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► **To cite this version:**

Christophe Bahadoran, T Bodineau. EXISTENCE OF A PLATEAU FOR THE FLUX OF TASEP WITH SITE DISORDER. 2016. hal-01275166v1

HAL Id: hal-01275166

<https://hal.science/hal-01275166v1>

Preprint submitted on 21 Feb 2016 (v1), last revised 8 Dec 2017 (v3)

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EXISTENCE OF A PLATEAU FOR THE FLUX OF TASEP WITH SITE DISORDER

C. BAHADORAN, T. BODINEAU

ABSTRACT. We prove that the flux function of the totally asymmetric simple exclusion process (TASEP) with site-disorder exhibits a plateau for sufficiently dilute disorder. The result is established under a decay assumption of the maximum current in finite boxes, which is implied in particular by a sufficiently slow power tail assumption on the disorder distribution near its minimum. To circumvent the absence of explicit invariant measures, we use a renormalization procedure and some ideas inspired by homogenization.

MSC: 60K35, 60K37, 82C22

Keywords: Totally asymmetric exclusion process; site disorder; flux function; plateau; phase transition; renormalization; homogenization

1. INTRODUCTION

The flux function, also called current-density relation in traffic-flow physics [12], is the most fundamental object to describe the macroscopic behavior of driven lattice gases, as well as, to some extent, their fluctuations around this mean behavior. The paradigmatic model in this class is the totally asymmetric simple exclusion process (TASEP), where particles on the one-dimensional integer lattice hop to the right at unit rate independently except that jumps are constrained to respect the exclusion rule. Mass is the only conserved quantity, and its local density $\rho \in (0, 1]$ has an associated current (or flux), that is defined as the amount of mass crossing a given site per unit time in a system with homogeneous density ρ . For TASEP, the flux function is explicitly given by

$$(1.1) \quad f(\rho) = \rho(1 - \rho).$$

In the hyperbolic scaling limit [31], the empirical particle density field is governed by entropy solutions of the scalar conservation law

$$(1.2) \quad \partial_t \rho(t, x) + \partial_x f(\rho(t, x)) = 0,$$

Date: February 22, 2016.

We would like to thank A. Sznitman for very useful discussions.

with f given by (1.1). This flux function is obtained as a steady state expectation of the microscopic flux:

$$(1.3) \quad f(\rho) := \int j_x(\eta) d\nu_\rho(\eta),$$

where

$$(1.4) \quad j_x(\eta) := \eta(x)[1 - \eta(x + 1)].$$

In (1.4), η denotes a particle configuration and $\eta(x) \in \{0, 1\}$ the number of particles at site x . The microscopic flux (1.4) of TASEP is simply the rate at which a particle hops to the right, and ν_ρ denotes the product Bernoulli measure ν_ρ with parameter $\rho \in [0, 1]$, that is extremal invariant for TASEP. This kind of result can be extended to a variety of asymmetric models [37, 32, 5], but when the invariant measures are not explicit, little can be said about the flux function. However, convexity or concavity can be obtained as an automatic byproduct of the variational approach set up in [36], which applies to totally asymmetric models with state-independent jump rates, like TASEP. But strict convexity or concavity, that is a central issue, requires new ideas. In fact, one does not even know whether an extremal measure exists for every possible density value. Nevertheless, the flux function can still be expressed by (1.3)–(1.4) for some densities, and the existence of a gap in a density interval implies that the flux in this interval is necessarily defined by linear interpolation [32, 5]. The latter case corresponds to a phenomenon of phase segregation at mesoscopic scale: if the local hydrodynamic density lies in a density gap, the system cannot reach a local equilibrium state at the corresponding density. Instead, the system reorganizes itself into a succession of mesoscopic stretches of lower and higher densities associated to the closest steady states. These mesoscopic density variations average out in a small macroscopic box to yield the hydrodynamic density. Conversely, the presence of a partial linear portion on the flux function suggests a phase transition. Although such a phenomenon is not believed to happen for homogenous systems, one has neither a proof nor a counterexample.

In this paper, we consider TASEP with i.i.d. site disorder, where the jump rate of each site has a random value whose distribution is supported in an interval $[r, 1]$, with $r \in (0, 1)$. Contrary to homogeneous systems, the existence of a phase transition in disordered driven lattice gases is a partially established fact. It has been proved mathematically for nearest-neighbor asymmetric site-disordered zero-range processes (ZRP), which exhibits constant flux on a density interval $[\rho_c, +\infty)$, where ρ_c is the density of the maximal invariant measure. The necessary and sufficient condition for the occurrence of a phase transition is a slow enough tail of the disorder distribution near its minimum value r . Microscopically, phase transition takes the form of Bose-Einstein condensation [15]. In a finite system of size L with periodic boundary conditions, a stationary state with mean density $\rho > \rho_c$ is achieved by completing the background state of density ρ_c with a macroscopic condensate of mass $L(\rho - \rho_c)$ at the slowest site.

In an infinite system with mean drift to the right, the excess mass is captured by the asymptotically slowest sites at $-\infty$, as proven rigorously in [3] for the totally asymmetric ZRP with constant rate, that is equivalent to TASEP with *particlewise* disorder, and for nearest-neighbor ZRP with more general rates [6] (see also [18] for partial results in higher dimension). The TASEP picture can be interpreted as a traffic-flow model with slow and fast vehicles. The phase transition then occurs on a density interval $[0, \rho'_c]$, where the flux is linear with a slope equal to the constant mean velocity of the system. This velocity is imposed by the slowest vehicles at $+\infty$. As one moves ahead, slowest and slowest vehicles are encountered, followed by a platoon of faster vehicles, and preceded by a gap before the next platoon [27]. The platoon is at equilibrium with critical density ρ'_c , while the gap is the TASEP version of the condensate in the ZRP picture. The hydrodynamic limit for disordered asymmetric ZRP was first established in [9] in any space dimension but only in the subcritical density interval. The full hydrodynamic limit (including the plateau) was proved in [27] for the TAZRP with constant rate, but remains open for more general ZRP.

The fundamental property of disordered ZRP, thanks to which the existence and location of the phase transition are known, is the existence of explicit invariant measures *despite* the presence of disorder, which allow exact computation of the flux function. This does not hold for site-disordered TASEP, and the occurrence of a phase transition is an open and a priori, a challenging problem, for which no mathematical results exist yet. On the other hand, a phase transition is firmly believed to hold in the physics community and this is supported by several heuristic arguments [24, 44, 22]. Independently of this, the existence of a hydrodynamic limit of the form (1.2) for TASEP with i.i.d. site disorder was established in [37], using last-passage percolation (LPP) and variational coupling. Consequently, the flux function was shown to be concave. More generally, the existence of a limit of the type (1.2) was obtained in [5] for asymmetric attractive systems in ergodic environment, based on the study of invariant measures. We refer also to [11, 33, 34, 40] for further rigorous results in a different class of disordered SEP.

The microscopic picture behind the phase transition in site-disordered TASEP is *jamming*. As one moves ahead along the disorder, slower and slower regions are encountered, with larger and larger stretches of minimal (or near minimal) rate sites. Locally a slow stretch of environment inside a typical region is expected to create a picture similar to the *slow bond* TASEP introduced in [23]. It is known that a slow bond with an arbitrarily small blockage [8] restricts the local current. On the hydrodynamic scale [38], this creates a traffic jam with a high density of queuing vehicles to the left and a low density to the right, that is an antishock for Burgers' equation. It also follows from [38] that the single slow bond generates a plateau in the asymptotic flux starting from a uniform density close to $1/2$. Diphasic versions of TASEP [20] and last-passage percolation [1] exhibit similar behaviour. However, these models are

perturbations of the homogeneous case, and their analysis reduces to the understanding of a single interface. In contrast, it is not clear if the quenched randomness will round off this first-order phase transition, as it is often the case in many equilibrium systems [2], or if the transition will remain in presence of disorder. Yet, one of the heuristic arguments put forward in [44] to justify the transition is the comparison with the so-called “fully segregated model”, which is nothing but a periodic version of [20], that is microscopically very different from the disordered one.

Our analysis is based on a renormalization scheme which controls rigorously this multi-scale recurring slow-bond picture. This scheme can also be interpreted in terms of homogenization of scalar conservation laws [7]. A major difficulty compared to the single slow bond is that as one moves to larger scales, the maximum current issued by the typical environment and the maximum current admitted by the slow regions converge to the same value $r/4$ when the jump rate distribution is supported by $[r, 1]$. The current limitation in both instances comes from large stretches of minimum rate r , which restrict the current to the maximum value $r/4$ of a homogeneous TASEP with jump rate r . In this context, one key problem is to make sure that the difference in the mean current between the environment and the slow parts exceeds the typical order of fluctuations at each scale, so that the slow-bond picture remains indeed valid at all scales. Heuristically, this infinite hierarchy of slow-bond blockages [19, 24] is expected to induce a dynamics of coalescing shocks and antishocks on mesoscopic scales depending on the disorder distribution. Similar behaviors can be found in many disordered systems (see for example chapter 6.1 of [41]) and multi-scale analysis has been a key tool for their study; we refer to [42, 45] for a general overview and to [10] for an approach related to ours.

Our main result is the existence of a plateau on the flux function of site-disordered TASEP for sufficiently dilute disorder. The dilution parameter $\varepsilon \in [0, 1]$ is defined by the fact that a site is “fast” with probability $1 - \varepsilon$, in which case it has rate 1, or “slow” with probability ε , in which case its rate has some distribution Q with support $(r, 1]$ for some $r \in (0, 1)$. A sufficient condition for our result is then that the tail of the Q -distributed random rate α near r decays like $o(\alpha - r)$. However, unlike the ZRP, we do not expect this condition to be necessary. In fact, we prove our result under a more general but abstract condition on the decay of the maximum current in a finite box. Heuristics suggest that this condition is always satisfied although we have not been able so far to prove this conjecture. In particular, the worst possible case $Q = \delta_r$, where the rates only take the two possible values 1 and r , is currently not covered by our result, but probably satisfies our decay assumption.

We achieve our renormalization scheme by formulating the problem in the exponential LPP framework. In this setting, we have a columnar disorder in wedge LPP, or diagonal disorder in corner-growth LPP, while the better understood columnar disorder

in the corner-growth model is equivalent to particlewise disorder in TASEP. In fact, our approach, like many shape theorems [30], partially extends to LPP with more general distributions. The core of this approach is to obtain a recursion between mean passage-times at two successive scales, and reduce the problem to bounding our passage-times by a family of “reference” passage-times, that correspond to trapezoidal flux functions. In this picture, the phase transition takes the form of a *pinning* transition for the optimal path [26]. Densities close to $1/2$ (i.e. large slopes of the optimal path in the wedge last-passage picture) are avoided on the mesoscopic scale. Indeed to increase its mean macroscopic slope, the path gets a better reward from vertical portions along slow parts of the disorder. As a result, in the linear phase, current fluctuations are expected to be governed by details of the disorder distribution, instead of the universal KPZ behavior arising from the service time distributions in homogeneous TASEP. A possibly related phenomenon is the disorder-dependent convergence speed in homogenization of conservation laws inside the plateau [4]. In the same spirit, it is also believed [21] that disorder-induced condensation will dominate the dynamics-induced [29] phase transition in non-attractive ZRP, where, in the absence of disorder, a uniformly located condensate appears in the supercritical phase.

The paper is organized as follows. In Section 2, we set up the notation and state our main result. In Section 3, we formulate the problem in the last-passage percolation framework and introduce the reference flux and the passage-time functions. In Section 4, we introduce the renormalization procedure and describe the main steps of the proof. In Section 5, we prove a recurrence relation linking passage-time bounds of two successive scales, that is the heart of the renormalization argument. In Section 6, we study this recurrence in detail and show that it propagates the bounds we need from one scale to another. In Section 7, we establish an important fluctuation estimate needed in Section 5. Finally, the proof is completed in Section 8.

2. NOTATION AND RESULTS

2.1. TASEP with site disorder. The disorder is modeled by $\alpha = (\alpha(x) : x \in \mathbb{Z}) \in \mathbf{A} := [0, 1]^{\mathbb{Z}}$, a stationary ergodic sequence of positive bounded random variables. The precise distribution of α will be defined in Section 2.2. For a given realization of α , we consider the TASEP on \mathbb{Z} with site disorder α . The dynamics is defined as follows. A site x is occupied by at most one particle which may jump with rate $\alpha(x)$ to the site $x + 1$ if it is empty. A particle configuration on \mathbb{Z} is of the form $\eta = (\eta(x) : x \in \mathbb{Z})$, where for $x \in \mathbb{Z}$, $\eta(x) \in \{0, 1\}$ is the number of particles at x . The state space is $\mathbf{X} := \{0, 1\}^{\mathbb{Z}}$. The generator of the process is given by

$$(2.1) \quad L^\alpha f(\eta) = \sum_{x \in \mathbb{Z}} \alpha(x) \eta(x) [1 - \eta(x + 1)] [f(\eta^{x, x+1}) - f(\eta)],$$

where $\eta^{x,x+1} = \eta - \delta_x + \delta_{x+1}$ denotes the new configuration after a particle has jumped from x to $x + 1$.

Current and flux function. The macroscopic flux function f can be defined as follows. We denote by $J_x^\alpha(t, \eta_0)$ the rightward current across site x up to time t , that is the number of jumps from x to $x + 1$ up to time t , in the TASEP $(\eta_t^\alpha)_{t \geq 0}$ starting from initial state η_0 , and evolving in environment α . For $\rho \in [0, 1]$, let η^ρ be an initial particle configuration with uniform density profile ρ in the following sense:

$$(2.2) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{x=0}^n \eta^\rho(x) = \rho = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{x=-n}^0 \eta^\rho(x).$$

We then set

$$(2.3) \quad f(\rho) := \lim_{t \rightarrow \infty} \frac{1}{t} J_x^\alpha(t, \eta^\rho),$$

where the limit is understood in probability with respect to the law of the quenched process. Equivalent definitions of the flux can be stated in terms of the steady state expectation of the microscopic flux function

$$(2.4) \quad j_x^\alpha(\eta) = \alpha(x) \eta(x) [1 - \eta(x + 1)]$$

through one of the following formulas:

$$(2.5) \quad f(\rho) := \lim_{\substack{N \rightarrow \infty, \\ k/N \rightarrow \rho}} \int j_x^\alpha(\eta) d\nu_{N,k}^\alpha(\eta),$$

$$(2.6) \quad f(\rho) := \int j_x^\alpha(\eta) d\nu_\rho^\alpha(\eta), \quad \rho \in \mathcal{R}.$$

In (2.5), $\nu_{N,k}^\alpha$ is the steady state for a system of k particles evolving according to the dynamics (2.1) restricted to a ring of size N instead of \mathbb{Z} . In (2.6) (see [5] for details), ν_ρ^α is the unique extremal invariant measure with mean density ρ for the system on \mathbb{Z} , $\mathcal{R} \supset \{0, 1\}$ is the (unknown) closed set of densities for which this measure exists, and f is interpolated linearly outside \mathcal{R} . We refer to [7] for the proof of equivalence of these three definitions in a more general framework.

It is shown in [37] that f is a concave function, see (3.8) below. Apart from concavity, little can be said about f . This is due to the fact that invariant measures for site-disordered TASEP are not explicit (in contrast with particle-disordered TASEP or equivalently site-disordered zero-range process, which has explicit product invariant measures). One of the key issues is whether f is strictly concave or not. It is conjectured in the physics literature (see e.g. [44]) that for i.i.d. disorder, the flux function f exhibits a plateau, that is an interval $[\rho_c, 1 - \rho_c]$ (with $0 \leq \rho_c < 1/2$) on which f is constant (see figure 1). The goal of this paper is to prove such a result for a class of disorder distributions. Our approach does not require the knowledge of invariant measures and proceeds via renormalization and homogenization ideas.

2.2. Existence of a plateau. From now on, we consider i.i.d disorder such that the support of the distribution of $\alpha(x)$ is contained in $[r, 1]$, where $r > 0$ is the infimum of this support. The following easy result comes from the fact that the current of the system dominates that of a homogeneous rate r TASEP, which has maximum current $r/4$, while existence of arbitrarily large stretches of environment close to r also limit the current to this same value. We refer to Appendix A for a detailed proof.

Proposition 2.1. *The maximum value of the flux function is given by*

$$\max_{\rho \in [0,1]} f(\rho) = r/4.$$

For our main result, we formulate additional assumptions on the distribution of the environment. We assume that the disorder is a perturbation of the homogeneous case with rate 1. Let $0 < r < R < 1$ and Q be a probability measure on $[r, R]$, such that r is the infimum of the support of Q . Given $\varepsilon \in (0, 1)$ a “small” parameter, we define the distribution of $\alpha(x)$ by

$$(2.7) \quad Q_\varepsilon = (1 - \varepsilon)\delta_1 + \varepsilon Q.$$

The law of $\alpha = (\alpha(x), x \in \mathbb{Z})$ is the product measure with marginal Q_ε at each site

$$\mathcal{P}_\varepsilon(d\alpha) := \bigotimes_{x \in \mathbb{Z}} Q_\varepsilon[d\alpha(x)].$$

Expectation with respect to \mathcal{P}_ε is denoted by \mathcal{E}_ε . We can interpret this by saying that each site is chosen independently at random to be, with probability $1 - \varepsilon$, a “fast” site with normal rate 1, or with probability ε to be a “defect” with rate distribution Q bounded above away from 1 and 0. Thus ε is the mean density of defects. For example if $Q = \delta_r$, then the defects are slow bonds with rate $r < 1$.

Let us denote by f_ε the flux function (2.3) for this disorder distribution. Our main result is the occurrence of a plateau in the flux for sufficiently dilute disorder. This result holds under a general assumption **(H)** on the disorder, which will be stated and explained in the next subsection. However we first state it under a concrete lower tail assumption on $\alpha(x)$, which easily implies the general assumption **(H)**.

Theorem 2.1. *Assume Q satisfies the lower tail assumption*

$$(2.8) \quad \text{for some } \kappa > 1, \quad Q((r, r + u)) = O(u^\kappa) \text{ as } u \rightarrow 0^+.$$

Then there exists $\varepsilon_0 > 0$ such that, for every $\varepsilon < \varepsilon_0$, the flux f_ε has a plateau with value $r/4$.

2.3. A general assumption. Let us now formulate the general condition **(H)** announced before Theorem 2.1. For this we need the following definition.

Maximal current in a finite domain. Let $B = [x_1, x_2]$ be an interval in \mathbb{Z} . In the following, $\alpha_B := (\alpha(x) : x \in B)$ denotes the environment restricted to B . Consider the TASEP in B with the following boundary dynamics: a particle enters at site x_1

with rate $\alpha(x_1 - 1)$ if this site is empty, and leaves from site x_2 with rate $\alpha(x_2)$ if this site is occupied. Note that this process, although spatially restricted to B , depends in fact on the restriction of the disorder to the larger box

$$(2.9) \quad B^\# := [x_1 - 1, x_2] \cap \mathbb{Z}$$

In the sequel, our notational choice will be to index all related objects by $B^\#$ (the domain of the relevant disorder variables) rather than B (the domain where particles evolve). The generator of this process is given by

$$(2.10) \quad \begin{aligned} L_{B^\#}^\alpha f(\eta) &:= \sum_{x=x_1}^{x_2-1} \alpha(x)\eta(x)[1 - \eta(x+1)] [f(\eta^{x,x+1}) - f(\eta)] \\ &+ \alpha(x_1 - 1)[1 - \eta(x_1)] [f(\eta + \delta_{x_1}) - f(\eta)] + \alpha(x_2)\eta(x_2) [f(\eta - \delta_{x_2}) - f(\eta)], \end{aligned}$$

where $\eta \pm \delta_x$ denotes creation/annihilation of a particle at x . We define now a notion of maximal current relative to B .

Definition 2.1. *The maximal current $j_{\infty, B^\#}(\alpha_{B^\#})$ is the stationary current in the open system defined above, i.e. (independently of $x = x_1, \dots, x_2 - 1$)*

$$(2.11) \quad \begin{aligned} j_{\infty, B^\#}(\alpha_{B^\#}) &= \int \alpha(x)\eta(x)[1 - \eta(x+1)] d\nu_{B^\#}^\alpha(\eta), \\ &= \int [1 - \eta(x_1)] d\nu_B^\alpha(\eta) = \int \alpha(x_2)\eta(x_2) d\nu_{B^\#}^\alpha(\eta), \end{aligned}$$

where $\nu_{B^\#}^\alpha$ is the unique invariant measure for the process on B with generator $L_{B^\#}^\alpha$.

Remark. One can see that the right-hand side of (2.11) is independent of x by writing that the expectation under $\nu_{B^\#}^\alpha$ of $L_{B^\#}^\alpha \eta(x)$ for $x \in [x_1, x_2] \cap \mathbb{Z}$ (which yields the difference of two consecutive integrals in (2.11)) is zero.

To simplify notation, we shall at times omit dependence on $\alpha_{B^\#}$ and write $j_{\infty, B^\#}$. It is well-known [14] that in the homogeneous case, i.e. when $\alpha(x) = r$ for all x in $[0, N]$ (with r a positive constant), then $j_{\infty, [1, N]}$ is no longer a random variable and

$$(2.12) \quad \lim_{N \rightarrow \infty} j_{\infty, [0, N]} = \inf_N j_{\infty, [0, N]} = \frac{r}{4}.$$

In fact, explicit computations [14] show that, for some constant $C > 0$,

$$(2.13) \quad j_{\infty, [0, N]} \geq \frac{r}{4} + \frac{C}{N}.$$

The constant environment with value r is a degenerate case of disorder for which the maximal current is $r/4$, but the explicitly known flux function $\rho \mapsto r\rho(1 - \rho)$ has no plateau. We will show that a slower rate of decay of $j_{\infty, [1, N]}$ than (2.13) is a sufficient condition for the existence of a plateau. To this end, we introduce an assumption on the disorder distribution, which quantifies the finite-size fluctuation of the maximal current as a function of disorder.

Assumption (H). *There exists $b \in (0, 2)$, $a > 0$, $c > 0$ and $\beta > 0$ such that, for ε small enough, the following holds for any N*

$$(2.14) \quad \mathcal{P}_\varepsilon \left(j_{\infty, [0, N]}(\alpha_{[0, N]}) \leq \frac{r}{4} + \frac{a}{N^{b/2}} \right) \leq \frac{c}{N^\beta}.$$

The main result of this paper is

Theorem 2.2. *Assume Q_ε satisfies assumption (H). Then there exists $\varepsilon_0 > 0$ such that, for every $\varepsilon < \varepsilon_0$, the flux f_ε has a plateau with value $r/4$*

$$\exists \rho_c \in]0, 1/2[, \quad \forall \rho \in [\rho_c, 1 - \rho_c], \quad f_\varepsilon(\rho) = r/4.$$

Remark 2.1. *A coupling argument (see Lemma 2.1 below) shows that the value of $j_{\infty, [0, N]}(\alpha_{[0, N]})$ for any environment $\alpha(\cdot) \geq r$ dominates the value in (2.13). This is true in particular for an environment with Bernoulli distribution taking only values r and 1 , that is $Q = \delta_r$ in (2.7). This lower bound, though very pessimistic (it assumes that all environment variables take the worst possible values, while in a box of size N the length of the longest stretch of defects is of order $\log N$), is only slightly worse than the one in Assumption (H). We believe that a refined analysis of $j_{\infty, [0, N]}$ (which we have not succeeded in carrying out so far) could lead to the conclusion that Assumption (H) is satisfied for Bernoulli disorder. Note that if indeed assumption (H) holds for such disorder, Lemma 2.1 below implies that it holds for any distribution Q in (2.7).*

Theorem 2.2 will be proved in Sections 3 to 8. To deduce Theorem 2.1, we only have to check that the tail assumption (2.8) implies (H). This is done using the following lemma, which relies on a standard coupling argument (see its proof in Appendix C).

Lemma 2.1. *Assume α and α' are two environments such that $\alpha \leq \alpha'$ in the sense of product order. Then, for every $N \in \mathbb{N}$, $j_{\infty, [0, N]}(\alpha_{[0, N]}) \leq j_{\infty, [0, N]}(\alpha'_{[0, N]})$.*

Proof of Theorem 2.1. Let $\alpha^* := \min_{x \in [0, N]} \alpha(x)$. It follows from Lemma 2.1 that

$$(2.15) \quad j_{\infty, [0, N]}(\alpha_{[0, N]}) \geq j_{\infty, [0, N]}(\alpha^*, \dots, \alpha^*),$$

where $j_{\infty, [0, N]}(\alpha^*, \dots, \alpha^*)$ stands for the current of a homogeneous TASEP in $[1, N]$ with bulk, exit and entrance rates α^* . By (2.13), it is larger than $\alpha^*/4$, so that $j_{\infty, [0, N]}(\alpha_{[0, N]}) \geq \alpha^*/4$. Thus Assumption (H) will be implied by a control of α^* . Using the tail of the distribution Q (2.8), we get

$$\mathcal{P}_\varepsilon \left(\min_{i \in [0, N]} \alpha(i) \leq r + \frac{a'}{N^{b/2}} \right) \leq NQ \left(r, r + \frac{a'}{N^{b/2}} \right) \leq \frac{c'}{N^\beta},$$

for some well chosen parameters $a' > 0$, $c' > 0$, $b \in (0, 2)$, $\beta > 0$. This follows from elementary computations. \square

3. LAST-PASSAGE PERCOLATION APPROACH

The proof of Theorem 2.2 relies on a reformulation of the problem in terms of last-passage percolation.

3.1. Wedge last-passage percolation. Let $Y = (Y_{i,j} : (i, j) \in \mathbb{Z} \times \mathbb{N})$ be an i.i.d family of exponential random variables with parameter 1 independent of the environment $(\alpha(i) : i \in \mathbb{Z})$. In the following, these variables will sometimes be called *service times*, in reference to the queuing interpretation of TASEP. The distribution of Y is denoted by \mathbb{P} , and expectation with respect to this distribution by \mathbb{E} . Let

$$\mathcal{W} := \{(i, j) \in \mathbb{Z}^2 : j \geq 0, i + j \geq 0\}.$$

Index i represents a site and index j a particle. Given two points (x, y) and (x', y') in $\mathbb{Z} \times \mathbb{N}$, we denote by $\Gamma((x, y), (x', y'))$ the set of paths $\gamma = (x_k, y_k)_{k=0, \dots, n}$ such that $(x_0, y_0) = (x, y)$, $(x_n, y_n) = (x', y')$, and $(x_{k+1} - x_k, y_{k+1} - y_k) \in \{(1, 0), (-1, 1)\}$ for every $k = 0, \dots, n-1$. Note that $\Gamma((x, y), (x', y')) = \emptyset$ if $(x' - x, y' - y) \notin \mathcal{W}$. Given a path $\gamma \in \Gamma((x, y), (x', y'))$, its passage time is defined by

$$(3.1) \quad T^\alpha(\gamma) := \sum_{k=0}^n \frac{Y_{x_k, y_k}}{\alpha(x_k)}.$$

The last-passage time between (x, y) and (x', y') is defined by

$$(3.2) \quad T^\alpha((x, y), (x', y')) := \max\{T^\alpha(\gamma) : \gamma \in \Gamma((x, y), (x', y'))\}.$$

We shall simply write $T^\alpha(x, y)$ for $T^\alpha((0, 0), (x, y))$. This quantity has the following particle interpretation. For $(t, x) \in [0, +\infty) \times \mathbb{Z}$, let

$$\begin{aligned} H^\alpha(t, x) &= \min\{y \in \mathbb{N} : T^\alpha(x, y) > t\}, \\ \eta_t^\alpha(x) &= H^\alpha(t, x-1) - H^\alpha(t, x). \end{aligned}$$

Then $(\eta_t^\alpha)_{t \geq 0}$ is a TASEP with generator (2.1) and initial configuration $\eta^* = \mathbf{1}_{\mathbb{Z} \cap (-\infty, 0]}$, and H^α is its height process. Besides, if we label particles initially so that the particle at $x \leq 0$ has label $-x$, then for $(x, y) \in \mathcal{W}$, $T^\alpha(x, y)$ is the time at which particle y reaches site $x + 1$. Let us recall the following result from [37]

Theorem 3.1. *Let $\mathcal{W}' := \{(x, y) \in \mathbb{R}^2 : y \geq 0, x + y \geq 0\}$. For \mathcal{P} -a.e. realization α of the disorder, the function*

$$(3.3) \quad (x, y) \in \mathcal{W}' \mapsto \tau(x, y) := \lim_{N \rightarrow \infty} \frac{1}{N} T^\alpha([Nx], [Ny])$$

is well-defined in the sense of a.s. convergence with respect to the distribution of Y . It is finite, positively 1-homogeneous and superadditive (thus concave). The function

$$(3.4) \quad (t, x) \in [0, +\infty) \times \mathbb{R} \mapsto h(t, x) := \lim_{N \rightarrow \infty} \frac{1}{N} H^\alpha([Nt], [Nx])$$

is well-defined in the sense of a.s. convergence with respect to the distribution of Y . It is finite, positively 1-homogeneous and subadditive (thus convex). These functions do not depend on α and are related through

$$(3.5) \quad h(t, x) = \inf\{y \in [0, +\infty) : \tau(x, y) > t\},$$

$$(3.6) \quad \tau(x, y) = \inf\{t \in [0, +\infty) : h(t, x) \geq y\}.$$

By homogeneity, the function h in (3.4) is of the form

$$(3.7) \quad h(t, x) = tk \left(\frac{x}{t} \right)$$

for some concave function $k : \mathbb{R} \rightarrow \mathbb{R}^+$. It is known that for homogeneous TASEP (that is $\alpha(x) = 1$ for all x), we have

$$\tau(x, y) = (\sqrt{x+y} + \sqrt{y})^2, \quad k(v) = \frac{(1-v)^2}{4} \mathbf{1}_{[-1,1]}(v) - v \mathbf{1}_{(-\infty, -1)}(v).$$

3.2. Reformulation of Theorem 2.2. In this section, we are going to rewrite the flux in the last-passage framework and show that Theorem 2.2 can be deduced from a statement on the passage time. It is shown in [37] that the macroscopic flux function f is related to k (defined in (3.7)) by the convex duality relation

$$(3.8) \quad f(\rho) := \inf_{v \in \mathbb{R}} [k(v) + v\rho], \quad \rho \in [0, 1]$$

which implies concavity of f . We now introduce a family of “reference” macroscopic flux functions and associated macroscopic passage-time and height functions. Let $0 \leq \rho_c \leq 1/2$ and $J \geq 0$. For $\rho \in [0, 1]$, we define

$$(3.9) \quad f^{\rho_c, J}(\rho) := J \min \left(\frac{\rho}{\rho_c}, \frac{1-\rho}{\rho_c}, 1 \right).$$

Given Proposition 2.1, Theorem 2.2 boils down to proving a lower bound for the flux, i.e. to show the existence of $\varepsilon_0 > 0$, $\rho_c \in [0, 1/2)$ such that the flux remains above $r/4$ for densities in $[\rho_c, 1 - \rho_c]$

$$(3.10) \quad \forall \varepsilon < \varepsilon_0, \quad f_\varepsilon(\rho) \geq f^{\rho_c, r/4}(\rho) \quad \text{with} \quad \rho \in [0, 1]$$

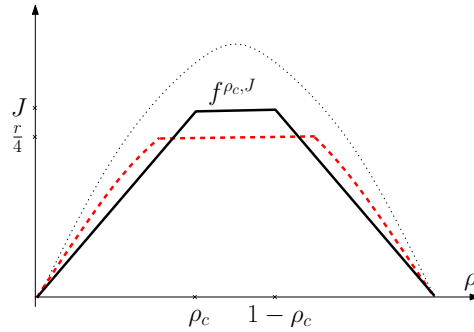


FIGURE 1. The graphs of three flux functions are depicted. The highest curve in dotted line is the graph of the homogeneous TASEP flux $\rho(1 - \rho)$. The middle curve is the truncated flux $f^{\rho_c, J}$ and the dashed curve is the conjectured flux f_ε for the disordered TASEP with a plateau at $r/4$.

The sufficient condition (3.10) for the plateau is actually necessary since, by Theorem 3.1 and (3.8)–(2.3), f_ε is a concave function such that $f_\varepsilon(0) = f_\varepsilon(1) = 0$. The convex

conjugate of $f^{\rho_c, J}$ through Legendre duality (3.8) is defined for $x \in \mathbb{R}$ by

$$(3.11) \quad \begin{aligned} k^{\rho_c, J}(x) &:= (-x)\mathbf{1}_{(-\infty, -J/\rho_c)}(x) \\ &+ [J - (1 - \rho_c)x]\mathbf{1}_{[-J/\rho_c, 0)}(x) + [J - \rho_c x]\mathbf{1}_{[0, J/\rho_c)}(x). \end{aligned}$$

Finally, one can associate to $k^{\rho_c, J}$ a passage-time function and a height function, related by (3.5)–(3.6), and defined for $x \in \mathbb{R}$ and $y \geq x^-$ by

$$(3.12) \quad \tau^{\rho_c, J}(x, y) := \frac{\rho_c x^+ - (1 - \rho_c)x^- + y}{J},$$

$$(3.13) \quad h^{\rho_c, J}(t, x) := tk^{\rho_c, J}(x/t),$$

where $x^+ = \max\{x, 0\}$ and $x^- = -\min\{x, 0\}$. Thus the lower bound (3.10) on the flux can be rephrased in terms of an upper bound on the last passage time:

Theorem 3.2. *Let τ_ε be the limiting passage-time defined by (3.3) when the environment has distribution \mathcal{P}_ε , then there exist $\varepsilon_0 > 0$ and $\rho_c < 1/2$ such that*

$$(3.14) \quad \forall \varepsilon < \varepsilon_0, \quad \tau_\varepsilon \leq \tau^{\rho_c, r/4}.$$

In particular, $\tau_\varepsilon(\cdot, y)$ has a cusp at $x = 0$.

The lower bound (3.10), and therefore Theorem 2.2, are a consequence of Theorem 3.2, which will be proved in the next sections.

Theorem 3.2 can be partially extended to LPP with general service-time distribution and heavier tails. In this case the particle interpretation is less standard, though the process can be viewed as a non-markovian TASEP (see e.g. [25]). Assume the service times $Y_{i,j}$ are i.i.d. variables with some distribution \mathbb{P} on \mathbb{R}^+ . Let M_n be the expectation of the maximum of n random variables with law \mathbb{P} . If $M_n = O(n^k)$ for some $k \in (0, 1/4)$, the conclusion of Theorem 3.2 still holds if the constant b in Assumption **(H)** satisfies $b < 2(1 - 4k)/(1 + 2k)$. In particular, if $M_n = O(n^k)$ for all $k > 0$, the conclusion holds for any $b \in (0, 2)$. Recall that for exponential service times, one has $M_n \sim \log n$. The tail of \mathbb{P} is reflected in the bound (7.16) and consequently on the form of the last (fluctuation) term on the r.h.s. of (4.12). The proof can be adapted with relatively minor modifications. Our approach (and the extension just explained) also applies to other LPP models with columnar disorder (in the wedge picture) or diagonal disorder (in the square picture). For instance, both Theorems 3.2 and 2.2 apply to the K -exclusion process [37].

3.3. Last-passage reformulation of Assumption (H). We will reformulate condition **(H)** in the last passage setting. To this end, we define restricted passage times. Let $B = [x_1, x_2] \cap \mathbb{Z}$ (where $x_1, x_2 \in \mathbb{Z}$) be a finite interval of \mathbb{Z} . If (x, y) and (x', y') are such that x and x' lie in B , we define $\Gamma_B((x, y), (x', y'))$ as the subset of $\Gamma((x, y), (x', y'))$

consisting of paths γ that lie entirely inside B in the sense that $x_k \in B$ for every $k = 0, \dots, n$. We then define

$$(3.15) \quad T_B^\alpha((x, y), (x', y')) := \max\{T^\alpha(\gamma) : \gamma \in \Gamma_B((x, y), (x', y'))\}$$

The counterpart of Definition 2.1 is

Lemma 3.1. *Let $B = [x_1, x_2] \cap \mathbb{Z}$. The limit*

$$(3.16) \quad T_{\infty, B}(\alpha_B) := \lim_{m \rightarrow \infty} \frac{1}{m} T_B^\alpha((x_0, 0), (x_0, m)) = \sup_{m \in \mathbb{N}^*} \mathbb{E} \left[\frac{1}{m} T_B^\alpha((x_0, 0), (x_0, m)) \right]$$

exists \mathbb{P} -a.s. for $x_0 \in B$, does not depend on the choice of x_0 , and defines a random variable depending only on the disorder restricted to B . Besides, we have

$$(3.17) \quad T_{\infty, B}(\alpha_B) = \frac{1}{j_{\infty, B}(\alpha_B)},$$

where and $j_{\infty, B}(\alpha_B)$ is the stationary current in the open system restricted to

$$(3.18) \quad B' := [x_1 + 1, x_2] \cap \mathbb{Z}$$

as in Definition 2.1.

Remark. Recall that in (2.11) we defined $j_{\infty, B^\#}(\alpha_{B^\#})$ as the maximum current for the TASEP in B . Since by (2.9) and (3.18), we have $(B')^\# = B$, the above lemma is therefore consistent with (2.11).

The proof of this lemma is postponed to Appendix B. To simplify notation, we shall at times omit α_B and write $T_{\infty, B}$, $j_{\infty, B}$. We can now restate condition **(H)** as follows:

Assumption (H). *There exists $b \in (0, 2)$, $a > 0$, $c > 0$ and $\beta > 0$ such that, for ε small enough,*

$$(3.19) \quad \mathcal{P}_\varepsilon \left(T_{\infty, [0, N]}(\alpha_{[0, N]}) \geq \frac{4}{r} - \frac{a}{N^{b/2}} \right) \leq \frac{c}{N^\beta}.$$

The constants a, c in (2.14) are different from those in (3.19), but b and β are the same.

4. RENORMALIZATION SCHEME

From now, we are going to focus on the last passage percolation model in order to prove Theorem 3.2. We first describe a renormalization procedure to show that a bound of the form (3.14) holds with high probability at every scales (see Proposition 4.1).

4.1. Definition of blocks. Let $n \in \mathbb{N} \setminus \{0\}$ be the renormalization “level” and $K_n = K_n(\varepsilon)$ the size of a renormalized block of order n (by block we mean a finite subinterval of \mathbb{Z}). For $n = 1$, we initialize $K_1 = K_1(\varepsilon)$, and define a block B of order 1 to be good if it contains no defect, i.e. $\alpha(x) = 1$ for every $x \in B$. Otherwise, the block is said to be bad.

For $n \geq 1$, we set $K_{n+1} = l_n K_n$, where $l_n = \lfloor K_n^\gamma \rfloor$ with $\gamma \in (0, 1)$. For $n \geq 1$, a block B_{n+1} of order $n + 1$ has size K_{n+1} and is partitioned into l_n disjoint blocks of order n . This block is called “good” if it contains at most one bad block of order n , and if condition (4.1) below holds

$$(4.1) \quad j_{\infty, B_{n+1}} \geq j_{n+1} \quad \text{with} \quad j_{n+1} := \frac{r}{4} + \frac{a}{K_{n+1}^{b/2}},$$

where the constants a, b were defined in (2.14). Otherwise B_{n+1} is said to be bad. We stress the fact that the status (good or bad) of B_{n+1} depends only on the disorder variables $\alpha_{B_{n+1}}$ in B_{n+1} and not on the exponential times $Y_{i,j}$.

The renormalization is built such that large blocks are good with high probability. Let $q_n(\varepsilon)$ denote the probability under \mathcal{P}_ε that the block $[0, K_n - 1] \cap \mathbb{Z}$, at level n , is bad.

Lemma 4.1. *Suppose that assumption (H) holds and set*

$$K^*(\varepsilon) := \left(\frac{2c}{\varepsilon} \right)^{\frac{1}{\beta+1}}, \quad K_* := (4c)^{\frac{1}{\beta-\gamma(\beta+2)}},$$

with the constants c, β appearing in (3.19)–(2.14). Then there exists $\gamma_0, \varepsilon_0 > 0$ such that for all $\gamma \in (0, \gamma_0)$ and $\varepsilon \leq \varepsilon_0$, the following holds for every $K_1 \in [K_*, K^*(\varepsilon)]$

$$(4.2) \quad \lim_{n \rightarrow \infty} q_n(\varepsilon) = 0,$$

where

$$(4.3) \quad \gamma_0 := \frac{\beta}{\beta+2}, \quad \varepsilon_0 := \min \left[1, 2c, (2c)(4c)^{-\frac{(\beta+1)}{\beta-\gamma(\beta+2)}} \right].$$

Proof. For $n \geq 1$, let $\zeta_n = \frac{c}{K_n^\beta}$ be the upper bound in (3.19). Then, by definition of good blocks and independence of the environment, one obtains the recursive inequality

$$\begin{aligned} q_1 &\leq K_1 \varepsilon, \\ q_{n+1} &\leq (l_n q_n)^2 + \zeta_{n+1}, \quad n \geq 1. \end{aligned}$$

Note that if for some n we have $q_n \leq 2\zeta_n$ and $\zeta_{n+1} \geq 4l_n^2 \zeta_n^2$, then $q_{n+1} \leq 2\zeta_{n+1}$. Thus if we have $q_1 \leq 2\zeta_1$ and $\zeta_{n+1} \geq 4l_n^2 \zeta_n^2$ for all $n \geq 1$, then $q_n \leq 2\zeta_n$ for all $n \geq 1$, implying $q_n(\varepsilon) \rightarrow 0$ as $n \rightarrow \infty$. On the one hand, $q_1 \leq 2\zeta_1$ follows from

$$(4.4) \quad K_1 \leq K^*(\varepsilon).$$

On the other hand, assuming $0 \leq \gamma < \frac{\beta}{\beta+2}$, then $\zeta_{n+1} \geq 4l_n^2 \zeta_n^2$ is equivalent to

$$K_n^{\beta-(\beta+2)\gamma} \geq 4c, \quad \forall n \geq 1.$$

Since K_n is increasing in n , the above inequality holds for all $n \geq 1$ if it holds for $n = 1$, which is equivalent to

$$(4.5) \quad K_1 \geq K_*.$$

Finally, $\max(1, K_*) \leq K^*(\varepsilon)$ if and only if $\varepsilon \leq \varepsilon_0$. \square

4.2. Mean passage-time in a block. The strategy to prove Theorem 2.2 is now as follows. To each block $B = [x_0, x_1 := x_0 + K_n - 1] \cap \mathbb{Z}$ of order n , we associate finite-size macroscopic restricted passage-time functions (in the left and right directions) taking as origin either extremity of the block:

$$(4.6) \quad \begin{cases} \tau_{n,B}^\alpha(1, y) = \mathbb{E} \left(\frac{1}{K_n} T_{B_n}^\alpha((x_0, 0), (x_1, [K_n y])) \right), & y \geq 0, \\ \tau_{n,B}^\alpha(-1, y) = \mathbb{E} \left(\frac{1}{K_n} T_{B_n}^\alpha((x_1, 0), (x_0, [K_n y])) \right), & y \geq 1, \end{cases}$$

which depend only on the disorder α_{B_n} . To keep compact notation, we will write both functions in the form $\tau_{B_n}^\alpha(\sigma, y)$ with $\sigma = \pm 1$ and $y \geq \sigma^- = -\min\{\sigma, 0\}$.

The main step to Theorem 3.2, stated in Proposition 4.1 below, is to prove that the mean passage time at each order n remains bounded by the reference function (3.12) with parameters ρ_n, J_n appropriately controlled to ensure that the plateau is preserved at each order. From now on, we will assume that $K_1 = K_1(\varepsilon)$ is chosen depending on ε so that

$$(4.7) \quad \lim_{\varepsilon \rightarrow 0} K_1(\varepsilon) = +\infty$$

and (4.2) holds. For instance, one can take the integer part

$$(4.8) \quad K_1(\varepsilon) := [K^*(\varepsilon)] = \left\lceil \left(\frac{2c}{\varepsilon} \right)^{\frac{1}{\beta+1}} \right\rceil,$$

where $K^*(\varepsilon)$ was defined in Lemma 4.1.

Proposition 4.1. *For small enough ε , there exist sequences $(\rho_n)_{n \geq 1} \in [0, 1]^{\mathbb{N}}$, $(J_n)_{n \geq 1} \in [0, +\infty)^{\mathbb{N}}$ such that:*

(i) *For any good block B of order n , every $\sigma \in \{+1, -1\}$, and every $y \geq \sigma^-$*

$$(4.9) \quad \tau_{n,B}^\alpha(\sigma, y) \leq \tau^{\rho_n, J_n}(\sigma, y),$$

(ii) *$\lim_{n \rightarrow \infty} J_n = r/4$, and $J_n > r/4$ for all $n \in \mathbb{N}$,*

(iii) *$\limsup_{n \rightarrow \infty} \rho_n < 1/2$.*

Once Proposition 4.1 is established, completing the proof of Theorem 3.2 (and thus Theorem 2.2) is a relatively simple task, the main part of which is to obtain a similar bound on *unrestricted* passage times. This will be done in Section 8. Let us now explain the structure of the proof of Proposition 4.1.

4.3. Coarse-graining and recursion. The strategy of the proof of Proposition 4.1 is based on a coarse-graining procedure. We assume that at scale n , we have found an upper bound of the form

$$\tau_{n,B}^\alpha(\sigma, y) \leq g_n(\sigma, y),$$

that holds for any good block B , where $g_n(\sigma, \cdot)$ is a concave function for each $\sigma \in \{-1, 1\}$. By decomposing a block of size K_{n+1} into subblocks of size K_n , we seek a relation between g_n and g_{n+1} so that a similar upper bound holds at scale $n+1$. In the next section, we will prove the following:

Proposition 4.2. *Let*

$$(4.10) \quad j_{n+1} := \frac{r}{4} + \frac{a}{K_{n+1}^{b/2}}, \quad l_n := \lfloor K_n^\gamma \rfloor \quad \text{and} \quad \delta_n := C \frac{(\log K_{n+1})^{3/2}}{\sqrt{K_n}},$$

where C is a large enough constant fixed once and for all. Then the sequence $(g_n)_{n \geq 1}$ defined on $[\sigma^-, +\infty)$ by

$$(4.11) \quad g_1 = \tau^{\rho_1, J_1}, \quad \text{with} \quad J_1 \in (r/4, 1/4), \quad \rho_1 = 2J_1,$$

and

$$(4.12) \quad g_{n+1}(\sigma, y) := \sup_{\sigma^- \leq \bar{y} \leq \frac{l_n}{(n-1)}y} \left\{ \left(1 - \frac{1}{l_n}\right) \left[g_n(\sigma, \bar{y}) - \frac{\bar{y}}{j_{n+1}} \right] \right\} + \frac{y}{j_{n+1}} + \frac{1 + \sigma}{2l_n j_{n+1}} + \delta_n \varphi(y),$$

where

$$(4.13) \quad \varphi(y) := \sqrt{\frac{\sigma}{2} + y} [2 + \log(1 + y)]^{3/2},$$

satisfies the bound

$$(4.14) \quad \tau_{n,B}^\alpha(\sigma, y) \leq g_n(\sigma, y),$$

for any good block B and $n \geq 1$.

The proof of this Proposition is postponed to Section 5. Roughly speaking, to obtain (4.12), a path at level $n+1$ is decomposed into subpaths contained in subblocks of size K_n . We then express the total passage-time as a maximum of a sum of the partial passage-times in each subblock, where the maximum is over all possible intermediate heights of the path at the interfaces. The first line on the r.h.s. of (4.12) comes from approximating each partial passage-time with its mean and using the induction hypothesis (4.14). The second line is a fluctuation estimate (see Proposition 5.1 below) on the difference between the expectation of the maximum of partial times and the maximum of the expectations.

Given the initialization $\rho_1 < 1/2$ from (4.11), we will show in Section 6 that

Proposition 4.3. *Assume **(H)** with $b \in [1, 2)$. Then for ε small enough, the sequence $(g_n)_{n \geq 1}$ defined in Proposition 4.2 satisfies the bound*

$$(4.15) \quad g_n(\sigma, y) \leq \frac{\rho_n^\sigma - \sigma^- + y}{J_n} = \tau^{\rho_n^\sigma, J_n}(\sigma, y),$$

with

$$(4.16) \quad J_n := j_{n+1} \quad \text{and} \quad \rho_n^\sigma := \sup_{y \geq \sigma^-} \left\{ j_{n+1} g_n(\sigma, y) - y \right\} + \sigma^-,$$

where ρ_n^σ satisfies

$$(4.17) \quad \rho_{n+1}^\sigma \leq \frac{j_{n+2}}{j_{n+1}} \left[\left(1 - \frac{1}{l_n} \right) \rho_n^\sigma + \frac{1}{l_n} + \Delta_n \right],$$

with $\Delta_n = \Delta_n(\varepsilon)$ which has the following property: there exist $\varepsilon > 0$ and $C > 0$ such that for every $0 < \varepsilon \leq \varepsilon_0$ and $n \geq 1$

$$(4.18) \quad \Delta_n \leq C j_{n+1} \frac{\delta_n^2}{2(j_{n+2}^{-1} - j_{n+1}^{-1})} \left[\log \left(\frac{\delta_n}{j_{n+2}^{-1} - j_{n+1}^{-1}} \right) \right]^3,$$

with δ_n as in (4.10). In the following, we will consider $\rho_n = \max\{\rho_n^1, \rho_n^{-1}\}$.

The role of Assumption **(H)** is to ensure that the decay of j_n to $r/4$ is slow enough so that the additional fluctuations of order Δ_n do not hinder property (ii) of Proposition 4.1. Note that (see Proposition 4.4 below) for the above proposition to be interesting, γ has to be chosen close to 0 so that the upper bound (4.18) vanishes in the limit $n \rightarrow \infty$. Recall that l_n , Δ_n and ρ_n actually depend on ε . To conclude the proof of Proposition 4.1, we will then show in Section 6 that

Proposition 4.4. *Assume **(H)** with $b \in [1, 2)$ and $\gamma < \inf\{\gamma_0, \frac{2}{b} - 1\}$, with γ_0 introduced in (4.3). Then for small enough ε , any sequence $(\rho_n)_{n \geq 1}$ satisfying (4.17) is such that*

$$\limsup_{n \rightarrow \infty} \rho_n < 1/2.$$

Note that a separate study of the case $b \in (0, 1)$ is not necessary: indeed, if assumption **(H)** is satisfied for some $b \in (0, 1)$, it is satisfied *a fortiori* for $b = 1$.

The seemingly natural route to derive Proposition 4.3 from Proposition 4.2 would be to plug the *ansatz* $g_n = \tau^{\rho_n, J_n}$ into (4.12), and try to obtain an induction on (ρ_n, J_n) . In fact, a more subtle treatment is required, because the induction obtained from such an *ansatz* turns out to be effective only for $b \in (0, 1)$, whereas it is not optimal for $b \in [1, 2)$. Although not necessary, let us indicate how to use the parametric form when $b \in (0, 1)$. In this case, Propositions 4.3 and 4.4 can be replaced by the following variant, that is simpler to prove. The price of this simpler approach is to restrict assumption **(H)** to values $b \in (0, 1)$.

Proposition 4.5. *Assume **(H)** with $b \in (0, 1)$, and $\gamma < \inf\{\gamma_0, \frac{1}{b} - 1\}$, with γ_0 introduced in (4.3). Then the following two assertions hold :*

(i) *There exists a sequence $\{\rho_n, J_n\}_{n \geq 1}$ with $\rho_1 < 1/2$ and $J_1 \in (r/4, 1/4)$, which satisfies Proposition 4.1, and the recursive bounds*

$$(4.19) \quad J_{n+1} \geq \min(J_n, j_{n+1}) - C\delta_n,$$

$$(4.20) \quad \rho_{n+1} \leq \left(1 - \frac{1}{l_n}\right) \rho_n + \frac{1}{l_n} + C\delta_n,$$

for some constant $C > 0$, with δ_n defined by (4.10).

(ii) *Any such sequence satisfies $\limsup_{n \rightarrow +\infty} \rho_n < 1/2$ and $J_n > r/4$.*

The derivation of Proposition 4.5 is postponed to Subsection 6.3. If $b \in [1, 2)$, statement (i) of Proposition 4.5 can be derived, but the second inequality of statement (ii) breaks down and Proposition 4.1 could not be established this way. In fact, the way (4.19)–(4.20) is obtained from (4.12) is not optimal, because it amounts to bounding the fluctuation term $\delta_n \sqrt{\sigma/2 + y[2 + \log(1 + y)]^{3/2}}$ by $\delta_n(1 + y)$, which is not accurate for large y , that corresponds to densities close to $1/2$ (hence the expected location of the plateau). In (4.19), fluctuations of J_n are controlled on the one hand by assumption **(H)**, which guarantees a margin of order $L^{-b/2}$ above $r/4$, and on the other hand by the fluctuations δ_n , which are essentially of order $L^{-1/2}$, where L is the block size. The latter annihilates the former if $b \geq 1$.

5. PROOF OF PROPOSITION 4.2

In this section, we prove the recursion in Proposition 4.2. To this end, we decompose a path of length K_{n+1} according to its traces on the interfaces between the subblocks of size K_n (see figure 2). The set of such traces will hereafter be called the “skeleton” of the path. The idea is to use (4.14) as an induction hypothesis for the subpaths in each block of size K_n . If we neglect the fluctuations of these subpaths, the “mean” computation reduces to optimizing the positions of the traces so as to maximize the total passage times of subpaths of order n . This “mean” induction relation is altered by an error term (see Proposition 5.1 below) arising from fluctuations of the subpaths as well as the entropy induced by the many possible skeletons.

5.1. Skeleton decomposition. We consider $B = [x, x' = x + K_{n+1} - 1] \cap \mathbb{Z}$ a block of order $n + 1$, where $x \in \mathbb{Z}$. Let $\gamma = ((x_k, y_k))_{k=0, \dots, m-1}$ be a path restricted to B connecting $(x, 0) = (x_0, y_0 = 0)$ to $(x', y' = [K_{n+1}y]) = (x_{m-1}, y_{m-1})$. We define the *skeleton* $s(\gamma) = \tilde{\gamma}$ of γ as follows (see figure 2). Let $k_0 = -1$ and $y_{-1} = 0$. For $i \in \mathbb{N}$, we set

$$\begin{aligned} k_{2i+1} &:= \min\{k > k_{2i} : x_k = x + (i + 1)K_n - 1\}, \\ k_{2i+2} &:= \max\{k \geq k_{2i+1} : x_k = x + (i + 1)K_n - 1\}. \end{aligned}$$

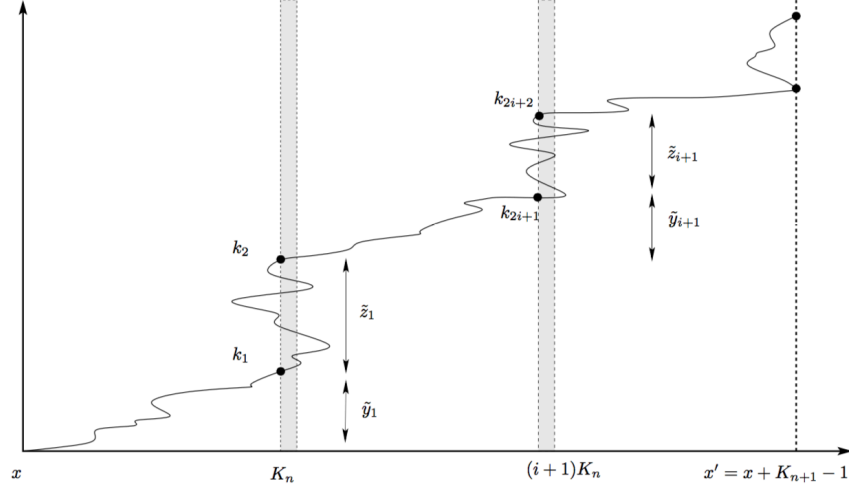


FIGURE 2. A block of level $n + 1$ is partitioned into blocks of length K_n (only 3 blocks have been depicted). The grey regions represent the boundaries between the blocks of order n which are separated by a microscopic length 1. A coarse grained path is depicted and the black dots denote the renewal points k_i .

Because $x_{k+1} - x_k \leq 1$, we necessarily have $x_{1+k_{2i+2}} = x + (i + 1)K_n$ and $y_{1+k_{2i+2}} = y_{k_{2i+2}}$. Note that $x_{k_{2l_n-1}} = x_{k_{2l_n}} = x'$ and $y_{k_{2l_n}} = y'$. Recall that the block B is made of $l_n = \lfloor K_n^\gamma \rfloor$ boxes of length K_n . The skeleton $s(\gamma)$ of γ is then the sequence $\tilde{\gamma} = (\tilde{y}_i, \tilde{z}_i)_{i=1, \dots, l_n} \in (\mathbb{N}^2)^{l_n}$ given by

$$(5.1) \quad \tilde{y}_i := y_{k_{2i-1}} - y_{k_{2i-2}},$$

$$(5.2) \quad \tilde{z}_i := y_{k_{2i}} - y_{k_{2i-1}}.$$

By definition, we have

$$(5.3) \quad \sum_{i=1}^{l_n} (\tilde{y}_i + \tilde{z}_i) = y' = \lfloor K_{n+1} y \rfloor.$$

In a similar way for the paths going from right to left, if $B = [x' = x - K_{n+1} + 1, x] \cap \mathbb{Z}$, we may define the skeleton of a path connecting $(x, 0) = (x_0, y_0 = 0)$ to $(x', y' = \lfloor K_{n+1} y \rfloor) = (x_{m-1}, y_{m-1})$. Let $k_0 = -1$ and $y_{-1} = -1$. For $i \in \mathbb{N}$, let

$$k_{2i+1} := \min\{k > k_{2i} : x_k = x - (i + 1)K_n + 1\},$$

$$k_{2i+2} := \max\{k \geq k_{2i+1} : x_k = x - (i + 1)K_n + 1\}.$$

Because $x_{k+1} - x_k \geq -1$, we necessarily have $x_{1+k_{2i+2}} = x - (i + 1)K_n$ and $y_{1+k_{2i+2}} = 1 + y_{k_{2i+2}}$. Note that $x_{k_{2l_n-1}} = x_{k_{2l_n}} = x'$ and $y_{k_{2l_n}} = y'$. The skeleton $s(\gamma)$ of γ is then the sequence $\tilde{\gamma} = (\tilde{y}_i, \tilde{z}_i)_{i=1, \dots, l_n}$ given by (5.1)–(5.2). Since allowed path increments are

$(1, 0)$ and $(-1, 1)$, this sequence must now satisfy the constraint $\tilde{y}_i \geq K_n$ for $i \geq 1$.

Let $\tilde{\Gamma}_n((x, 0), (x', y'))$ denote the set of skeletons of all paths γ restricted to B connecting $(x, 0)$ and (x', y') , that is the set of sequences $\tilde{\gamma} = (\tilde{y}_i, \tilde{z}_i)_{i=1, \dots, l_n} \in (\mathbb{N}^2)^{\{1, \dots, l_n\}}$ satisfying (5.3), with the constraint $\tilde{y}_i \geq K_n$ in the case $x' < x$. We will simply write $\tilde{\Gamma}_n$ when the endpoints are obvious from the context.

5.2. Passage-time decomposition. Let $\sigma = \pm 1$ denote as in (4.6) the direction of the paths. To encompass both cases $\sigma = \pm 1$, we will use the following simplifying convention: an interval can be written $[a, b]$ even if $a > b$, in which case it actually means $[b, a]$. We consider the block $B = [0, \sigma(K_{n+1} - 1)]$. For $l \in \{1, \dots, l_n\}$, we denote by $B_l := [\sigma(l-1)K_n, \sigma(lK_n - 1)] \cap \mathbb{Z}$ the l -th subblock of order n in the decomposition of B . For a path skeleton $\tilde{\gamma} = (\tilde{y}_l, \tilde{z}_l)_{l=1, \dots, l_n} \in \tilde{\Gamma}_n$, define

$$\tilde{h}_i := \sum_{j=1}^{i-1} [\tilde{y}_j + \tilde{z}_j]$$

if $i \geq 2$, and $\tilde{h}_1 = 0$. The quantity \tilde{h}_i represents the height at which a path with skeleton $\tilde{\gamma}$ enters block i . For a path $\gamma \in \Gamma_B((0, 0), (\sigma(K_{n+1} - 1), y'))$ with skeleton $\tilde{\gamma}$,

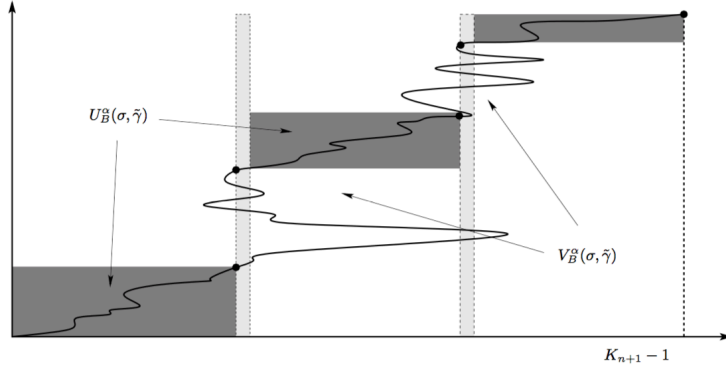


FIGURE 3. A coarse grained path is depicted in a block of order $n + 1$. The horizontal crossings through each block B_l are restricted to the dark grey regions. The passage time $U_B^\alpha(\sigma, \tilde{\gamma})$ depends only on the variables $\{Y_{i,j}\}$ inside the grey regions which are disjoint from the regions used by the vertical paths contributing to $V_B^\alpha(\sigma, \tilde{\gamma})$.

we have that

$$(5.4) \quad T_B^\alpha(\gamma) \leq U_B^\alpha(\sigma, \tilde{\gamma}) + V_B^\alpha(\sigma, \tilde{\gamma}) \leq T_B^\alpha((0, 0), (\sigma(K_{n+1} - 1), y')),$$

where

$$U_B^\alpha(\sigma, \tilde{\gamma}) := \sum_{l=1}^{l_n} U_{B,l}^\alpha(\sigma, \tilde{\gamma}), \quad V_B^\alpha(\sigma, \tilde{\gamma}) := \sum_{l=1}^{l_n} V_{B,l}^\alpha(\sigma, \tilde{\gamma}),$$

with

$$\begin{aligned} U_{B,l}^\alpha(\sigma, \tilde{\gamma}) &:= T_{B_l}^\alpha(\sigma(l-1)K_n, \tilde{h}_l, (\sigma(lK_n-2), \tilde{h}_l + \tilde{y}_l + \sigma - 1)), \\ V_{B,l}^\alpha(\sigma, \tilde{\gamma}) &:= T_B^\alpha((\sigma(lK_n-1), \tilde{h}_l + \tilde{y}_l + \frac{\sigma-1}{2}), (\sigma(lK_n-1), \tilde{h}_l + \tilde{y}_l + \frac{\sigma-1}{2} + \tilde{z}_l)), \end{aligned} \quad (5.5)$$

where $U_B^\alpha(\tilde{\gamma})$ is the contribution of the horizontal crossings in the blocks B_l and $V_B^\alpha(\tilde{\gamma})$ the contribution of the vertical paths at the junction of the blocks B_l (see figure 3). Noticing that the second inequality in (5.4) is an equality if and only if $\tilde{\gamma}$ is the skeleton of the optimal path, we get

$$(5.6) \quad T_B^\alpha((0,0), (\sigma(K_{n+1}-1), y')) = \max_{\tilde{\gamma} \in \tilde{\Gamma}_n((0,0), (\sigma(K_{n+1}-1), y'))} \{U_B^\alpha(\sigma, \tilde{\gamma}) + V_B^\alpha(\sigma, \tilde{\gamma})\}.$$

To derive Proposition 4.1, we have to estimate

$$(5.7) \quad \tau_{n+1,B}^\alpha(\sigma, y) := \frac{1}{K_{n+1}} \mathbb{E} \left(T_B^\alpha((0,0), (\sigma(K_{n+1}-1), y')) \right)$$

with $y' = [K_{n+1}y]$ and we decompose this expectation into the sum of two components:

$$(5.8) \quad \max_{\tilde{\gamma} \in \tilde{\Gamma}_n((0,0), (\sigma(K_{n+1}-1), y'))} \left\{ \mathbb{E} \left(\frac{U_B^\alpha(\sigma, \tilde{\gamma})}{K_{n+1}} \right) + \mathbb{E} \left(\frac{V_B^\alpha(\sigma, \tilde{\gamma})}{K_{n+1}} \right) \right\},$$

that is the ‘‘mean optimization problem’’ and a ‘‘fluctuation part’’ defined for $y' = [K_{n+1}y]$ as

$$(5.9) \quad \begin{aligned} \mathcal{F}_n(y) &= \mathbb{E} \left(\frac{1}{K_{n+1}} \max_{\tilde{\gamma} \in \tilde{\Gamma}_n((0,0), (\sigma(K_{n+1}-1), y'))} \{U_B^\alpha(\sigma, \tilde{\gamma}) + V_B^\alpha(\sigma, \tilde{\gamma})\} \right) \\ &\quad - \max_{\tilde{\gamma} \in \tilde{\Gamma}_n((0,0), (\sigma(K_{n+1}-1), y'))} \left\{ \mathbb{E} \left(\frac{U_B^\alpha(\sigma, \tilde{\gamma})}{K_{n+1}} \right) + \mathbb{E} \left(\frac{V_B^\alpha(\sigma, \tilde{\gamma})}{K_{n+1}} \right) \right\}. \end{aligned}$$

The term (5.8), which involves known information from subblocks, will give the main recursion structure, while (5.9) will be an error term. The latter will be controlled by fluctuations and entropy of paths. The precise result that will be established in the Section 7 is the following:

Proposition 5.1. *With the notation (5.9), one has uniformly in y*

$$(5.10) \quad \mathcal{F}_n(y) \leq \delta_n \sqrt{\frac{\sigma}{2} + y} (1 + \log(1 + y))^{3/2},$$

with δ_n defined in (4.10).

We will in fact replace the upper bound in (5.10) by a slightly worse one for the sole purpose of making it a concave function of y , which is important for us. We therefore observe that

$$(5.11) \quad \mathcal{F}_n(y) \leq \mathcal{G}_n(y) := \delta_n \varphi(y),$$

where φ is the (concave) function defined by (4.13).

5.3. The main recursion (4.12). Using the skeleton decomposition, we are now going to derive Proposition 4.2.

Let us explain the choice (4.11) of g_1 in Proposition 4.2. To initiate the induction relation, we need a bound at level 1 for ρ_1 and J_1 . For $n = 1$, a good block of order 1 contains only rates $\alpha(x) = 1$. Since the restricted passage times are smaller than the unrestricted ones, the values for ρ_1 and J_1 can be inferred from the asymptotic shape (3.7)–(3.8) of the homogeneous last passage percolation and the corresponding exact upper bound for the expected passage time, due to superadditivity:

$$\tau_{1,B}^\alpha(\sigma, y) \leq (\sqrt{\sigma + y} + \sqrt{y})^2 \leq 4 \left(\frac{1}{2}\sigma + y \right) \leq \tau^{2J_1, J_1}(\sigma, y),$$

for every $\sigma \in \{-1, 1\}$ and any $J_1 \in (0, 1/4]$, where the last inequality follows from (3.12). However, we restrict to $J_1 \in (r/4, 1/4)$ in view of condition (ii) of Proposition 4.1. By construction g_1 is concave. Note that, if $g_n(\sigma, \cdot)$ is concave, then $g_{n+1}(\sigma, \cdot)$ defined by (4.12) inherits this property.

Suppose now that the inequality (4.14)

$$\tau_{n,B}^\alpha(\sigma, y) \leq g_n(\sigma, y)$$

holds at step n and that g_n is concave. We will show that the recursion is valid at step $n + 1$ with g_{n+1} defined as in (4.12).

We first focus on the mean optimization problem (5.8) and consider a good block $B = [0, \sigma(K_{n+1} - 1)]$ at level $n + 1$. For a fixed disorder α , note that, by superadditivity and uniformity of α in the y -direction,

$$(5.12) \quad \mathbb{E} [V_{B,l}^\alpha(\sigma, \tilde{\gamma})] \leq \frac{1}{j_{\infty,B}} \tilde{z}_l.$$

Since B is a good block, $j_{\infty,B}$ satisfies (4.1). Thus

$$(5.13) \quad j_{\infty,B} \geq \frac{r}{4} + \frac{a}{K_{n+1}^{b/2}} =: j_{n+1},$$

(recall that j_{n+1} was introduced in (4.1) as one of the conditions defining a good block). As B is a good block, the subblocks B_l are good for all values of $l = 1, \dots, l_n$ except

for possibly one bad subblock with index i_0 . The recurrence hypothesis (4.14) at order n implies that the mean passage time on a good subblock B_l is bounded by

$$(5.14) \quad \mathbb{E} [U_{B,l}^\alpha(\sigma, \tilde{\gamma})] = K_n \tau_{n,B_l}^\alpha \left(\sigma, \frac{\tilde{y}_l}{K_n} \right) \leq K_n g_n \left(\sigma, \frac{\tilde{y}_l}{K_n} \right).$$

For the possibly remaining value i_0 such that B_{i_0} is a bad block, we use a crude upper bound by artificially extending the path in order to compare its cost to the one of a vertical connection:

$$U_{B,i_0}^\alpha(\sigma, \tilde{\gamma}) \leq T_{B_{i_0}}^\alpha \left(\left(\sigma(i_0 - 1)K_n, \tilde{h}_{i_0} \right), \left(\sigma(i_0 - 1)K_n, \tilde{h}_{i_0} + \tilde{y}_{i_0} + \frac{1 + \sigma}{2} K_n \right) \right),$$

which yields, as in (5.12),

$$(5.15) \quad \mathbb{E} [U_{B,i_0}^\alpha(\sigma, \tilde{\gamma})] \leq \frac{1}{j_{\infty, B_{i_0}}} \left(\tilde{y}_{i_0} + \frac{1 + \sigma}{2} K_n \right) \leq \frac{1}{j_{n+1}} \left(\tilde{y}_{i_0} + \frac{1 + \sigma}{2} K_n \right).$$

Note that there may sometimes not exist any bad subblock, but even in this case we may apply (5.15) to an arbitrarily chosen subblock to avoid distinguishing this seemingly better case (by which distinction we would anyway gain nothing in the sequel). Combining the above expectation bounds, we obtain

$$(5.16) \quad \mathbb{E} [U_B^\alpha(\sigma, \tilde{\gamma}) + V_B^\alpha(\sigma, \tilde{\gamma})] \leq K_{n+1} g_{n+1}^{(1)}(\sigma, y, \tilde{\gamma}),$$

where

$$(5.17) \quad g_{n+1}^{(1)}(\sigma, y, \tilde{\gamma}) := \frac{1}{l_n} \left\{ \sum_{l=1, l \neq i_0}^{l_n} g_n(\sigma, \bar{y}_l) + \frac{1}{j_{n+1}} \left[\frac{1 + \sigma}{2} + \bar{y}_{i_0} + \sum_{l=1}^{l_n} \bar{z}_l \right] \right\},$$

where $(\bar{y}_l, \bar{z}_l)_{l=1, \dots, l_n} \in [0, +\infty)^{2l_n}$ is the rescaled skeleton defined by $\bar{y}_l = K_n^{-1} \tilde{y}_l$ and $\bar{z}_l = K_n^{-1} \tilde{z}_l$, which satisfies the constraint (5.3), whence

$$(5.18) \quad \sum_{l=1}^{l_n} (\bar{y}_l + \bar{z}_l) \leq l_n y, \quad \text{with } \bar{y}_l \geq \sigma^-.$$

Define

$$(5.19) \quad \sigma^- \leq \bar{y} := \frac{1}{l_n - 1} \sum_{l=1, \dots, l_n: l \neq i_0} \bar{y}_l \leq \frac{l_n}{l_n - 1} y,$$

so that from (5.18), we have

$$(5.20) \quad \bar{y}_{i_0} + \sum_{l=1}^{l_n} \bar{z}_l \leq l_n y - (l_n - 1) \bar{y}.$$

By concavity of g_n , (5.16)–(5.17) and (5.20), we obtain the following upper bound for (5.8):

$$(5.21) \quad \max_{\tilde{\gamma} \in \tilde{\Gamma}_n((0,0),(\sigma(K_{n+1}-1),[K_{n+1}y]))} \left\{ \mathbb{E} \left(\frac{U_B^\alpha(\sigma, \tilde{\gamma})}{K_{n+1}} \right) + \mathbb{E} \left(\frac{V_B^\alpha(\sigma, \tilde{\gamma})}{K_{n+1}} \right) \right\} \\ \leq \sup_{\sigma^- \leq \bar{y} \leq \frac{l_n}{l_n-1}y} \left\{ \left(1 - \frac{1}{l_n} \right) \left[g_n(\sigma, \bar{y}) - \frac{\bar{y}}{j_{n+1}} \right] \right\} + \frac{y}{j_{n+1}} + \frac{1+\sigma}{2l_n j_{n+1}},$$

where the value of \bar{y} in (5.19) has been replaced by a supremum. To bound from above $\tau_{n+1,B}^\alpha(\sigma, y)$ (see (5.7)), it is enough to combine (5.21) and Proposition 5.1. This completes the proof of Proposition 4.2.

6. CONSEQUENCES OF THE MAIN RECURSION

In this section, we prove Propositions 4.3, 4.4 and 4.5.

6.1. Proof of Proposition 4.3. As $g_1(\sigma, \cdot)$ is concave, the recursion (4.12) implies that $g_n(\sigma, \cdot)$ is a concave function for all n . For notational simplicity, we shall write details of the proof for $\sigma = 1$, and in this case simply write $g_n(\cdot)$ for $g_n(\sigma, \cdot)$. We will only briefly indicate what changes are involved for $\sigma = -1$. We consider the sequence $(g_n)_{n \geq 1}$ given by the recursion (4.12) and set

$$(6.1) \quad y_n := \inf \left\{ y \geq 0 : g'_n(y) \leq \frac{1}{j_{n+1}} \right\},$$

where g'_n stands for the right derivative of the concave function. Thus, if $y \geq (1 - l_n^{-1})y_n$

$$(6.2) \quad g_{n+1}(y) = \left(1 - \frac{1}{l_n} \right) \left[g_n(y_n) - \frac{y_n}{j_{n+1}} \right] + \frac{y}{j_{n+1}} + \frac{1}{l_n j_{n+1}} + \delta_n \varphi(y),$$

and if $y \leq (1 - l_n^{-1})y_n$

$$(6.3) \quad g_{n+1}(y) = \left(1 - \frac{1}{l_n} \right) \left[g_n \left(\frac{l_n}{l_n-1} y \right) - \frac{l_n}{l_n-1} \frac{y}{j_{n+1}} \right] + \frac{y}{j_{n+1}} + \frac{1}{l_n j_{n+1}} + \delta_n \varphi(y).$$

Lemma 6.1. *Assume (H) with $b \in [1, 2)$. Then for ε small enough, the sequence $(y_n)_{n \geq 1}$ satisfies*

$$(6.4) \quad \forall n \geq 2, \quad y_{n-1} \leq y_n < \infty \quad \text{and} \quad \varphi'(y_n) = \frac{j_{n+1}^{-1} - j_n^{-1}}{\delta_{n-1}},$$

with φ as in (4.13).

Proof. For $n \geq 1$, we set

$$(6.5) \quad t_{n+1} := \frac{j_{n+2}^{-1} - j_{n+1}^{-1}}{\delta_n} = \psi_3(K_n),$$

with

$$\begin{aligned} \psi_3(K) &:= (1+\gamma)^{-3/2} \frac{K^{1/2}}{(\log K)^{3/2}} \left\{ \left(\frac{r}{4} + \frac{a}{K^{\frac{b}{2}(1+\gamma)^2}} \right)^{-1} - \left(\frac{r}{4} + \frac{a}{K^{\frac{b}{2}(1+\gamma)}} \right)^{-1} \right\} \\ &\stackrel{K \rightarrow +\infty}{\sim} (1+\gamma)^{-3/2} \frac{16a}{r^2} K^{\frac{1}{2} - \frac{b}{2}(1+\gamma)} \xrightarrow{K \rightarrow +\infty} 0. \end{aligned}$$

Since $b \geq 1$, using (4.7), we conclude that

$$(6.6) \quad \limsup_{\varepsilon \rightarrow 0} \liminf_{n \geq 1} t_{n+1}(\varepsilon) = 0.$$

Recall that for $n = 1$, g_1 is defined by (4.11). Thanks to (4.7), we have $\lim_{\varepsilon \rightarrow 0} j_2(\varepsilon) = r/4$. Thus, for small enough ε , we have $j_2(\varepsilon) < J_1$. It follows that $y_1 = 0$ and, from (6.2), we get that for every $y \geq 0$,

$$(6.7) \quad g_2(y) = \left(1 - \frac{1}{l_1}\right) \frac{\rho_1}{J_1} + \frac{y}{j_2} + \frac{1}{l_1 j_2} + \delta_1 \varphi(y).$$

Thus for $n = 2$, (6.1) becomes

$$y_2 = \inf \left\{ y \geq 0 : \varphi'(y) \leq \frac{j_3^{-1} - j_2^{-1}}{\delta_1} \right\}.$$

Since φ is strictly concave and $\lim_{y \rightarrow +\infty} \varphi'(y) = 0$, (6.6) implies that for ε small enough, y_2 is the unique solution of $\varphi'(y_2) = t_2$. Thus identity (6.4) holds for $n = 2$.

We are going to prove the claim by induction. Suppose that (6.4) is valid up to rank n . To show $y_{n+1} \geq y_n$, it is enough to check that

$$g'_{n+1}(y_n) > j_{n+2}^{-1}.$$

Since $y_n > (1 - l_n^{-1})y_n$, the above derivative is computed from the expression (6.2). Thus, using the induction hypothesis (6.4), we get for $n \geq 2$

$$\begin{aligned} g'_{n+1}(y_n) - j_{n+2}^{-1} &= j_{n+1}^{-1} - j_{n+2}^{-1} + \delta_n \varphi'(y_n) \\ &= j_{n+1}^{-1} - j_{n+2}^{-1} + \frac{\delta_n}{\delta_{n-1}} (j_{n+1}^{-1} - j_n^{-1}) \\ (6.8) \quad &= \psi(K_{n-1}), \end{aligned}$$

where, since $K_n = K_{n-1}^{1+\gamma}$,

$$\begin{aligned} \psi(K) &= \left(\frac{r}{4} + \frac{a}{K^{\frac{b}{2}(1+\gamma)^2}} \right)^{-1} - \left(\frac{r}{4} + \frac{a}{K^{\frac{b}{2}(1+\gamma)^3}} \right)^{-1} \\ (6.9) \quad &+ (1+\gamma)^{3/2} K^{-\gamma/2} \left[\left(\frac{r}{4} + \frac{a}{K^{\frac{b}{2}(1+\gamma)^2}} \right)^{-1} - \left(\frac{r}{4} + \frac{a}{K^{\frac{b}{2}(1+\gamma)}} \right)^{-1} \right]. \end{aligned}$$

Let us respectively denote by $\psi_1(K)$ and $\psi_2(K)$ the first and second line on the r.h.s. of (6.9). Then as $K \rightarrow +\infty$,

$$\psi_1(K) \sim -16ar^{-2} K^{-b(1+\gamma)^2/2}, \quad \psi_2(K) \sim 16a(1+\gamma)^{3/2} r^{-2} K^{-b(1+\gamma)/2 - \gamma/2}.$$

Since for $b \geq 1$ and $\gamma > 0$ we have

$$\frac{b}{2}(1 + \gamma) + \frac{\gamma}{2} < \frac{b}{2}(1 + \gamma)^2.$$

It follows that $\psi(K) > 0$ for K large enough. As $K_1(\varepsilon)$ diverges when ε tends to 0 (see (4.7)), we have that for small enough ε , $y_{n+1} \geq y_n \geq (1 - l_n^{-1})y_n$ holds for all $n \geq 2$.

As $g'_{n+1}(y_{n+1})$ is given by the derivative of (6.2) and φ is strictly concave, we have to solve

$$(6.10) \quad g'_{n+1}(y_{n+1}) = j_{n+1}^{-1} + \delta_n \varphi'(y_{n+1}) = j_{n+2}^{-1} \quad \Rightarrow \quad \varphi'(y_{n+1}) = \frac{j_{n+2}^{-1} - j_{n+1}^{-1}}{\delta_n} = t_{n+1}.$$

As above, (6.6) implies that, for ε small enough, a solution of (6.10) exists for all $n \geq 2$. This proves the second part of the claim (6.4).

In the case $\sigma = -1$, the infimum (6.1) defining y_n is over $y \geq 1$. The initialization step differs slightly since we then have

$$g_1(-1, y) = \frac{\rho_1 - 1 + y}{J_1}$$

and $y_1 = 1$. Therefore the inequality in (6.4) is no longer trivial for $n = 2$. However, since $\varphi'(y_2) = t_2$, this inequality will be true if $t_2 \leq \varphi'(1)$, which holds for ε small enough thanks to (6.6). \square

Using Lemma 6.1, we can now complete the proof of Proposition 4.3. The definition (4.16) of ρ_n leads immediately to the inequality (4.15)

$$g_n(y) \leq \tau^{\rho_n, J_n}(1, y).$$

Thus it remains only to prove the inequality (4.17) satisfied by ρ_n . By definition (6.1) of y_n , we have

$$\rho_n = j_{n+1} g_n(y_n) - y_n.$$

We are going to obtain a recursion for ρ_n . To this end, consider

$$\rho_{n+1} = j_{n+2} g_{n+1}(y_{n+1}) - y_{n+1}.$$

By Lemma 6.1, $y_{n+1} \geq y_n > (1 - l_n^{-1})y_n$, so $g_{n+1}(y_{n+1})$ is obtained from (6.2). Thus

$$(6.11) \quad \begin{aligned} \rho_{n+1} &= j_{n+2} \left(1 - \frac{1}{l_n}\right) \left[g_n(y_n) - \frac{y_n}{j_{n+1}} \right] + \frac{j_{n+2}}{l_n j_{n+1}} + \left(\frac{j_{n+2}}{j_{n+1}} - 1 \right) y_{n+1} + j_{n+2} \delta_n \varphi(y_{n+1}) \\ &\leq \frac{j_{n+2}}{j_{n+1}} \left(\left(1 - \frac{1}{l_n}\right) [j_{n+1} g_n(y_n) - y_n] + \frac{1}{l_n} + j_{n+1} \delta_n \varphi(y_{n+1}) \right), \end{aligned}$$

where on the second line we have used $j_{n+2} \leq j_{n+1}$. Setting $\Delta_n := j_{n+1} \delta_n \varphi(y_{n+1})$, we recovered the inequality (4.17), and it remains to verify (4.18). Starting from

$$\varphi'(y) \underset{y \rightarrow +\infty}{\sim} \frac{1}{2\sqrt{y}} (\log y)^{3/2},$$

we see that

$$(6.12) \quad \varphi[\varphi'^{-1}(t)] \stackrel{t \rightarrow 0}{\sim} \frac{1}{2t} \left(\log \frac{1}{4t^2} \right)^3.$$

Recall that by (6.10), $y_{n+1} = \varphi'^{-1}(t_{n+1})$, where t_n is defined by (6.5) and satisfies (6.6). Thus, there exist $C', C'' > 0$ and $\varepsilon_2 > 0$ such that, for every $0 < \varepsilon \leq \varepsilon_2$ and $n \geq 1$

$$\varphi(y_{n+1}) = \varphi[\varphi'^{-1}(t_{n+1})] \leq C'' \frac{1}{t_{n+1}} |\log t_{n+1}|^3 \leq C' \frac{\delta_n}{2(j_{n+2}^{-1} - j_{n+1}^{-1})} \left[\log \left(\frac{\delta_n}{j_{n+2}^{-1} - j_{n+1}^{-1}} \right) \right]^3.$$

This implies (4.18) with $\Delta_n = j_{n+1} \delta_n \varphi(y_{n+1})$.

For $\sigma = -1$, we set

$$\rho_n - 1 = \sup_{y \geq 1} \{j_{n+1} g_n(-1, y) - y\}$$

and we get a recursion similar to (6.11)

$$\rho_{n+1} - 1 \leq \frac{j_{n+2}}{j_{n+1}} \left(\left(1 - \frac{1}{l_n} \right) [\rho_n - 1] + j_{n+1} \delta_n \varphi(y_{n+1}) \right),$$

which can be rewritten

$$\rho_{n+1} \leq \frac{j_{n+2}}{j_{n+1}} \left(\left(1 - \frac{1}{l_n} \right) \rho_n + \frac{1}{l_n} + j_{n+1} \delta_n \varphi(y_{n+1}) + \frac{j_{n+1}}{j_{n+2}} - 1 \right).$$

For $b \geq 1$, the remainder $\frac{j_{n+1}}{j_{n+2}} - 1$ can be bounded by Δ_n so that the same type of inequality is also valid for $\sigma = -1$.

6.2. Proof of Proposition 4.4. Let $a_n = 1 - \frac{1}{l_n}$. Then one can see by induction that (4.17) implies

$$(6.13) \quad \begin{aligned} \rho_n &\leq \rho_1 \prod_{i=1}^{n-1} a_i + \left(1 - \prod_{i=1}^{n-1} a_i \right) + \sum_{i=1}^{n-1} \Delta_i \prod_{j=i+1}^{n-1} a_j \\ &\leq \rho_1 \prod_{i=1}^{n-1} a_i + \left(1 - \prod_{i=1}^{n-1} a_i \right) + \sum_{i=1}^{n-1} \Delta_i, \end{aligned}$$

where we used that $\frac{j_{i+1}}{j_i} \leq 1$ for any $i \geq 1$. Remember that the quantities $\rho_n, j_n, a_n, \Delta_n$ actually depend on ε . The proof of Proposition 4.4 is then concluded thanks to lemma 6.2. \square

Lemma 6.2. *Assume (H) with $b \in [1, 2)$, and K_1 given by (4.8). With the notation of Lemma 4.1, we fix*

$$\gamma < \min \left\{ \gamma_0, \frac{2}{b} - 1 \right\}.$$

Then

- (1) $\lim_{\varepsilon \rightarrow 0} \prod_{n=1}^{+\infty} a_n(\varepsilon) = 1$,
- (2) $\lim_{\varepsilon \rightarrow 0} \sum_{n=1}^{+\infty} \Delta_n(\varepsilon) = 0$.

Proof.

Proof of (1). We have to show that

$$(6.14) \quad \lim_{\varepsilon \rightarrow 0} \sum_{n=1}^{+\infty} \log \left(1 - \frac{1}{l_n(\varepsilon)} \right) = 0.$$

Since

$$l_n(\varepsilon) = \exp \left[\log (K_1(\varepsilon)) \gamma (1 + \gamma)^{n-1} \right] \geq \exp \left[\gamma (1 + \gamma)^{n-1} \right],$$

we have, for $n \geq 2$, $l_n(\varepsilon)^{-1} \leq C(\gamma) := e^{-\gamma(1+\gamma)} < 1$. Hence, for $n \geq 2$,

$$0 \leq -\log \left(1 - \frac{1}{l_n(\varepsilon)} \right) \leq \frac{1}{l_n(\varepsilon)} + \frac{C'(\gamma)}{l_n(\varepsilon)^2} \leq (1 + C'(\gamma)) \exp \left[-\gamma (1 + \gamma)^{n-1} \right].$$

The limit (6.14) then follows from dominated convergence, and $\lim_{\varepsilon \rightarrow 0} K_1(\varepsilon) = +\infty$, which implies $\lim_{n \rightarrow \infty} l_n(\varepsilon) = 0$ for any $n \geq 1$.

Proof of (2). Here we can write $\Delta_n \simeq \psi_0(K_n)$, where

$$(6.15) \quad \begin{aligned} \psi_0(K) &:= \frac{(\log K)^3}{K} \left[\left(\frac{r}{4} + \frac{a}{K^{\frac{b}{2}(1+\gamma)^2}} \right)^{-1} - \left(\frac{r}{4} + \frac{a}{K^{\frac{b}{2}(1+\gamma)}} \right)^{-1} \right]^{-1} \\ &\times \left(\log \left\{ \frac{(\log K)^{3/2}}{\sqrt{K}} \left[\left(\frac{r}{4} + \frac{a}{K^{\frac{b}{2}(1+\gamma)^2}} \right)^{-1} - \left(\frac{r}{4} + \frac{a}{K^{\frac{b}{2}(1+\gamma)}} \right)^{-1} \right]^{-1} \right\} \right)^3 \\ &\stackrel{K \rightarrow +\infty}{\sim} C''' (\log K)^6 K^{\frac{b}{2}(1+\gamma)-1}. \end{aligned}$$

for some constant $C''' > 0$. The assumption on b and the choice of γ imply that $c := 1 - \frac{b}{2}(1 + \gamma) > 0$ (equation (6.15) is the main reason for restricting to the case $b < 2$). By (4.7), there exists $\varepsilon_1 > 0$ such that $K_n(\varepsilon) \geq 2$ for every $n \geq 1$ and $\varepsilon \in [0, \varepsilon_1]$. Thus, by (6.15), there exists a constant $D > 0$ such that, for such n and ε ,

$$\Delta_n(\varepsilon) \leq \frac{D}{K_n(\varepsilon)^c} \leq \frac{D}{K_1(\varepsilon)^{c(1+\gamma)^n}} \leq \frac{D}{2^{c(1+\gamma)^n}}.$$

Since $\lim_{\varepsilon \rightarrow 0} K_1(\varepsilon) = 0$, the result follows again from dominated convergence. \square

6.3. Proof of Proposition 4.5 for $b \in (0, 1)$. Assume that $g_n(\sigma, y) \leq \tau^{\rho_n, J_n}(\sigma, y)$ for some $\rho_n \in [0, 1/2[$ and $J_n \in (r/4, 1/4]$. We use the recursion (4.12) to obtain $g_{n+1}(\sigma, y) \leq \tau^{\rho_{n+1}, J_{n+1}}(\sigma, y)$ for a new pair (ρ_{n+1}, J_{n+1}) defined as a function of (ρ_n, J_n) . We start with

$$(6.16) \quad g_n(\sigma, y) = \frac{\rho_n \sigma^+ - (1 - \rho_n) \sigma^- + y}{J_n} \leq \frac{\rho_n \sigma^+ - (1 - \rho_n) \sigma^- + y}{J_{n+1}^{(1)}} =: \tilde{g}_n(\sigma, y),$$

where

$$J_{n+1}^{(1)} := \min(J_n, j_{n+1}).$$

Replacing $g_n(\sigma, y)$ in (4.12) with $\tilde{g}_n(\sigma, y)$, since $J_{n+1}^{(1)} \leq j_{n+1}$, the supremum of the new expression is achieved for $\bar{y} = l_n y / (l_n - 1)$. On the other hand, there is a constant $B > 0$ such that

$$\varphi(y) = \sqrt{\sigma/2 + y} [2 + \log(1 + y)]^{3/2} \leq B(1 + y).$$

This yields

$$(6.17) \quad g_{n+1}(\sigma, y) \leq \tau^{\rho_{n+1}^{(1)}, J_{n+1}^{(1)}}(\sigma, y) + B\delta_n(1 + y),$$

where

$$J_{n+1}^{(1)} := \min(J_n, j_{n+1}) \quad \text{and} \quad \rho_{n+1}^{(1)} := \left(1 - \frac{1}{l_n}\right) \rho_n + \frac{1}{l_n}.$$

A simple computation shows that

$$\tau^{\rho, J}(x, y) + \delta(x + y) \leq \tau^{\rho', J'}(x, y),$$

with

$$\rho' := \frac{\rho + \delta J}{1 + \delta J}, \quad J' := \frac{J}{1 + \delta J}.$$

Note that $\rho' \leq \rho + \delta$ and $J' \geq J - \delta$ for $J \leq 1$. Since $J_n \leq 1$ for all $n \geq 1$, we obtain $g_n(\sigma, y) \leq \tau^{\rho_n, J_n}(\sigma, y)$, with (ρ_n, J_n) satisfying (4.19)–(4.20).

Let us now prove that the two statements in (ii) of Proposition 4.1 hold. On the one hand, (4.20) is similar to (6.13) with δ_n instead of Δ_n , the first limit in statement (ii) of Proposition 4.5 is proved exactly like Proposition 4.4, since 2) of Lemma 6.2 can be also obtained for δ_n instead of Δ_n . On the other hand, (4.19) implies

$$(6.18) \quad J_n \geq \min \left\{ J_1 - \sum_{j=1}^{n-1} \delta_j; \min_{k=1, \dots, n-1} \left[j_{k+1} - \sum_{j=k}^{n-1} \delta_j \right] \right\}.$$

The inequality $J_n > r/4$ then follows from (6.18) and

Lemma 6.3. *Assume **(H)** with $b \in (0, 1)$, and K_1 given by (4.8). With the notation of Lemma 4.1, we fix*

$$\varepsilon \leq \varepsilon_0, \quad \gamma < \min \left\{ \gamma_0, \frac{1}{b} - 1 \right\}, \quad J_1 \in \left(\frac{r}{4}, \frac{1}{4} \right).$$

Then the sequence $\delta_n(\varepsilon) := C \frac{(\log K_{n+1})^{3/2}}{\sqrt{K_n}}$ (where the dependence on ε is encoded in K_n) satisfies the properties:

(1) *There exists $\varepsilon_1 \in (0, \varepsilon_0)$ such that, for all $k \geq 1$ and $\varepsilon \in (0, \varepsilon_1)$,*

$$\sum_{j=k}^{+\infty} \delta_j(\varepsilon) \leq 2\delta_k(\varepsilon).$$

(2) *There exists $\varepsilon_2 \in (0, \varepsilon_1)$ such that, for all $\varepsilon \in (0, \varepsilon_2)$ and all $k \geq 1$,*

$$j_{k+1}(\varepsilon) - 2\delta_k(\varepsilon) > r/4, \quad \text{and} \quad J_1 - 2\delta_1(\varepsilon) > r/4.$$

Proof.

Proof of (1). Since

$$\frac{\delta_{n+1}(\varepsilon)}{\delta_n(\varepsilon)} = (1 + \gamma)^{3/2} K_n^{-\gamma/2} \leq (1 + \gamma)^{3/2} K_1(\varepsilon)^{-\gamma/2},$$

then for ε small enough so that $(1 + \gamma)^{3/2} K_1(\varepsilon)^{-\gamma/2} < 1/2$, we obtain $\sum_{j=k}^{+\infty} \delta_j(\varepsilon) \leq 2\delta_k(\varepsilon)$.

Proof of (2). Recall that

$$j_{k+1} = \frac{r}{4} + \frac{a}{K_{k+1}^{b'/2}} = \frac{r}{4} + \frac{a}{K_k^{b'/2}},$$

where a is the constant in (2.14) and $b' = (1 + \gamma)b$. The assumptions on b and γ imply $b' \in (0, 1)$. Thus, there exists $x_0 \geq 1$ such that

$$\frac{a}{x^{b'/2}} > 4C \frac{(\ln x)^{3/2}}{\sqrt{x}},$$

for all $x \geq x_0$, where C is the constant in the definition of δ_n . Since $K_n(\varepsilon)$ increases with n , in view of (4.7), it suffices to choose ε small enough such that $K_1(\varepsilon) \geq x_0$. Since $\lim_{\varepsilon \rightarrow 0} \delta_1(\varepsilon) = 0$, the second inequality follows for ε small enough from the choice $J_1 > r/4$. \square

7. FLUCTUATION BOUNDS : PROOF OF PROPOSITION 5.1

In this section, we prove Proposition 5.1. We start in Subsection 7.1 by stating preliminary estimates, and apply these in Subsection 7.2, which is the body of the proof.

7.1. Concentration estimates. We shall need a classical gaussian concentration inequality for last-passage times (see e.g. [30]). In the following lemma, it is assumed that the service times $Y_{i,j}$ involved in the definition (3.1)–(3.2) of last passage times are i.i.d. random variables *bounded* by M instead of being exponentially distributed. To avoid confusion with the previous notation, the corresponding probability \mathbb{P}_M and expectation \mathbb{E}_M are denoted below by an index M .

Lemma 7.1. *Assume that $Y = (Y_{i,j} : (i, j) \in \mathbb{Z} \times \mathbb{N})$ is a vector of non negative independent random variables bounded from above by rM . Let (x_1, y_1) and (x_2, y_2) in $\mathbb{Z} \times \mathbb{N}$ such that $(x_2 - x_1, y_2 - y_1) \in \mathcal{W}$. Then*

$$T^\alpha((x_1, y_1), (x_2, y_2)) = \mathbb{E}_M [T^\alpha((x_1, y_1), (x_2, y_2))] + 8M \sqrt{L((x_1, y_1), (x_2, y_2))} Z,$$

where $L((x_1, y_1), (x_2, y_2)) := (x_2 - x_1) + 2(y_2 - y_1)$ is the length of any path connecting (x_1, y_1) to (x_2, y_2) , and Z is a random variable with subgaussian tail

$$\forall t \geq 0, \quad \mathbb{P}_M(|Z| \geq t) \leq \exp(-t^2).$$

We stress the fact that Gaussian bounds on last passage times are by no means optimal in the case of exponential service times, for which more refined (but also more specific) gaussian-exponential estimates are available (see e.g. [43]). However, for our purpose, they have the advantage of being both simple and sufficient, while also extending to service distributions with heavier tails, as a result of the cutoff procedure introduced in Subsection 7.2.

The above concentration inequality will be combined with the following result.

Lemma 7.2. *Let \mathcal{A} and \mathcal{I} be finite sets. Assume that for each $a \in \mathcal{A}$, we have a family $(\mathcal{Y}_{a,i})_{i \in \mathcal{I}}$ of independent random variables such that, for every $i \in \mathcal{I}$,*

$$(7.1) \quad \mathcal{Y}_{a,i} = \mathbb{E}(\mathcal{Y}_{a,i}) + \sqrt{V_{a,i}}Z_{a,i},$$

where $V_{a,i} > 0$, and $Z_{a,i}$ is a random variable such that

$$(7.2) \quad \mathbb{P}(Z_{a,i} \geq t) \leq e^{-t^2},$$

for every $t \geq 0$. Then

$$(7.3) \quad \mathbb{E} \left(\max_{a \in \mathcal{A}} \sum_{i \in \mathcal{I}} \mathcal{Y}_{a,i} \right) \leq \max_{a \in \mathcal{A}} \sum_{i \in \mathcal{I}} \mathbb{E}(\mathcal{Y}_{a,i}) + \left(\max_{a \in \mathcal{A}} \sum_{i \in \mathcal{I}} V_{a,i} \right)^{\frac{1}{2}} \left(\sqrt{\pi} \sqrt{|\mathcal{I}|} + \sqrt{\pi} \sqrt{A} + \sqrt{A} \sqrt{\log |\mathcal{A}|} \right),$$

where $|\cdot|$ denotes the cardinality, and A is the same constant as in (ii) of Lemma 7.3.

The proof of Lemma 7.2 relies on the following elementary estimates.

Lemma 7.3.

(i) *Let Y be a random variable such that $\mathbb{P}(Y \geq t) \leq Ce^{-t^2/V}$ for all $t \geq 0$, where $C \geq 1$ and $V > 0$. Then, we have*

$$Y = \sqrt{V \log C} + \sqrt{V}X,$$

where $\mathbb{P}(X \geq t) \leq e^{-t^2}$.

(ii) *There exists a positive constant A such that the following holds. Let $(X_k)_{k=1, \dots, n}$ be nonnegative independent random variables such that $\mathbb{P}(X_k \geq t) \leq e^{-t^2}$, and $(V_k)_{k=1, \dots, n}$ be nonnegative numbers. Then*

$$\sum_{k=1}^n \sqrt{V_k} X_k = \sqrt{\pi} \sum_{k=1}^n \sqrt{V_k} + \left(A \sum_{k=1}^n V_k \right)^{1/2} Z,$$

where Z is a r.v. such that $\mathbb{P}(Z \geq t) \leq e^{-t^2}$.

Proof of Lemma 7.3. (i) Follows from an immediate computation. To obtain (ii) we note that, for $\theta \geq 0$,

$$\begin{aligned} \mathbb{E}(e^{\theta X_k}) &\leq 1 + \int_0^{+\infty} \theta e^{\theta t} \mathbb{P}(X_k \geq t) dt \\ &\leq 1 + \theta e^{\theta^2/4} \int_{-\theta/2}^{+\infty} e^{-t^2} dt \leq 1 + \sqrt{\pi} \theta e^{\theta^2/4}. \end{aligned}$$

Setting $Y_k = X_k - \sqrt{\pi}$, we have, for $\theta \geq 0$,

$$\Lambda(\theta) := \log \mathbb{E}(e^{\theta Y_k}) \leq \log \left[1 + \sqrt{\pi} \theta e^{\theta^2/4} \right] - \sqrt{\pi} \theta.$$

Thus there exists $A > 0$ such that $\Lambda(\theta) \leq A\theta^2/4$ for $\theta \geq 0$. Hence, by independence of the r.v.'s X_k ,

$$\log \mathbb{E} \left[\exp \left(\theta \left(\sum_{k=1}^n \sqrt{V_k} X_k - \sqrt{\pi} \sum_{k=1}^n \sqrt{V_k} \right) \right) \right] \leq \frac{A}{4} \theta^2 \sum_{k=1}^n V_k.$$

The result then follows by Cramer's argument. \square

Proof of Lemma 7.2. By (ii) of Lemma 7.3, for every $a \in \mathcal{A}$, we have

$$(7.4) \quad \sum_{i \in \mathcal{I}} \mathcal{Y}_{a,i} = \sum_{i \in \mathcal{I}} \mathbb{E}(\mathcal{Y}_{a,i}) + \sqrt{\pi} \sum_{i \in \mathcal{I}} \sqrt{V_{a,i}} + \left(A \sum_{i \in \mathcal{I}} V_{a,i} \right)^{1/2} Z_a,$$

where Z_a is a random variable satisfying $\mathbb{P}(Z_a \geq t) \leq e^{-t^2}$ for all $t \geq 0$. On the other hand, by Cauchy-Schwarz inequality,

$$(7.5) \quad \sum_{i \in \mathcal{I}} \sqrt{V_{a,i}} \leq \sqrt{|\mathcal{I}|} \left(\sum_{i \in \mathcal{I}} V_{a,i} \right)^{1/2}.$$

Thus, for every $a \in \mathcal{A}$,

$$(7.6) \quad \sum_{i \in \mathcal{I}} \mathcal{Y}_{a,i} \leq m + (AV)^{1/2} Z_a^+,$$

where

$$V := \max_{a \in \mathcal{A}} \sum_{i \in \mathcal{I}} V_{a,i} \quad \text{and} \quad m := \sqrt{\pi} \sqrt{|\mathcal{I}|} \sqrt{V} + \max_{a \in \mathcal{A}} \sum_{i \in \mathcal{I}} \mathbb{E}(Y_{a,i}).$$

Next, for any $t \geq 0$, we have

$$\begin{aligned} \mathbb{P} \left(\max_{a \in \mathcal{A}} \sum_{i \in \mathcal{I}} \mathcal{Y}_{a,i} \geq m + t \right) &\leq \mathbb{P} \left(\bigcup_{a \in \mathcal{A}} \left\{ \sum_{i \in \mathcal{I}} \mathcal{Y}_{a,i} \geq m + t \right\} \right) \\ &\leq \sum_{a \in \mathcal{A}} \mathbb{P} \left((AV)^{1/2} Z_a^+ \geq t \right) \leq |\mathcal{A}| e^{-\frac{t^2}{AV}}. \end{aligned}$$

It follows from (i) of Lemma 7.3 that

$$\max_{a \in \mathcal{A}} \sum_{i \in \mathcal{I}} \mathcal{Y}_{a,i} = m + \sqrt{\log |\mathcal{A}|} (AV)^{1/2} + (AV)^{1/2} Z,$$

where Z is a random variable satisfying $\mathbb{P}(Z \geq t) \leq e^{-t^2}$ for all $t \geq 0$. The result then follows from

$$\mathbb{E}(Z) \leq \mathbb{E}(Z^+) \leq \int_0^{+\infty} \mathbb{P}(Z^+ \geq t) dt \leq \int_0^{+\infty} e^{-t^2} dt = \sqrt{\pi}.$$

□

7.2. Path renormalization: fluctuation and entropy. We now proceed in three steps. In step one, we define a cutoff procedure for the service times $Y_{i,j}$, by conditioning on their maximum, in order to replace them with bounded variables, to which the results of Subsection 7.1 apply. In step two, we apply Lemma 7.2 to passage-times in subblocks. This yields for the cutoff service times a result similar to the statement of the proposition 5.1, but without whole logarithmic correction. Finally, in step three, we remove the cutoff and use a bound on the expectation of the maximum of exponential variables, to obtain a quasi-gaussian estimate with a logarithmic correction.

Step 1. Notation and conditional measure. Pick γ such that

$$(7.7) \quad 0 < \gamma < \min \{ \gamma_0, (2/b) - 1 \},$$

with γ_0 introduced in Lemma 4.1, and b in (3.19)–(2.14). Let $B = \mathbb{Z} \cap [0, \sigma(K_{n+1} - 1)]$ be a block of order $n + 1$ and partition B into the subblocks of order n denoted by $B_l = [\sigma(l - 1)K_n, \sigma(lK_n - 1)] \cap \mathbb{Z}$, where $l = 1, \dots, l_n$.

Set $y' = [K_{n+1}y]$, $\tilde{\Gamma}_n = \tilde{\Gamma}_n((0, 0), (\sigma(K_{n+1} - 1), y'))$ and define

$$M_B(y) := \max \{ Y_{i,j} : i \in B, j = 0, \dots, y' = [K_{n+1}y] \}.$$

Given $M > 0$, denote by $\mathbb{P}_{B,M,y'}$ the distribution of $(X_{i,j} : i \in B, j = 0, \dots, y')$, where $X_{i,j}$ are i.i.d. and have the same distribution as $Y_{i,j}$ conditioned on $Y_{i,j} \leq rM$. (Note that after conditioning by rM , the percolation paths have weights $\frac{Y_{i,j}}{\alpha} \leq M$ as $\alpha \geq r$). Denote by $\mathbb{P}'_{B,M,y'}$ the distribution of $(Y'_{i,j} : i \in B, j = 0, \dots, y')$ conditioned on $M_B(y) = rM$. A vector $(Y'_{i,j} : (i, j) \in B \times \{0, \dots, y'\})$ with distribution $\mathbb{P}'_{B,M,y'}$ is obtained as follows. Pick a uniformly distributed (i_0, j_0) in $B \times \{0, \dots, y'\}$; then give value rM to Y'_{i_0, j_0} , and let the other $Y'_{i,j}$ for $(i, j) \neq (i_0, j_0)$ be independent with the same distribution as the above $X_{i,j}$.

Step 2. Fluctuation and entropy bounds. Given α , the random variables $\{U_{B,l'}^\alpha(\sigma, \tilde{\gamma}), V_{B,l}^\alpha(\sigma, \tilde{\gamma})\}_{l,l'}$ (defined in (5.5)) are independent under $\mathbb{P}_{B,M,y'}$, because they depend on

disjoint subvectors of Y (see figure 3). On the other hand, by Lemma 7.1, we get

$$(7.8) \quad \begin{cases} U_{B,l}^\alpha(\sigma, \tilde{\gamma}) = \mathbb{E}_M [U_{B,l}^\alpha(\sigma, \tilde{\gamma})] + 8M\sqrt{\sigma K_n + 2\tilde{y}_l} Z_l^{(1)}, \\ V_{B,l}^\alpha(\sigma, \tilde{\gamma}) = \mathbb{E}_M [V_{B,l}^\alpha(\sigma, \tilde{\gamma})] + 8M\sqrt{2\tilde{z}_l} Z_l^{(2)}, \end{cases}$$

where $(Z_l^{(i)})_{l=1, \dots, l_n; i=1,2}$ is a family of r.v.'s independent under $\mathbb{P}_{B,M,y'}$ and such that

$$(7.9) \quad \mathbb{P}_{B,M,y'} \left(Z_l^{(i)} \geq t \right) \leq \exp(-t^2),$$

for all $t \geq 0$. To apply Lemma 7.2 to the random variables in (7.8), we take $\mathcal{A} = \tilde{\Gamma}_n((0,0), (\sigma(K_{n+1} - 1), y'))$ with $\mathcal{I} = \{1, \dots, 2l_n\}$, and for $a = \tilde{\gamma} \in \mathcal{A}$, we set

$$l \in \{1, \dots, l_n\}, \quad \mathcal{Y}_{a,2l-1} = U_{B,l}^\alpha(\sigma, \tilde{\gamma}) \quad \text{and} \quad \mathcal{Y}_{a,2l} = V_{B,l}^\alpha(\sigma, \tilde{\gamma}).$$

Thus in (7.3) we have $|\mathcal{I}| = 2l_n = 2K_{n+1}/K_n$, and (cf. (7.8) and (5.3))

$$\sum_{i \in \mathcal{I}} V_{a,i} = 64M^2(\sigma K_{n+1} + 2y') \leq 64M^2 K_{n+1}(\sigma + 2y).$$

To estimate the cardinal $|\mathcal{A}|$ of the skeletons, we need the following

Lemma 7.4. *For every $y' \in \mathbb{N}$, one has*

$$\log |\mathcal{A}| = \log |\tilde{\Gamma}_n((0,0), (\pm(K_{n+1} - 1), y'))| \leq 2 \frac{K_{n+1}}{K_n} [1 + \log(1 + K_n y)].$$

Proof. The number of such skeletons satisfies the inequality

$$(7.10) \quad \sigma \in \{-1, 1\}, \quad \left| \tilde{\Gamma}_n((0,0), (\sigma(K_{n+1} - 1), y')) \right| \leq \binom{2l_n + y' - 1}{2l_n - 1}.$$

The previous upper bound follows by noticing that choosing a skeleton amounts to choose $2l_n - 1$ heights corresponding to the different renewal times to reach the total height y' . In fact, when $\sigma = 1$, some of these heights can be equal if $\tilde{y}_i = 0$ or $\tilde{z}_i = 0$ for some $i \leq 2l_n - 1$. To take this into account, it is enough to choose a collection of distinct heights in the larger set $2l_n + y' - 1$. Thus estimate (7.10) is actually an equality if $\sigma = 1$.

Recall the inequalities:

$$(7.11) \quad \log \binom{N}{k} \leq Nh \left(\frac{k}{N} \right),$$

where h is defined on $(0, 1)$ by

$$-h(x) := x \log x + (1 - x) \log(1 - x),$$

and equal to 0 on $\{0, 1\}$. Furthermore

$$uh(1/u) \leq 1 + \log u$$

for $u \geq 1$, and

$$\frac{2l_n + y' - 1}{2l_n - 1} \leq 1 + \frac{y'}{l_n} = 1 + \frac{K_{n+1}}{l_n} y,$$

(the inequality follows from $l_n \geq 1$). This completes the Lemma 7.4.

The bound (7.11) follows e.g. from Cramer's exact large deviation upper bound applied to a sum of i.i.d. Bernoulli variables with parameter $1/2$ denoted by $(\zeta_i)_{i=1,\dots,n}$, since for $k \geq N/2$,

$$\binom{N}{k} \leq 2^N \mathbb{P} \left(\frac{1}{N} \sum_{i=1}^N \zeta_i \geq \frac{k}{N} \right).$$

□

Combining (7.3) with the entropy estimate of Lemma 7.4, we obtain

$$(7.12) \quad \mathbb{E}_M \left(\max_{\tilde{\gamma} \in \tilde{\Gamma}_n((0,0),(\sigma(K_{n+1}-1),[K_{n+1}y]))} \left\{ U_B^\alpha(\sigma, \tilde{\gamma}) + V_B^\alpha(\sigma, \tilde{\gamma}) \right\} \right) \\ \leq \max_{\tilde{\gamma} \in \tilde{\Gamma}_n((0,0),(\sigma(K_{n+1}-1),[K_{n+1}y]))} \left\{ \mathbb{E}_M (U_B^\alpha(\sigma, \tilde{\gamma}) + V_B^\alpha(\sigma, \tilde{\gamma})) \right\} \\ + 8M \sqrt{K_{n+1}} \sqrt{\sigma + 2y} \sqrt{2 \frac{K_{n+1}}{K_n}} \left(\sqrt{\pi} + \sqrt{A} + \sqrt{A} \sqrt{[1 + \ln(1 + K_n y)]} \right),$$

where we used that $2 \frac{K_{n+1}}{K_n} \geq \pi$.

Step 3. Removing the cut-off on Y . The random variables $\{U_{B,l}^\alpha(\sigma, \tilde{\gamma}), V_{B,l}^\alpha(\sigma, \tilde{\gamma})\}_{l,l'}$ are nondecreasing functions of $Y = (Y_{i,j} : (i,j) \in \mathbb{Z} \times \mathbb{N})$ with respect to the product order. Therefore, their distributions under $\mathbb{P}_{B,M,y'}$ are stochastically dominated by their distributions under \mathbb{P} and one has

$$(7.13) \quad \mathbb{E} [U_{B,l}^\alpha(\sigma, \tilde{\gamma})] \geq \mathbb{E}_M [U_{B,l}^\alpha(\sigma, \tilde{\gamma})], \quad \mathbb{E} [V_{B,l}^\alpha(\sigma, \tilde{\gamma})] \geq \mathbb{E}_M [V_{B,l}^\alpha(\sigma, \tilde{\gamma})].$$

On the other hand, a coupling argument shows that the distribution of $T_B^\alpha((0,0),(\sigma(K_{n+1}-1),y'))$ under $\mathbb{P}'_{B,M,y'}$ is stochastically dominated by the distribution of $T_B^\alpha((0,0),(\sigma(K_{n+1}-1),y')) + M$ under $\mathbb{P}_{B,M,y'}$. This property combined with (7.13) and (7.12) yields

$$(7.14) \quad \mathbb{E}'_M \left(\max_{\tilde{\gamma} \in \tilde{\Gamma}_n((0,0),(\sigma(K_{n+1}-1),[K_{n+1}y]))} \left\{ U_B^\alpha(\sigma, \tilde{\gamma}) + V_B^\alpha(\sigma, \tilde{\gamma}) \right\} \right) \\ \leq \max_{\tilde{\gamma} \in \tilde{\Gamma}_n((0,0),(\sigma(K_{n+1}-1),[K_{n+1}y]))} \left\{ \mathbb{E} [U_B^\alpha(\sigma, \tilde{\gamma}) + V_B^\alpha(\sigma, \tilde{\gamma})] \right\} + M \\ + 8MK_{n+1} \sqrt{\sigma + 2y} \sqrt{\frac{2}{K_n}} \left(\sqrt{\pi} + \sqrt{A} + \sqrt{A} \sqrt{[1 + \log(1 + K_n y)]} \right).$$

Recall that \mathbb{E}'_M on the left-hand side of (7.14) stands for the expectation with respect to \mathbb{P} conditioned on the maximum $M_B(y) = M$. We can now remove this conditioning by integrating both sides of (7.14) with respect to the law of $M_B(y)$. We first write

$$(7.15) \quad \mathbb{E} [M_B(y)] = m([yK_{n+1}]K_{n+1}),$$

where the function $t \in [0, +\infty) \mapsto m(t)$ is defined as the expectation of the maximum of $1 + [t]$ i.i.d. exponential variables of rate 1. In particular, we have

$$(7.16) \quad m(t) \leq C[1 + \log(1 + t)],$$

for some constant $C > 0$. Thus, after conditioning on $M_B(y)$, we obtain

$$(7.17) \quad \begin{aligned} & \frac{1}{K_{n+1}} \mathbb{E} \left(\max_{\tilde{\gamma} \in \tilde{\Gamma}_n((0,0), (\sigma(K_{n+1}-1), [K_{n+1}y]))} \left\{ U_B^\alpha(\sigma, \tilde{\gamma}) + V_B^\alpha(\sigma, \tilde{\gamma}) \right\} \right) \\ & \leq \frac{1}{K_{n+1}} \max_{\tilde{\gamma} \in \tilde{\Gamma}_n((0,0), (\sigma(K_{n+1}-1), [K_{n+1}y]))} \left\{ \mathbb{E} (U_B^\alpha(\sigma, \tilde{\gamma}) + V_B^\alpha(\sigma, \tilde{\gamma})) \right\} \\ & \quad + m([yK_{n+1}]K_{n+1}) \Delta_n(y), \end{aligned}$$

where

$$\tilde{\Delta}_n(y) := \frac{1}{K_{n+1}} + 8 \frac{\sqrt{\sigma + 2y}}{\sqrt{K_n}} \left(\sqrt{A} \sqrt{\pi} + \sqrt{2\pi} + \sqrt{A} \sqrt{1 + \log(1 + K_n y)} \right).$$

A simple computation shows that

$$m([yK_{n+1}]K_{n+1}) \tilde{\Delta}_n(y) \leq \delta_n \sqrt{\sigma/2 + y} [1 + \log(1 + y)]^{3/2},$$

with δ_n given by (4.10). Using the notation of (5.9), we get

$$\mathcal{F}_n(y) \leq \delta_n \sqrt{\frac{\sigma}{2} + y} [1 + \log(1 + y)]^{3/2}.$$

This completes the proof of Proposition 5.1. \square

8. COMPLETION OF PROOF OF THEOREM 2.2

In this section, we complete the remaining parts in the proof of Theorem 2.2. In Subsection 8.1, we deduce from Proposition 4.1 a similar statement for unrestricted passage times (that is, when the paths are not restricted to the box defined by the endpoints). Finally, in Subsection 8.2, we conclude the proof of Theorem 2.2 using the fact that most boxes are good.

8.1. Bounds on unrestricted passage times. To obtain Theorem 2.2 from Proposition 4.1, we first deduce from Proposition 4.1 the following result for unrestricted passage times, i.e. passage times obtained by maximizing over paths not bound to stay in the interval between the two endpoints (see figure 4).

Given the sequence $(\rho_n)_{n \geq 1}$ of Proposition 4.4, we set

$$(8.1) \quad \rho_c := \limsup_{n \rightarrow \infty} \rho_n \in [0, 1/2).$$

Corollary 8.1. *For $\sigma \in \{-1, 1\}$ and $y \geq \sigma^-$, we consider the unrestricted passage-time*

$$(8.2) \quad \tau_n^\alpha(\sigma, y) := \mathbb{E} \left[\frac{1}{K_n} T^\alpha(\sigma K_n, [K_n y]) \right].$$

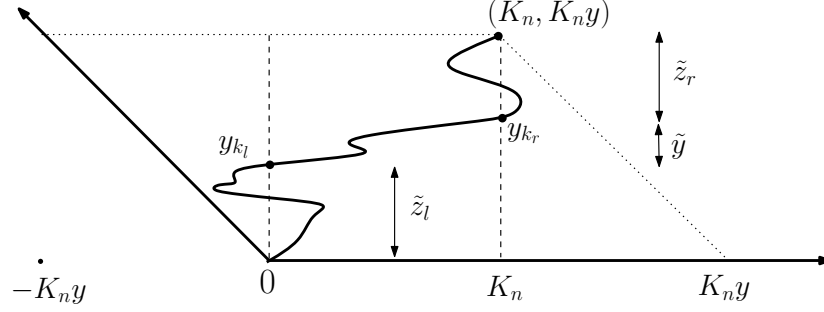


FIGURE 4. The optimal path is not restricted to the box $[0, K_n] \times [0, K_n y]$ but can wander around the whole parallelogram marked by the dotted line. The optimal path is split into 3 parts $(0, y_{k_l})$, (y_{k_l}, y_{k_r}) and $(y_{k_r}, (K_n, [K_n y]))$.

Then there are functions $e_n(\sigma, y)$ such that, for all $n \in \mathbb{N}^*$ and environments α for which $[0, \sigma(K_n - 1)]$ is a good block,

$$(8.3) \quad \tau_n^\alpha(\sigma, y) \leq \tau^{\rho_c, r/4}(\sigma, y) + e_n(\sigma, y)$$

and for each $\sigma \in \{-1, 1\}$, $e_n(\sigma, \cdot)$ does not depend on α and converges locally uniformly to 0 on $[\sigma^-, +\infty)$ as $n \rightarrow +\infty$.

Proof of Corollary 8.1. The unrestricted passage time $T^\alpha(\sigma K_n, [K_n y])$ may use paths that do not stay in $B := [0, \sigma K_n]$. To control the contribution outside B , we use a decomposition of the path in the same spirit as Section 5. The problem here is simpler because there is no more renormalization, and there are only three regions to consider for the path according to its x -coordinate (recall the simplifying notational convention $[a, b] = [b, a]$), namely the interval $[0, \sigma(K_n - 1)]$ and the two intervals on either side of it, which are also bounded by the fact that the only possible increments are $(1, 0)$ and $(-1, 1)$. If $\sigma = 1$, these intervals are $[-[K_n y], -1]$ and $[K_n, K_n + [K_n y]]$. If $\sigma = -1$ then $y \geq 1$ and these intervals are $[-[K_n y], -K_n]$ and $[1, -K_n + [K_n y]]$. We thus define a simpler path skeleton $(\tilde{z}_1, \tilde{y}, \tilde{z}_2) = \tilde{\gamma}$ as described below.

Let $\gamma = (x_k, y_k)_{k=0, \dots, m-1}$ be a path connecting $(0, 0) = (x_0, y_0)$ to $(x_{m-1}, y_{m-1}) = (\sigma K_n, [K_n y] = y')$. We set

$$\begin{aligned} k_1 &:= 1 + \max\{k = 0, \dots, m-1 : \sigma x_k < 0\}, \\ k_2 &:= \min\{k = k_1, \dots, m-1 : x_k = \sigma K_n\}, \end{aligned}$$

with the convention that the max is -1 if the corresponding set is empty. Since allowed path increments are $(1, 0)$ and $(-1, 1)$, we have $x_{k_1} = 0$. We then define (see figure 4)

$$\tilde{z}_1 := y_{k_1}, \quad \tilde{y} := y_{k_2} - y_{k_1}, \quad \tilde{z}_2 := y' - y_{k_2}.$$

Let $\tilde{\Gamma}_n$ denote the set of these new ‘‘skeletons’’, that is the set of triples $\tilde{\gamma} = (\tilde{z}_1, \tilde{y}, \tilde{z}_2)$ such that

$$(8.4) \quad \tilde{z}_1 + \tilde{y} + \tilde{z}_2 = [K_n y] =: y', \quad (\tilde{z}_1, \tilde{y}, \tilde{z}_2) \in \mathbb{N} \times (\mathbb{N} \cap [\sigma^- K_n, +\infty)) \times \mathbb{N}.$$

The path between k_1 and k_2 corresponds to the restricted part which has already been studied in the previous sections. As in (5.6), we write

$$(8.5) \quad T^\alpha((0, 0), (K_n, [K_n y])) = \max_{(\tilde{z}_1, \tilde{y}, \tilde{z}_2) \in \tilde{\Gamma}_n} [V_1^\alpha(\sigma, \tilde{\gamma}) + U_B^\alpha(\sigma, \tilde{\gamma}) + V_2^\alpha(\sigma, \tilde{\gamma})],$$

where $B := [0, \sigma K_n - 1]$, and

$$\begin{aligned} V_1^\alpha(\sigma, \tilde{\gamma}) &:= T^\alpha((0, 0), (0, \tilde{z}_1)), \\ U_B^\alpha(\sigma, \tilde{\gamma}) &:= T_B^\alpha\left(\left(\sigma, \tilde{z}_1 + \frac{1 - \sigma}{2}\right), \left(\sigma(K_n - 1), \tilde{z}_1 + \tilde{y} - \frac{1 - \sigma}{2}\right)\right), \\ V_2^\alpha(\sigma, \tilde{\gamma}) &:= T^\alpha((\sigma K_n, \tilde{z}_1 + \tilde{y}), (\sigma K_n, \tilde{z}_1 + \tilde{y} + \tilde{z}_2)). \end{aligned}$$

Note that the second passage time in (8.5) is restricted to B by definition of the skeleton. We then proceed as in Section 5 by studying the mean optimization problem (that is the maximum of the expectations of the three terms in (8.5)) and estimating the error due to this approximation.

By definition of restricted passage times τ_B^α and superadditivity bounds for vertical passage times,

$$\begin{aligned} \mathbb{E}\left(V_1^\alpha(\sigma, \tilde{\gamma})\right) &\leq \frac{4}{r} \tilde{z}_1, & \mathbb{E}\left(V_2^\alpha(\sigma, \tilde{\gamma})\right) &\leq \frac{4}{r} \tilde{z}_2, \\ \mathbb{E}\left(U_B^\alpha(\sigma, \tilde{\gamma})\right) &\leq K_n \tau_B^\alpha(\sigma, K_n^{-1} \tilde{y}) \leq K_n \tau^{\rho_n, J_n}(\sigma, K_n^{-1} \tilde{y}), \end{aligned}$$

where the last inequality follows from Propositions 4.2 and 4.3. From the definition (3.12) of $\tau^{\rho, J}$, we get for $i = 1, 2$

$$\frac{4}{r} \tilde{z}_i = \tau^{\rho_c, r/4}(0, \tilde{z}_i) \quad \text{and} \quad \frac{1}{J_n} \tilde{z}_i = \tau^{\rho_n, J_n}(0, \tilde{z}_i).$$

Thus we deduce that

$$(8.6) \quad \begin{aligned} \frac{1}{K_n} \left(\frac{4}{r} \tilde{z}_1 + \frac{4}{r} \tilde{z}_2 + K_n \tau^{\rho_n, J_n}(\sigma, K_n^{-1} \tilde{y}) \right) &\leq \tau^{\rho_c, r/4}(\sigma, y) \\ &+ \tau^{\rho_n, J_n}(\sigma, y) - \tau^{\rho_c, r/4}(\sigma, y) + \left| \frac{4}{r} - \frac{1}{J_n} \right| y, \end{aligned}$$

where we used that $\tilde{z}_1 + \tilde{z}_2 \leq K_n y$. Let us denote by $e_n^{(1)}(\sigma, y)$ the sum of the second and third terms on the r.h.s. of (8.6). We have thus shown that

$$\frac{1}{K_n} \max_{(\tilde{z}_1, \tilde{y}, \tilde{z}_2) \in \tilde{\Gamma}_n} \left\{ \mathbb{E}\left(V_1^\alpha(\sigma, \tilde{\gamma})\right) + \mathbb{E}\left(U_B^\alpha(\sigma, \tilde{\gamma})\right) + \mathbb{E}\left(V_2^\alpha(\sigma, \tilde{\gamma})\right) \right\} \leq \tau^{\rho_c, r/4}(\sigma, y) + e_n^{(1)}(\sigma, y),$$

where by the definition of $\tau^{\rho, J}$ (3.12), of ρ_c (8.1) and the convergence of J_n to $r/4$, we deduce the (locally uniform) convergence

$$\lim_{n \rightarrow \infty} e_n^{(1)}(\sigma, \cdot) = 0.$$

We conclude by controlling the error thanks to Proposition 8.1, in the same spirit as Proposition 5.1. \square

Proposition 8.1. *For $\sigma \in \{-1, 1\}$, there exist functions $e_n^{(2)}(\sigma, y)$ such that*

$$(8.7) \quad \mathbb{E} \left(\frac{1}{K_n} \max_{(\tilde{z}_1, \tilde{y}, \tilde{z}_2) \in \tilde{\Gamma}_n} \left\{ V_1^\alpha(\sigma, \tilde{\gamma}) + U_B^\alpha(\sigma, \tilde{\gamma}) + V_2^\alpha(\sigma, \tilde{\gamma}) \right\} \right) \\ - \max_{(\tilde{z}_1, \tilde{y}, \tilde{z}_2) \in \tilde{\Gamma}_n} \left\{ \frac{1}{K_n} \left[\mathbb{E}(V_1^\alpha(\sigma, \tilde{\gamma})) + \mathbb{E}(U_B^\alpha(\sigma, \tilde{\gamma})) + \mathbb{E}(V_2^\alpha(\sigma, \tilde{\gamma})) \right] \right\} \leq e_n^{(2)}(\sigma, y),$$

and $e_n^{(2)}(\sigma, \cdot)$ converges locally uniformly to 0 on $[\sigma^-, +\infty)$ as $n \rightarrow +\infty$.

Proof. We proceed in three steps as in the proof of Proposition 5.1.

Step 1. Cutoff. We again use a truncation procedure for the service times $Y_{i,j}$ as in step one of Subsection 7.2. Here, we define $\mathbb{P}'_{n,M,y'}$ as the distribution of the family $(Y_{i,j} : i \in [-y', \sigma K_n + y'], j \in [0, y'])$ conditioned on their maximum $M_B(y)$ being rM , and $\mathbb{P}_{n,M,y'}$ as the distribution of the family $(X_{i,j} : i \in [-\sigma y', \sigma K_n + y'], j \in [0, y'])$, where $X_{i,j}$ are i.i.d. random variables, and the law of $X_{i,j}$ is the law of $Y_{i,j}$ conditioned on $Y_{i,j} \leq rM$. For notational simplicity we will only write \mathbb{P}_M and \mathbb{P}'_M for these distributions.

Step 2. Fluctuations under cutoff. Applying Lemma 7.1 under $\mathbb{P}_{n,M,y'}$, we have

$$(8.8) \quad \begin{aligned} V_1^\alpha(\sigma, \tilde{\gamma}) &\leq \mathbb{E}_M(V_1^\alpha(\sigma, \tilde{\gamma})) + 8M\sqrt{2\tilde{z}_1}Z_1, \\ V_2^\alpha(\sigma, \tilde{\gamma}) &\leq \mathbb{E}_M(V_2^\alpha(\sigma, \tilde{\gamma})) + 8M\sqrt{2\tilde{z}_2}Z_2, \\ U_B^\alpha(\sigma, \tilde{\gamma}) &\leq \mathbb{E}_M(U_B^\alpha(\sigma, \tilde{\gamma})) + 8M\sqrt{\sigma K_n + 2\tilde{y}}Z_0, \end{aligned}$$

where Z_1, Z_2 and Z_0 are independent random variables such that

$$\mathbb{P}_{n,M,y}(Z_k \geq t) \leq e^{-t^2},$$

for $k \in \{0, 1, 2\}$. We now apply Lemma 7.2 with $\mathcal{A} = \tilde{\Gamma}_n$, $\mathcal{I} = \{1, 2, B\}$, and for $a = \tilde{\gamma} \in \tilde{\Gamma}_n$, $\mathcal{Y}_{a,1} = V_1^\alpha(\sigma, \tilde{\gamma})$, $\mathcal{Y}_{a,2} = V_2^\alpha(\sigma, \tilde{\gamma})$, $\mathcal{Y}_{a,B} = U_B^\alpha(\sigma, \tilde{\gamma})$, $V_{a,1} = 2\tilde{z}_1$, $V_{a,2} = 2\tilde{z}_2$, $V_{a,B} = \sigma K_n + 2\tilde{y}$. Since (see (8.4))

$$(8.9) \quad |\tilde{\Gamma}_n| = \binom{2 + [K_n y] - \sigma^- K_n}{2} \leq K_n^2 (1 + y)^2,$$

we obtain

$$\begin{aligned}
(8.10) \quad & \mathbb{E}_M \left(\frac{1}{K_n} \max_{(\tilde{z}_1, \tilde{y}, \tilde{z}_2) \in \tilde{\Gamma}_n} \left\{ V_1^\alpha(\sigma, \tilde{\gamma}) + U_B^\alpha(\sigma, \tilde{\gamma}) + V_2^\alpha(\sigma, \tilde{\gamma}) \right\} \right) \\
& \leq \max_{(\tilde{z}_1, \tilde{y}, \tilde{z}_2) \in \tilde{\Gamma}_n} \left\{ \frac{1}{K_n} \left[\mathbb{E}_M(V_1^\alpha(\sigma, \tilde{\gamma})) + \mathbb{E}_M(U_B^\alpha(\sigma, \tilde{\gamma})) + \mathbb{E}_M(V_2^\alpha(\sigma, \tilde{\gamma})) \right] \right\} \\
& \quad + 8M \frac{\sqrt{\sigma + 2y}}{\sqrt{K_n}} \left(\sqrt{3\pi} + \sqrt{\pi}\sqrt{A} + \sqrt{A}\sqrt{\log |\tilde{\Gamma}_n|} \right) \\
& \leq \max_{(\tilde{z}_1, \tilde{y}, \tilde{z}_2) \in \tilde{\Gamma}_n} \left\{ \frac{1}{K_n} \left[\mathbb{E}(V_1^\alpha(\sigma, \tilde{\gamma})) + \mathbb{E}(U_B^\alpha(\sigma, \tilde{\gamma})) + \mathbb{E}(V_2^\alpha(\sigma, \tilde{\gamma})) \right] \right\} \\
& \quad + 8M \frac{\sqrt{\sigma + 2y}}{\sqrt{K_n}} \left(\sqrt{3\pi} + \sqrt{\pi}\sqrt{A} + \sqrt{A}\sqrt{\log |\tilde{\Gamma}_n|} \right).
\end{aligned}$$

In the last inequality, we have used the fact that the passage times under \mathbb{P}_M are stochastically dominated by the passage times under \mathbb{P} .

Step 3. Removing the cutoff. As in step three of the proof of Lemma 7.2, a coupling argument shows that the distribution under \mathbb{P}'_M of any passage-time T depending only on the previous set of $Y_{i,j}$ is dominated by the distribution under \mathbb{P}_M of $T + M$. Therefore

$$\begin{aligned}
& \mathbb{E}'_M \left(\frac{1}{K_n} \max_{(\tilde{z}_1, \tilde{y}, \tilde{z}_2) \in \tilde{\Gamma}_n} \left\{ V_1^\alpha(\sigma, \tilde{\gamma}) + U_B^\alpha(\sigma, \tilde{\gamma}) + V_2^\alpha(\sigma, \tilde{\gamma}) \right\} \right) \\
& \leq \max_{(\tilde{z}_1, \tilde{y}, \tilde{z}_2) \in \tilde{\Gamma}_n} \left\{ \frac{1}{K_n} \left(\mathbb{E}(V_1^\alpha(\sigma, \tilde{\gamma})) + \mathbb{E}(U_B^\alpha(\sigma, \tilde{\gamma})) + \mathbb{E}(V_2^\alpha(\sigma, \tilde{\gamma})) \right) \right\} \\
& \quad + \frac{M}{K_n} + 8M \frac{\sqrt{\sigma + 2y}}{\sqrt{K_n}} \left(\sqrt{3\pi} + \sqrt{\pi}\sqrt{A} + \sqrt{A}\sqrt{\log |\tilde{\Gamma}_n|} \right).
\end{aligned}$$

Integrating the above inequality with respect to the distribution of $M_B(y)$ yields (8.7), with

$$e_n^{(2)}(\sigma, y) := m([K_n y](\sigma K_n + 2[K_n y])) E_n(\sigma, y),$$

where $m(\cdot)$ satisfies the bound (7.16), and

$$E_n(\sigma, y) := \frac{1}{K_n} + 8 \frac{\sqrt{\sigma + 2y}}{\sqrt{K_n}} \left(\sqrt{3\pi} + \sqrt{\pi}\sqrt{A} + \sqrt{A}\sqrt{\log |\tilde{\Gamma}_n|} \right),$$

from which one can see that $e_n^{(2)}(\sigma, \cdot)$ converges locally uniformly to 0. \square

8.2. Proof of Theorems 2.2 and 3.2. Theorem 2.2 is a consequence of Theorem 3.2 which we prove now. Given $\sigma \in \{-1, 1\}$, by Theorem 3.1, we have

$$\tau_\varepsilon(\sigma, y) = \lim_{n \rightarrow \infty} \mathcal{E}_\varepsilon \times \mathbb{E} \left(\frac{1}{K_n} T(\sigma K_n, [K_n y]) \right) = \lim_{n \rightarrow \infty} \mathcal{E}_\varepsilon (\tau_n^\alpha(\sigma, y)),$$

where \mathcal{E}_ε stands for the expectation with respect to the disorder α . Note that the above limit does not follow directly from Theorem 3.1, which yields an a.s. limit. However,

the convergence in Theorem 3.1 holds also in L^1 . This follows from a quasi-Gaussian tail estimate for the passage-time $T(\sigma K_n, [K_n y])$, obtained from Lemma 7.1 and cutoff as in step four of Subsection 7.2. Let $\mathbf{G}_n(\sigma)$ be the set of environments α for which $[0, \sigma(K_n - 1)]$ is a good block. The mean passage time $\tau_\varepsilon(\sigma, y)$ can be decomposed as

$$\mathcal{E}_\varepsilon [\tau_n^\alpha(\sigma, y) \mathbf{1}_{\mathbf{G}_n(\sigma)}] + \mathcal{E}_\varepsilon [\tau_n^\alpha(\sigma, y) \mathbf{1}_{\mathbf{A} \setminus \mathbf{G}_n(\sigma)}],$$

where $\mathbf{A} := [0, 1]^\mathbb{Z}$ is the set of environments. By Corollary 8.1, the lim sup of the first term is bounded above by $\tau^{\rho_c, r/4}(1, y)$. On the other hand, the second term is bounded above by

$$\mathcal{E}_\varepsilon [(\tau_n^\alpha(\sigma, y))^2]^{1/2} \mathcal{P}_\varepsilon [\mathbf{A} \setminus \mathbf{G}_n(\sigma)]^{1/2}.$$

The \mathcal{P}_ε -probability vanishes as $n \rightarrow \infty$ by Lemma 4.1, while for the expectation of the squared passage time we can take as upper bound

$$\mathcal{E}_\varepsilon [(\tau_n^\alpha(\sigma, y))^2] \leq \tau_n(\sigma, y)^2,$$

where $\tau_n(\sigma, y)$ is defined as (8.2) for a homogeneous environment $\alpha(x) \equiv 1$ (that is for rate 1 homogeneous TASEP). The limit $\tau_n(\sigma, y) \rightarrow (\sqrt{\sigma + y} + \sqrt{y})^2$ as $n \rightarrow \infty$, which follows from the above remark on L^1 -convergence of rescaled passage times in Theorem 3.1, implies $\tau_n(\sigma, y)^2 \rightarrow (\sqrt{\sigma + y} + \sqrt{y})^4$ as $n \rightarrow \infty$. We finally get

$$(8.11) \quad \tau_\varepsilon(x, y) \leq \tau^{\rho_c, r/4}(x, y),$$

for every $x \in \{-1, 1\}$ and $y \geq x^-$. Since τ_ε and $\tau^{\rho_c, r/4}$ are homogeneous functions, (3.14) follows.

APPENDIX A. PROOF OF PROPOSITION 2.1.

To show that the maximum value of the flux is at least $r/4$, we can use Definition (2.3) and couple the process $(\eta_t^\alpha)_{t \geq 0}$ with generator (2.1) with a homogeneous rate r TASEP denoted by $(\eta_t^r)_{t \geq 0}$. A standard coupling argument (see proof of Lemma 2.1) shows that $J_x^\alpha(t, \eta^\rho) \geq J_x^r(t, \eta^\rho)$, where J_x^r denotes the current in the homogeneous TASEP. It is known (see e.g. [36]) that

$$\lim_{t \rightarrow +\infty} \frac{1}{t} J_x^r(t, \eta^\rho) = r\rho(1 - \rho)$$

and it is maximum for $\rho = 1/2$.

We now prove that $f(\rho) \leq r/4$ for all $\rho \in [0, 1]$. It is enough to consider $\rho \in \mathcal{R}$, where \mathcal{R} is the density set in (2.6), since the flux is interpolated linearly outside this set. Since the random variables $\alpha(x)$ are i.i.d. and the infimum of their support is r , for \mathcal{P} -a.e. environment $\alpha \in \mathbf{A}$, there exist sequences $(x_N)_{N \geq 1}$, $(y_N)_{N \geq 1}$ and $(\varepsilon_N)_{N \geq 1}$ such that $x_N \in \{0, \dots, N-1\}$, $y_N \in \{0, \dots, N-1\}$, $\lim_{N \rightarrow \infty} x_N = +\infty$, $\lim_{N \rightarrow \infty} [y_N - x_N] = +\infty$, $\lim_{N \rightarrow \infty} \varepsilon_N = 0$, and

$$(A.1) \quad r \leq \min_{x=x_N, \dots, y_N} \alpha(x) \leq \max_{x=x_N, \dots, y_N} \alpha(x) \leq r + \varepsilon_N.$$

Set

$$a_N = \frac{2x_N + y_N}{3}, \quad b_N = \frac{x_N + 2y_N}{3},$$

which satisfy $x_N \leq a_N \leq b_N \leq y_N$ and $b_N - a_N \rightarrow +\infty$. By (2.6),

$$(A.2) \quad f(\rho) = \frac{1}{b_N - a_N + 1} \sum_{x=a_N}^{b_N} \int_{\mathbf{X}} j_x^\alpha(\eta) d\nu_\rho^\alpha(\eta) \leq [r + \varepsilon_N] \int_{\mathbf{X}} \tilde{j}(\eta) d\mu_N(\eta),$$

where $\tilde{j}(\eta) = \eta(0)[1 - \eta(1)]$, and

$$(A.3) \quad \mu_N := \frac{1}{b_N - a_N + 1} \sum_{x=a_N}^{b_N} \tau_x \nu_\rho^\alpha.$$

The sequence $(\mu_N)_{N \in \mathbb{N}^*}$ of probability on the compact space \mathbf{X} is tight. Let μ^* be one of its limit points. It follows from (A.3) that μ^* is shift invariant, i.e. $\tau_x \mu^* = \mu^*$ for all $x \in \mathbb{Z}$. We claim and prove below that μ^* is an invariant measure for the homogeneous TASEP, that is the process with generator (2.1) with $\alpha(x) \equiv 1$. By Liggett's characterization result [17] for shift-invariant stationary measures, μ^* is then of the form

$$\mu^* = \int_{[0,1]} \nu_\rho \gamma(d\rho).$$

where γ is a probability measure on $[0, 1]$, and ν_ρ is the product Bernoulli measure on \mathbf{X} with parameter ρ . Thus

$$\int_{\mathbf{X}} \tilde{j}(\eta) d\mu^*(\eta) = \int_{[0,1]} \rho(1 - \rho) d\gamma(\rho) \leq \frac{1}{4}.$$

Letting $N \rightarrow \infty$ in (A.2) implies $f(\rho) \leq r/4$.

We now prove that μ^* is an invariant measure for the homogeneous TASEP. Let $g : \mathbf{X} \rightarrow \mathbb{R}$ be a local function that depends on η only through sites $x \in \mathbb{Z}$ such that $|x| \leq \Delta$, where $\Delta \in \mathbb{N}$. Take N large enough so that $\Delta < (y_N - x_N)/3$. Notice that the generator L^α defined in (2.1) satisfies the commutation relation

$$(A.4) \quad \tau_x L^{\tau_x \alpha} f = L^\alpha(\tau_x f).$$

It follows that

$$\int_{\mathbf{X}} L^{\tau_x \alpha} g d(\tau_x \nu_\rho^\alpha) = \int_{\mathbf{X}} L^\alpha(\tau_x g) d\nu_\rho^\alpha = 0.$$

The last equality follows from invariance of ν_ρ^α . On the other hand, for $x \in [a_N, b_N]$ and $|y| \leq (y_N - x_N)/3$, $\tau_x \alpha(y) \in [r, r + \varepsilon_N]$. Let L denote the generator of the homogeneous TASEP on \mathbb{Z} , that is the one obtained from (2.1) when $\alpha(x) \equiv 1$. Since

$$\left| L^{\tau_x \alpha} g(\eta) - r L g(\eta) \right| \leq 2 \|g\|_\infty \varepsilon_N \sum_{y=-\Delta-1}^{\Delta} |\tau_x \alpha(y) - r|,$$

it follows that

$$\lim_{N \rightarrow \infty} \max_{x=a_N, \dots, b_N} \sup_{\eta \in \mathbf{X}} |L^{\tau_x \alpha} g(\eta) - r L g(\eta)| = 0.$$

Hence

$$\int_{\mathbf{X}} L g(\eta) d\mu^*(\eta) = \lim_{N \rightarrow \infty} \int_{\mathbf{X}} L g(\eta) d\mu_N(\eta) = \lim_{N \rightarrow +\infty} \frac{1}{b_N - a_N} \sum_{x=a_N}^{b_N} \int_{\mathbf{X}} \frac{1}{r} L^\alpha(\tau_x g) d\nu_\rho^\alpha = 0$$

holds for every local function g .

APPENDIX B. PROOF OF LEMMA 3.1

Before deriving Lemma 3.1, we first explain a mapping between the restricted passage-times in a box B and the TASEP restricted to B with reservoirs.

B.1. Last-passage times in a finite domain. Let $B := [x_1, x_2] \cap \mathbb{Z}$. The purpose of this subsection is to give an interpretation of the passage-times (3.15) restricted to B in terms of an open disordered TASEP on $B' := [x_1 + 1, x_2] \cap \mathbb{Z}$ with generator L_B^α , see (2.10) (recall from (2.9) and (3.18) that $(B')^\# = B$). It is convenient to view the dynamics generated by (2.10) as follows. We add an infinite stack of particles (reservoir) at site x_1 , and a site $x_2 + 1$ where the number of particles is not restricted. Particles enter B' from the stack at x_1 , and when they leave, they stay at $x_2 + 1$ forever. We are going to check that $T_B^\alpha((x_1, 0), (i, j))$ has the same distribution as the time when particle j reaches site $i + 1$ in the process generated by L_B^α , if the initial state is given by

$$(B.1) \quad \sigma_0(j) = x_1 \mathbf{1}_{\{j \geq 0\}} + (x_2 + 1) \mathbf{1}_{\{j \leq -1\}}.$$

where $\sigma_0(j)$ denotes the initial position of the particle with label j , and particles are numbered increasingly from right to left. In fact, we may define passage-times associated with more general labeled initial configurations in B' . By this we mean that σ_0 , instead of being defined by (B.1), can be any nonincreasing function σ_0 from \mathbb{Z} to $[x_1, x_2 + 1] \cap \mathbb{Z}$. Let

$$(B.2) \quad \tilde{B} := \{(i, j) \in B \times \mathbb{Z} : i \geq \sigma_0(j)\}$$

$$(B.3) \quad \bar{B} := \{(i, j) \in B \times \mathbb{Z} : i < \sigma_0(j)\}$$

For $(i, j) \in B \times \mathbb{Z}$, let $T_{B, \sigma_0}^\alpha(i, j)$ denote the time at which particle j reaches site $i + 1$. These passage times are determined by the boundary condition

$$(B.4) \quad T_{B, \sigma_0}^\alpha(i, j) = 0 \text{ for } (i, j) \in \bar{B}$$

together with the following recursions:

$$(B.5) \quad T_{B, \sigma_0}^\alpha(i, j) = \frac{Y_{i,j}}{\alpha(i)} + \max[T_{B, \sigma_0}^\alpha(i - 1, j), T_{B, \sigma_0}^\alpha(i + 1, j - 1)]$$

for $(i, j) \in \tilde{B}$ such that $x_1 < i < x_2$,

$$(B.6) \quad T_{B, \sigma_0}^\alpha(i, j) = \frac{Y_{i,j}}{\alpha(i)} + T_{B, \sigma_0}^\alpha(i-1, j),$$

for $(i, j) \in \tilde{B}$ such that $i = x_2$,

$$(B.7) \quad T_{B, \sigma_0}^\alpha(i, j) = \frac{Y_{i,j}}{\alpha(i)} + T_{B, \sigma_0}^\alpha(i+1, j-1)$$

for $(i, j) \in \tilde{B}$ such that $i = x_1$. In the special case (B.1), we have

$$(B.8) \quad \tilde{B} = [x_1, x_2] \times \mathbb{N}. \quad \bar{B} = [x_1, x_2] \times (\mathbb{Z} \setminus \mathbb{N})$$

By plugging (B.8) into (B.5), one recovers

$$T_{B, \sigma_0}^\alpha(i, j) = T_B^\alpha((x_1, 0), (i, j))$$

where the r.h.s. was defined in (3.15). For notational simplicity, in the sequel of this subsection, we omit dependence on α , B and σ_0 , and write $T(i, j)$ instead of $T_{B, \sigma_0}^\alpha(i, j)$. The position of particle j at time t , denoted by $\sigma_t(j) \in [x_1, x_2 + 1]$, is given by

$$(B.9) \quad \sigma_t(j) = \begin{cases} x_1 & \text{if } T(x_1, j) > t \\ x_2 + 1 & \text{if } T(x_2, j) \leq t \\ i \in [x_1 + 1, x_2] \cap \mathbb{Z} & \text{if } T(i-1, j) \leq t < T(i, j) \end{cases}$$

$$(B.10) \quad T(i, j) = \sup\{t \geq 0 : \sigma_t(j) \leq i\}.$$

The particle process $(\sigma_t)_{t \geq 0}$ is equivalent to the growing cluster process $(\mathcal{C}_t)_{t \geq 0}$ defined by

$$\mathcal{C}_t := \{(i, j) \in [x_1, x_2] \times \mathbb{Z} : T(i, j) \leq t\} = \{(i, j) \in B \times \mathbb{Z} : i < \sigma_t(j)\}$$

whose initial state is $\mathcal{C}_0 = \bar{B}$. One can proceed as in [39] to show that both processes are Markovian and that the undistinguishable particle process $(\eta_t)_{t \geq 0}$ defined from $(\sigma_t)_{t \geq 0}$ via

$$(B.11) \quad \eta_t(x) := \sum_{j \in \mathbb{Z}} \mathbf{1}_{\{\sigma_t(j)=x\}}$$

is Markov with generator L_B^α .

B.2. Proof of Lemma 3.1.

Step 1. We prove that definition (3.16) does not depend on x_0 . Let $x_0, x'_0 \in B$ with $x_0 < x'_0$. Then (3.15) implies

$$(B.12) \quad T_B^\alpha((x_0, 0), (x_0, m + x'_0 - x_0)) \geq T_B^\alpha((x'_0, 0), (x'_0, m)) \geq T_B^\alpha((x_0, x'_0 - x_0), (x_0, m)).$$

Since the sequence $(Y_{i,j} : i \in \mathbb{Z}, j \geq 0)$ is stationary with respect to shifts of j , the expectation of the last quantity is equal to that of $T_B^\alpha((x_0, 0), (x_0, m - x'_0 + x_0))$. Thus taking expectations, dividing by m and letting $m \rightarrow \infty$ yields the result.

Step 2. *Proof of (3.17).* Given this statement, let us denote by $J_x^{\alpha,B}(t, \eta_0)$ the current up to time t across site $x \in B'$, in the open system on B' , when starting from η_0 . Assume η_0 is the occupation configuration associated with σ_0 via (B.1)–(B.11). Then it is clear that

$$J_x^{\alpha,B}(t, \eta_0) = \min\{j \in \mathbb{Z} : T_B^\alpha(x, j) > t\},$$

which implies the \mathbb{P} a.s. limit

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{1}{t} J_x^{\alpha,B}(t, \eta_0) &= \frac{1}{T_{\infty,B}} = \lim_{t \rightarrow \infty} \mathbb{E} \left(\frac{1}{t} J_x^{\alpha,B}(t, \eta_0) \right) = \lim_{t \rightarrow \infty} \mathbb{E} \left(\frac{1}{t} \int_0^t j_x^{\alpha,B}(\eta_s) ds \right) \\ &= \lim_{t \rightarrow \infty} \int j_x^{\alpha,B}(\eta) d\nu_t(\eta) = \int j_x^{\alpha,B}(\eta) d\nu_B^\alpha(\eta), \end{aligned}$$

where

$$j_x^{\alpha,B}(\eta) = \begin{cases} \alpha(x)\eta(x)[1 - \eta(x+1)] & \text{if } x_1 + 1 < x \leq x_2 - 1, \\ \alpha(x_2)\eta(x_2) & \text{if } x = x_2, \\ 1 - \eta(x_1 + 1) & \text{if } x = x_1 + 1, \end{cases}$$

and

$$\nu_t := \frac{1}{t} \int_0^t \delta_{\eta_0} e^{sL_B^\alpha} ds.$$

The second equality follows from the fact that the family of random variables $(\frac{1}{t} J_x^{\alpha,B}(t, \eta_0))_{t \geq 0}$ is uniformly integrable, because $(J_x^{\alpha,B}(t, \eta_0))_{t \geq 0}$ is dominated in distribution by a Poisson random variable with parameter t . The last equality follows from the fact that ν_t converges to the invariant measure ν_B^α as $t \rightarrow \infty$. \square

APPENDIX C. PROOF OF LEMMA 2.1

We consider the coupled processes $(\eta_t^\alpha)_{t \geq 0}$ and $(\eta_t^{\alpha'})_{t \geq 0}$ corresponding to the environments α, α' . To define the coupling, use the construction of TASEP from a space-time Poisson point measure where an atom at $(t, x) \in (0, +\infty) \times (\mathbb{Z} \cap [0, N])$ generates an attempt for a potential particle at site x and time $t-$ to jump to site $x+1$ at time t . Besides, at each Poissonian event, an additional $\mathcal{U}(0, 1)$ random variable U is generated, these r.v.'s being mutually independent and independent of the Poisson measure. The jump is realized for the process in environment α (resp. α') iff. at time $t-$, the following conditions are fulfilled: site x does contain a particle, site $x+1$ is vacant at time $t-$, and $U \leq \alpha$ (resp. $U \leq \alpha'$). The same realization of the Poisson measure and set of uniform variables is used for both processes. We consider that initially site 0 has an infinite stack of particles in both processes, that these particles are labeled from 0 upward, and that the particle that jumps from 0 to 1 is the one with lowest label among particles at this site. On the other hand, we take the initial configuration in $[1, N]$ to be empty for both systems. Thus the initial configuration is given by (B.1) with $x_1 = 0$ and $x_2 = N$. Let $\sigma_i(t)$ denote the position at time t of particle i in η_t^α , and write similarly $\sigma'_i(i)$ for the other process. Then it holds that

$$(C.1) \quad \forall i \in \mathbb{N}, \quad \sigma_t(i) \leq \sigma'_t(i)$$

(indeed it is easy to see that property (C.1) is preserved by each Poisson event). Let $J_0(t)$ and $J'_0(t)$ denote the current across 0 in each system, that is the number of jumps from 0 to 1. In either process, this current is the highest label of a particle having left site 0 up to time t . It follows from (C.1) that $J_0(t) \leq J'_0(t)$ for every $t \geq 0$. By the ergodic theorem, one has the a.s. limit

$$j_{\infty,[0,N]}(\alpha) = \lim_{t \rightarrow +\infty} \frac{1}{t} J_0(t)$$

and similarly for $j_{\infty,[0,N]}(\alpha')$. The result follows.

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