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To cite this version:

Mikhail Zubkov. On the absence of equilibrium chiral magnetic effect. 2016. hal-01275180v2

HAL Id: hal-01275180

https://hal.science/hal-01275180v2

Preprint submitted on 19 Feb 2016 (v2), last revised 15 Apr 2016 (v3)

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On the absence of equilibrium chiral magnetic effect

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(Dated: February 19, 2016)

We demonstrate how in the fermionic systems the linear response to the external electromagnetic field is related to momentum space topology. We use derivative expansion applied to the Wigner transform of the two - point Green function. It appears, that the response of electric current to the field strength is proportional to momentum space topological invariant. For the $2+1$ D systems this allows to reproduce the well - known expression for Hall conductivity. Next, we analyse for the wide class of the $3+1$ D fermionic systems the equilibrium chiral magnetic effect (CME). For the lattice regularization of continuum quantum field theory (and also for the Dirac semimetals) the mentioned topological invariant does not depend on the value of the chiral chemical potential. That means, that the bulk equilibrium CME is absent in those systems.

PACS numbers: 11.40.Ha, 11.15.Ha, 73.43.-f, 03.65.Vf

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I. INTRODUCTION.

The chiral magnetic effect (CME) has been widely discussed recently in different contexts both within the continuous quantum field theory and in the condensed matter physics. The CME for the case, when the left - handed and the right - handed fermions are truly separated was first discussed in [1]. In the context of quantum field theory the existence of chiral magnetic effect was considered in [2], followed by a number of papers (see, for example, [3, 6] and references therein). In particular, CME has been discussed using a different technique in the Fermi liquids [4].

The possible existence of the chiral magnetic contribution to conductivity was proposed in [5], and was discussed later in a number of papers. The experimental observation of this contribution to conductivity in the recently discovered Dirac semimetals was reported in [7]. Notice, that from our point of view such a chiral magnetic contribution to conductivity should be distinguished from the equilibrium CME [2]. In the former the chiral imbalance appears as a pure kinetic phenomenon, and the final expression for the CME current is proportional to the squared magnetic field and, in addition, to electric field. At the same time in the latter the nondissipative current is linear in magnetic field and is predicted to appear without any external electric field. Therefore, the linear response theory to be considered in the present paper, strictly speaking, does not describe chiral magnetic contribution to conductivity. Thus we will concentrate on the equilibrium CME.

The family of the non - dissipative transport effects being the cousins of the CME has also been widely discussed recently both in the context of the high energy physics and in the context of condensed matter theory [8–15]. The possible appearance of such effects in the recently discovered Dirac and Weyl semimetals has been considered [16–22]. In the context of the high energy physics the possibility to observe CME in relativistic heavy - ion collisions was widely discussed (see, for example, [3, 23, 24] and references therein). Certain lattice calculations seem to confirm indirectly this possibility [25].

In several publications the existence of equilibrium CME was questioned. In particular, in [12–15] using different numerical methods the CME current was investigated in the context of lattice field theory. It was argued, that the equilibrium bulk CME does not exist, but close to the boundary of the system the nonzero CME current may appear. It was demonstrated, that in the given systems the integrated total CME current remains zero. The similar conclusion was drawn in [10] basing on the consderation of the system of finite size with the special boundary conditions in the direction of the external magnetic field. The consideration of [10], however, does not refer to the systems, which do not have boundaries or, say, have the form of a circle with magnetic field directed along the circle. In the context of condensed matter theory the absence of CME was reported within the particular model of Weyl semimetal [34]. Besides, it was argued, that the equilibrium CME may contradict to the no - go Bloch theorem [35].

In the present paper we consider CME on the basis of Wigner transformation technique [30, 31] applied to Green functions. First of all, we demonstrate, that the derivative expansion within this technique allows to reduce the expression for the linear response of electric current to the external field strength to the momentum space topological invariant. The power of this method is demonstrated on the example of the 2+1 D quantum Hall effect (QHE), where it allows to derive in a simple way the conventional expression for Hall conductivity [26]. Momentum space topology is a powerful method, which was developed earlier mainly within condensed matter theory. In addition to the ordinary quantum Hall effect it allows to describe in a simple way a lot of the other effects (for the review see [26, 27]). Recently certain aspects of momentum space topology were discussed in the framework of the four - dimensional lattice gauge theory [28, 29]. Here we derive the expression for the linear response of the electric current to the external magnetic field in the wide class of the $3+1$ D fermionic systems, which includes popular lattice regularizations of continuum quantum field theory and the models of discovered recently Dirac semimetals.

Strictly speaking, our calculations remain unambiguous only for the systems with the Green functions that do not have poles (or zeros). It appears, that like in the $2+1$ D case the resulting $3+1$ D response of electric current to the external magnetic field is proportional to the topological invariant in momentum space. Unlike the case of the naive continuum fermions for the lattice regularized quantum field theory (or for the real Dirac semimetals) with the infrared regularization via the nonzero Dirac mass the value of the mentioned topological invariant does not depend on the chiral chemical potential. Moreover, the presence of chiral chemical potential eliminates poles of the Green function even if the bare Dirac mass is zero. Therefore, the limit, when the physical Dirac mass tends to zero, does not change our conclusion. This means, that the equilibrium bulk CME current is absent for the lattice regularized quantum field theory and in the real Dirac semimetals.

The paper is organized as follows. In Sect. II we start the discussion of the linear response of electric current to external electromagnetic field using continuum formulation. The Wigner transform of the two - point Green function is defined in Sect. II A. The main equation obeyed by this object is proved in Appendix A. In Section II B we present the gradient expansion for the Wigner transform of the Green function. The linear response of the electric current to external gauge field is considered in Sect. II C. It appears, that the resulting expression is divergent and requires regularization. In Sect. III we consider lattice regularization. In Sect. III A we discuss lattice theory of general form, which allows to describe not only the lattice regularization of the continuum quantum field theory, but also the tight

- binding like models of solid state physics. We propose the unusual way to introduce the external gauge field to the lattice model. This method allows to deal with the theory written in momentum space, which is important for our further considerations. The proposed method is manifestly gauge invariant, and it is obviously reduced to the conventional minimal connection of theory with the gauge field in continuum limit. Therefore, it allows to introduce effectively the gauge field both into the lattice regularization of quantum field theory and to the models of the solid state physics. In Sect. III B the Wigner transform of the lattice two - point Green function in momentum space is discussed. It appears, that it obeys the same equation as its continuum counterpart. This is proved in Appendix B. In Sect. III C the linear response of the electric current to the external gauge field is derived for the lattice theory. It appears, that the resulting expression represents the direct lattice discretization of the corresponding continuum expression as expected. This expression is, in turn, a topological invariant in momentum space. This is proved in Appendix C. In Sect. IV the celebrated expression for the Hall current is reproduced using the proposed technique. In Sect. V we finally discuss the chiral magnetic effect, and prove basing on the proposed methodology, that it is absent for the lattice regularized theory and for the solid state systems like topological insulators and Dirac semimetals. In Sect. VI we end with the conclusions.

II. CONTINUUM THEORY

A. Wigner transform of the Green function.

In the present section we recall some of the basic notions of the Wigner (Weyl) transform. For the more details see, for example, [37] and Appendix B in [36]. Next, we will apply those notions to the two point Green function of a non - interacting fermion system in the presence of external gauge field.

Let us consider the $d+1=D$ dimensional continuum model with the fermionic Green function $\mathcal{G}(\mathbf{p})$ that depends on the D vector $\mathbf{p}=(p_1,...,p_D)$ of Euclidean momentum. (The Wick rotation has been performed.) When interactions between the fermions are neglected, the external electromagnetic field $\mathbf{A}(\mathbf{r})$ may be taken into account through the Hermitian operator - valued function $\hat{\mathcal{Q}}(\mathbf{r},\hat{\mathbf{p}})=\mathcal{G}^{-1}(\hat{\mathbf{p}}-\mathbf{A}(\mathbf{r}))$, where $\hat{\mathbf{p}}=-i\partial_{\mathbf{r}}$. Operators $\hat{p}_i-A_i(\mathbf{r})$ and $\hat{p}_j-A_j(\mathbf{r})$ do not commute for $i\neq j$. Therefore, we should point out the way of their ordering inside $\hat{\mathcal{Q}}$. We choose the following way for definiteness: each product $p_{i_1}...p_{i_n}$ in the expansion of $\mathcal{G}^{-1}$ is substituted by the symmetrized product $\frac{1}{n!}\sum_{\text{permutations}}(\hat{p}_{i_1}-A_{i_1})...(\hat{p}_{i_n}-A_{i_n})$. This way of ordering corresponds to the so - called symmetrical (or, Wigner) quantization according to [37]. The resulting function $\hat{\mathcal{Q}}$ enters the functional integral representation for the Euclidean partition function

$$Z=\int D\bar{\Psi}D\Psi\exp\Big(-\int d^D\mathbf{r}\bar{\Psi}(\mathbf{r})\hat{Q}(\mathbf{r},\hat{\mathbf{p}})\Psi(\mathbf{r})\Big) \tag{1}$$

Here Ψ, $\bar{\Psi}$ are the Grassmann - valued continuum fermionic fields. In the presence of the gauge field the Green function appears as a correlator

$$\begin{aligned}G(\mathbf{r}_1,\mathbf{r}_2)=\frac{1}{Z}\int D\bar{\Psi}D\Psi\,\bar{\Psi}(\mathbf{r}_2)\Psi(\mathbf{r}_1)\\ \exp\Big(-\int d^D\mathbf{r}\bar{\Psi}(\mathbf{r})\hat{Q}(\mathbf{r},\hat{\mathbf{p}})\Psi(\mathbf{r})\Big)\end{aligned} \tag{2}$$

It obeys equation

$$\hat{\mathcal{Q}}(\mathbf{r}_1,-i\partial_{\mathbf{r}_1})G(\mathbf{r}_1,\mathbf{r}_2)=\delta^{(D)}(\mathbf{r}_1-\mathbf{r}_2) \tag{3}$$

Wigner transform [30] of the Green function is defined as

$$\tilde{G}(\mathbf{R},\mathbf{p})=\int d^D\mathbf{r}e^{-i\mathbf{pr}}G(\mathbf{R}+\mathbf{r}/2,\mathbf{R}-\mathbf{r}/2) \tag{4}$$

In Appendix A the Groenewold equation [36] for the function $\tilde{G}$ is derived

$$\begin{aligned}1=\mathcal{Q}(\mathbf{R},\mathbf{p})*\tilde{G}(\mathbf{R},\mathbf{p})\\ \equiv\mathcal{Q}(\mathbf{R},\mathbf{p})e^{\frac{i}{2}(\overleftarrow{\partial}_{\mathbf{R}}\overrightarrow{\partial}_{\mathbf{p}}-\overleftarrow{\partial}_{\mathbf{p}}\overrightarrow{\partial}_{\mathbf{R}})}\tilde{G}(\mathbf{R},\mathbf{p})\end{aligned} \tag{5}$$

Here function $\mathcal{Q}$ represents the so - called Weyl symbol of operator $\hat{\mathcal{Q}}$ being the Wigner transform of its matrix elements [36, 37]. It depends on the real numbers rather than on the operators. The explicit form of the relation

between $\mathcal{Q}$ and $\hat{\mathcal{Q}}$ is given in Appendix A. Here we will need only the following property of the correspondence between $\hat{\mathcal{Q}}$ and $\mathcal{Q}$. If $\hat{\mathcal{Q}}$ has the form of a function $\mathcal{G}^{-1}$ of the combination $(\hat{\mathbf{p}} - \mathbf{A}(\mathbf{r}))$ with a gauge potential $\mathbf{A}(\mathbf{r})$, i.e.

$$\hat{\mathcal{Q}}(\mathbf{r}, \hat{\mathbf{p}}) = \mathcal{G}^{-1}(\hat{\mathbf{p}} - \mathbf{A}(\mathbf{r})) \tag{6}$$

then we have

$$\mathcal{Q}(\mathbf{r}, \mathbf{p}) = \mathcal{G}^{-1}(\mathbf{p} - \mathbf{A}(\mathbf{r})) + O([\partial_i A_j]^2) \tag{7}$$

Here $O([\partial_i A_j]^2)$ may contain the terms with the second power of the derivative of $\mathbf{A}$, with the squared derivative of $\mathbf{A}$, and the terms higher order in derivatives. In principle, the restrictions on the term $O([\partial_i A_j]^2)$ may be more strong[1], but for our purposes it will be enough, that the terms linear in the derivatives of $\mathbf{A}$ are absent in $O([\partial_i A_j]^2)$ (which is proved in Appendix A and Appendix B).

Notice, that the star product entering Eq. (5) is widely used in deformation quantisation [32, 37] and also in some other applications (see, for example, [31] and references therein).

B. Gradient expansion for the Wigner transform of the Green function in the presence of external gauge field.

Here we apply the formalism of Wigner transform to the $d+1 = \mathrm{D}$ dimensional fermionic systems. Let us consider the model with the Green function $\mathcal{G}(\mathbf{p})$ that depends on the D - vector $\mathbf{p} = (p_1, p_2, ..., p_4)$ of Euclidean momentum. We introduce the slowly varying external $U(1)$ vector gauge field $\mathbf{A}(\mathbf{r})$ defining operator function $\hat{\mathcal{Q}}$ of Eq. (6). The Wigner transform of the Green function Eq. (4) satisfies Eq. (5).

We apply the gradient expansion and come to

$$\tilde{G}(\mathbf{R}, \mathbf{p}) = \tilde{G}^{(0)}(\mathbf{R}, \mathbf{p}) + \tilde{G}^{(1)}(\mathbf{R}, \mathbf{p}) + ... \tag{8}$$

$$\tilde{G}^{(1)} = -\frac{i}{2}\tilde{G}^{(0)}\frac{\partial\left[\tilde{G}^{(0)}\right]^{-1}}{\partial p_i}\tilde{G}^{(0)}\frac{\partial\left[\tilde{G}^{(0)}\right]^{-1}}{\partial p_j}\tilde{G}^{(0)}A_{ij}(\mathbf{R})$$

Here $\tilde{G}^{(0)}(\mathbf{R}, \mathbf{p})$ is defined as the Green function with the field strength $A_{ij} = \partial_i A_j - \partial_j A_i$ neglected. It is given by

$$\tilde{G}^{(0)}(\mathbf{R}, \mathbf{p}) = \mathcal{G}(\mathbf{p} - \mathbf{A}(\mathbf{R})) \tag{9}$$

C. Linear response of electric current to the strength of external gauge field.

The components of vector $U(1)$ current in the system of non - interacting fermions may be expressed as:

$$j^k(\mathbf{R}) = -\mathrm{Tr}\,\frac{\partial\mathcal{G}^{-1}(-i\partial_{\mathbf{r}_1} - \mathbf{A}(\mathbf{r}_1))}{\partial A_k}\,G(\mathbf{r}_1, \mathbf{r}_2)\Big|_{\mathbf{r}_1, \mathbf{r}_2 \to \mathbf{R}}$$
$$= \int \frac{d^D\mathbf{p}}{(2\pi)^D}\,\mathrm{Tr}\,\tilde{G}(\mathbf{R}, \mathbf{p})\frac{\partial\left[\tilde{G}^{(0)}(\mathbf{R}, \mathbf{p})\right]^{-1}}{\partial p_k} \tag{10}$$

For the derivation of the second row in this expression we applied expressions of Appendix A. Also, this expression follows as a continuum limit of the corresponding formula to be derived in the next section. We also advise the reader to consult Appendix B of [36], where many useful relations are collected, including those, which give rise to Eq. (10).

In the $3+1$ D systems the contribution to electric current originated from $\tilde{G}^{(1)}$ is given by

$$j^{(1)k}(\mathbf{R}) = \frac{1}{4\pi^2}\epsilon^{ijkl}\mathcal{M}_l A_{ij}(\mathbf{R}), \tag{11}$$

$$\mathcal{M}_l = \int \mathrm{Tr}\,\nu_l\, d^4p \tag{12}$$

$$\nu_l = -\frac{i}{3!\,8\pi^2}\,\epsilon_{ijkl}\left[\mathcal{G}\frac{\partial\mathcal{G}^{-1}}{\partial p_i}\frac{\partial\mathcal{G}}{\partial p_j}\frac{\partial\mathcal{G}^{-1}}{\partial p_k}\right] \tag{13}$$

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[1] For example, the Weyl symbol of the operator $f(-i\partial_r - Hr)$ for the one - dimensional problem ($r \in \mathbf{R}^1$) is given by $f(p - Hr)$ exactly [37].

In the linear response theory we should substitute $\mathbf{A} = 0$ into the expression for $\mathcal{M}_l$. Therefore, in Eq. (12) we substitute $\mathcal{G}$ instead of $\tilde{G}^{(0)}$.

In order to understand how Eq. (12) works let us consider the single massless Dirac fermion, which is the couple of the left - handed and the right - handed Weyl fermions. The corresponding expression for the Green function in the presence of chiral chemical potential μ_5 is given by

$$\mathcal{G}(\mathbf{p}) = \Big(\sum_k \gamma^k p_k + i\gamma^4\gamma^5\mu_5\Big)^{-1} \tag{14}$$

with the Euclidean gamma - matrices that satisfy $\{\gamma^a, \gamma^b\} = 2\delta^{ab}$ and the γ^5 matrix given by $\mathrm{diag}(1, 1, -1, -1)$ in chiral representation. In this situation the Green function contains poles. Besides, the integral in Eq. (12) is divergent at infinite values of $\mathbf{p}$. Therefore, in order to apply the above expressions the regularization is needed.

The obvious expectation about Eq. (13) is that in lattice regularization we need simply to substitute into it the lattice Green function defined as a function of lattice momentum, and integrate in Eq. (12) over the compact momentum space $\mathcal{M}$. Below we will see, that this is indeed what should be done.

III. LATTICE REGULARIZED THEORY

A. A way to introduce the external gauge field to lattice model.

Again, let us consider the $d + 1 = D$ dimensional model with the fermionic Green function $\mathcal{G}(\mathbf{p})$ that depends on the D vector $\mathbf{p} = (p_1, ..., p_D)$ of Euclidean momentum. Now we assume, that momentum space is compact and has the form of the product $\mathcal{M} = S^1 \otimes \Omega$, where Ω is the d - dimensional Brillouin zone while S^1 is the circle of p^D. Notice, that the lattice momentum $\mathbf{p}$ does not appear as the eigenvalue of the operator $-i\partial_{\mathbf{r}}$. The same refers also to the solid state models.

In the absence of the external electromagnetic field the partition function of the theory under consideration may be written as

$$Z = \int D\bar{\Psi} D\Psi \exp\Big(-\int_{\mathcal{M}} \frac{d^D\mathbf{p}}{|\mathcal{M}|}\bar{\Psi}(\mathbf{p})\mathcal{G}^{-1}(\mathbf{p})\Psi(\mathbf{p})\Big) \tag{15}$$

where $|\mathcal{M}|$ is the volume of momentum space $\mathcal{M}$. (We neglect here those interactions, which are not taken into account by the form of the two point Green function $\mathcal{G}(\mathbf{p})$. Otherwise, the interaction terms with higher powers of Ψ or with the additional dynamical fields would have been written.)

We assume, that the theory to be dealt with has the form of the lattice regularization of the continuum quantum field theory, or the form of the solid state tight - binding like model. In both cases the theory is defined in discrete coordinate space. We assume, that the dynamical variables Ψ of this theory are attached to the lattice sites $\mathbf{r}_n$.

The fields in coordinate space are related to the fields in momentum space as follows

$$\Psi(\mathbf{r}) = \int_{\mathcal{M}} \frac{d^D\mathbf{p}}{|\mathcal{M}|} e^{i\mathbf{p}\mathbf{r}}\Psi(\mathbf{p}) \tag{16}$$

At the discrete values of $\mathbf{r}$ corresponding to the points of the lattice this expression gives the values of the fermionic field at these points, i.e. the dynamical variables of the original lattice model. However, Eq. (16) allows to define formally the values of fields at any other values of $\mathbf{r}$. The partition function may be rewritten in the form

$$Z = \int D\bar{\Psi} D\Psi \exp\Big(-\sum_{\mathbf{r}_n}\bar{\Psi}(\mathbf{r}_n)\Big[\mathcal{G}^{-1}(-i\partial_{\mathbf{r}})\Psi(\mathbf{r})\Big]_{\mathbf{r}=\mathbf{r}_n}\Big) \tag{17}$$

Here the sum in the exponent is over the discrete coordinates $\mathbf{r}_n$. However, the operator $-i\partial_{\mathbf{r}}$ acts on the function $\Psi(\mathbf{r})$ defined using Eq. (16). In order to derive Eq. (17) we use identity

$$\sum_{\mathbf{r}} e^{i\mathbf{p}\mathbf{r}} = |\mathcal{M}|\delta(\mathbf{p}) \tag{18}$$

Gauge transformation of the original lattice field

$$\Psi(\mathbf{r}_n) \to e^{i\alpha(\mathbf{r}_n)}\Psi(\mathbf{r}_n) \tag{19}$$

now may be understood as the gauge transformation of the field Ψ defined for any values of $\mathbf{r}$: we simply extend the definition of the function $\alpha(\mathbf{r})$ to the function, which is defined at any values of $\mathbf{r}$ and take the original values at the discrete lattice points. This prompts the following way to introduce the external gauge field to our lattice model. We consider the partition function of the form

$$Z = \int D\bar{\Psi} D\Psi \exp\Big(-\frac{1}{2} \sum_{\mathbf{r}=\mathbf{r}_n} \Big[\bar{\Psi}(\mathbf{r}) \mathcal{G}^{-1}(-i\partial_{\mathbf{r}} - \mathbf{A}(\mathbf{r}))\Psi(\mathbf{r}) + (h.c.) \Big] \Big) \tag{20}$$

Here by $(h.c.)$ we denote the Hermitian conjugation, which is defined as follows. First of all, it relates the components of Grassmann variable Ψ with the corresponding components of $\bar{\Psi}$. Besides, it inverses the ordering of operators and the variables $\bar{\Psi}, \Psi$, and substitutes each operator by its Hermitian conjugated. For example, a conjugation of $\bar{\Psi}\hat{B}(i\partial_{r^{i_1}})...(i\partial_{r^{i_n}})\Psi$ for a certain operator (in internal space) $\hat{B}$ is given by $\Big[(-i\partial_{r^{i_1}})...(-i\partial_{r^{i_n}})\bar{\Psi}\Big]\hat{B}^{+}\Psi$. As well as in continuum theory operators $\hat{p}_i - A_i(\mathbf{r})$ and $\hat{p}_j - A_j(\mathbf{r})$ do not commute for $i \neq j$. Therefore, we should point out the way of their ordering inside $\mathcal{G}^{-1}(-i\partial_{\mathbf{r}} - \mathbf{A}(\mathbf{r}))$. We choose the following way for definiteness: each product $p_{i_1}...p_{i_n}$ in the expansion of $\mathcal{G}^{-1}$ is substituted by the symmetrized product $\frac{1}{n!}\sum_{\text{permutations}}(\hat{p}_{i_1} - A_{i_1})...(\hat{p}_{i_n} - A_{i_n})$. This method of introducing the gauge field to the lattice model differs from the more conventional ways, but it is manifestly gauge invariant, and it is obviously reduced to the conventional way the gauge field is to be introduced in the naive continuum limit. Therefore, it satisfies all requirements to be fulfilled by the introduction of the gauge field in lattice regularization of quantum field theory.

Now let us come back to momentum space:

$$Z = \int D\bar{\Psi} D\Psi \exp\Big(-\frac{1}{2} \int_{\mathcal{M}} \frac{d^D\mathbf{p}}{|\mathcal{M}|} \Big[\bar{\Psi}(\mathbf{p}) \mathcal{Q}_{right}(i\partial_{\mathbf{p}}, \mathbf{p})\Psi(\mathbf{p}) + \bar{\Psi}(\mathbf{p}) \mathcal{Q}_{left}(i\partial_{\mathbf{p}}, \mathbf{p})\Psi(\mathbf{p}) \Big] \Big) \tag{21}$$

Here by $\mathcal{Q}_{right}$ we denote the function, that is constructed of $\mathcal{G}^{-1}$ as follows. We represent $\mathcal{G}^{-1}(-i\partial_{\mathbf{r}} - \mathbf{A}(\mathbf{r}))$ as a series in powers of $-i\partial_{\mathbf{r}}$ and $\mathbf{A}(\mathbf{r})$ such that in each term $\mathbf{A}(\mathbf{r})$ stand right to $-i\partial_{\mathbf{r}}$. For example, we represent $(-i\partial_{\mathbf{r}} - A(\mathbf{r}))^2$ as $(-i\partial_{\mathbf{r}})^2 - 2(-i\partial_{\mathbf{r}})\mathbf{A}(\mathbf{r}) + \mathbf{A}^2(\mathbf{r}) - i(\partial\mathbf{A})$. Next, we substitute the argument of $\mathbf{A}$ by $i\partial_{\mathbf{p}}$ and $-i\partial_{\mathbf{r}}$ by $\mathbf{p}$. Correspondingly, $\mathcal{Q}_{left}$ is defined with the inverse ordering.

Let us introduce the following notation

$$\hat{\mathcal{Q}} = \frac{1}{2}\Big[\mathcal{Q}_{right}(i\partial_{\mathbf{p}}, \mathbf{p}) + \mathcal{Q}_{left}(i\partial_{\mathbf{p}}, \mathbf{p}) \Big] \tag{22}$$

Since the commutators $[-i\partial_{r^i}, r^j] = i\delta_i^j$ and $[p_i, i\partial_{p_j}] = i\delta_i^j$ are equal to each other, the actual expression for $\hat{\mathcal{Q}}$ is given by

$$\hat{\mathcal{Q}} = \mathcal{G}^{-1}(\mathbf{p} - \mathbf{A}(i\partial_{\mathbf{p}})) \tag{23}$$

The Green function of our system in momentum space satisfies equation

$$\hat{\mathcal{Q}}(i\partial_{\mathbf{p}_1}, \mathbf{p}_1) G(\mathbf{p}_1, \mathbf{p}_2) = |\mathcal{M}| \delta^{(D)}(\mathbf{p}_1 - \mathbf{p}_2) \tag{24}$$

B. Wigner transform in momentum space

According to the proposed above way to introduce the gauge field the Green function appears as a correlator

$$G(\mathbf{p}_1, \mathbf{p}_2) = \frac{1}{Z} \int D\bar{\Psi} D\Psi \; \bar{\Psi}(\mathbf{p}_2)\Psi(\mathbf{p}_1) \exp\Big(-\int \frac{d^D\mathbf{p}}{|\mathcal{M}|} \bar{\Psi}(\mathbf{p}) \hat{Q}(i\partial_{\mathbf{p}}, \mathbf{p})\Psi(\mathbf{p}) \Big) \tag{25}$$

It obeys equation Eq. (24). Wigner transform [30] of the Green function may be defined as

$$\tilde{G}(\mathbf{R}, \mathbf{p}) = \int \frac{d^D\mathbf{P}}{|\mathcal{M}|} e^{i\mathbf{P}\mathbf{R}} G(\mathbf{p} + \mathbf{P}/2, \mathbf{p} - \mathbf{P}/2) \tag{26}$$

In terms of the Green function in coordinate space this Green function is expressed as:

$$\tilde{G}(\mathbf{R},\mathbf{p}) = \sum_{\mathbf{r}=\mathbf{r}_n} e^{-i\mathbf{p}\mathbf{r}} G(\mathbf{R}+\mathbf{r}/2, \mathbf{R}-\mathbf{r}/2) \tag{27}$$

which is the direct analogue of Eq. (4). In Appendix B we prove, that this Green function obeys the same equation as the one of the continuum theory:

$$\begin{aligned} 1 &= \mathcal{Q}(\mathbf{R},\mathbf{p}) * \tilde{G}(\mathbf{R},\mathbf{p}) \\ &\equiv \mathcal{Q}(\mathbf{R},\mathbf{p}) e^{\frac{i}{2}(\overleftarrow{\partial}_{\mathbf{R}}\overrightarrow{\partial}_{\mathbf{p}} - \overleftarrow{\partial}_{\mathbf{p}}\overrightarrow{\partial}_{\mathbf{R}})} \tilde{G}(\mathbf{R},\mathbf{p}) \end{aligned} \tag{28}$$

As well as in continuum case the Weyl symbol of operator $\hat{\mathcal{Q}}$ is given by function $\mathcal{Q}$ that depends on the real numbers rather than on the operators. As it is explained in Appendix B, if $\hat{\mathcal{Q}}$ has the form of a function $\mathcal{G}^{-1}$ of the combination $(\mathbf{p}-\mathbf{A}(\hat{\mathbf{r}}))$ with a gauge potential $\mathbf{A}(\hat{\mathbf{r}})$, i.e.

$$\hat{\mathcal{Q}}(\mathbf{r},\hat{\mathbf{p}}) = \mathcal{G}^{-1}(\mathbf{p}-\mathbf{A}(i\partial_{\mathbf{p}})) \tag{29}$$

then we have

$$\mathcal{Q}(\mathbf{r},\mathbf{p}) = \mathcal{G}^{-1}(\mathbf{p}-\mathbf{A}(\mathbf{r})) + O([\partial_i A_j]^2) \tag{30}$$

Here $O([\partial_i A_j]^2)$ does not contain terms independent of the derivatives of $\mathbf{A}$ and the terms linear in those derivatives, i.e. it is higher order in derivatives. In certain particular cases the restrictions on the term $O(([\partial_i A_j]^2)$ may be more strong, or it may even vanish at all [37].

C. Linear response of electric current to the strength of external gauge field.

In our lattice formalism the derivative expansion for the Wigner transform of the Green function is still given by Eq. (8), where $\tilde{G}^{(0)}(\mathbf{R},\mathbf{p}) = \mathcal{G}(\mathbf{p}-\mathbf{A}(\mathbf{R}))$. Suppose, that we modified the external gauge field as $\mathbf{A} \to \mathbf{A} + \delta\mathbf{A}$. The response to this extra contribution to gauge potential gives electric current. Let us calculate this response basing on the description of the system given by Eq. (21):

$$\begin{aligned} \delta \log Z &= -\frac{1}{Z}\int D\bar{\Psi} D\Psi \exp\Big(-\int_{\mathcal{M}} \frac{d^D\mathbf{p}}{|\mathcal{M}|}\bar{\Psi}(\mathbf{p})\hat{\mathcal{Q}}(i\partial_{\mathbf{p}},\mathbf{p})\Psi(\mathbf{p})\Big)\int_{\mathcal{M}} \frac{d^D\mathbf{p}}{|\mathcal{M}|}\bar{\Psi}(\mathbf{p})\Big[\delta\hat{\mathcal{Q}}(i\partial_{\mathbf{p}},\mathbf{p})\Big]\Psi(\mathbf{p}) \\ &= -\int_{\mathcal{M}} \frac{d^D\mathbf{p}}{|\mathcal{M}|}\,\mathrm{Tr}\left[\delta\hat{\mathcal{Q}}(i\partial_{\mathbf{p}_1},\mathbf{p}_1)\right] G(\mathbf{p}_1,\mathbf{p}_2)\Big|_{\mathbf{p}_1=\mathbf{p}_2=\mathbf{p}} \\ &= -\sum_{\mathbf{R}=\mathbf{R}_n}\int_{\mathcal{M}} \frac{d^D\mathbf{p}}{|\mathcal{M}|}\,\mathrm{Tr}\left[\delta\hat{\mathcal{Q}}(i\partial_{\mathbf{P}}+i\partial_{\mathbf{p}}/2,\mathbf{p}+\mathbf{P}/2)\right] e^{-i\mathbf{P}\mathbf{R}}\tilde{G}(\mathbf{R},\mathbf{p})\Big|_{\mathbf{P}=0} \end{aligned} \tag{31}$$

In Appendix B we introduce function $\mathcal{Q}(\mathbf{r},\mathbf{p})$ of real - valued arguments entering Eq. (28). Notice, that $2\mathbf{p}$ and $\mathbf{P}$ enter the expression inside $\hat{\mathcal{Q}}$ in a symmetric way. This allows to use Eq. (58). The form of Eq. (31) demonstrates, that the above expression for the electric current may also be written through the function $\mathcal{Q}$:

$$\begin{aligned} \delta \log Z &= -\sum_{\mathbf{R}=\mathbf{R}_n}\int_{\mathcal{M}} \frac{d^D\mathbf{p}}{|\mathcal{M}|}\,\mathrm{Tr}\left[\delta\mathcal{Q}(i\overrightarrow{\partial}_{\mathbf{P}} - i\overleftarrow{\partial}_{\mathbf{p}}/2,\mathbf{p}+\mathbf{P}/2)\right] \\ &\quad e^{-i\mathbf{P}\mathbf{R}}\tilde{G}(\mathbf{R},\mathbf{p})\Big|_{\mathbf{P}=0} \\ &= -\sum_{\mathbf{R}=\mathbf{R}_n}\int_{\mathcal{M}} \frac{d^D\mathbf{p}}{|\mathcal{M}|}\,\mathrm{Tr}\left[\delta\mathcal{Q}(\mathbf{R},\mathbf{p}+\mathbf{P}/2)\right] \\ &\quad e^{-i\mathbf{P}\mathbf{R}}\tilde{G}(\mathbf{R},\mathbf{p})\Big|_{\mathbf{P}=0} \end{aligned} \tag{32}$$

According to the notations of Appendix B the arrows above the derivatives mean, that those derivatives act only outside of $\mathcal{Q}$, and do not act on the arguments of $\mathcal{Q}$, i.e. $\overleftarrow{\partial}_{\mathbf{p}}$ acts on the function equal to 1 standing left to the function $\mathcal{Q}$ while $\overrightarrow{\partial}_{\mathbf{P}}$ acts on the exponent $e^{-i\mathbf{P}\mathbf{R}}$.

As a result of the above mentioned manipulations we come to the following simple expression for the electric current per unit volume of coordinate space, which follows from the relation $\delta \log Z = \sum_{\mathbf{R}=\mathbf{R}_n} \mathbf{j}^k(\mathbf{R})\delta A_k(\mathbf{R})|\mathcal{V}|$:

$$j^k(\mathbf{R}) = \int_{\mathcal{M}} \frac{d^D\mathbf{p}}{|\mathcal{V}||\mathcal{M}|} \operatorname{Tr} \tilde{G}(\mathbf{R},\mathbf{p}) \frac{\partial}{\partial p_k} \left[\tilde{G}^{(0)}(\mathbf{R},\mathbf{p})\right]^{-1} \tag{33}$$

Here by $|\mathcal{V}|$ we denote the volume of the unit cell understood as the ratio of the total volume of the system to the number of lattice points at which the field Ψ is defined. For the ordinary hypercubic lattice the product of the two volumes is obviously equal to $(2\pi)^D$. One might think, that for the lattices of more complicated symmetry the product of the momentum space volume and the defined above volume of the lattice cell may differ from this expression. Nevertheless, this is not so, and in general case the given product is always equal to $(2\pi)^D$ exactly [2]. In general case of an arbitrary crystal the direct proof is rather complicated. However, the result for the product of the two volumes may be found from the simple field theoretical correspondence: the limit of the microscopic model described by the effective low energy theory should correspond to the product of the two volumes equal to $(2\pi)^D$. Notice, that the construction of the unit cell in the original lattice should be performed with care. One has to count only those sites of the original crystal lattice, at which the dynamical variables of the model described by Eq. (20) are incident. (This was illustrated above by the case of graphene, where we surrounded by this unit cell only the A (or B) - atoms.) Thus Eq. (33) coincides with the continuum expression Eq. (10).

Let us apply the gradient expansion to Eq. (33). It results in the following expression for the electric cureent:

$$j^k(\mathbf{R}) = j^{(0)k}(\mathbf{R}) + j^{(1)k}(\mathbf{R}) + ...$$

$$j^{(0)k}(\mathbf{R}) = \int \frac{d^D\mathbf{p}}{(2\pi)^D} \operatorname{Tr} \tilde{G}^{(0)}(\mathbf{R},\mathbf{p}) \frac{\partial \left[\tilde{G}^{(0)}(\mathbf{R},\mathbf{p})\right]^{-1}}{\partial p_k} \tag{34}$$

Notice, that the second row of this expression represents the topological invariant as long as we deal with the system with regular Green functions, which do not have poles or zeros, i.e. this expression is unchanged while we are continuously deforming the Green function. We will not need this expression below since it does not contain the linear response to the external field strength.

In the $3+1$ D systems the contribution to this current originated from $\tilde{G}^{(1)}$ is given by

$$j^{(1)k}(\mathbf{R}) = \frac{1}{4\pi^2}\epsilon^{ijkl}\mathcal{M}_l A_{ij}(\mathbf{R}), \tag{35}$$

$$\mathcal{M}_l = \int \operatorname{Tr} \nu_l \, d^4p \tag{36}$$

$$\nu_l = -\frac{i}{3!\,8\pi^2}\,\epsilon_{ijkl}\left[\mathcal{G}\frac{\partial \mathcal{G}^{-1}}{\partial p_i}\frac{\partial \mathcal{G}}{\partial p_j}\frac{\partial \mathcal{G}^{-1}}{\partial p_k}\right] \tag{37}$$

In the linear response theory we should substitute $\mathbf{A} = 0$ into the expression for $\mathcal{M}_l$. Therefore, in Eq. (37) we substitute $\mathcal{G}$ instead of $\tilde{G}^{(0)}$. Further we will be interested in the component $\mathcal{M}_4$, which is the topological invariant, i.e. it is robust to any variations of the Green function $\tilde{G}$ as long as the singularities are not encountered (for the proof see Appendix C).

In the similar way for the $2+1$ D systems we get

$$j^{(1)k}(\mathbf{R}) = \frac{1}{2\pi}\epsilon^{ijk}\tilde{\mathcal{N}}_3 A_{ij}(\mathbf{R}), \tag{38}$$

where the topological invariant (denoted by $\tilde{\mathcal{N}}_3$ according to the classification of [26]) is to be calculated for the original system with vanishing background gauge field:

$$\tilde{\mathcal{N}}_3 = \frac{1}{24\pi^2}\operatorname{Tr}\int \mathcal{G}^{-1}d\mathcal{G}\wedge d\mathcal{G}^{-1}\wedge d\mathcal{G} \tag{39}$$

Notice, that the above expression for $\mathcal{M}_l$ is the direct 4D generalization of the invariant $\tilde{\mathcal{N}}_3$. The proof that $\tilde{\mathcal{N}}_3$ is the topological invariant also follows from Appendix C.

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[2] For the purpose of illustration let us consider the 2D lattice of graphene [38]. In coordinate space we should take the hexagons that are formed by the atoms of sublattice A (or B) because the resulting two component spinor is composed of the variables incident at the two sublattices. The resulting unit cell of the lattice is the hexagon surrounding each atom of the sublattice A. The length of its size is a, where a is the distance between the adjacent A and B atoms. The area of the hexagon is equal to $|\mathcal{V}| = \frac{3\sqrt{3}}{2}a^2$. The Brillouin zone has also the form of the hexagon with the side length $\frac{2}{3\sqrt{3}}\frac{2\pi}{a}$. Its volume is $|\mathcal{M}| = \frac{2}{3\sqrt{3}}\frac{(2\pi)^2}{a^2}$. One can see, that the product is given by $(2\pi)^2$ as it should.

IV. $2+1$ D QUANTUM HALL EFFECT

In this section we demonstrate how the technique developed in the previous sections allows to reproduce the well - known expression for the Hall current in the gapped systems. Let us consider the $2+1$ D model with the gapped fermions. In the presence of external electric field $\mathbf{E} = (E_1, E_2)$ we substitute $A_{4k} = -iE_k$ into Eq. (38). This results in the following expression for the Hall current

$$j^k_{Hall} = \frac{1}{2\pi}\tilde{\mathcal{N}}_3\,\epsilon^{ki}E_i, \tag{40}$$

Thus the well - known expression for the $2+1$ D QHE is reproduced (see Eqs. (11.1) and (21.12) of [26]). For one of the previous derivations of this result see, for example, [33]. The following remark is in order. In the real systems of finite sizes the total current is still given by this expression integrated over the direction of electric field, but the local current is concentrated close to the boundary. Our analysis based on the consideration of the systems of infinite volume does not allow to distinguish this inhomogeneity of current in coordinate space.

V. BULK CHIRAL MAGNETIC CURRENT.

Now let us concentrate on the $3+1$ D systems. We consider the situation, when vector gauge field $A_k(\mathbf{R})$ has the nonzero components with $k = 1, 2, 3$ that do not depend on (imaginary) time. The conventional expression for the CME reads

$$j^k_{CME} = \frac{\mu_5}{4\pi^2}\,\xi_{CME}\,\epsilon^{ijk4}\,A_{ij} \tag{41}$$

where ξ_{CME} is integer number while μ_5 is the chiral chemical potential. Such an expression should follow from Eq. (35): we need to substitute $\mathbf{A} = 0$ into Eq. (35) in the linear response approximation. Then one might expect that $\mathcal{M}_4 = \mu_5\xi_{CME}$. However, below we will demonstrate, that ξ_{CME} calculated in this way vanishes identically (for the reasonable choice of the way the chiral chemical potential is introduced) if we deal with compact momentum space and regular Green functions.

There may exist many different definitions of μ_5. Possibly, the most straightforward way is to consider the following expression for the fermion Green function:

$$\mathcal{G}(\mathbf{p}) = \Big(\sum_k \gamma^k g_k(\mathbf{p}) + i\gamma^4\gamma^5\mu_5 - im(\mathbf{p})\Big)^{-1} \tag{42}$$

In the limit of vanishing chiral chemical potential it is reduced to

$$\mathcal{G}(\mathbf{p})\Big|_{\mu_5=0} = \Big(\sum_k \gamma^k g_k(\mathbf{p}) - im(\mathbf{p})\Big)^{-1} \tag{43}$$

where γ^k are Euclidean Dirac matrices while $g_k(\mathbf{p})$ and $m(\mathbf{p})$ are the real - valued functions, $k = 1, 2, 3, 4$. Here we define γ^5 in chiral representation as $\mathrm{diag}(1, 1, -1, -1)$. It can be easily seen, that the consideration of the previous sections may be applied to the Green function, which has this form for nonzero value of μ_5. Therefore, we may substitute $\mathcal{G}$ of Eq. (42) into Eq. (35) instead of $\tilde{G}(\mathbf{R}, \mathbf{p})$ while dealing with the linear response to the external magnetic field.

We are considering the theory with compact momentum space that can be represented as $S^1 \otimes \Omega$, where Ω is the compact $3D$ Brillouin zone. We assume, that the Green functions do not have zeros or poles, which means, that the fermions are massive. However, at the end of the calculations the limit of vanishing mass may always be considered. We need, though, that the inclusion of the chiral chemical potential does not produce poles in the Green function. For the Green function of the form of Eq. (42) this may be proved as follows. The poles of the Green function appear as the zeros of $\det\mathcal{G}^{-1}(\mathbf{p})\Big[\mathcal{G}^{-1}(\mathbf{p})\Big]^+$. The latter zeros are found as the solutions of the following equation

$$g_4^2(\mathbf{p}) + \Big(\mu_5 \pm \sqrt{g_1^2(\mathbf{p}) + g_2^2(\mathbf{p}) + g_3^2(\mathbf{p})}\Big)^2 + m^2(\mathbf{p}) = 0 \tag{44}$$

One can easily see, that if functions $m(\mathbf{p})$ and $g_4(\mathbf{p})$ never vanish simultaneously (which is the common case if the original Green function Eq. (43) does not have poles), then the inclusion of the chiral chemical potential in the form

of Eq. (42) cannot lead to the appearance of the poles of $\mathcal{G}$. (This is in contrast to the case of the ordinary chemical potential, in which the poles of the Green function may appear if chemical potential exceeds the gap.) In the following we will need this property to be obeyed by the inclusion of the chiral chemical potential, and we will not need the particular form of Eq. (42). To illustrate this general pattern let us recall, however, the form of the Green function for the free lattice Wilson fermions:

$$g_k(\mathbf{p}) = \sin p_k, \quad m(\mathbf{p}) = m^{(0)} + \sum_{a=1,2,3,4} (1 - \cos p_a) \tag{45}$$

One can see, that for nonzero bare mass parameter $m^{(0)}$ the values of $g_4(\mathbf{p})$ and $m(\mathbf{p})$ do not vanish at the same value of $\mathbf{p}$. The counter - examples are considered here as marginal.

As it was mentioned above, $\mathcal{M}_4$ is topological invariant, i.e. it is robust to any variations of the Green function as long as the singularities are not encountered (for the proof see Appendix B). The introduction of chiral chemical potential is the particular case of such a variation. Therefore, $\mathcal{M}_4$ does not depend on μ_5. Actually, for the Green function of the form of Eq. (43) $\mathcal{M}_4 = 0$. However, we do not need here the particular form of the Green function. We only need that momentum space is compact and can be represented as $S^1 \otimes \Omega$, where Ω is the compact $3D$ Brillouin zone.

The absence of the dependence on chiral chemical potential means, that there is no CME as long as we deal with compact momentum space and regular Green function. The finite value of chiral chemical potential does not change the situation because as we mentioned above such a finite value cannot provide the Green function of the form of Eq. (43) with the pole (at least, when the form of $\mathcal{G}$ is not marginal).

We did not considered here yet the case of gapless fermions. In that case the pole of the Green function appears and expression for $\mathcal{M}_4$ becomes ambiguous for $\mu_5 = 0$. However, even in that case for $\mu_5 \neq 0$ the poles disappear, the expression for $\mathcal{M}_4$ becomes well - defined and independent of μ_5. This is demonstrated in clear way by the consideration of the case of Wilson fermions Eq. (45) with zero bare mass parameter $m^{(0)} = 0$: in this example Eq. (44) does not have a solution for $\mu_5 \neq 0$. Therefore, even for the gapless fermions our analysis gives the expression for the linear response of the electric current to the magnetic field that does not depend on chiral chemical potential as long as the latter is nonzero. This means, that the equilibrium CME is absent even for the fermions with zero mass.

VI. CONCLUSIONS AND DISCUSSION.

In the present paper we propose the methodology, which allows to reduce the consideration of the linear response of electric current (to external gauge field) to the discussion of momentum space topology. We propose the original method to introduce the slow varying external gauge field to the lattice models. Although the proposed method looks unusual, it is manifestly gauge invariant, and it is obviously reduced in continuum limit to the conventional minimal connection of the fermionic theory with the gauge field. Therefore, it allows to introduce effectively the gauge field both into the lattice regularization of quantum field theory and to the models of the solid state physics. Since the proposed formalism in momentum space utilizes the pseudo - differential operators, i.e. the argument of the gauge field $\mathbf{A}(i\partial_{\mathbf{p}})$ is substituted by the differential operator, this formalism is not useful for the numerical simulations. However, it appears as a powerful tool for the analytical derivations. The power of this methodology was demonstrated by the consideration of the $2+1$ D quantum Hall effect, where the conventional expression of the Hall conductivity through the topological invariant in momentum space is reproduced.

Further, we apply the same technique to the analysis of the equilibrium chiral magnetic effect. We demonstrated that the corresponding current is also proportional to the momentum space topological invariant. It does not depend on the value of chiral chemical potential for the systems with compact momentum space (that can be represented as $S^1 \otimes \Omega$, where Ω is the compact $3D$ Brillouin zone) and without poles or zeros of the Green function. This is the case of the lattice regularized quantum field theory and the case of the discovered recently Dirac semimetals. The given topological invariant does not depend on the value of the chiral chemical potential. Therefore, we conclude, that the properly regularized quantum field theory does not possess the equilibrium bulk chiral magnetic effect. The limit of the zero fermion mass may be taken at the end of the calculations, or it may be taken from the very beginning, but then the nonzero chiral chemical potential should be introduced to prevent the appearance of the poles of the Green function.

We calculated the response to the external magnetic field in the system of the non - interacting fermions. However, according to the general properties of the topological invariants, they cannot be changed by the continuous deformation of the system. Therefore, the mentioned above conclusion on the topological contribution to the CME remains the same if we turn on the self - interactions. The expression for the topological contribution to the considered current remains unchanged until the phase transition is encountered. Our considerations do not exclude that the self - interactions cause the non - topological contribution to electric current proportional to chiral chemical potential and

external magnetic field. However, since such a contribution is not related to topology, it must be dissipative, which excludes its appearance because magnetic field cannot cause heat.

Our conclusion on the absence of bulk equilibrium CME current is in accordance with the recent lattice calculations of the CME by Buividovich and co - authors [13–15], it is also in accordance with the consideration of the particular model of Weyl semimetals [34] and with the no - go Bloch theorem [35]. (The formulations used in [35], however, seem to the author rather distant from the setup of the present paper.) Besides, our conclusion is in line with the discussion of the CME using the continuous model with special boundary conditions in the direction of magnetic field [10]. However, the methodology presented in the present paper is rather general, and it allows to draw the conclusion on the absence of the equilibrium CME for the wide class of systems, which is not limited to the particular models considered in the mentioned above papers.

Notice, that our conclusion refers to the equilibrium states only. The contribution to the conductivity in the presence of both electric and magnetic fields [5, 7]) due to the chiral chemical potential induced by chiral anomaly may avoid the restrictions imposed on the CME by momentum space topology. This may be related to essentially non - equilibrium nature of this phenomenon. The notion of the chiral chemical potential generated by the interplay of chiral anomaly and the quasiparticle interactions with the change of chirality may differ from the notion of the chiral chemical potential of equilibrium theory. Actually, the given contribution to conductivity is to be described by the higher orders of the derivative expansion for the Wigner transform of the Green function just because it is expected to be proportional to the magnetic field squared. This could restore the CME. Besides, the experience of the quantum Hall effect prompts, that inclusion into consideration of boundaries may be important. These issues are, however, out of the scope of the present paper.

ACKNOWLEDGEMENTS

The author is greateful to G.E. Volovik, who pointed out that the equilibrium CME in solids may contradict to Bloch theorem, and thus changed the point of view of the author on the subject of the present paper. The author also kindly acknowledges useful discussions with M.N.Chernodub.

APPENDIX A. WIGNER TRANSFORM OF THE GREEN FUNCTION

Let us consider the $d+1=D$ dimensional model with the Green function $\mathcal{G}(\mathbf{r}_1,\mathbf{r}_2)$ that obeys equation

$$\hat{\mathcal{Q}}(\mathbf{r}_1,-i\partial_{\mathbf{r}_1})G(\mathbf{r}_1,\mathbf{r}_2)=\delta^{(D)}(\mathbf{r}_1-\mathbf{r}_2) \tag{46}$$

for some Hermitian operator - valued function $\mathcal{Q}$. Let us apply Wigner decomposition

$$\tilde{G}(\mathbf{R},\mathbf{p})=\int d^D\mathbf{r}e^{-i\mathbf{pr}}G(\mathbf{R}+\mathbf{r}/2,\mathbf{R}-\mathbf{r}/2) \tag{47}$$

Below we will prove the following identity

$$\mathcal{Q}(\mathbf{R},\mathbf{p})e^{\frac{i}{2}(\overleftarrow{\partial}_{\mathbf{R}}\overrightarrow{\partial}_{\mathbf{p}}-\overleftarrow{\partial}_{\mathbf{p}}\overrightarrow{\partial}_{\mathbf{R}})}\tilde{G}(\mathbf{R},\mathbf{p})=1 \tag{48}$$

Here the function $\mathcal{Q}$ depends on the real numbers rather than on the operators. $\mathcal{Q}$ is called the Weyl symbol of the operator $\hat{\mathcal{Q}}$ [36]. We determine relation between the function $\mathcal{Q}(\mathbf{r},\mathbf{p})$ (of real - valued vectors $\mathbf{r}$ and $\mathbf{p}$) and the function $\hat{\mathcal{Q}}(\mathbf{r},\hat{\mathbf{p}})$ (of the operators $\mathbf{p}$ and $\hat{\mathbf{p}}=-i\partial_{\mathbf{r}}$) through the identity

$$\begin{aligned}&\int d^D\mathbf{r}\, f(\mathbf{r},\mathbf{R})\,\mathcal{Q}(\mathbf{R}+\frac{\mathbf{r}}{2},i\overleftarrow{\partial}_{\mathbf{r}}-\frac{i}{2}\overrightarrow{\partial}_{\mathbf{R}})\,h(\mathbf{r},\mathbf{R})\\&=\int d^D\mathbf{r}\, f(\mathbf{r},\mathbf{R})\hat{\mathcal{Q}}\Big(\mathbf{R}+\frac{\mathbf{r}}{2},-i\frac{\partial}{\partial(\mathbf{R}+\frac{\mathbf{r}}{2})}\Big)\,h(\mathbf{r},\mathbf{R})\end{aligned} \tag{49}$$

which works for arbitrary functions $f(\mathbf{r},\mathbf{R})$ and $h(\mathbf{r},\mathbf{R})$ that decrease sufficiently fast at infinity. The important point concerning this expression is that the derivatives $\overrightarrow{\partial}_{\mathbf{R}}$ and $\overleftarrow{\partial}_{\mathbf{r}}$ inside the arguments of $\mathcal{Q}$ act only outside of this function, i.e. $\overleftarrow{\partial}_{\mathbf{r}}$ acts on $f(\mathbf{r},\mathbf{R})$ while $\overrightarrow{\partial}_{\mathbf{R}}$ acts on $h(\mathbf{r},\mathbf{R})$. At the same time the derivatives without arrows act as usual operators, i.e. not only right to the function $\hat{\mathcal{Q}}$, but inside it as well. Notice, that $\frac{\partial}{\partial(\mathbf{R}+\frac{\mathbf{r}}{2})}=\partial_{\mathbf{r}}+\frac{1}{2}\partial_{\mathbf{R}}$.

The given correspondence looks rather complicated. However, it takes the simple form in certain particular cases. For example, if $\hat{\mathcal{Q}} = (\hat{\mathbf{p}} - \mathbf{A}(\mathbf{r}))^2 = \hat{\mathbf{p}}^2 + \mathbf{A}^2(\mathbf{r}) + i\Big(\partial^k A_k(\mathbf{r})\Big) - 2\mathbf{A}(\mathbf{r})\hat{\mathbf{p}}$, then $\mathcal{Q} = \mathbf{p}^2 + \mathbf{A}^2(\mathbf{r}) - 2\mathbf{A}(\mathbf{r})\mathbf{p}$. Besides, one can easily check, that if $\hat{\mathcal{Q}}$ has the form

$$\hat{\mathcal{Q}}(\mathbf{r}, \hat{\mathbf{p}}) = \mathcal{F}(\hat{\mathbf{p}} - \mathbf{A}(\mathbf{r})) \tag{50}$$

then we have

$$\mathcal{Q}(\mathbf{r}, \hat{\mathbf{p}}) = \mathcal{F}(\mathbf{p} - \mathbf{A}(\mathbf{r})) + O([\partial_i A_j]^2) \tag{51}$$

Here $O([\partial_i A_j]^2)$ may contain the terms with the second power of the derivatives of $\mathbf{A}$ and the terms higher order in derivatives. In order to prove Eq. (51) it is necessary to consider the function $\mathcal{F}(\hat{\mathbf{p}} - \mathbf{A}(\mathbf{r})) = \sum_n \mathcal{F}_{i_1 \ldots i_n}(-i\partial_{i_1} - A_{i_1}(\mathbf{r}))\ldots(-i\partial_{i_n} - A_{i_n}(\mathbf{r}))$ as a series in powers of its arguments ($\mathcal{F}_{i_1 \ldots i_n}$ are the Hermitian operator - valued coefficients), and apply the correspondence of Eq. (49) to each term. The details of the consideration are similar to that of Appendix B, where the Wigner transform in momentum space is discussed. Therefore, we do not represent them here and advise the reader to follow Appendix B.

In order to prove Eq. (48) let us substitute Eq. (47) into it. Argument of the exponent in Eq. (48) acts on $\mathcal{Q}$ as follows:

$$\begin{aligned} 1 = \int d^D\mathbf{r}\mathcal{Q}(\mathbf{R} + \frac{i}{2}\overrightarrow{\partial}_{\mathbf{p}}, \mathbf{p} - \frac{i}{2}\overrightarrow{\partial}_{\mathbf{R}}) \\ e^{-i\mathbf{pr}}G(\mathbf{R} + \mathbf{r}/2, \mathbf{R} - \mathbf{r}/2) \end{aligned} \tag{52}$$

The important point concerning this expression is that the derivatives $\overrightarrow{\partial}_{\mathbf{R}}$ and $\overrightarrow{\partial}_{\mathbf{p}}$ inside the arguments of $\mathcal{Q}$ act only outside of this function, i.e. on $e^{-i\mathbf{pr}}G(\mathbf{R} + \mathbf{r}/2, \mathbf{R} - \mathbf{r}/2)$ and do not act inside the function $\mathcal{Q}$, i.e. on $\mathbf{p}$ and $\mathbf{R}$ in its arguments. This gives

$$\begin{aligned} 1 = \int d^D\mathbf{r}e^{-i\mathbf{pr}}\mathcal{Q}(\mathbf{R} + \frac{\mathbf{r}}{2}, i\overleftarrow{\partial}_{\mathbf{r}} - \frac{i}{2}\overrightarrow{\partial}_{\mathbf{R}}) \\ G(\mathbf{R} + \mathbf{r}/2, \mathbf{R} - \mathbf{r}/2) \end{aligned} \tag{53}$$

Up to the boundary terms (which are assumed to be absent) we arrive at

$$\int d^D\mathbf{r}e^{-i\mathbf{pr}}\hat{\mathcal{Q}}(\mathbf{R} + \frac{\mathbf{r}}{2}, -i\partial_{\mathbf{r}} - \frac{i}{2}\partial_{\mathbf{R}})G(\mathbf{R} + \mathbf{r}/2, \mathbf{R} - \mathbf{r}/2) = 1$$

Now it is clear why we should order operators $\hat{\mathbf{p}}$ and $\mathbf{r}$ in $\hat{\mathcal{Q}}$ according to Eq. (49) in order to obtain function $\mathcal{Q}$. Applying the inverse Wigner transform we finally arrive at Eq. (46).

Notice, that the Weyl symbol $\mathcal{Q}$ of the operator $\hat{\mathcal{Q}}$ may also be defined as [36, 37] the Wigner transform of the matrix elements of $\hat{\mathcal{Q}}$:

$$\mathcal{Q}(\mathbf{R}, \mathbf{p}) = \int d^D\mathbf{x}d^D\mathbf{r}e^{-i\mathbf{pr}}\delta(\mathbf{R} - \mathbf{r}/2 - \mathbf{x})\hat{\mathcal{Q}}(\mathbf{x}, -i\partial_{\mathbf{x}})\delta(\mathbf{R} + \mathbf{r}/2 - \mathbf{x}) \tag{54}$$

APPENDIX B. WIGNER TRANSFORM OF THE GREEN FUNCTION (LATTICE VERSION)

We consider the $d + 1 = D$ dimensional model with the Green function $\mathcal{G}(\mathbf{p}_1, \mathbf{p}_2)$ that obeys equation

$$\hat{\mathcal{Q}}(i\partial_{\mathbf{p}_1}, \mathbf{p}_1)G(\mathbf{p}_1, \mathbf{p}_2) = |\mathcal{M}|\delta^{(D)}(\mathbf{p}_1 - \mathbf{p}_2) \tag{55}$$

for some Hermitian operator - valued function $\hat{\mathcal{Q}}$. Let us apply Wigner decomposition in momentum space

$$\tilde{G}(\mathbf{R}, \mathbf{p}) = \int \frac{d^D\mathbf{P}}{|\mathcal{M}|}e^{i\mathbf{PR}}G(\mathbf{p} + \mathbf{P}/2, \mathbf{p} - \mathbf{P}/2) \tag{56}$$

We will prove identity

$$\mathcal{Q}(\mathbf{R}, \mathbf{p})e^{\frac{i}{2}(\overleftarrow{\partial}_{\mathbf{R}}\overrightarrow{\partial}_{\mathbf{p}} - \overleftarrow{\partial}_{\mathbf{p}}\overrightarrow{\partial}_{\mathbf{R}})}\tilde{G}(\mathbf{R}, \mathbf{p}) = 1 \tag{57}$$

Weyl symbol $\mathcal{Q}$ of the operator $\hat{\mathcal{Q}}$ is the function of real numbers rather than the operators. Similar to the continuum case we determine relation between the function $\mathcal{Q}(\mathbf{r},\mathbf{p})$ (of real - valued vectors $\mathbf{r}$ and $\mathbf{p}$) and the function $\hat{\mathcal{Q}}(\hat{\mathbf{r}},\mathbf{p})$ (of the operators $\mathbf{p}$ and $\hat{\mathbf{r}} = i\partial_{\mathbf{p}}$) through the identity

$$\int d^D\mathbf{X}d^D\mathbf{Y}\, f(\mathbf{X},\mathbf{Y})\, \mathcal{Q}(-i\overleftarrow{\partial}_{\mathbf{Y}} + i\overrightarrow{\partial}_{\mathbf{X}}, \mathbf{X}/2+\mathbf{Y}/2)\, h(\mathbf{X},\mathbf{Y})$$
$$= \int d^D\mathbf{X}d^D\mathbf{Y}\, f(\mathbf{X},\mathbf{Y})\hat{\mathcal{Q}}\Big(i\partial_{\mathbf{X}} + i\partial_{\mathbf{Y}}, \mathbf{X}/2+\mathbf{Y}/2\Big)\, h(\mathbf{X},\mathbf{Y}) \tag{58}$$

which works for arbitrary functions $f(\mathbf{X},\mathbf{Y})$ and $h(\mathbf{X},\mathbf{Y})$ defined on compact momentum space $\mathbf{X},\mathbf{Y}\in\mathcal{M}$. The important point concerning this expression is that the derivatives $\overrightarrow{\partial}_{\mathbf{X}}$ and $\overleftarrow{\partial}_{\mathbf{Y}}$ inside the arguments of $\mathcal{Q}$ act only outside of this function, i.e. $\overleftarrow{\partial}_{\mathbf{Y}}$ acts on $f(\mathbf{X},\mathbf{Y})$ while $\overrightarrow{\partial}_{\mathbf{X}}$ acts on $h(\mathbf{X},\mathbf{Y})$. At the same time the derivatives without arrows act as usual operators, i.e. not only right to the function $\hat{\mathcal{Q}}$, but inside it as well. Notice, that $\frac{\partial}{\partial(\mathbf{X}/2+\mathbf{Y}/2)} = \partial_{\mathbf{Y}} + \partial_{\mathbf{X}}$ and $\frac{\partial}{\partial(\mathbf{X}/2-\mathbf{Y}/2)} = \partial_{\mathbf{X}} - \partial_{\mathbf{Y}}$. Therefore, we may rewrite Eq. (58) as

$$\int d^D\mathbf{X}d^D\mathbf{Y}\, f(\mathbf{X},\mathbf{Y})\, \mathcal{Q}(-i\overleftarrow{\partial}_{\mathbf{Y}} + i\overrightarrow{\partial}_{\mathbf{X}}, \mathbf{X}/2+\mathbf{Y}/2)\, h(\mathbf{X},\mathbf{Y})$$
$$= -2\int d^D\mathbf{Q}d^D\mathbf{K}\, f(\mathbf{Q}+\mathbf{K},\mathbf{Q}-\mathbf{K})\hat{\mathcal{Q}}\Big(i\partial_{\mathbf{Q}}, \mathbf{Q}\Big)\, h(\mathbf{Q}+\mathbf{K},\mathbf{Q}-\mathbf{K}) \tag{59}$$

The given correspondence takes the simple form in certain particular cases. For example, if $\hat{\mathcal{Q}} = (\mathbf{p}-\mathbf{A}(\hat{\mathbf{r}}))^2 = \mathbf{p}^2 + \mathbf{A}^2(\hat{\mathbf{r}}) + i\Big(\partial^k A_k(\hat{\mathbf{r}})\Big) - 2\mathbf{A}(\hat{\mathbf{r}})\mathbf{p}$ (recall, that $\hat{\mathbf{r}}$ is operator equal to $i\partial_{\mathbf{p}}$), then $\mathcal{Q} = \mathbf{p}^2 + \mathbf{A}^2(\mathbf{r}) - 2\mathbf{A}(\mathbf{r})\mathbf{p}$. Besides, one can easily check, that if $\hat{\mathcal{Q}}$ has the form

$$\hat{\mathcal{Q}}(\hat{\mathbf{r}},\mathbf{p}) = \mathcal{F}(\mathbf{p}-\mathbf{A}(\hat{\mathbf{r}})) \tag{60}$$

then we have

$$\mathcal{Q}(\mathbf{r},\mathbf{p}) = \mathcal{F}(\mathbf{p}-\mathbf{A}(\mathbf{r})) + O([\partial_i A_j]^2) \tag{61}$$

Here $O([\partial_i A_j]^2)$ may contain the terms with the second power of the derivatives of $\mathbf{A}$ and the terms higher order in derivatives. Let us prove Eq. (61). First of all, this is necessary to consider the function $\mathcal{F}(\mathbf{p}-\mathbf{A}(\hat{\mathbf{r}})) = \sum_n \mathcal{F}_{i_1\ldots i_n}(p_{i_1} - A_{i_1}(i\partial_{\mathbf{p}}))\ldots(p_{i_n} - A_{i_n}(i\partial_{\mathbf{p}}))$ as a series in powers of its arguments ($\mathcal{F}_{i_1\ldots i_n}$ are Hermitian operators that do not depend on $\mathbf{p}$). Operator $\hat{Q}$ is Hermitian, therefore, the kernel of the first row in Eq. (59) should also be Hermitian. It may be represented as follows. Suppose, that function $\mathcal{Q}$ is expanded in powers of $\mathbf{Q} = (\mathbf{X}+\mathbf{Y})/2$ and $-i\overleftarrow{\partial}_{\mathbf{Y}} + i\overrightarrow{\partial}_{\mathbf{X}}$ as follows

$$\mathcal{Q}(-i\overleftarrow{\partial}_{\mathbf{Y}} + i\overrightarrow{\partial}_{\mathbf{X}}, \mathbf{X}/2+\mathbf{Y}/2) = \sum q_{i_1\ldots i_n;j_1\ldots j_m;k_1\ldots k_l}(-i\overleftarrow{\partial}_{Y_{i_1}})\ldots(-i\overleftarrow{\partial}_{Y_{i_n}})Q_{j_1}\ldots Q_{j_m}(i\overrightarrow{\partial}_{X_{k_1}})(i\overrightarrow{\partial}_{X_{k_l}}) \tag{62}$$

In this expression inside the first row of Eq. (58) we may be substitute $-i\overleftarrow{\partial}_{Y_i}$ by $i\partial_{Y_i}$ and $i\overrightarrow{\partial}_{X_k}$ by $i\partial_{X_k}$. Because the second row in Eq. (58) is symmetric under the interchange of $\mathbf{X}$ and $\mathbf{Y}$, we have $q_{i_1\ldots i_n;j_1\ldots j_m;k_1\ldots k_l} = q_{k_1\ldots k_l;j_1\ldots j_m;i_1\ldots i_n}$. For the same reason Eq. (62) is invariant under the interchange $\mathbf{X}\leftrightarrow\mathbf{Y}$. Then the change of $q_{\ldots}$ by its Hermitian conjugate $q^+_{\ldots}$ is equivalent to the Hermitian conjugation of the whole expression. This demonstrates that coefficients $q_{\ldots}$ are Hermitian. Now let us suppose, that $\mathcal{Q}$ is linear in the derivative of $\mathbf{A}$. Algebraically the linear term appears as a product of a certain combination of $\mathcal{F}_{\ldots}$ and the commutator $[p_k, \mathbf{A}(i\partial_{\mathbf{p}})] = -i(\partial_k\mathbf{A})$. Therefore, it would lead to the appearance of imaginary unity in the expression for $q_{\ldots}$ as a combination of $\mathcal{F}_{\ldots}$, which means that $q_{\ldots}$ is not Hermitian. The contradiction proves the non - appearance of the terms linear in the derivatives of $\mathbf{A}$ in the expression for $\mathcal{Q}(\mathbf{r},\mathbf{p})$.

In order to prove Eq. (57) let us substitute Eq. (56) into it. Argument of the exponent in Eq. (57) acts on $\mathcal{Q}$ as follows:

$$1 = \int \frac{d^D\mathbf{P}}{|\mathcal{M}|}\mathcal{Q}(\mathbf{R} + \frac{i}{2}\overrightarrow{\partial}_{\mathbf{p}}, \mathbf{p} - \frac{i}{2}\overrightarrow{\partial}_{\mathbf{R}})$$
$$e^{i\mathbf{PR}}G(\mathbf{p}+\mathbf{P}/2, \mathbf{p}-\mathbf{P}/2) \tag{63}$$

In this expression the derivatives $\overrightarrow{\partial}_{\mathbf{p}}$ and $\overrightarrow{\partial}_{\mathbf{R}}$ inside the arguments of $\mathcal{Q}$ act only outside of this function, i.e. on $e^{i\mathbf{PR}}G(\mathbf{p}+\mathbf{P}/2,\mathbf{p}-\mathbf{P}/2)$ and do not act inside the function $\mathcal{Q}$, i.e. on $\mathbf{p}$ and $\mathbf{R}$ in its arguments. This gives

$$1=\int\frac{d^D\mathbf{P}}{|\mathcal{M}|}e^{i\mathbf{PR}}\mathcal{Q}(-i\overleftarrow{\partial}_{\mathbf{P}}+\frac{i}{2}\overrightarrow{\partial}_{\mathbf{p}},\mathbf{p}+\frac{\mathbf{P}}{2})\\ G(\mathbf{p}+\mathbf{P}/2,\mathbf{p}-\mathbf{P}/2) \tag{64}$$

Because of the absence of boundary of $\mathcal{M}$ we arrive at

$$\int\frac{d^D\mathbf{P}}{|\mathcal{M}|}e^{i\mathbf{PR}}\hat{\mathcal{Q}}(i\partial_{\mathbf{P}}+\frac{i}{2}\partial_{\mathbf{p}},\mathbf{p}+\frac{\mathbf{P}}{2})G(\mathbf{p}+\mathbf{P}/2,\mathbf{p}-\mathbf{P}/2)=1$$

where we applied Eq. (58). Now it is clear why we should order operators $\mathbf{p}$ and $\mathbf{r}$ in $\hat{\mathcal{Q}}$ according to Eq. (58) in order to obtain function $\mathcal{Q}$. Taking into account Eq. (59) and applying the inverse Wigner transform we finally arrive at Eq. (55).

Finally, let us notice, that the Weyl symbol $\mathcal{Q}$ of the operator $\hat{\mathcal{Q}}$ may also be defined following that of Appendix A (see also [36, 37]) as

$$\mathcal{Q}(\mathbf{R},\mathbf{p})=\int d^D\mathbf{K}d^D\mathbf{P}e^{i\mathbf{PR}}\delta(\mathbf{p}-\mathbf{P}/2-\mathbf{K})\hat{\mathcal{Q}}(i\partial_{\mathbf{K}},\mathbf{K})\delta(\mathbf{p}+\mathbf{P}/2-\mathbf{K}) \tag{65}$$

APPENDIX C. TOPOLOGICAL INVARIANT RESPONSIBLE FOR THE LINEAR RESPONSE OF ELECTRIC CURRENT TO MAGNETIC FIELD

In the main text we encountered Eq. (36) for the coefficient entering the linear response of electric current to external magnetic field. If momentum space $\mathcal{M}$ has the form of the product $S^1\otimes\Omega$, where Ω is the 3D Brillouin zone, while S^1 is the circle of the values of Matsubara frequencies (at zero temperature), then for $l=4$ we may rewrite this quantity as follows:

$$\mathcal{M}_4=-\frac{1}{2}\int dp^4\tilde{\mathcal{N}}_3(p^4),\\ \tilde{\mathcal{N}}_3(p^4)=\frac{i}{24\pi^2}\epsilon_{ijk4}\mathrm{Tr}\int_\Omega d^3p\Big(\mathcal{G}\partial^i\mathcal{G}^{-1}\Big)\\ \Big(\mathcal{G}\partial^j\mathcal{G}^{-1}\Big)\Big(\mathcal{G}\partial^k\mathcal{G}^{-1}\Big) \tag{66}$$

Here for the fixed value of p^4 we encounter the expression for the topological invariant in the 3D Brillouin zone. Green function $\mathcal{G}$ should be considered here as the function of the 3 arguments p^1,p^2,p^3 while p^4 is to be considered as a parameter.

Notice, that for the Green function of the form of Eq. (43) the value of $\tilde{\mathcal{N}}_3(p^4)$ is equal to zero. At the same time for the Green function of general form this invariant may be nonzero. This explains the quantization of Hall conductivity as has been explained in Sect. IV. One might naively think, that the deviation of the Green function from the form of Eq. (43) - say, of the form of Eq. (42) may change the expressions for $\tilde{\mathcal{N}}_3$ and $\mathcal{M}_4$. Below we will demonstrate, that this does not occur as long as we deal with the compact Brillouin zone and regular Green functions.

Let us consider arbitrary variation of the Green function: $\mathcal{G}\to\mathcal{G}+\delta\mathcal{G}$. Then expression for $\tilde{\mathcal{N}}_3$ is changed as follows:

$$\delta\tilde{\mathcal{N}}_3=\frac{3}{24i\pi^2}\int\mathrm{Tr}\Big(([\delta\mathcal{G}]d\mathcal{G}^{-1}+\mathcal{G}d[\delta\mathcal{G}^{-1}])\wedge\mathcal{G}d\mathcal{G}^{-1}\\ \wedge\mathcal{G}d\mathcal{G}^{-1}\Big)\\ =-\frac{3}{24i\pi^2}\int d\,\mathrm{Tr}\left(([\delta\mathcal{G}^{-1}]\mathcal{G})d\mathcal{G}^{-1}\wedge d\mathcal{G}\right)=0 \tag{67}$$

Thus we proved that $\tilde{\mathcal{N}}_3$ is the topological invariant.

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