixed or variable electorate social choice rules, PART necessitates a variable electorate. As a result, PART requires a connection between how a social rule behaves in electorates of different sizes but MON does not. This renders the construction of a social choice rule which satisfies MON but fails PART very easy. In fact, the example we use in Proposition 1 to show that MON does not imply PART even with two alternatives exploits this ease.

A fairer question is whether MON implies PART under mild consistency requirements over the behavior of social choice rules in different electorates. One such condition is reinforcement, also known as consistency, which requires that alternatives which are separately chosen by both of two disjoint electorates must form the choice made by the union of these electorates (Smith [1973], Young [1974, 1975]). A much milder version of reinforcement is homogeneity which requires that an alternative which is chosen by some given electorate must also be chosen when this electorate is replicated.⁴ We show that in the two-alternative case, under the homogeneity assumption, MON implies PART.

With three or more alternatives, Moulin [1988], while establishing the logical independence between PART and reinforcement, uses a threshold scoring rule to exemplify the satisfaction of reinforcement and the failure of PART. As all threshold scoring rules satisfy MON, the example also shows that even when homogeneity is replaced by reinforcement, MON fails to imply PART.

Inspired by this example we devote further attention to threshold scoring rules⁵ and ask whether they always fail PART. The answer is almost affirmative: we show that, except one member, the class of threshold scoring rules fails PART.

We also consider a weaker version of participation (WPART) as the absence of a stronger version of the no-show paradox (Pérez [2001]) where a voter, by abstaining, can enforce his most preferred alternative as the social outcome. We show that MON, even when homogeneity is assumed, does not imply WPART. On the

⁴Without omitting to note some borderline counter examples in Fishburn [1977], we can nevertheless say that almost all social choice rules considered in the literature are homogeneous.

⁵One can see Saari [1990] for an analysis of these rules.

other hand, reinforcement, when combined with a weak unanimity condition, implies WPART.

Regarding the implication of monotonicity by PART, we introduce a weaker version of monotonicity (WMON) and prove that PART implies WMON. Moreover, we note that WMON is sufficiently strong to discriminate among social choice rules that fail MON: we show that plurality with a runoff even fails WMON while Campbell and Kelly [2002]’s examples that fail MON satisfy WMON.

Section 3 carries our analysis to the fixed-electorate interpretation of PART. Our findings differ from those obtained under the variable-electorate interpretation. We show that with two alternatives, PART and MON are logically equivalent. Moreover, when three or more alternatives are available MON implies WPART and PART implies WMON.

Section 4 makes some closing remarks.

2. THE VARIABLE POPULATION CASE

We consider a finite set of alternatives A with $\#A \geq 2$. \mathbf{N} denotes the set of natural numbers. For each $n \in \mathbf{N}$, we define $N = \{1, \dots, n\} \subset \mathbf{N}$ as the n -voter electorate, where each $v \in N$ is a voter. Π stands for the set of linear orders over A . $P_v \in \Pi$ is the preference of $v \in N$ over A , where for any distinct $x, y \in A$, $xP_v y$ indicates that voter v prefers x to y .⁶ We write $P_N = \{P_v\}_{v \in N}$ for a preference profile over A . A social choice rule (SCR) is a mapping F that returns, for each $N \in \mathbf{N}$ and each $P_N \in \Pi^N$, a single alternative $F(P_N) \in A$. So the SCRs we consider are variable-electorate in the sense of being defined for every number of voters and they satisfy the full domain condition in the sense that given any electorate, they are defined for every possible preference profile.

For any two electorates $N = \{1, \dots, n\}$ and $M = \{1, \dots, m\}$, we define the joint electorate $M \oplus N = \{1, \dots, m+n\}$. Note that \oplus is commutative. Now letting $m \leq n$, for any two profiles R_N, Q_M , we let $P_{N \oplus M} = (R_N, Q_M)$ stand for the profile of $M \oplus N$ where $P_v = R_v \forall v \in \{1, \dots, n\}$ and $P_{n+v} = Q_v \forall v \in \{1, \dots, m\}$. Note that when $m < n$, $P_{M \oplus N}$ is uniquely defined by setting the first n voters as the voters of N and the remaining m voters as the voters of M . Abusing notation, when $M = \{v\}$, we write (R_N, Q_v) to denote the profile obtained from R_N by adding the preference Q_v of voter v . Given any $N \in \mathbf{N}$ and $v \in N$, we let $N^{-v} = N \setminus \{v\}$.

Definition 1. A SCR F satisfies participation (PART) iff $\forall N$ with $n \geq 2$, $\forall v \in N$, $\forall P_{N^{-v}}, \forall P_v$,

$$F(P_{N^{-v}}, P_v) \neq F(P_{N^{-v}}) \implies F(P_{N^{-v}}, P_v) P_v F(P_{N^{-v}}).$$

⁶Since P_v is a linear order it is complete, asymmetric and transitive. So, by completeness, for any distinct $x, y \in A$, we have $xP_v y$ or $yP_v x$. Moreover, since P_v is asymmetric, $\forall x, y \in A$ $xP_v y \implies y \neg P_v x$. Furthermore, if $xP_v y$ and $yP_v z$ then $xP_v z$ by transitivity.

Definition 2. Given any N , any x and any P_N, P'_N such that $P_v \neq P'_v$ for some $v \in N$ and $P_w = P'_w \forall w \in N \setminus \{v\}$, we say that P_N is an improvement for x w.r.t. P'_N if

- (1) $xP'_v y \implies xP_v y$ for every $y \in A \setminus \{x\}$,
- (2) $yP'_v z \iff yP_v z$ for every $y, z \in A \setminus \{x\}$.

Definition 3. A SCR F is monotonic (MON) iff given $x \in A$, $P_N, P'_N \in \Pi^N$ such that P_N is an improvement for x w.r.t. P'_N

$$x = F(P'_N) \implies x = F(P_N).^7$$

2.1. The case of two alternatives. The logical independence between MON and PART vanishes when there are only two alternatives. In fact, in this case PART implies MON, as stated by the following theorem.

Theorem 1. Let $\#A := 2$. If a SCR F satisfies PART, then it satisfies MON.

Proof. Let $A := \{x, y\}$. Take some F which satisfies PART but fails MON. So $\exists N, v$, P_{N-v}, P_v, P'_v with $xP_v y$, $yP'_v x$ while $F(P_{N-v}, P_v) = y$ and $F(P_{N-v}, P'_v) = x$. However, by PART, $F(P_{N-v}, P_v) = y$ implies $F(P_{N-v}) = y$ and $F(P_{N-v}, P'_v) = x$ implies $F(P_{N-v}) = x$, giving a contradiction. **Q.E.D.**

The reverse implication holds, when a mild homogeneity condition is assumed. For any positive integer m , and any profile P_N , we write mP_N for any of the profiles obtained from P_N by replacing each single voter v of P_N with m voters having the same preference as v .⁸

Definition 4. A SCR F satisfies homogeneity (HOM) if $\forall N, \forall P_N, \forall m \in \mathbf{N}, \forall mP_N, F(mP_N) = F(P_N)$.

As the following theorem shows, any homogeneous SCR which satisfies MON satisfies PART as well.

Theorem 2. Let $\#A := 2$. If a SCR F satisfies HOM and MON, then it satisfies PART.

Proof. Let $A := \{x, y\}$. Assume that F satisfies HOM and MON but fails PART. Since F fails PART, there is some N with $\#N := n \geq 2$, some profile P_N and some voter $v \in N$ with $xP_v y$ while $F(P_{N-v}) = x$ and $F(P_N) = y$. Let $\#\{v \in N \mid xP_v y\} := k$ and $\#\{v \in N \mid yP_v x\} := n - k$. Consider now the profiles nP_{N-v} and $(n-1)P_N$. Due to HOM,

⁷The definition of MON applies just to profiles that differ by a single voter's preference, since our focus is on its relation to PART which is defined with respect to the addition of a single voter. However, it should be noted that our definition is equivalent to the more common definition in the literature where MON applies also to profiles which possibly differ in a group of voters' preferences.

⁸Note that mP_N is equivalent to $P_{N \oplus N \oplus \dots \oplus N}$. As here, \oplus is applied to sets of equal size, mP_N is not uniquely defined.

it follows that $F(nP_{N-v}) = x$ and $F((n-1)P_N) = y$. Moreover, both profiles have the same number $n(n-1)$ of voters. Yet, they differ on the number of voters who prefer x to y : there are $nk - n$ voters who prefer x to y in nP_{N-v} and $nk - k$ in $(n-1)P_N$. Hence, there is some profile $(n-1)P_N$ which is an improvement for x w.r.t. some profile nP_{N-v} . Since $F(nP_{N-v}) = x$, applying MON it follows that $F((n-1)P_N) = x$, giving a contradiction. **Q.E.D.**

However, it should be noted that MON does not imply PART without HOM, as the following proposition shows.

Proposition 1. *Let $\#A := 2$. There exists some SCR F that satisfies MON and fails PART.*

Proof. Let $A := \{x, y\}$. We construct some F that satisfies MON but fails PART (and HOM, by Theorem 2). When $\#N$ is even, let $F(P_N) = x$ iff $\#\{v \in N \mid xP_v y\} \geq \#\{v \in N \mid yP_v x\}$. When $\#N$ is odd, let $F(P_N) = x$ iff $\#\{v \in N \mid xP_v y\} = \#N$. In other words, when $\#N$ is even, F is the majority rule biased towards x in the event of tie and when $\#N$ is odd F is the unanimity rule biased towards y in the absence of a unanimously agreed alternative. It is clear that F satisfies MON. Let P_N be a profile with two out of three voters who prefer x to y and one voter who prefers y to x . Thus, there is some $P_v \in \Pi$ with $xP_v y$. It follows that $F(P_{N-v}) = x$ and $F(P_N) = y$. Since $F(P_{N-v})P_v F(P_N)$, F violates PART as desired. To see that F fails HOM, note that $F(2(P_N)) = x$ whereas $F(P_N) = y$. **Q.E.D.**

2.2. The case of three or more alternatives.

2.2.1. *On the implication of PART by MON.* We start by showing that PART is not implied by MON even under the following reinforcement condition.⁹

Definition 5. *A SCR F satisfies reinforcement (REIN) if for any pair of electorates M and N , for any P_M, P_N and for any $x \in A$,*

$$F(P_M) = F(P_N) = x \implies F(P_{M \oplus N}) = x.$$

As we show our claim through a threshold scoring rule, we introduce the general class:

For any preference P_v and any alternative x , the rank of x in P_v equals $r(x; P_v) = 1 + \#\{y \in A \mid yP_v x\}$. A score vector $s = (s_1, \dots, s_{\#A})$ is an $\#A$ -dimensional vector with $s_1 \geq s_2 \geq \dots \geq s_{\#A}$ and $s_1 > s_{\#A}$. Under a score vector s , the score of alternative x

⁹Reinforcement has a strong version (Smith [1973], Young [1974, 1975], Myerson [1995]) and a weak version (Saari [1990]) which coincide for single-valued SCRs. For multi-valued SCRs, Saari [1990] discusses the notion of weak reinforcement and shows that threshold scoring rules represent a class of SCRs that fail strong reinforcement but satisfy its weak version.

at the preference profile P_N equals $S(x; P_N; s) = \sum_{v \in N} s_{r(x; P_v)}$. For any electorate N , any set A of alternatives and any score vector s , we define a threshold $t(N, A, s) := \frac{\#N}{\#A} \sum_{i=1}^{\#A} s_i$.

Definition 6. Every score vector s induces a threshold scoring rule F which is defined for every N and every P_N as $F(P_N) = \{x \in A \mid S(x; P_N; s) \geq t(N, A, s)\}$. When $F(P_N)$ is multi-valued, ties are broken according to an exogenous (alphabetical) linear order.

Threshold scoring rules are well-defined since, by the choice of $t(N, A, s)$, $F(P_N)$ is not empty for every P_N . The next lemma proves this point formally.

Lemma 1. If F is a threshold scoring rule, $F(P_N)$ is not empty for every P_N .

Proof. Assume by contradiction that there is some profile P_N for which $F(P_N)$ is empty. Thus, $S(x; P_N, s) < t(N, A, s)$ for any $x \in A$. It follows that $\sum_{x \in A} S(x; P_N, s) < \sum_{x \in A} t(N, A, s)$. However, the left part of the inequality equals $\#N \sum_{i=1}^{\#A} s_i$ whereas the right part equals $\#A t(N, A, s)$ which contradicts the definition of $t(N, A, s)$. **Q.E.D.**

Proposition 2. Let $\#A \geq 3$. There exists some SCR F that satisfies *MON* and *REIN* and fails *PART*.

Proof. Let $A := \{x, y, z\}$ and consider F to be the threshold scoring rule with $s = (1, 1, 0)$. Hence, $t(N, A, s) = \frac{2}{3}n$. F satisfies *MON* and *REIN*. To see that F fails *PART*, let $N = \{1, \dots, 6\}$ and take the preference profile P_N with $aP_v bP_v c$ for $v = 1, 2$, $cP_v bP_v a$ for $v = 3, 4$ and $aP_v cP_v b$ for $v = 5$, and $bP_v aP_v c$ for $v = 6$. It follows that for $v = 6$, $F(P_{N-v}) = b$ and $F(P_N) = a$ while $F(P_{N-v})P_v F(P_N)$, proving that *PART* fails.¹⁰ **Q.E.D.**

We now ask whether all threshold scoring rules fail *PART*. The answer is almost affirmative as the theorem below shows.

Theorem 3. Let $\#A \geq 3$. Let F be a threshold scoring rule induced by a score vector s . F satisfies *PART* if and only if

$$s = (s_1, \frac{s_1 + s_{\#A}}{2}, \frac{s_1 + s_{\#A}}{2}, \dots, \frac{s_1 + s_{\#A}}{2}, s_{\#A}).$$

Proof. Take some threshold scoring rule F with score vector s . Suppose *PART* fails so $F(P_{N-v}) = x$ and $F(P_{N-v}, P_v) = y$ with $xP_v y$ for some N , v , P_{N-v} and P_v . This can occur under one of the following two exhaustive cases.

Case 1: $S(x, P_{N-v}, s) \geq t(N^{-v}, A, s)$ and $S(x, P_N, s) < t(N, A, s)$.

¹⁰This example is equivalent to the one discussed by Moulin [1988] in p.62 showing that *REIN* and *PART* are logically independent.

Case 2: $\exists y \neq x$ with $S(y, P_{N-v}, s) < t(N^{-v}, A, s)$ and $S(y, P_N, s) \geq t(N, A, s)$.

Note that $t(N, A, s) - t(N^{-v}, A, s) = \frac{\sum_{i=1}^{\#A} s_i}{\#A}$.

A necessary and sufficient condition to avoid case 1 is that the lowest additional score that x receives with the arrival of v is at least $\frac{\sum_{i=1}^{\#A} s_i}{\#A}$. As $xP_v y$, hence $r(x; P_v) \geq \#A - 1$, this is ensured by setting $s_{\#A-1} \geq \frac{\sum_{i=1}^{\#A} s_i}{\#A}$.

A necessary and sufficient condition to avoid case 2, it is that the highest additional score that y gets with the arrival of v does not exceed $\frac{\sum_{i=1}^{\#A} s_i}{\#A}$. As $xP_v y$, hence $r(y; P_v) \leq 2$, this is ensured by setting $s_2 \leq \frac{\sum_{i=1}^{\#A} s_i}{\#A}$.

The two inequalities combined with $s_2 \leq s_3 \leq \dots \leq s_{\#A-1}$ imply $s_j = \frac{s_1 + s_{\#A}}{2}$ for all $j = 2, \dots, \#A - 1$. **Q.E.D.**

Having shown that MON does not imply PART even under REIN, we now adopt a weaker version of participation, introduced by Pérez [2001], as the absence of a stronger version of the no-show paradox where a voter, by abstaining, can enforce his most preferred alternative as the social outcome.

Definition 7. A SCR F satisfies weak participation (WPART) iff $\forall N, \forall v \in N, \forall P_{N-v}, \forall P_v$,

$$F(P_{N-v}, P_v) \neq F(P_{N-v}) \implies \exists x \in A \text{ s.t. } xP_v F(P_{N-v}).$$

We now show that MON does not imply WPART either, even when combined with HOM.

Proposition 3. Let $\#A \geq 3$. There exists some SCR F that satisfies HOM and MON but fails WPART.

Proof. Let $A := \{x, y, z\}$ and consider the threshold scoring rule with $s = (1, 1, 0)$. This rule satisfies REIN and MON. Note that REIN implies HOM. In order to see why it fails WPART, consider the example used in the proof of Proposition 2. Since $b = F(P_{N-v})P_v F(P_{N-v}, P_v) = a$ with $bP_v aP_v c$, this shows that WPART fails. **Q.E.D.**

We now ask whether MON would imply WPART under REIN. However, REIN is sufficiently strong to imply WPART almost by its own. In fact, combined with the following weak unanimity condition imposed over singleton electorates, REIN implies WPART.

Definition 8. A SCR F is weakly unanimous iff $\forall v \in N, \forall P_v$,

$$xP_v y \forall y \neq x \implies F(P_v) = x.$$

Proposition 4. Let $\#A \geq 3$. If a weakly unanimous SCR F satisfies REIN, then it satisfies WPART.

Proof. Take some weakly unanimous F that satisfies REIN. If it fails WPART, there exist P_{N-v}, P_v such that $F(P_{N-v}, P_v) = x \neq F(P_{N-v}) = z$ with $z P_v y \forall y \in A \setminus \{z\}$. As z is first ranked in P_v , by weak unanimity, $F(P_v) = z$. By REIN, it follows that $F(P_{N-v}, P_v) = z$, which leads to a contradiction. **Q.E.D.**

2.2.2. On the implication of MON by PART. We ask whether MON has reasonable weakenings implied by PART. We first strengthen the definition of an improvement, by asking that the lifted alternative must be raised from the bottom of the ranking.

Definition 9. Given any N , any x and any P_N, P'_N such that $P_v \neq P'_v$ for some $v \in N$ and $P_w = P'_w \forall w \in N \setminus \{v\}$, we say that P_N is a strong improvement for x w.r.t. P'_N if

- (1) $y P'_v x$ for every $y \in A \setminus \{x\}$,
- (2) $y P'_v z \iff y P_v z$ for every $y, z \in A \setminus \{x\}$.

The following is a weakening of MON because the definition of improvement is strengthened but also because it allows alternatives above the lifted alternative to be chosen.

Definition 10. A SCR F is weakly monotonic (WMON) iff given $x \in A$, $P_N, P'_N \in \Pi^N$ such that P_N is a strong improvement for x w.r.t. P'_N :

$$x = F(P'_N) \text{ and } F(P_N) \neq F(P'_N) \implies F(P_N) P_v x.$$

Theorem 4. Let $\#A \geq 3$. If a SCR F satisfies PART, then it satisfies WMON.

Proof. Take some F that satisfies PART but violates WMON. Since WMON fails, there exist some N , $v \in N$, P_N, P'_N with $P'_v \neq P_v$ where (P_{N-v}, P_v) is a strong improvement for x with respect to (P_{N-v}, P'_v) , while $F(P_{N-v}, P'_v) = x$ and $F(P_{N-v}, P_v) = y$ with $x P_v y$. Due to PART, $F(P_{N-v}, P'_v) = x$ implies $F(P_{N-v}) = x$, as otherwise $F(P_{N-v}) P'_v x$ would hold, violating PART. Since $F(P_{N-v}) = x$, then by PART again, we have $F(P_{N-v}, P_v) \neq y$, giving a contradiction. **Q.E.D.**

It should be noted that WMON is not too weak: it is able to discriminate among the SCRs that fail MON. For instance, one can check that the examples described in Campbell and Kelly [2002] which fail MON do satisfy WMON, which is the case by Theorem 4, as they all satisfy PART. On the other hand, plurality with a runoff, well known to fail MON, fails WMON as well, as we illustrate through the example below.

Example 1: Let $A := \{x, y, z\}$ and consider two profiles P'_N, P_N with eight voters such that P_N is a strong improvement for z w.r.t. to P'_N :

$$\begin{aligned}
P_N & : \#\{v \in N \mid xP_v y P_v z\} = 2, \#\{v \in N \mid yP_v z P_v x\} = 2, \#\{v \in N \mid zP_v x P_v y\} = 4, \\
P'_N & : \#\{v \in N \mid xP'_v y P'_v z\} = 3, \#\{v \in N \mid yP'_v z P'_v x\} = 2, \#\{v \in N \mid zP'_v x P'_v y\} = 3.
\end{aligned}$$

Under plurality with a runoff where ties are broken in favor of y , at P'_N , x and z which are both first ranked by three voters go for a runoff and, since there is a majority of voters who prefer z to x , we have $F(P'_N) = z$. At P_N , z is first ranked by four voters whereas both x and y are ranked first by two voters each. As ties are broken in favor of y , y and z go for a runoff where $F(P_N) = y$. However, this violates WMON since P_N is a strong improvement for z w.r.t. to P'_N and $F(P'_N) = z$.

3. THE FIXED ELECTORATE CASE

We now consider the case where the electorate N is of fixed size $n \geq 2$. A voter v is now allowed to have as preference a linear order $P_v \in \Pi$ or to abstain, i.e. have full indifference over the whole set of alternatives. This indifference is denoted by the null preference R_0 where xR_0y holds for any $x, y \in A$. We let $\bar{\Pi} := \Pi \cup \{R_0\}$. The profile $(P_{N-v}, R_0) \in \bar{\Pi}^N$ is the profile in which voter v abstains and the rest of voters' preferences are as in P_{N-v} .

The social choice rules that we consider in this section are defined for a fixed size of the electorate. So a fixed-size social choice rule (FSCR) is a mapping \bar{F} that returns, for some $N \in \mathbf{N}$ and each $P_N \in \bar{\Pi}^N$, a single alternative $\bar{F}(P_N) \in A$. Note that the full domain assumption prevails, i.e. given the fixed electorate N , \bar{F} is defined for every possible preference profile P_N .

We now define PART and MON in this framework.

Definition 11. A FSCR \bar{F} satisfies participation (PART) iff $\forall v \in N, \forall P_{N-v} \in \bar{\Pi}^{N-v}, \forall P_v \in \bar{\Pi}$,

$$\bar{F}(P_{N-v}, P_v) \neq \bar{F}(P_{N-v}, R_0) \implies \bar{F}(P_{N-v}, P_v) P_v \bar{F}(P_{N-v}, R_0).$$

We now define MON under the possibility of abstention in individual preferences.

Definition 12. Given any x and any P_N, P'_N with $P_v \neq P'_v$ for some $v \in N$ and $P_w = P'_w \forall w \in N \setminus \{v\}$,

If $P'_v \neq R_0$, then P_N is an improvement for x w.r.t. P'_N if

- (1) $xP'_v y \implies xP_v y$ for every $y \in A \setminus \{x\}$,
- (2) $yP'_v z \iff yP_v z$ for every $y, z \in A \setminus \{x\}$.

If $P'_v = R_0$, then P_N is an improvement for x w.r.t. P'_N if

- (1) $xP_v y$ for every $y \in A \setminus \{x\}$.

Definition 13. A FSCR \bar{F} is monotonic (MON) iff given $x \in A$, $P_N, P'_N \in \bar{\Pi}^N$ such that P_N is an improvement for x w.r.t. P'_N ,

$$x = \bar{F}(P'_N) \implies x = \bar{F}(P_N).$$

As the following theorem states, for FSCRs, MON and PART are equivalent when there are two alternatives only.

Theorem 5. Let $\#A := 2$. A FSCR \bar{F} satisfies MON if and only if it satisfies PART.

Proof. Let $A := \{x, y\}$. Take some \bar{F} that satisfies PART but fails MON. Since \bar{F} fails MON, w.l.o.g., one of the following two exhaustive cases holds.

Case 1: there exist some profile $P_{N-v} \in \bar{\Pi}^{N-v}$ and some pair P'_v, P_v with $yP'_v x$ and $xP_v y$ with $\bar{F}(P_{N-v}, P'_v) = x$ and $\bar{F}(P_{N-v}, P_v) = y$. However, due to PART, $\bar{F}(P_{N-v}, P'_v) = x$ implies $\bar{F}(P_{N-v}, R_0) = x$ which in turn implies $\bar{F}(P_{N-v}, P_v) = x$, giving a contradiction.

Case 2: there exist some profile $P_{N-v} \in \bar{\Pi}^{N-v}$ and some pair $P'_v = R_0, P_v$ with $xP_v y$ with $\bar{F}(P_{N-v}, P'_v) = x$ and $\bar{F}(P_{N-v}, P_v) = y$. However, as $\bar{F}(P_{N-v}, R_0) = x$, PART implies $\bar{F}(P_{N-v}, P_v) = x$, giving a contradiction. We leave the reader to check that MON implies PART. **Q.E.D.**

We now define WPART in the fixed electorate framework.

Definition 14. A FSCR \bar{F} satisfies weak participation (WPART) iff $\forall v \in N, \forall P_{N-v} \in \bar{\Pi}^{n-1}, \forall P_v \in \bar{\Pi}$,

$$\bar{F}(P_{N-v}, P_v) \neq \bar{F}(P_{N-v}, R_0) \implies \exists x \in A \text{ s.t. } xP_v \bar{F}(P_{N-v}, R_0).$$

Theorem 6. Let $\#A \geq 3$. If a FSCR \bar{F} satisfies MON, then it satisfies WPART.

Proof. Take some \bar{F} that fails WPART. Since \bar{F} fails WPART, there must some profile $P_{N-v} \in \bar{\Pi}^{N-v}$ and some pair $P'_v, P_v = R_0$ with $xP'_v z \forall z \in A \setminus \{x\}$ while $\bar{F}(P_{N-v}, P_v) = x$ and $\bar{F}(P_{N-v}, P'_v) \neq x$. As (P_{N-v}, P'_v) is an improvement for x w.r.t. (P_{N-v}, P_v) , this violates MON and concludes the proof. **Q.E.D.**

The following is the weakening of MON in the same spirit as the weakening introduced in Section 2.

Definition 15. Given any x and any P_N, P'_N such that $P_v \neq P'_v$ for some $v \in N$ and $P_w = P'_w \forall w \in N \setminus \{v\}$,

If $P'_v \neq R_0$, then P_N is a strong improvement for x w.r.t. P'_N if

- (1) $yP'_v x$ for every $y \in A \setminus \{x\}$,
- (2) $yP'_v z \iff yP_v z$ for every $y, z \in A \setminus \{x\}$.

If $P'_v = R_0$, then P_N is a strong improvement for x w.r.t. P'_N if

(1) $xP_v y$ for every $y \in A \setminus \{x\}$.

Definition 16. A FSCR \bar{F} is weakly monotonic (WMON) iff given $x \in A$, $P_N, P'_N \in \bar{\Pi}^N$ such that P_N is a strong improvement for x w.r.t. P'_N

$$x = \bar{F}(P'_N) \text{ and } \bar{F}(P_N) \neq \bar{F}(P'_N) \implies \bar{F}(P_N)P_v x.$$

Theorem 7. Let $\#A \geq 3$. If a FSCR \bar{F} satisfies PART, then it satisfies WMON.

Proof. Take some \bar{F} that satisfies PART but fails WMON. Since \bar{F} fails WMON, one of the following two exhaustive cases holds.

Case 1: there exist some profile $P_{N-v} \in \bar{\Pi}^{N-v}$ and some pair P'_v, P_v with $zP'_v x \forall z \in A \setminus \{x\}$ and $xP_v y$ with $\bar{F}(P_{N-v}, P'_v) = x$ and $\bar{F}(P_{N-v}, P_v) = y$. However, due to PART, $\bar{F}(P_{N-v}, P'_v) = x$ implies $\bar{F}(P_{N-v}, R_0) = x$ which in turn implies $\bar{F}(P_{N-v}, P_v) \neq y$, giving a contradiction.

Case 2: there exist some profile $P_{N-v} \in \bar{\Pi}^{N-v}$ and some pair $P'_v = R_0, P_v$ with $xP_v z \forall z \neq x$ while $\bar{F}(P_{N-v}, P'_v) = x$ and $\bar{F}(P_{N-v}, P_v) = y$. However, due to PART, $\bar{F}(P_{N-v}, R_0) = x$ implies $\bar{F}(P_{N-v}, P_v) = x$, giving a contradiction.

Thus, there is no \bar{F} satisfying PART but failing WMON, which concludes the proof. **Q.E.D.**

4. CONCLUDING REMARKS

Although the logical independence between the no-show paradox and the failure of monotonicity has already been observed, our findings suggest that this observation does not mean a major conceptual gap between the two conditions. In fact, under two different interpretations of "a new-comer to the society", we are able to present instances where participation and monotonicity get very close to each other - in some particular cases to the extent that the established general logical independence between them vanishes.

This closeness is rather expected to us because, as we discuss in the introduction, both conditions are related to the manipulability of SCRs. In fact, until their logical independence was established by Nurmi [1999] and Campbell and Kelly [2002], there was a prevailing intuition that the two conditions were somehow related. That MON does not imply PART was viewed mostly as a consequence of the fact that PART is a condition for SCRs defined over variable size societies while MON is not (see, for example, Sanver and Zwicker [2009]). A comparative look at our results on the implication of PART by MON under two different interpretations of abstention points to a wisdom in this view. That PART does not imply MON has been established by Campbell and Kelly [2002] until which the general intuition was that the said implication would hold (see, for example, Nurmi [1999], p.62).

It is worth noting that PART and MON get more closer under the fixed electorate interpretation. This is also rather expected because, again as discussed in the introduction, under this interpretation the link between the no-show paradox and manipulability of SCRs is more direct.

A point we wish to emphasize is the equivalence between PART and MON under the fixed-electorate interpretation, when there are two alternatives. In this framework, majority rules with quorums are known to fail MON and they are supposed to give room to manipulation by abstention (see Houy [2009]). Our Theorem 5 is a formal expression of this supposition.

Finally, we wish to remark that PART has been mostly considered in the literature for single-valued SCRs which led our analysis to be held in this framework. However, there are a relatively few considerations of PART for multi-valued SCRs, such as Jimeno et al. [2009], and how our analysis would carry to that framework remains as an open question.

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