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# On uniqueness of measure-valued solutions to Liouville's equation of Hamiltonian PDEs

Zied Ammari and Quentin Liard\*

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## Abstract

In this paper, the Cauchy problem of classical Hamiltonian PDEs is recast into a Liouville's equation with measure-valued solutions. Then a uniqueness property for the latter equation is proved under some natural assumptions. Our result extends the method of characteristics to Hamiltonian systems with infinite degrees of freedom and it applies to a large variety of Hamiltonian PDEs (Hartree, Klein-Gordon, Schrödinger, Wave, Yukawa . . . ). The main arguments in the proof are a projective point of view and a probabilistic representation of measure-valued solutions to continuity equations in finite dimension.

*Keywords:* Continuity equation, method of characteristics, measure-valued solutions, nonlinear PDEs.

2010 Mathematics subject classification: 35Q82, 35A02, 35Q55, 35Q61, 37K05, 28A33

## 1 Introduction

Liouville's equation is a fundamental equation of statistical mechanics which describes the time evolution of phase-space distribution functions. Consider for instance a Hamiltonian system  $H(p, q) = H(p_1, \dots, p_n, q_1, \dots, q_n)$  of finite degrees of freedom where  $(q_1, \dots, q_n, p_1, \dots, p_n)$  are the position-momentum canonical coordinates. Then, the time evolution of a probability density function  $\varrho(p, q, t)$  describing the system at time  $t$  is governed by the Liouville's equation,

$$(1) \quad \frac{\partial \varrho}{\partial t} + \{\varrho, H\} = 0,$$

with the Poisson bracket defined as follows,

$$\{\varrho, H\} = \sum_{i=1}^n \left[ \frac{\partial H}{\partial p_i} \frac{\partial \varrho}{\partial q^i} - \frac{\partial H}{\partial q^i} \frac{\partial \varrho}{\partial p_i} \right].$$

By formally differentiating  $\varrho(p_t, q_t, t)$  with respect to time, when  $(p_t, q_t)$  are solutions of the Hamiltonian equations, we recover the Liouville's theorem as stated by Gibbs "The distribution function is constant along any trajectory in phase space", i.e.,

$$\frac{d}{dt} \varrho(p_t, q_t, t) = 0.$$

The method of characteristics says indeed that if the Hamiltonian is sufficiently smooth and generates a unique Hamiltonian flow  $\Phi_t$  on the phase-space, then the density function  $\varrho(p, q, t)$  is uniquely determined by its initial value  $\varrho(p, q, 0)$  and it is given as the backward propagation along the characteristics, i.e.,

$$\varrho(p, q, t) = \varrho(\Phi_t^{-1}(p, q), 0).$$

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It is known that Liouville's theorem holds true in a broader context than those of Hamiltonian systems. Consider a differential equation,

$$(2) \quad \frac{d}{dt}X = F(X), \quad X(t=0) = X_0,$$

with  $X = (X_1, \dots, X_n) \in \mathbb{R}^n$  and  $F = (F_1, \dots, F_n) : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is a given smooth vector field such that a unique flow map  $\Phi_t : \mathbb{R}^n \mapsto \mathbb{R}^n$  exists and solves the ODE (2). If the system (2) is at an initial statistical state described by a probability density function  $\varrho(X, 0)$  at  $t = 0$ , then under the flow map  $\Phi_t$ , the evolution of this state is described by a density  $\varrho(X, t)$ , which is the pull-back of the initial one,

$$(3) \quad \varrho(X, t) = \varrho(\Phi_t^{-1}(X), 0).$$

If the vector field  $F$  satisfies the Liouville's property, which is the following divergence-free condition,

$$\operatorname{div}(F) = \sum_{j=1}^n \frac{\partial F_j}{\partial X_j} = 0,$$

then the flow map  $\Phi_t$  is volume preserving (i.e. Lebesgue measure preserving) on the phase space and for all times the density  $\varrho(X, t)$  verifies the Liouville's equation,

$$(4) \quad \frac{\partial \varrho}{\partial t} + F \cdot \nabla_X \varrho = 0.$$

Once again, when the vector field  $F$  is sufficiently smooth the theory of characteristics says that (3) is the unique solution of the Liouville's equation (4) with the initial value  $\varrho(X, 0)$ . This enlightens the relationship between individual solutions of the ODE (2) and statistical (probability measure) solutions of the Liouville's equation (4). Hence, one can easily believe that those results reflect a fundamental relation that may extend to non-smooth vector fields or to dynamical systems with infinite degrees of freedom. Actually, the non-smooth framework has been extensively studied and uniqueness of probability measure solutions of Liouville's equation is established via a general superposition principle, see e.g. [2, 5, 16, 24, 25, 26, 40, 42] and also [14, 23]. In contrast, the extension to dynamical systems with infinite degrees of freedom is less investigated. There are indeed fewer results [3, 34, 48] and as far as we understand those attempts do not apply to classical PDEs. However, in [12, Appendix C] the authors established a general uniqueness result for measure-valued solutions to Liouville's equation of Hamiltonian PDEs and used it to derive the mean-field limit of Bose gases. Our aim in this article is to improve the result in [12, Appendix C], to give a detailed and accessible presentation, and to provide some applications to nonlinear classical PDEs like Hartree, Klein-Gordon, Schrödinger, Wave, Yukawa equations.

Beyond the fact that Liouville's equation is the natural ground for a statistical theory of Hamiltonian PDEs that will be fruitful to develop in a general and systematic way (see e.g. [17, 18, 19, 35, 38]); there is another concrete reason to address the previous uniqueness property. In fact, when we study the relationship between quantum field theories and classical PDEs we encounter such uniqueness problem, see [7, 12, 13]. Roughly speaking, the quantum counterpart of Liouville's equation is the von Neumann equation describing the time evolution of quantum states of (linear) Hamiltonian systems. If we consider the classical limit,  $\hbar \rightarrow 0$  where  $\hbar$  is an effective "Planck constant" which depends on the scaling of the system at hand, then quantum states transform in the limit  $\hbar \rightarrow 0$  into probability measures satisfying a Liouville equation related to a nonlinear Hamiltonian PDE, see [9, 10, 11, 12]. Therefore, the uniqueness property for probability measure solutions of Liouville's equation is a crucial step towards a rigorous justification of the classical limit or the so-called Bohr's correspondence principle for quantum field theories [7, 8, 12, 36, 37].

It is not so clear how to generalize the characteristics method for Hamiltonian systems with infinite degrees of freedom [48]. One of the difficulties for instance is the lack of translation-invariant measures on infinite dimensional normed spaces. Nevertheless, there is an interesting

approach [5, Chapter 8] related to optimal transport theory that improves the standard characteristics method by using a regularization argument and the differential structure of spaces of probability measures. In particular, this approach avoids the use of a reference measure and it is suitable for generalization to systems with infinite degrees of freedom. This was exploited in [12, Appendix C] to prove a uniqueness property for Liouville's equation considered in a weak sense,

$$(5) \quad \partial_t \mu_t + \nabla^T(F \cdot \mu_t) = 0.$$

Here  $t \rightarrow \mu_t$  are probability measure-valued solutions and  $F$  is a non-autonomous vector field, related to a Hamiltonian PDE, and defined on a rigged Hilbert space  $\mathcal{X}_1 \subset \mathcal{X}_0 \subset \mathcal{X}'_1$  (an example is given by Sobolev spaces  $H^1(\mathbb{R}^d) \subset L^2(\mathbb{R}^d) \subset H^{-1}(\mathbb{R}^d)$ ). The precise meaning of the equation (5) will be explained in the next section. The aforementioned result in [12, Appendix C] uses essentially the existence of a continuous Hamiltonian flow on the space  $\mathcal{X}_1$  with the following assumption on the vector field  $F$ ,

$$(6) \quad \forall T > 0, \exists C > 0, \quad \int_{-T}^T \left[ \int_{\mathcal{X}_1} \|F(t, z)\|_{\mathcal{X}'_1}^2 d\mu_t(z) \right]^{\frac{1}{2}} dt \leq C.$$

In the present article, we simplify the proof in [12, Appendix C] by avoiding the use of Wasserstein distances. In fact, we exclusively rely on the weak narrow topology, which is more flexible. More importantly, we relax the above scalar velocity estimate (6) so that the required assumption is now:

$$(7) \quad \forall T > 0, \exists C > 0, \quad \int_{-T}^T \int_{\mathcal{X}_1} \|F(t, z)\|_{\mathcal{X}'_1} d\mu_t(z) dt \leq C.$$

This means that the vector field  $F$  maps  $\mathcal{X}_1$  into  $\mathcal{X}'_1$  while before  $F : \mathcal{X}_1 \rightarrow \mathcal{X}_1$ . Moreover, we have replaced the  $L^2$  norm with a  $L^1$  norm and respectively the  $\mathcal{X}_1$  norm by  $\mathcal{X}'_1$ . In particular, the assumption (7) allows to consider more singular nonlinearities. To enlighten the type of results we obtain here, we consider the following example. Let  $\mathcal{X}_0 = L^2(\mathbb{R})$ ,  $\mathcal{X}_1 = H^1(\mathbb{R})$  and consider the one dimensional nonlinear Schrödinger (NLS) equation,

$$(8) \quad \begin{cases} i\partial_t z_t = F(z_t), \\ z|_{t=0} = z_0, \end{cases}$$

with  $F : H^1(\mathbb{R}) \rightarrow H^{-1}(\mathbb{R})$ ,  $F(z) = -\Delta z + |z|z$ , an autonomous vector field defined on the energy space  $H^1(\mathbb{R})$ . By working in the equivalent interaction representation, we obtain a non-autonomous vector field  $F(t, z) = e^{-it\Delta} (|e^{it\Delta} z|^2 e^{it\Delta} z)$  and a differential equation similar to (8). It is well-known that the initial value problem (8) is globally well-posed on  $H^1(\mathbb{R})$ . Moreover, Sobolev-Gagliardo-Nirenberg inequality gives the existence of a constant  $C > 0$  such that for any  $z_1, z_2 \in H^1(\mathbb{R})$  and for any  $t \in \mathbb{R}$ ,

$$\|F(t, z_1) - F(t, z_2)\|_{L^2(\mathbb{R})} \leq C(\|z_1\|_{H^1(\mathbb{R})}^2 + \|z_2\|_{H^1(\mathbb{R})}^2) \|z_1 - z_2\|_{L^2(\mathbb{R})}.$$

Consider now any measure-valued solution  $(\mu_t)_{t \in \mathbb{R}}$  of the Liouville equation (5) with  $F$  given by (8) and suppose that the following a priori estimate,

$$(9) \quad \int_{H^1(\mathbb{R})} \|z\|_{H^1(\mathbb{R})}^2 (1 + \|z\|_{L^2(\mathbb{R})}) d\mu_t \leq C,$$

holds true for some time-independent constant  $C > 0$ . Then the assumption (7) is satisfied and our main Theorem 2.4 says that  $\mu_t$  is the push-forward (or the image measure) of  $\mu_0$  by the NLS flow map, i.e.  $\mu_t = \Phi(t, 0)_\# \mu_0$  where  $\Phi(t, 0)$  is the global flow of (8) (see Section 2 and 3 for more details). Remark that the requirement (9) says essentially that the energy mean with respect to  $\mu_t$  is finite. Because of energy and mass conservation, in this case, the estimate (9)

holds true for all times if we assume it at time  $t = 0$ . Notice also that in several other examples, some are provided in Section 3, the assumption (6) can not be verified while (7) is satisfied. In particular, the improvement provided in this article allows to show general and stronger results in the mean-field theory of quantum many-body dynamics, see [36].

Our proof of the uniqueness property relies on a probabilistic representation for measure-valued solutions to continuity equations in finite dimension due to S. Maniglia [40] who extended a previous result of L. Ambrosio, N. Gigli and G. Savaré [5, Chapter 8]. To handle the infinite dimensional case, we utilize a projective argument employed in [12] and adapted from [5, Chapter 8]. We believe that the methods used in this paper are not widely known although they seem quite fundamental. For this reason, we attempt to give a detailed and self-contained exposition addressed for a wider audience and we hope that this will pave the way for further development towards a general and consistent statistical theory of Hamiltonian PDEs.

*Outline:* Our main results, Theorem 2.3 and 2.4, are stated in section 2. Then several examples of nonlinear PDEs are discussed in Section 3. The proof is detailed in Section 4. For reader's convenience a short appendix collecting useful notions in measure theory is provided (tightness, equi-integrability, Dunford-Pettis theorem, disintegration).

## 2 Main Results

Consider a rigged Hilbert space  $\mathcal{Z}_1 \subset \mathcal{Z}_0 \subset \mathcal{Z}'_1$  such that  $(\mathcal{Z}_1, \mathcal{Z}_0)$  is a pair of complex *separable* Hilbert spaces,  $\mathcal{Z}'_1$  is densely continuously embedded in  $\mathcal{Z}_0$  and  $\mathcal{Z}'_1$  is the dual of  $\mathcal{Z}_1$  with respect to the duality bracket extending the inner product  $\langle \cdot, \cdot \rangle_{\mathcal{Z}_0}$ . A significant example is provided by Sobolev spaces  $H^s(\mathbb{R}^d) \subset L^2(\mathbb{R}^d) \subset H^{-s}(\mathbb{R}^d)$  with  $s > 0$ .

*The initial value problem:* Let  $v : \mathbb{R} \times \mathcal{Z}_1 \rightarrow \mathcal{Z}'_1$  be a non-autonomous *continuous* vector field, i.e.  $v \in C(\mathbb{R} \times \mathcal{Z}_1, \mathcal{Z}'_1)$ , such that  $v$  is bounded on bounded sets of  $\mathcal{Z}_1$ . We shall consider the following initial value (or Cauchy) problem on an open interval  $I \subset \mathbb{R}$ :

$$(10) \quad \begin{cases} \dot{\gamma}(t) = v(t, \gamma(t)), \\ \gamma(s) = x \in \mathcal{Z}_1, \quad s \in I, \end{cases}$$

We are interested in the notion of weak and strong  $\mathcal{Z}_1$ -valued solutions.

**Definition 2.1.** (i) A weak solution of the above initial value problem on  $I$  is a function  $I \ni t \rightarrow \gamma(t)$  belonging to the space  $L^\infty(I, \mathcal{Z}_1) \cap W^{1,\infty}(I, \mathcal{Z}'_1)$  satisfying (10) for a.e.  $t \in I$  and for some  $s \in I$ .

(ii) A strong solution of the above initial value problem on  $I$  is a function  $I \ni t \rightarrow \gamma(t)$  belonging to the space  $\gamma \in C(I, \mathcal{Z}_1) \cap C^1(I, \mathcal{Z}'_1)$  satisfying (10) for all  $t \in I$  and for some  $s \in I$ .

Here  $W^{1,p}(I, \mathcal{Z}'_1)$ , for  $1 \leq p \leq \infty$ , denote the Sobolev spaces of classes of functions in  $L^p(I, \mathcal{Z}'_1)$  with distributional first derivatives in  $L^p(I, \mathcal{Z}'_1)$ . It is well known that the elements  $\gamma$  of  $W^{1,p}(I, \mathcal{Z}'_1)$  are absolutely continuous curves in  $\mathcal{Z}'_1$  with almost everywhere defined derivatives in  $\mathcal{Z}'_1$  satisfying  $\dot{\gamma} \in L^p(I, \mathcal{Z}'_1)$ . Moreover, if  $I$  is a bounded open interval, the following embeddings hold true:

$$\begin{aligned} W^{1,\infty}(I, \mathcal{Z}'_1) &\subset W^{1,p}(I, \mathcal{Z}'_1) \subset C^{0,\alpha}(\bar{I}, \mathcal{Z}'_1), \quad \text{with } \alpha = \frac{p-1}{p}, \text{ for } 1 < p < \infty, \\ W^{1,\infty}(I, \mathcal{Z}'_1) &\subset W^{1,1}(I, \mathcal{Z}'_1) \subset C_{u,b}(\bar{I}, \mathcal{Z}'_1), \end{aligned}$$

where  $C_{u,b}$  stands for uniformly continuous bounded functions and  $C^{0,\alpha}$  for Hölder continuous functions. In particular, if  $\gamma$  is a weak solution of (10) then  $\gamma : \bar{I} \rightarrow \mathcal{Z}_1$  is weakly continuous,  $\gamma$  is differentiable almost everywhere on  $I$  and  $\dot{\gamma}(t) = v(t, \gamma(t)) \in \mathcal{Z}'_1$ , for a.e.  $t \in I$ . Hence, the initial value problem (10) makes sense in the space  $L^\infty(I, \mathcal{Z}_1) \cap W^{1,\infty}(I, \mathcal{Z}'_1)$ . Furthermore, it

is easy to check using the assumptions on the vector field  $v$  that any function  $\gamma \in L^\infty(I, \mathcal{X}_1)$  satisfying the Duhamel formula,

$$(11) \quad \gamma(t) = x + \int_s^t v(\tau, \gamma(\tau)) ds, \text{ for a.e. } t \in I,$$

is a weak solution of (10). Conversely, any weak solution  $\gamma$  of (10) satisfies (11) since  $\gamma$  is absolutely continuous with an almost everywhere derivative in  $L^\infty(I, \mathcal{X}'_1)$ . Similarly, strong solutions of (10) on  $I$  are exactly continuous curves in  $C(I, \mathcal{X}_1)$  satisfying the Duhamel formula (11) for all  $t \in I$  (see [20, 21] for more details). In the following, we precise the meaning of local and global well posedness of the initial value problem (10) on  $\mathcal{X}_1$ .

**Definition 2.2.** *Let  $v : \mathbb{R} \times \mathcal{X}_1 \rightarrow \mathcal{X}'_1$  be a continuous vector field that is bounded on bounded sets. We say that the initial value problem (10) is locally well posed (LWP) in  $\mathcal{X}_1$  if:*

- (i) *Weak uniqueness: Any two weak solutions of (10), defined on the same open interval  $I$  and satisfying the same initial condition, coincide.*
- (ii) *Strong existence: For any  $x \in \mathcal{X}_1$  and  $s \in \mathbb{R}$ , there exists a non-empty open interval  $I$  containing  $s$  such that a strong solution of (10) defined on  $I$  exists.*
- (iii) *Blowup alternative: Let  $(T_{min}(x, s), T_{max}(x, s))$  be the maximal interval of existence of a strong solution of (10). If  $T_f = T_{max}(x, s) < +\infty$  (resp.  $T_i = T_{min}(x, s) > -\infty$ ) then,*

$$\lim_{t \uparrow T_f} \|\gamma(t)\|_{\mathcal{X}_1} = +\infty, \quad (\text{resp. } \lim_{t \downarrow T_i} \|\gamma(t)\|_{\mathcal{X}_1} = +\infty).$$

- (iv) *Continuous dependence on initial data: If  $x_n \rightarrow x$  in  $\mathcal{X}_1$  and  $J \subset (T_{min}(x, s), T_{max}(x, s))$  is a closed interval, then for  $n$  large enough the strong solutions  $\gamma_n$  of (10) provided by (ii) with  $\gamma_n(s) = x_n$  are defined on  $J$  and satisfy  $\gamma_n \xrightarrow{n \rightarrow \infty} \gamma$  in  $C(J, \mathcal{X}_1)$ .*

If  $I = \mathbb{R}$  in (ii) for any  $x \in \mathcal{X}_1$  and any  $s \in \mathbb{R}$ , we say that the initial value problem is globally well-posed (GWP).

The above notion of (LWP) fits better our purpose of using energy techniques when considering applications to Hamiltonian PDEs. Notice that (i)-(ii) imply the existence of a unique maximal strong solution of the initial value problem (10) defined on an interval  $(T_{min}(x, s), T_{max}(x, s))$ , containing  $s$ , for each initial datum  $x \in \mathcal{X}_1$ . Notice also that by (iv) the maps  $x \rightarrow T_{min}(x, s)$  and  $x \rightarrow T_{max}(x, s)$  are respectively upper and lower semicontinuous. Furthermore, the local flow  $\Phi : \mathbb{R} \times \mathbb{R} \times \mathcal{X}_1 \rightarrow \mathcal{X}_1$ , with domain  $\mathcal{D} = \{T_{min}(x, s) < t < T_{max}(x, s), s \in \mathbb{R}, x \in \mathcal{X}_1\}$ , is well defined with  $t \rightarrow \Phi(t, s)(x)$  being the unique maximal strong solution of (10). Another consequence of (i)-(iv) is that the map  $\Phi(\cdot, s) : B \rightarrow C(J, \mathcal{X}_1)$ ,  $x \rightarrow \Phi(\cdot, s)(x) \in C(J, \mathcal{X}_1)$  is continuous for a ball  $B$  of  $\mathcal{X}_1$  and  $J$  a closed interval such that  $J \subset (T_{min}(x, s), T_{max}(x, s))$  for each  $x \in B$ . Moreover, the following local group law holds true for any  $x \in \mathcal{X}_1$ ,  $s \in \mathbb{R}$  and  $t, r \in (T_{min}(x, s), T_{max}(x, s))$ ,

$$\begin{aligned} \Phi(s, s)(x) &= x, \\ \Phi(t, r) \circ \Phi(r, s)(x) &= \Phi(t, s)(x). \end{aligned}$$

*The Liouville equation:* In this paragraph we give a precise meaning of the Liouville's equation in infinite dimension. Indeed, we formulate the equation (5) in a weak sense using a convenient space of cylindrical test functions over  $\mathcal{X}'_1$  (see e.g. [5, Chapter 5]).

Let  $\mathcal{Z}$  be a complex separable Hilbert space endowed with its euclidian structure  $\text{Re}\langle \cdot, \cdot \rangle_{\mathcal{Z}}$ , denoted for shortness by  $\langle \cdot, \cdot \rangle_{\mathcal{Z}, \mathbb{R}}$ . Consider  $\mathcal{Z}_{\mathbb{R}} := \mathcal{Z}$  as a real Hilbert space and let  $\Pi_n(\mathcal{Z}_{\mathbb{R}})$  be the set of all projections  $\pi : \mathcal{Z}_{\mathbb{R}} \rightarrow \mathbb{R}^n$  defined by

$$(12) \quad \pi(x) = (\langle x, e_1 \rangle_{\mathcal{Z}, \mathbb{R}}, \dots, \langle x, e_n \rangle_{\mathcal{Z}, \mathbb{R}}),$$

where  $\{e_1, \dots, e_n\}$  is any orthonormal family of  $\mathcal{Z}_{\mathbb{R}}$ . We denote by  $\mathcal{C}_{0,cyl}^{\infty}(\mathcal{Z})$  the space of functions  $\varphi = \psi \circ \pi$  with  $\pi \in \Pi_n(\mathcal{Z}_{\mathbb{R}})$  for some  $n \in \mathbb{N}$  and  $\psi \in \mathcal{C}_0^{\infty}(\mathbb{R}^n)$ . In particular, one can check that the gradient (or the  $\mathbb{R}$ -differential) of  $\varphi$  is equal to

$$\nabla \varphi = \pi^T \circ \nabla \psi \circ \pi,$$

where  $\pi^T$  denotes the transpose map of  $\pi$ . We equally define, for any open interval  $I \subset \mathbb{R}$ , the space  $\mathcal{C}_{0,cyl}^{\infty}(I \times \mathcal{Z})$  as the set of functions  $\varphi(t, x) = \psi(t, \pi(x))$  with  $\psi \in \mathcal{C}_0^{\infty}(\mathbb{R}^{n+1})$  and  $\pi \in \Pi_n(\mathcal{Z}_{\mathbb{R}})$ .

The non-compactness of closed balls in a separable (infinite dimensional) Hilbert space  $\mathcal{Z}$  suggests the introduction of a norm in  $\mathcal{Z}$  that ensures relative compactness of bounded sets. Let  $(e_n)_{n \in \mathbb{N}^*}$  be a Hilbert basis of  $\mathcal{Z}_{\mathbb{R}}$  and define the following norm over  $\mathcal{Z}$ ,

$$(13) \quad \|z\|_{\mathcal{Z}_w}^2 = \sum_{n \in \mathbb{N}^*} \frac{1}{n^2} |\langle z, e_n \rangle_{\mathcal{Z}_{\mathbb{R}}}|^2, \quad \forall z \in \mathcal{Z}.$$

We simply denote by  $\mathcal{Z}_w$  the space  $\mathcal{Z}$  endowed with the above norm. Remark that the weak topology on  $\mathcal{Z}$  and the one induced by the norm  $\|\cdot\|_{\mathcal{Z}_w}$  coincide on bounded sets. Moreover, the Borel  $\sigma$ -algebra of  $\mathcal{Z}$  is the same with respect to the norm, weak or  $\|\cdot\|_{\mathcal{Z}_w}$  topology.

The space of Borel probability measures on a Hilbert space  $\mathcal{Z}$  will be denoted by  $\mathfrak{P}(\mathcal{Z})$  and it is naturally endowed with a strong or weak narrow convergence topology. Indeed, a curve  $t \in I \rightarrow \mu_t \in \mathfrak{P}(\mathcal{Z})$  is said strongly (resp. weakly) narrowly continuous if the real-valued map,

$$t \in I \rightarrow \int_{\mathcal{Z}} \varphi(x) d\mu_t(x) \in \mathbb{R},$$

is continuous for every bounded continuous function  $\varphi \in C_b((\mathcal{Z}, \|\cdot\|_{\mathcal{Z}}), \mathbb{R})$  (resp.  $\varphi \in C_b((\mathcal{Z}, \|\cdot\|_{\mathcal{Z}_w}), \mathbb{R})$ ). Let  $v : \mathbb{R} \times \mathcal{Z}'_1 \rightarrow \mathcal{Z}'_1$  a continuous vector field. We consider the following Liouville's equation defined in a bounded open interval  $I \subset \mathbb{R}$ ,

$$\partial_t \mu_t + \nabla^T(v \cdot \mu_t) = 0,$$

understood, in a weak sense, as the integral equation,

$$(14) \quad \int_I \int_{\mathcal{Z}'_1} \partial_t \varphi(t, x) + \operatorname{Re} \langle v(t, x), \nabla \varphi(t, x) \rangle_{\mathcal{Z}'_1} d\mu_t(x) dt = 0, \quad \forall \varphi \in \mathcal{C}_{0,cyl}^{\infty}(I \times \mathcal{Z}'_1).$$

In order that the above problem makes sense we assume that  $\mu_t \in \mathfrak{P}(\mathcal{Z}'_1)$  for all  $t \in I$ . So the integration with respect to  $\mu_t$  is taken on the set  $\mathcal{Z}'_1$  where the integrand is well defined. We also assume two more conditions on  $t \rightarrow \mu_t$ , namely we require that

$$(15) \quad \int_I \int_{\mathcal{Z}'_1} \|v(t, x)\|_{\mathcal{Z}'_1} d\mu_t(x) dt < \infty,$$

and the curve  $I \ni t \rightarrow \mu_t$  is weakly narrowly continuous in  $\mathfrak{P}(\mathcal{Z}'_1)$ . The latter assumption is a mild requirement slightly better than assuming  $(\mu_t)_{t \in I}$  to be a Borel family in  $\mathfrak{P}(\mathcal{Z}'_1)$ , in the sense that  $t \rightarrow \mu_t(A)$  is Borel for any Borel set  $A \subset \mathcal{Z}'_1$ , see [5, Lemma 8.1.2]. Consequently, the integral with respect to time in (14) is well defined and finite thanks to the assumption (15) which ensures the integrability.

We are now in position to announce our main results which provide a naturel link between the solutions of the Liouville's equation (14) and the initial value problem (10). Theorem 2.3 and 2.4 significantly improve the former result in [12] which previously extended the characteristics theory to Hamiltonian PDEs.

**Theorem 2.3.** *Let  $v : \mathbb{R} \times \mathcal{Z}'_1 \rightarrow \mathcal{Z}'_1$  be a (non-autonomous) continuous vector field such that  $v$  is bounded on bounded sets. Let  $t \in I \rightarrow \mu_t \in \mathfrak{P}(\mathcal{Z}'_1)$  be a weakly narrowly continuous solution*

in  $\mathfrak{P}(\mathcal{Z}'_1)$  of the Liouville equation (14) defined on an open bounded interval  $I$  with the vector field satisfying the scalar velocity estimate:

$$(A) \quad \int_I \int_{\mathcal{Z}'_1} \|v(t, x)\|_{\mathcal{Z}'_1} d\mu_t(x) dt < \infty,$$

Assume additionally that:

- (i) There exists a ball  $B$  of  $\mathcal{Z}'_1$  such that  $\mu_t(B) = 1$  for all  $t \in I$ .
- (ii) The initial value problem (10) is (LWP) in  $\mathcal{Z}'_1$ .

Then for any  $s \in I$ , the maximal existence interval  $(T_{\min}(x, s), T_{\max}(x, s)) \supseteq \bar{I}$  for  $\mu_s$ -almost every  $x \in \mathcal{Z}'_1$ . Moreover,  $\mu_t = \Phi(t, s)_\# \mu_s$  for all  $t \in I$  with  $\Phi(t, s)$  is the local flow of the initial value problem (10). Additionally, if the curve  $t \rightarrow \mu_t$  is defined on  $\mathbb{R}$  and the above assumptions still satisfied for any arbitrary bounded open interval  $I \subset \mathbb{R}$ , then  $\mu_t = \Phi(t, s)_\# \mu_s$  for all  $t, s \in \mathbb{R}$ .

The assumption (i) in Theorem 2.3 requires a concentration of the measure  $\mu_t$  on a bounded set of  $\mathcal{Z}'_1$  for all times in the interval  $I$ . This is a rather implicit condition which may not be so practical for the applications that we have in mind [7, 8, 13, 36]. In particular, we are interested in extending the previous result to measures  $\mu_t$  that are not concentrated in a ball of  $\mathcal{Z}'_1$  but rather having a second finite moment in  $\mathcal{Z}'_1$ . Of course, in order to do so we require a stronger assumption in the vector field.

**Theorem 2.4.** *Let  $v : \mathbb{R} \times \mathcal{Z}'_1 \rightarrow \mathcal{Z}'_0$  be a (non-autonomous) continuous vector field such that for any  $M > 0$  and any bounded interval  $J \subset \mathbb{R}$ , there exists  $C(M, J) > 0$  satisfying:*

$$(16) \quad \|v(t, x) - v(t, y)\|_{\mathcal{Z}'_0} \leq C(M, J) (\|x\|_{\mathcal{Z}'_1}^2 + \|y\|_{\mathcal{Z}'_1}^2) \|x - y\|_{\mathcal{Z}'_0},$$

for all  $t \in J$  and  $x, y \in \mathcal{Z}'_1$  such that  $\|x\|_{\mathcal{Z}'_0}, \|y\|_{\mathcal{Z}'_0} \leq M$ . Let  $t \in I \rightarrow \mu_t \in \mathfrak{P}(\mathcal{Z}'_1)$  be a weakly narrowly continuous solution in  $\mathfrak{P}(\mathcal{Z}'_1)$  of the Liouville equation (14) defined on an open bounded interval  $I$ . Assume additionally that:

- (i) There exists  $C > 0$  such that  $\int_I \int_{\mathcal{Z}'_1} \|x\|_{\mathcal{Z}'_1}^2 d\mu_t(x) dt \leq C$ .
- (ii) There exists an open Ball  $B$  of  $\mathcal{Z}'_0$  such that  $\mu_t(B) = 1$  for all  $t \in I$ .
- (iii) For  $s \in I$  and any  $x \in \mathcal{Z}'_1 \cap B$  there exists a strong solution of (10) defined on  $\bar{I}$  with Definition 2.2-(iv) satisfied.

Then  $\mu_t = \Phi(t, s)_\# \mu_s$  for all  $t \in I$  with  $\Phi(t, s)$  is the local flow of the initial value problem (10). Additionally, if the curve  $t \rightarrow \mu_t$  is defined on  $\mathbb{R}$  and the above assumptions still satisfied for any arbitrary bounded open interval  $I \subset \mathbb{R}$ , then  $\mu_t = \Phi(t, s)_\# \mu_s$  for all  $t, s \in \mathbb{R}$ .

**Remark 2.5.** *Here some useful comments on the above theorems.*

1. Both Thm. 2.3 and 2.4 rely on a probabilistic representation result given in Proposition 4.1 with some concentration arguments.
2. Observe that  $\{x \in \mathcal{Z}'_1 : (T_{\min}(x, s), T_{\max}(x, s)) \supseteq \bar{I}\}$  in Thm. 2.3 is an open subset of  $\mathcal{Z}'_1$  thanks to the semi-continuity of the maps  $x \rightarrow T_{\min}(x, s), T_{\max}(x, s)$ .
3. The existence in Thm. 2.3 of a non-trivial solution on  $I$  of the Liouville's equation (14) implies the existence of a non-trivial strong solution of the initial value problem (10) defined on  $I$ .
4. The condition (16) implies uniqueness of weak solutions of the initial value problem (10).



5. Thm. 2.4 can also be used with  $B = \mathcal{Z}_0$ . Of course, in this case (ii) is trivial but one have to check in addition the scalar velocity estimate:

$$(A') \quad \int_I \int_{\mathcal{Z}_1} \|v(t, x)\|_{\mathcal{Z}_0} d\mu_t(x) dt < \infty,$$

which is automatically satisfied if  $B \subsetneq \mathcal{Z}_0$  thanks to the estimate (16) and (i).

6. The set  $E = B \cap \mathcal{Z}_1$  in Thm. 2.4 is  $\Phi(t, s)$ -invariant modulo  $\mu_s$  for any  $t \in I$ , i.e.  $\mu_s(E \Delta \Phi(t, s)^{-1}(E)) = 0$ .

7. Thm. 2.4 is oriented towards some specific applications related to the author's interest. However, the proof is rather flexible and interested reader may work out a different form.

### 3 Application to Hamiltonian PDEs

Consider a Hamiltonian PDE with a real-valued energy functional having the general form,

$$(17) \quad h(z, \bar{z}) = \langle z, Az \rangle_{\mathcal{Z}_0} + h_I(z, \bar{z}),$$

where  $\mathcal{Z}_0$  is a complex separable Hilbert space,  $A$  is a non-negative self-adjoint operator,  $h_I(z, \bar{z})$  is a nonlinear functional and  $(z, \bar{z})$  are the complex classical fields of the Hamiltonian theory. One has a natural rigged Hilbert space  $\mathcal{Z}_1 \subset \mathcal{Z}_0 \subset \mathcal{Z}'_1$  with the energy space  $\mathcal{Z}_1 = Q(A)$ , which is the form domain of  $A$  equipped with the graph norm,

$$(18) \quad \|z\|_{\mathcal{Z}_1}^2 = \langle z, (A + 1)z \rangle_{\mathcal{Z}_0},$$

and  $\mathcal{Z}'_1$  is the dual of  $\mathcal{Z}_1$  with respect to the inner product of  $\mathcal{Z}_0$ . It is not necessary, but one can assume that  $\mathcal{Z}_0$  is endowed with an (anti-linear) conjugation  $c : z \rightarrow \bar{z}$ , such that  $\langle u, \bar{v} \rangle = \langle v, \bar{u} \rangle$ , keeping invariant  $\mathcal{Z}_1$  and commuting with  $A$  (see [12, Appendix C]). A detailed discussion on the derivation of Liouville's equation (14) and its relationship with the Poisson structure of Hamiltonian systems is given in [12].

Assume that the energy (17) is well-defined on  $\mathcal{Z}_1$  and that  $h$  admits directional derivatives,

$$\partial_{\bar{z}} h(x, \bar{x})[u] := \frac{d}{d\lambda} h(x + \lambda u, \overline{x + \lambda u})|_{\lambda=0},$$

such that the map  $x \in \mathcal{Z}_1 \rightarrow \partial_{\bar{z}} h(x, \bar{x}) \in \mathcal{Z}'_1$  is continuous and bounded on bounded sets. The Hamiltonian equation (or equation of motion) reads,

$$(19) \quad i\partial_t u = \partial_{\bar{z}} h(u, \bar{u}).$$

So, this Hamiltonian system enters naturally into the framework of Theorem 2.3 with a time-independent vector field  $v_1(t, x) = -i\partial_{\bar{z}} h(x, \bar{x})$  defined as a continuous map  $v_1 : \mathbb{R} \times \mathcal{Z}_1 \rightarrow \mathcal{Z}'_1$  bounded on bounded sets. Thus, Theorem 2.3 can be applied to the Hamiltonian equation (19) if either (LWP) or (GWP) holds true in the energy space  $\mathcal{Z}_1$ . Remark that no conservation law is directly used to establish the propagation along characteristics.

In order to apply Theorem 2.4, one needs to work in the interaction representation since the vector field  $v_1$  may take its values outside  $\mathcal{Z}_0$ . Indeed, by differentiating  $\tilde{u} := e^{itA}u$  with respect to time, where  $u$  is a (strong or weak) solution of the Hamiltonian equation (19), one obtains

$$(20) \quad i\partial_t \tilde{u} = e^{itA} \partial_{\bar{z}} h_I(e^{-itA} \tilde{u}, \overline{e^{-itA} \tilde{u}}).$$

The initial value problems (19) and (20) are equivalent, in the sense that  $u$  is a strong or weak solution of (19) if and only if  $\tilde{u} := e^{itA}u$  is a strong or weak solution of (20) respectively. Hence, if the non-autonomous vector field,

$$v_2(t, z) := -ie^{itA} \partial_{\bar{z}} h_I(e^{-itA} z, \overline{e^{-itA} z}),$$

is well-defined as a continuous map  $v_2 : \mathbb{R} \times \mathcal{Z}_1 \rightarrow \mathcal{Z}_0$  and satisfies the inequality (16), then one can apply Theorem 2.4 to the Hamiltonian system (17) whenever a (LWP) or (GWP) result is known for the initial value problem (19) in the energy space  $\mathcal{Z}_1$ .

It is clear from the above discussion that Theorem 2.3 and 2.4 apply to various PDEs. We illustrate this with few examples that are related to the authors interest.

**Example 1** (Nonlinear Schrödinger equation). *Consider the NLS equation in dimension  $d$  with energy functional,*

$$(21) \quad h(z, \bar{z}) = \langle z, -\Delta_x z \rangle_{L^2(\mathbb{R}^d)} + \frac{2\lambda}{2 + \alpha} \int_{\mathbb{R}^d} |z(x)|^{\alpha+2} dx,$$

such that  $2 \leq \alpha < \frac{4}{d-2}$  ( $2 \leq \alpha < \infty$  if  $d = 1, 2$ ) and  $\lambda \in \mathbb{C}$ . According to [20, Theorem 4.4.1] the related initial value problem is (LWP) in  $H^1(\mathbb{R}^d)$ . Hence, Theorem 2.3 applies to this case. The derivation of such equation from quantum many-body dynamics, for  $\alpha = 2$ , is proved for instance in [1, 6, 28].

**Example 2** (Non-relativistic Hartree equation). *The energy functional of the Hartree equation is*

$$(22) \quad h(z, \bar{z}) = \langle z, -\Delta_x + V(x) z \rangle_{L^2(\mathbb{R}^d)} + \iint_{\mathbb{R}^d \times \mathbb{R}^d} |z(x)| |z(y)|^2 W(x - y) dx dy,$$

where  $W : \mathbb{R}^d \rightarrow \mathbb{R}$  is an even measurable function and  $V$  is a real-valued potential both satisfying the following assumptions for some  $p$  and  $q$ ,

$$\begin{aligned} V &\in L^p(\mathbb{R}^d) + L^\infty(\mathbb{R}^d), \quad p \geq 1, \quad p > \frac{d}{2}, \\ W &\in L^q(\mathbb{R}^d) + L^\infty(\mathbb{R}^d), \quad q \geq 1, \quad q \geq \frac{d}{2} \text{ (and } q > 1 \text{ if } d = 2). \end{aligned}$$

The vector field  $v(t, z) := W * |z|^2 z : Q(A) \rightarrow L^2(\mathbb{R}^d)$  verifies, by Hölder, Young and Sobolev-Gagliardo-Nirenberg's inequalities, the estimate (16). The global well-posedness on  $Q(A)$ , conservation of energy and charge of the Hartree equation

$$\begin{cases} i\partial_t z = -\Delta z + Vz + W * |z|^2 z \\ z_{t=0} = z_0, \end{cases}$$

are proved in [20, Corollary 4.3.3 and 6.1.2]. Therefore, Theorem 2.4 applies here. Remark that the assumption on  $W$  are satisfied by the Coulomb type potentials  $\frac{\lambda}{|x|^\alpha}$  when  $\alpha < 2$ ,  $\lambda \in \mathbb{R}$  and  $d = 3$ . The derivation of such equation from quantum many-body dynamics is extensively investigated, see for instance [12, 15, 28, 29, 30, 32, 33, 46].

**Example 3** (Klein-Gordon equation). *One of the prominent examples of relativistic quantum field theory is the  $\varphi^4$  field theory, see e.g. [44, 45]. Consider the classical Klein-Gordon energy functional*

$$(23) \quad \mathcal{H}(\varphi, \pi) = \frac{1}{2} \langle \varphi, -\Delta + m^2 \varphi \rangle_{L^2(\mathbb{R}^3)}^2 + \frac{1}{2} \|\pi(x)\|_{L^2(\mathbb{R}^3)}^2 + \frac{1}{4} \int_{\mathbb{R}^3} \varphi^4(x) dx$$

where  $\varphi, \pi$  are real fields and  $m > 0$  a given parameter. Writing the above Hamiltonian system with complex fields  $z(\cdot), \bar{z}(\cdot)$ , such that:

$$\begin{aligned} \varphi(x) &= \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} (\bar{z}(k) e^{-ikx} + z(k) e^{-ikx}) \frac{dk}{\sqrt{2\omega(k)}}, & \text{and} & \quad \omega(k) = \sqrt{k^2 + m^2}, \\ \pi(x) &= \frac{i}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} (\bar{z}(k) e^{-ikx} - z(k) e^{-ikx}) \sqrt{\frac{\omega(k)}{2}} dk, \end{aligned}$$

we obtain the equivalent PDE,

$$(24) \quad i\partial_t z = \omega(k)z + \frac{1}{\sqrt{2\omega(k)}} \mathcal{F}(\varphi^3),$$

where  $\mathcal{F}$  denotes the Fourier transform and  $\varphi$  depends on  $z, \bar{z}$  as above. The energy space of the equation (24) is the form domain  $\mathcal{Z}_1 = Q(\omega)$  of  $\omega$  considered as an unbounded multiplication operator on  $\mathcal{Z}_0 = L^2(\mathbb{R}^3)$ . According to [31, Proposition 3.2], we have (GWP) of the Klein-Gordon equation (24) in the space  $\mathcal{Z}_1$  and hence Theorem 2.3 is applicable in this case. Note that the derivation of a Klein-Gordon equation with nonlocal nonlinearity from the  $P(\varphi)_2$  quantum field theory is established for instance in [13, 27, 32].

**Example 4** (Schrödinger-Klein-Gordon system). *The Schrödinger-Klein-Gordon system with Yukawa interaction is defined by:*

$$(S-KG) \quad \begin{cases} i\partial_t u = -\frac{\Delta}{2M}u + \varphi u, \\ (\square + m^2)\varphi = -|u|^2, \end{cases}$$

where  $(u, \varphi)$  are the unknowns and  $M, m > 0$  are given parameters. If we introduce the complex fields  $\alpha, \bar{\alpha}$ , defined according to the formula,

$$(25) \quad \varphi(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{\mathbb{R}^3} \frac{1}{\sqrt{2\omega(k)}} (\bar{\alpha}(k)e^{-ik \cdot x} + \alpha(k)e^{ik \cdot x}) dk, \quad \omega(k) = \sqrt{k^2 + m^2},$$

we can rewrite (S-KG) as the equivalent system:

$$(S-KG_\alpha) \quad \begin{cases} i\partial_t u = -\frac{\Delta}{2M}u + \varphi u \\ i\partial_t \alpha = \omega\alpha + \frac{1}{\sqrt{2\omega}} \mathcal{F}(|u|^2) \end{cases},$$

where  $\mathcal{F}$  denotes the Fourier transform. It is known that the Cauchy problem for the Schrödinger-Klein-Gordon system (S-KG) is globally well posed on the energy space  $\mathcal{Z}_1 = H^1(\mathbb{R}^3) \oplus Q(\omega)$  where  $Q(\omega)$  is the form domain of the operator  $\omega$  equipped with the graph norm (18), see for instance [22, 41] and references therein. Moreover, the vector field  $v : H^1(\mathbb{R}^3) \oplus Q(\omega) \rightarrow L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)$  satisfies by Sobolev-Gagliardo-Nirenberg's inequality the estimate (16). Hence, Theorem 2.4 is applicable. Remark that the derivation of such equation from a quantum field theory, describing a nucleon-meson field theory, is studied in [7, 8].

## 4 Proof of the main results

For a normed vector space  $E$  and an open bounded interval  $I$ , we denote by  $\Gamma_I(E)$  the space of all continuous curves from  $\bar{I}$  into  $(E, \|\cdot\|_E)$  endowed with the sup norm,

$$\|\gamma\|_{\Gamma_I(E)} = \sup_{t \in \bar{I}} \|\gamma(t)\|_E.$$

We will use these notations in two cases  $E = \mathbb{R}^d$  and  $E = \mathcal{Z}'_1$ . In particular, the metric space

$$(26) \quad \mathfrak{X} = (\mathcal{Z}'_1 \times \Gamma_I(\mathcal{Z}'_1), \|\cdot\|_{\mathcal{Z}'_1, w} + \|\cdot\|_{\Gamma_I(\mathcal{Z}'_1, w)})$$

will play an important role. To be precise here  $\Gamma_I(\mathcal{Z}'_1)$  is the space of continuous functions with respect to the norm  $\|\cdot\|_{\mathcal{Z}'_1}$  while  $\mathfrak{X}$  is endowed with the product weak norm related to (13). Notice also that we will follow the setting of Section 2 without further specification. For each  $t \in I$ , we define the continuous evaluation map,

$$e_t : (x, \gamma) \in E \times \Gamma_I(E) \mapsto \gamma(t) \in E.$$

As a first step, we prove a probabilistic representation similar to the one proved in finite dimension by S. Maniglia in [40, Theorem 4.1]. The proof is inspired by [5, Theorem 8.2.1 and 8.3.2] and the extension to infinite dimension in [12, Proposition C.2].

**Proposition 4.1.** *Let  $v : \mathbb{R} \times \mathcal{Z}_1 \rightarrow \mathcal{Z}'_1$  be a (non-autonomous) Borel vector field such that  $v$  is bounded on bounded sets. Let  $t \in I \rightarrow \mu_t \in \mathfrak{P}(\mathcal{Z}'_1)$  be a weakly narrowly continuous solution in  $\mathfrak{P}(\mathcal{Z}'_1)$  of the Liouville equation (14) defined on an open bounded interval  $I$  with a vector field satisfying the scalar velocity estimate (A). Then there exists a Borel probability measure  $\eta$ , on the space  $\mathfrak{X}$  given in (26), satisfying:*

- (i)  $\eta$  is concentrated on the set of  $(x, \gamma) \in \mathcal{Z}_1 \times \Gamma_I(\mathcal{Z}'_1)$  such that the curves  $\gamma \in W^{1,1}(I, \mathcal{Z}'_1)$  are solutions of the initial value problem  $\dot{\gamma}(t) = v(t, \gamma(t))$  for a.e.  $t \in I$  and  $\gamma(t) \in \mathcal{Z}'_1$  for a.e.  $t \in I$  with  $\gamma(s) = x \in \mathcal{Z}_1$  for some fixed  $s \in I$ .
- (ii)  $\mu_t = (e_t)_\# \eta$  for any  $t \in I$ .

*Proof.* Let  $(e_n)_{n \in \mathbb{N}^*}$  be a Hilbert basis of  $\mathcal{Z}'_{1, \mathbb{R}}$  and consider the following commutative diagram,

$$\begin{array}{ccc} \mathcal{Z}'_1 & \xrightarrow{\pi^d} & \mathbb{R}^d \\ & \searrow \hat{\pi}^d & \downarrow \pi^{d,T} \\ & & \mathcal{Z}'_1 \end{array}$$

with  $\pi^d(x) = (\langle e_1, x \rangle_{\mathcal{Z}'_{1, \mathbb{R}}}, \dots, \langle e_d, x \rangle_{\mathcal{Z}'_{1, \mathbb{R}}})$ ,  $\pi^{d,T}(y_1, y_2, \dots, y_d) = \sum_{j=1}^d y_j e_j$  and  $\hat{\pi}^d = \pi^{d,T} \circ \pi^d$ .

The proof splits into several steps. We first consider the image measures  $\mu_t^d := (\pi^d)_\# \mu_t$  with respect to the sequence of projections  $\pi^d$  defined above. It turns that the curve  $t \in I \rightarrow \mu_t^d$  solves a continuity equation on  $\mathbb{R}^d$  with a given vector field  $v^d : \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}^d$  satisfying a scalar velocity estimate similar to (A). Then using the result of S. Maniglia [40, Theorem 4.1], we deduce the existence of a probability measure  $\eta^d$  on  $\mathbb{R}^d \times \Gamma_I(\mathbb{R}^d)$  satisfying the relation  $\mu_t^d = (e_t)_\# \eta^d$  and concentrated on the characteristics  $\dot{\gamma}(t) = v^d(t, \gamma(t))$ . The last two steps are the proof of tightness of the family  $\{\eta^d\}_{d \in \mathbb{N}^*}$  and the derivation of the properties (i)-(ii) for any limit point  $\eta$  of the sequence  $\{\eta^d\}_{d \in \mathbb{N}^*}$ .

Reduction to finite dimension: Define the probability measures  $\mu_t^d \in \mathfrak{P}(\mathbb{R}^d)$  and  $\hat{\mu}_t^d \in \mathfrak{P}(\mathcal{Z}'_1)$  as the image measures of  $\mu_t$  with respect to the projections  $\pi^d$  and  $\hat{\pi}^d$  respectively, i.e.:

$$(27) \quad \mu_t^d := \pi^d_\# \mu_t, \quad \hat{\mu}_t^d := \hat{\pi}^d_\# \mu_t.$$

By using the Liouville equation (14) with  $\varphi(t, x) = \chi(t)\phi(x)$ , such that  $\chi \in \mathcal{C}_0^\infty(I)$  and  $\phi = \psi \circ \pi^d$ ,  $\psi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ , one obtains in the sense of distribution  $\mathcal{D}'(I, \mathbb{R})$ ,

$$(28) \quad \begin{aligned} \frac{d}{dt} \int_{\mathcal{Z}'_1} \phi(x) d\mu_t(x) &= \int_{\mathcal{Z}'_1} \operatorname{Re} \langle v(t, x), \nabla \phi(x) \rangle_{\mathcal{Z}'_1} d\mu_t(x) \\ &= \int_{\mathcal{Z}'_1} \operatorname{Re} \langle \pi^d \circ v(t, x), \nabla \psi \circ \pi^d(x) \rangle_{\mathcal{Z}'_1} d\mu_t(x), \end{aligned}$$

since  $\nabla \phi = \pi^{d,T} \circ \nabla \psi \circ \pi^d$ . One also notices that  $t \in I \rightarrow g(t) = \int_{\mathcal{Z}'_1} \phi(x) d\mu_t(x)$  is continuous thanks to the weak narrow continuity of the curve  $t \in I \rightarrow \mu_t \in \mathfrak{P}(\mathcal{Z}'_1)$ . Moreover, the velocity estimate (A) implies that the right hand side of (28) is integrable. Hence,  $g$  is an absolutely continuous function in  $W^{1,1}(I, \mathbb{R})$  and (28) holds a.e.  $t \in I$ . Since  $(\mathcal{Z}'_1, \|\cdot\|_{\mathcal{Z}'_{1,w}})$  is a Radon separable space, we can apply the disintegration theorem (see Appendix B and Theorem E.1). Hence, there exists a  $\mu_t^d$ -a.e. determined family of measures  $\{\mu_{t,y}, y \in \mathbb{R}^d\} \subset \mathfrak{P}(\mathcal{Z}'_1)$  such that  $\mu_{t,y}(\mathcal{Z}'_1 \setminus (\pi^d)^{-1}(y)) = 0$ , for  $\mu_t^d$ -a.e.  $y \in \mathbb{R}^d$ , and

$$(29) \quad \begin{aligned} \frac{d}{dt} \int_{\mathcal{Z}'_1} \phi(x) d\mu_t(x) &= \int_{\mathbb{R}^d} \left( \int_{(\pi^d)^{-1}(y)} \langle \pi^d \circ v(t, x), \nabla \psi(y) \rangle_{\mathbb{R}^d} d\mu_{t,y}(x) \right) d\mu_t^d(y) \\ &= \int_{\mathbb{R}^d} \langle v^d(t, y), \nabla \psi(y) \rangle_{\mathbb{R}^d} d\mu_t^d(y), \end{aligned}$$

with the vector field  $v^d$  defined as,

$$(30) \quad v^d(t, y) := \int_{(\pi^d)^{-1}(y)} \pi^d \circ v(t, x) d\mu_{t,y}^d(x), \quad \text{for } \mu_t^d - \text{a.e. } y \in \mathbb{R}^d \text{ and a.e. } t \in I.$$

Moreover, repeating the same computation as in the r.h.s of (28)-(29), one shows,

$$(31) \quad \left| \int_{\mathbb{R}^d} \langle v^d(t, y), f(y) \rangle_{\mathbb{R}^d} d\mu_t^d(y) \right| \leq \left| \int_{\mathcal{X}'_1} \langle \pi^d \circ v(t, x), f \circ \pi^d(x) \rangle_{\mathcal{X}'_1} d\mu_t(x) \right| \\ \leq \|v(t, \cdot)\|_{L^1(\mathcal{X}'_1, \mu_t)} \|f\|_{L^\infty(\mathbb{R}^d, \mu_t^d)},$$

for any  $f \in L^\infty(\mathbb{R}^d, \mu_t^d)$ . Hence, the vector field  $v^d$  satisfies the scalar velocity estimate,

$$\int_I \int_{\mathbb{R}^d} \|v^d(t, y)\|_{\mathbb{R}^d} d\mu_t^d(y) dt \leq \int_I \int_{\mathcal{X}'_1} \|v(t, x)\|_{\mathcal{X}'_1} d\mu_t(x) dt < \infty.$$

Notice also that the curve  $t \in I \rightarrow \mu_t^d \in \mathfrak{P}(\mathbb{R}^d)$  is narrowly continuous. Indeed, for any bounded continuous function  $\varphi \in C_b(\mathbb{R}^d, \mathbb{R})$  we have that  $\varphi \circ \pi^d \in C_b(\mathcal{X}'_{1,w}, \mathbb{R})$ . So, the map  $t \rightarrow \int_{\mathbb{R}^d} \varphi(y) d\mu_t^d(y) = \int_{\mathcal{X}'_1} \varphi \circ \pi^d(x) d\mu_t(x)$  is continuous since  $t \in I \rightarrow \mu_t^d \in \mathfrak{P}(\mathcal{X}'_1)$  is weakly narrowly continuous. Furthermore, by multiplying (29) with  $\chi \in \mathcal{C}_0^\infty(I)$  and integrating with respect to time, one obtains the following Liouville equation,

$$\int_I \int_{\mathbb{R}^d} \partial_t \varphi(t, y) + \langle v^d(t, y), \nabla \varphi(t, y) \rangle_{\mathbb{R}^d} d\mu_t^d(y) dt = 0,$$

for all  $\varphi(t, y) = \chi(t)\psi(y) \in \mathcal{C}_0^\infty(I \times \mathbb{R}^d)$ . The latter equation extends to all  $\varphi \in \mathcal{C}_0^\infty(I \times \mathbb{R}^d)$  by a density argument. Therefore, we have at hand all the ingredients to use the probabilistic representation of [40, Theorem 4.1]. Hence, for each  $d \in \mathbb{N}^*$  there exists a finite measure  $\eta^d \in \mathfrak{P}(\mathbb{R}^d \times \Gamma_I(\mathbb{R}^d))$  satisfying:

- (a)  $\eta^d$  is concentrated on the set of curves  $\gamma \in W^{1,1}(I, \mathbb{R}^d)$  that are solutions of the initial value problem  $\dot{\gamma}(t) = v^d(t, \gamma(t))$  for a.e.  $t \in I$  with  $\gamma(s) = x \in \mathbb{R}^d$  and  $s \in I$  fixed.
- (b)  $\mu_t^d = (e_t)_\# \eta^d$  for any  $t \in I$ .

Tightness: Recall that the space  $\mathfrak{X}$  denotes  $\mathcal{X}'_1 \times \Gamma_I(\mathcal{X}'_1)$  endowed with the product norm,

$$\|\cdot\|_{\mathcal{X}'_{1,w}} + \|\cdot\|_{\Gamma_I(\mathcal{X}'_{1,w})}.$$

Remark that  $\mathfrak{X}$  is a separable metric space. Consider in the following the family of measures  $\hat{\eta}^d \in \mathfrak{P}(\mathfrak{X})$  defined by the relation,

$$\hat{\eta}^d := (\pi^{d,T} \times \pi^{d,T})_\# \eta^d.$$

In particular, for any bounded Borel function  $\varphi : \mathcal{X}'_1 \rightarrow \mathbb{R}$  and  $t \in I$ , we have:

$$(32) \quad \int_{\mathcal{X}'_1} \varphi d\hat{\mu}_t^d = \int_{\mathbb{R}^d} \varphi \circ \pi^{d,T} d\mu_t^d = \int_{\mathbb{R}^d \times \Gamma_I(\mathbb{R}^d)} \varphi(\pi^{d,T} \circ \gamma(t)) d\eta^d = \int_{\mathfrak{X}} \varphi(\gamma(t)) d\hat{\eta}^d.$$

We claim that the sequence  $\{\hat{\eta}^d\}_{d \in \mathbb{N}^*}$  is tight in  $\mathfrak{P}(\mathfrak{X})$ . To prove this fact we use a criterion taken from [5] and recalled in the appendix Lemma B.2. Choose the maps  $r^1$  and  $r^2$  defined respectively on  $\mathfrak{X}$  as

$$r^1 : (x, \gamma) \in \mathfrak{X} \mapsto x \in \mathcal{X}'_1 \quad \text{and} \quad r^2 : (x, \gamma) \in \mathfrak{X} \mapsto \gamma - x \in \Gamma_I(\mathcal{X}'_1).$$

Notice that since the map  $r = r^1 \times r^2 : \mathfrak{X} \rightarrow \mathfrak{X}$  is a homeomorphism then  $r$  is proper, in the sense that the inverse images of compact subsets are compact. Using the concentration property

(a) and (32), one shows that  $(r^1)_\# \hat{\eta}^d = \hat{\mu}_s^d$  for any  $d \in \mathbb{N}^*$  and for some fixed  $s \in I$ . Besides, the family  $\{\hat{\mu}_s^d\}_{d \in \mathbb{N}^*}$  is (weakly) tight in  $\mathfrak{P}(\mathcal{Z}'_1)$ , since  $\hat{\mu}_s^d \xrightarrow{d \rightarrow \infty} \mu_s$  weakly narrowly in  $\mathfrak{P}(\mathcal{Z}'_1)$  and  $\mathcal{Z}'_{1,w}$  is a separable Radon space (see Appendix B).

Using Dunford-Pettis Theorem D.2 and equi-integrability, the scalar velocity estimate (A) leads to the existence of a nondecreasing super-linear convex function  $\theta : \mathbb{R}^+ \rightarrow [0, \infty]$  such that:

$$\int_I \int_{\mathcal{Z}'_1} \theta(\|v(t, x)\|_{\mathcal{Z}'_1}) d\mu_t(x) dt < +\infty.$$

Indeed, by setting for all  $\alpha, \beta \in I$  and for every Borel subset  $E$  of  $\mathcal{Z}'_1$

$$(33) \quad \nu([\alpha, \beta] \times E) := \int_\alpha^\beta \mu_t(E) dt,$$

we have the equality

$$\int_I \int_{\mathcal{Z}'_1} \|v(t, x)\|_{\mathcal{Z}'_1} d\mu_t(x) dt = \int_{I \times \mathcal{Z}'_1} \|v(t, x)\|_{\mathcal{Z}'_1} d\nu.$$

Since the singleton  $\{v\}$  is a compact set, Dunford-Pettis Theorem D.2 ensures that  $\{v\}$  is equi-integrable. Hence, Lemma D.1 in the appendix leads to the existence of the aforementioned function  $\theta$ . We are now in position to prove the tightness of the family  $\{(r^2)_\# \hat{\eta}^d\}_{d \in \mathbb{N}^*}$ . For that we consider the functional

$$g(\gamma) = \begin{cases} \int_I \theta(\|\dot{\gamma}(t)\|_{\mathcal{Z}'_1}) dt, & \text{if } \gamma \in W^{1,1}(I, \mathcal{Z}'_1), \text{ and } \gamma(s) = 0, \\ +\infty & \text{if } \gamma \notin W^{1,1}(I, \mathcal{Z}'_1), \text{ or } \gamma(s) \neq 0. \end{cases}$$

Therefore, using the concentration property (a),

$$(34) \quad \int_{\Gamma_I(\mathcal{Z}'_{1,w})} g(\gamma) d(r^2)_\# \hat{\eta}^d(\gamma) = \int_I \int_{\mathfrak{X}} \theta(\|\dot{\gamma}(t)\|_{\mathcal{Z}'_1}) d\hat{\eta}^d dt = \int_I \int_{\mathbb{R}^d} \theta(\|\pi^{d,T} \circ v^d(t, x)\|_{\mathcal{Z}'_1}) d\mu_t^d dt.$$

Using the monotonicity of the function  $\theta$ , (30) and Jensen's inequality, one shows

$$\begin{aligned} \theta(\|\pi^{d,T} \circ v^d(t, x)\|_{\mathcal{Z}'_1}) &\leq \theta(\|v^d(t, x)\|_{\mathbb{R}^d}) \leq \theta\left(\left\| \int_{(\pi^d)^{-1}(x)} \pi^d \circ v(t, y) d\mu_{t,x}(y) \right\|_{\mathbb{R}^d}\right) \\ &\leq \int_{(\pi^d)^{-1}(x)} \theta(\|v(t, y)\|_{\mathcal{Z}'_1}) d\mu_{t,x}(y). \end{aligned}$$

So, the disintegration Theorem E.1 and (34) yield the estimate,

$$\int_{\Gamma_I(\mathcal{Z}'_{1,w})} g(\gamma) d(r^2)_\# \hat{\eta}^d(\gamma) \leq \int_I \int_{\mathcal{Z}'_1} \theta(\|v(t, x)\|_{\mathcal{Z}'_1}) d\mu_t(x) dt < \infty.$$

Then, by Lemma B.1 the family  $\{(r^2)_\# \hat{\eta}^d\}_{d \in \mathbb{N}}$  is (weakly) tight if we prove that the functional  $g$  has relatively compact sublevels on  $(\Gamma_I(\mathcal{Z}'_1), \|\cdot\|_{\Gamma_I(\mathcal{Z}'_{1,w})})$ . Indeed, consider the set  $\mathcal{A} = \{\gamma \in \Gamma_I(\mathcal{Z}'_1), g(\gamma) \leq c\}$  for some  $c \geq 0$ . Then by the Arzelà-Ascoli theorem  $\mathcal{A}$  is a relatively compact set of  $(\Gamma_I(\mathcal{Z}'_{1,w}), \|\cdot\|_{\Gamma_I(\mathcal{Z}'_{1,w})})$  since:

- For any given  $t \in \bar{I}$ , the set  $\mathcal{A}(t) := \{\gamma(t), \gamma \in \mathcal{A}\}$  is bounded in  $\mathcal{Z}'_1$ . In fact, by Jensen's inequality, we have

$$\theta(\|\gamma(t)\|_{\mathcal{Z}'_1}) \leq \theta\left(\int_s^t \|\dot{\gamma}(\tau)\|_{\mathcal{Z}'_1} d\tau\right) \leq g(\gamma) \leq c.$$

Remember that  $\theta$  is superlinear. Hence, the set  $\mathcal{A}(t)$  is relatively compact in  $\mathcal{Z}'_{1,w}$ .

- Equicontinuity: For any  $t_0 \in I$ ,  $\gamma \in \mathcal{A}$  and  $M > 0$ , we have

$$\|\gamma(t) - \gamma(t_0)\|_{\mathcal{X}'_{1,w}} \leq \int_{t_0}^t \|\dot{\gamma}(\tau)\|_{\mathcal{X}'_1} d\tau \leq \int_{\{\|\dot{\gamma}(\tau)\|_{\mathcal{X}'_1} \leq M\} \cap [t_0, t]} M d\tau + \int_{\{\|\dot{\gamma}(\tau)\|_{\mathcal{X}'_1} > M\}} \|\dot{\gamma}(\tau)\|_{\mathcal{X}'_1} d\tau.$$

Hence, using Lemma D.1, one gets the equicontinuity of the set  $\mathcal{A}$ .

It still to check that  $\mathcal{A}$  is relatively (sequentially) compact in  $(\Gamma_I(\mathcal{Z}'_1), \|\cdot\|_{\Gamma_I(\mathcal{Z}'_{1,w})})$ . For that consider a sequence  $(\gamma_n)_{n \in \mathbb{N}^*}$  in  $\mathcal{A}$  and notice by Lemma D.1 that the family  $\mathcal{F} = \{t \rightarrow \|\dot{\gamma}_n(t)\|_{\mathcal{X}'_1}, n \in \mathbb{N}^*\}$  is bounded and equi-integrable in  $L^1(I, dt)$ . Hence, by Dunford-Pettis Theorem D.2,  $\mathcal{F}$  is relatively sequentially compact in  $L^1(I, dt)$  for the weak topology  $\sigma(L^1, L^\infty)$ . So, there exists a subsequence, still denoted by  $(\|\dot{\gamma}_n(\cdot)\|_{\mathcal{X}'_1})_{n \in \mathbb{N}^*}$ , such that it converges to  $m(\cdot) \in L^1(I, dt)$ . Moreover, we have the following estimate for any  $t, t_0 \in \bar{I}$ ,

$$(35) \quad \|\gamma_n(t) - \gamma_n(t_0)\|_{\mathcal{X}'_1} \leq \int_{t_0}^t \|\dot{\gamma}_n(s)\|_{\mathcal{X}'_1} ds.$$

Using the relative (sequential) compactness of  $\mathcal{A}$  in  $(\Gamma_I(\mathcal{Z}'_{1,w}), \|\cdot\|_{\Gamma_I(\mathcal{Z}'_{1,w})})$  proved before, we get a convergent subsequence  $(\gamma_{n_k})_{k \in \mathbb{N}^*}$  to a given  $\gamma \in \Gamma_I(\mathcal{Z}'_{1,w})$ . In particular, for each  $t \in \bar{I}$  and  $j \in \mathbb{N}^*$  we have

$$\langle \gamma_{n_k}(t) - \gamma(t), e_j \rangle_{\mathcal{X}'_{1,\mathbb{R}}} \xrightarrow{k \rightarrow \infty} 0.$$

Now, the Fatou's lemma yields

$$\|\gamma(t) - \gamma(t_0)\|_{\mathcal{X}'_1}^2 = \sum_{j=1}^{\infty} \liminf_{k \rightarrow \infty} |\langle \gamma_{n_k}(t) - \gamma_{n_k}(t_0), e_j \rangle_{\mathcal{X}'_{1,\mathbb{R}}}|^2 \leq \liminf_{k \rightarrow \infty} \|\gamma_{n_k}(t) - \gamma_{n_k}(t_0)\|_{\mathcal{X}'_1}^2,$$

and subsequently (35) gives

$$\|\gamma(t) - \gamma(t_0)\|_{\mathcal{X}'_1} \leq \int_{t_0}^t m(s) ds.$$

So, we conclude that the limit point  $\gamma \in \Gamma_I(\mathcal{Z}'_1)$ . Applying now Lemma B.1, one obtains the tightness of the family  $\{(r^2)_{\#} \hat{\eta}^d\}_{d \in \mathbb{N}}$  in  $\mathfrak{P}(\Gamma_I(\mathcal{Z}'_1))$ . Let  $\eta$  be any (weak) narrow limit point of  $\hat{\eta}^d$ . Hence taking the limit  $d_i \rightarrow +\infty$  in (32) and using the weak narrow convergence  $\hat{\eta}^{d_i} \xrightarrow{i \rightarrow \infty} \eta$ , one proves

$$\int_{\mathcal{X}'_1} \varphi d\mu_t = \int_{\mathfrak{X}} \varphi(\gamma(t)) d\eta(x, \gamma) = \int_{\mathcal{X}'_1} \varphi d(e_t)_{\#} \eta,$$

for all  $\varphi \in C_b(\mathcal{X}'_{1,w}, \mathbb{R})$  and  $t \in I$ . Since  $\mathcal{X}'_{1,w}$  is a Suslin space the above identity extends to any bounded Borel function on  $\mathcal{X}'_1$  (see Appendix B and C). This proves (ii).

The concentration property: Let  $v_1, v_2$  two Borel extensions of the vector field  $v$  to  $\mathbb{R} \times \mathcal{X}'_1$  such that  $v_1(t, x) = x_0$ , with  $0 \neq x_0 \in \mathcal{X}'_1$  and  $v_2(t, x) = 0$  for any  $x \notin \mathcal{Z}_1$ . By (ii), we remark

$$\int_{\mathfrak{X}} \int_I \|v_1(t, \gamma(t)) - v_2(t, \gamma(t))\|_{\mathcal{X}'_1} dt d\eta = \int_I \int_{\mathcal{X}'_1} \|v_1(t, x) - v_2(t, x)\|_{\mathcal{X}'_1} d\mu_t(x) dt = 0.$$

So,  $\int_I 1_{\{\gamma(t) \notin \mathcal{Z}_1\}} dt = 0$  for  $\eta$ -a.e., which means that  $\gamma(t) \in \mathcal{Z}_1$  for a.e.  $t \in I$  and  $\eta$ -a.e.  $(x, \gamma) \in \mathfrak{X}$ . A similar estimate yields,

$$(36) \quad \int_{\mathfrak{X}} \int_I \|v(t, \gamma(t))\|_{\mathcal{X}'_1} dt d\eta = \int_I \int_{\mathcal{X}'_1} \|v(t, x)\|_{\mathcal{X}'_1} d\mu_t(x) dt < \infty,$$

which means that  $\|v(t, \gamma(t))\|_{\mathcal{X}'_1} \in L^1(I, dt)$  for  $\eta$ -a.e.

Let  $w^{d'} : \mathbb{R} \times \mathbb{R}^{d'} \rightarrow \mathbb{R}^{d'}$  be a bounded uniformly continuous vector field and define  $\hat{w} : \mathbb{R} \times \mathcal{X}'_1 \rightarrow \mathcal{X}'_1$  such that  $\hat{w}(\tau, x) = \pi^{d',T} \circ w^{d'}(\tau, \pi^{d'}(x))$ . Notice that  $\hat{w}(t, \pi^{d',T} \circ \pi^d(x)) = \hat{w}(t, x)$  for any  $d \geq d'$ . We also define, as in (30), the projected vector field,

$$\hat{w}^d(t, y) := \int_{(\pi^d)^{-1}(y)} \pi^d \circ \hat{w}(t, x) d\mu_{t,y}(x), \quad \text{for } \mu_t^d - a.e. y \in \mathbb{R}^d \text{ and } a.e. t \in I.$$

Since  $\mu_{t,y}((\pi^d)^{-1}(y)) = 1$  for  $\mu_t^d$ -almost everywhere  $y \in \mathbb{R}^d$ , one remarks

$$\hat{w}^d(t, y) = \int_{(\pi^d)^{-1}(y) \ni x} \pi^d \circ \hat{w}(t, \pi^{d',T} \circ \pi^d(x)) d\mu_{t,y}(x) = \pi^d \circ \hat{w}(t, \pi^{d',T}(y)).$$

Then using the concentration property (a), one shows for any  $d \geq d'$ ,

$$\begin{aligned} \int_{\mathfrak{X}} \|\gamma(t) - x - \int_s^t \hat{w}(\tau, \gamma(\tau)) d\tau\|_{\mathcal{X}'_{1,w}} d\hat{\eta}^d(x, \gamma) &\leq \int_{\mathfrak{X}_d} \|\gamma(t) - x - \int_s^t \hat{w}^d(\tau, \gamma(\tau)) d\tau\|_{\mathbb{R}^d} d\eta^d \\ &\leq \int_s^t \int_{\mathbb{R}^d} \|v^d(\tau, x) - \hat{w}^d(\tau, x)\|_{\mathbb{R}^d} d\mu_\tau^d d\tau. \end{aligned}$$

Here  $\mathfrak{X}_d$  denotes  $\mathbb{R}^d \times \Gamma_I(\mathbb{R}^d)$ . Using the disintegration Theorem E.1 and a similar estimate as (31), one proves

$$\int_{\mathbb{R}^d} \|v^d(\tau, x) - \hat{w}^d(\tau, x)\|_{\mathbb{R}^d} d\mu_\tau^d(x) \leq \int_{\mathcal{X}'_1} \|v(\tau, x) - \hat{w}(\tau, x)\|_{\mathcal{X}'_1} d\mu_\tau(x).$$

Notice that the map  $(x, \gamma) \rightarrow \|\gamma(t) - x - \int_s^t \hat{w}(\tau, \gamma(\tau)) d\tau\|_{\mathcal{X}'_{1,w}}$  is continuous on  $\mathfrak{X}$ . Hence, by the narrow convergence  $\hat{\eta}^d \rightarrow \eta$ , one obtains

$$(37) \quad \int_{\mathfrak{X}} \|\gamma(t) - x - \int_s^t \hat{w}(\tau, \gamma(\tau)) d\tau\|_{\mathcal{X}'_{1,w}} d\eta \leq \int_s^t \int_{\mathcal{X}'_1} \|v(\tau, x) - \hat{w}(\tau, x)\|_{\mathcal{X}'_1} d\mu_\tau d\tau.$$

Now, we use an approximation argument. For any  $\varepsilon > 0$  there exists, by dominated convergence, a  $d \in \mathbb{N}^*$  such that

$$(38) \quad \int_s^t \int_{\mathcal{X}'_1} \|v(\tau, x) - \hat{\pi}^d \circ v(\tau, x)\|_{\mathcal{X}'_1} d\mu_\tau d\tau \leq \varepsilon.$$

Remember that  $\hat{\pi}^d \circ v(\tau, x) = \sum_{i=1}^d \langle v(\tau, x), e_i \rangle_{\mathcal{X}'_1} e_i$  with  $\langle v(\tau, x), e_i \rangle_{\mathcal{X}'_1} \in L^1([s, t] \times \mathcal{X}'_{1,w}, \nu)$  where the measure  $\nu$  is given by (33). Remark that the metric space  $[s, t] \times \mathcal{X}'_{1,w}$  is Suslin. Hence,  $\nu$  is a Radon finite measure on  $[s, t] \times \mathcal{X}'_{1,w}$ . According to Corollary C.4 the space of Lipschitz bounded functions  $\text{Lip}_b([s, t] \times \mathcal{X}'_{1,w})$  is dense in  $L^1([s, t] \times \mathcal{X}'_{1,w}, \nu)$ . So, there exist  $d$  Lipschitz bounded vector fields  $w_i : [s, t] \times \mathcal{X}'_{1,w} \rightarrow \mathbb{R}$  such that for  $i = 1, \dots, d$ ,

$$(39) \quad \int_{[s,t] \times \mathcal{X}'_1} |\langle v(\tau, x), e_i \rangle_{\mathcal{X}'_1} - w_i(\tau, x)| d\nu(t, x) \leq \frac{\varepsilon}{\sqrt{d}}.$$

Again, by dominated convergence, there exists  $d' \in \mathbb{N}^*$  such that for  $i = 1, \dots, d$ ,

$$(40) \quad \int_{[s,t] \times \mathcal{X}'_1} |w_i(\tau, x) - w_i(\tau, \hat{\pi}^{d'}(x))| d\nu(t, x) \leq \frac{\varepsilon}{\sqrt{d}}.$$

Notice that  $\hat{\pi}^{d'}(x) \rightarrow x$  in  $\mathcal{X}'_{1,w}$  when  $d' \rightarrow \infty$ . Define the following vector field,

$$(41) \quad \hat{w}(\tau, x) = \sum_{i=1}^d w_i(\tau, \hat{\pi}^{d'}(x)) e_i.$$



Then it is easy to see that  $\hat{w} : [s, t] \times \mathcal{Z}'_1 \rightarrow \mathcal{Z}'_1$  is bounded Lipschitz since  $\mathcal{Z}'_1$  is continuously embedded into  $\mathcal{Z}'_{1,w}$ . So, we can apply the estimate (37) with the choice (41). Indeed, one easily checks that  $\hat{\pi}^n \circ \hat{w}(t, \hat{\pi}^n(x)) = \hat{w}(t, x)$  for any  $n \geq d_0 = \max(d, d')$  (this determines a bounded uniformly continuous vector field  $w^{d_0} : [s, t] \times \mathbb{R}^{d_0} \rightarrow \mathbb{R}^{d_0}$ ). The triangle inequality with (37) and (ii), yield

$$\int_{\mathfrak{X}} \|\gamma(t) - x - \int_s^t v(\tau, \gamma(\tau)) d\tau\|_{\mathcal{Z}'_{1,w}} d\eta \leq 2 \int_s^t \int_{\mathcal{Z}'_1} \|v(\tau, x) - \hat{w}(\tau, x)\|_{\mathcal{Z}'_1} d\mu_\tau d\tau.$$

Estimating the right hand side with (38)-(40), one obtains

$$\int_s^t \int_{\mathcal{Z}'_1} \|v(\tau, x) - \hat{w}(\tau, x)\|_{\mathcal{Z}'_1} d\mu_\tau d\tau \leq 3\varepsilon.$$

So, for any  $t \in I$ , there exists an  $\eta$ -negligible set  $\mathcal{N}$  such that

$$(42) \quad \gamma(t) = x + \int_s^t v(\tau, \gamma(\tau)) d\tau, \quad \text{for } (x, \gamma) \in \mathfrak{X} \setminus \mathcal{N}.$$

Remember that  $\gamma$  are continuous curves in  $\mathcal{Z}'_1$  with  $v(\tau, \gamma(\tau)) \in L^1(I, \mathcal{Z}'_1)$  for  $\eta$ -a.e. according to (36). So, taking a dense sequence  $(t_i)_{i \in \mathbb{N}^*} \in I$  and using the latter properties, one can obtain an  $\eta$ -negligible  $\mathcal{N}_0$  set independent of  $t$  such that for all  $(x, \gamma) \in \mathfrak{X} \setminus \mathcal{N}_0$  we have  $\gamma(t) = x + \int_s^t v(\tau, \gamma(\tau)) d\tau$  for all  $t \in I$ . Hence,  $\gamma$  is an absolutely continuous curve in  $W^{1,1}(I, \mathcal{Z}'_1)$  with  $\dot{\gamma}(t) = v(t, \gamma(t))$  a.e.  $t \in I$  and  $\gamma(s) = x$ .  $\square$

**Proof of Theorem 2.3:** Suppose that  $\mu_t(B_{\mathcal{Z}'_1}(0, R)) = 1$  for some  $R > 0$ . We will use the properties of the measure  $\eta$  provided by Proposition 4.1. In particular, by Hölder inequality and Proposition 4.1-(ii), one shows for any  $p \geq 1$ ,

$$(43) \quad \int_{\mathfrak{X}} \left( \int_I \|\gamma(t)\|_{\mathcal{Z}'_1}^p dt \right)^{\frac{1}{p}} d\eta \leq \left( \int_I \int_{\mathfrak{X}} \|\gamma(t)\|_{\mathcal{Z}'_1}^p d\eta dt \right)^{\frac{1}{p}} = \left( \int_I \int_{\mathcal{Z}'_1} \|x\|_{\mathcal{Z}'_1}^p d\mu_t dt \right)^{\frac{1}{p}} \leq R|I|^{\frac{1}{p}}.$$

Hence, Fatou's lemma gives

$$\int_{\mathfrak{X}} \|\gamma(t)\|_{L^\infty(I, \mathcal{Z}'_1)} d\eta = \int_{\mathfrak{X}} \liminf_{p \rightarrow \infty} \left( \int_I \|\gamma(t)\|_{\mathcal{Z}'_1}^p dt \right)^{\frac{1}{p}} d\eta \leq R.$$

Thus, there exists an  $\eta$ -negligible set  $\mathcal{N}$  such that for all  $(x, \gamma) \in \mathfrak{X} \setminus \mathcal{N}$ , the norm  $\|\gamma(t)\|_{L^\infty(I, \mathcal{Z}'_1)}$  is finite and the Duhamel formula (42) holds true for all  $t \in I$ . Since the vector field is bounded on bounded sets of  $\mathcal{Z}'_1$ , one sees that  $\dot{\gamma}(t) = v(t, \gamma(t)) \in L^\infty(I, \mathcal{Z}'_1)$ . So,  $\eta$  concentrates on the set  $\mathcal{A} \subset \mathfrak{X}$  of weak solutions  $\gamma \in L^\infty(I, \mathcal{Z}'_1) \cap W^{1,\infty}(I, \mathcal{Z}'_1)$  of the initial value problem (10) satisfying  $\gamma(s) = x$  for some fixed  $s \in I$ . Consider now the subset

$$\mathcal{B} = \{(x, \gamma) \in \mathcal{Z}'_1 \times \Gamma_I(\mathcal{Z}'_1) : T_{\min}(x, s) \text{ or } T_{\max}(x, s) \in \bar{I}\},$$

where  $T_{\min}(x, s)$  and  $T_{\max}(x, s)$  are the minimal and maximal time of existence of the strong solution of (10) with initial condition  $\gamma(s) = x \in \mathcal{Z}'_1$ . Using the Definition 2.2 of local well posedness and the blowup alternative, we see that  $\mathcal{A} \cap \mathcal{B} = \emptyset$ ; since we can not find a weak solution that extends a strong solution beyond its maximal interval of existence. So, the set  $\mathcal{B} \subset \mathcal{A}^c$  is  $\eta$ -negligible and  $\eta$  concentrates on  $\mathcal{B}^c \cap \mathcal{A}$  a subset of strong solutions of the initial value problem (10) defined at least on the whole interval  $\bar{I}$ . Hence, we conclude that for  $\eta$ -almost everywhere  $\gamma(t) = \Phi(t, s)\gamma(s)$  for any  $t \in \bar{I}$ , with  $\Phi$  is the local flow provided by (LWP). Moreover, if we take  $\varphi = 1_{\mathcal{B}_0}$ , we get

$$\int_{\mathcal{Z}'_1} 1_{\mathcal{B}_0}(x) d\mu_s = \int_{\mathfrak{X}} 1_{\mathcal{B}_0}(\gamma(s)) d\eta = \int_{\mathcal{B}^c \cap \mathcal{A}} 1_{\mathcal{B}_0}(\gamma(s)) d\eta = 1,$$

with  $\mathcal{B}_0 = \{x \in \mathcal{X}_1 : (T_{min}(x, s), T_{max}(x, s)) \supseteq \bar{I}\}$ . This proves that  $(T_{min}(s, x), T_{max}(s, x)) \supseteq \bar{I}$  for  $\mu_s$ -almost everywhere  $x \in \mathcal{X}_1$ . Using these concentration properties of  $\eta$  with Proposition 4.1-(ii), we deduce for any  $t \in I$  and any bounded Borel function  $\varphi : \mathcal{X}_1 \rightarrow \mathbb{R}$ ,

$$(44) \quad \int_{\mathcal{X}_1} \varphi d\mu_t = \int_{\bar{x}} \varphi(\gamma(t)) d\eta = \int_{\bar{x}} \varphi \circ \Phi(t, s)[\gamma(s)] d\eta = \int_{\mathcal{X}_1} \varphi \circ \Phi(t, s)(x) d\mu_s.$$

The map  $\Phi(t, s) : \mathcal{B}_0 \rightarrow \mathcal{X}_1$  is Borel thanks to the Definition 2.2-(iv). In particular, we obtain that  $\mu_t = \Phi(t, s)_\# \mu_s$  for all  $t \in I$ .  $\square$

**Proof of Theorem 2.4.** Recall that  $v$  in this case is a continuous vector field  $v : \mathbb{R} \times \mathcal{X}_1 \rightarrow \mathcal{X}_0 \subset \mathcal{X}'_1$  satisfying the scalar velocity estimate (A'). Using Proposition 4.1-(ii), one proves

$$\int_{\bar{x}} \int_I \|v(t, \gamma(t))\|_{\mathcal{X}_0} dt d\eta = \int_I \int_{\mathcal{X}_0} \|v(t, x)\|_{\mathcal{X}_0} d\mu_t(x) dt < \infty.$$

So, we deduce that  $v(t, \gamma(t)) \in L^1(I, \mathcal{X}_0)$  for  $\eta$ -a.e. Then the Duhamel formula (42) implies that  $\eta$  concentrates actually on absolutely continuous curves  $\gamma \in W^{1,1}(I, \mathcal{X}_0)$ . Furthermore, using the estimate (43), with  $p = 2$ , we see also that  $\gamma \in L^2(I, \mathcal{X}_1)$  for  $\eta$ -a.e. Therefore, the measure  $\eta$  concentrates on the solutions  $\gamma \in L^2(I, \mathcal{X}_1) \cap W^{1,1}(I, \mathcal{X}_0)$  of the initial value problem (10). Now, we claim that (16) implies the uniqueness of those "weak" solutions. Let  $\gamma_1, \gamma_2 \in L^2(I, \mathcal{X}_1) \cap W^{1,1}(I, \mathcal{X}_0)$  two solutions of (10) such that for some fixed  $s \in I$ ,  $\gamma_1(s) = \gamma_2(s)$ . Since  $\gamma_i, i = 1, 2$ , are continuous  $\mathcal{X}_0$ -valued functions on  $\bar{I}$ , we take

$$M = \max_{i=1,2} \left( \sup_{t \in \bar{I}} \|\gamma_i\|_{\mathcal{X}_0} \right).$$

Notice that the case  $M = 0$  is trivial. The hypothesis (16) with the Duhamel formula (42), give the existence of a constant  $C(M, \bar{I}) > 0$  such that for any  $t \in \bar{I}$ ,

$$(45) \quad \begin{aligned} \|\gamma_1(t) - \gamma_2(t)\|_{\mathcal{X}_0} &\leq \int_s^t \|v(\tau, \gamma_1(\tau)) - v(\tau, \gamma_2(\tau))\|_{\mathcal{X}_0} d\tau \\ &\leq C(M, \bar{I}) \int_s^t (\|\gamma_1(\tau)\|_{\mathcal{X}_1}^2 + \|\gamma_2(\tau)\|_{\mathcal{X}_1}^2) \|\gamma_1(\tau) - \gamma_2(\tau)\|_{\mathcal{X}_0} d\tau. \end{aligned}$$

For each  $t \in \bar{I}$  one can find a nontrivial interval  $I(t) \subset \bar{I}$  containing  $t$  such that

$$C(M, \bar{I}) \int_{I(t)} (\|\gamma_1(\tau)\|_{\mathcal{X}_1}^2 + \|\gamma_2(\tau)\|_{\mathcal{X}_1}^2) d\tau < 1.$$

Moreover, one can choose all the  $I(t)$  to be open sets of  $\bar{I}$ . Hence, the cover  $(I(t))_{t \in \bar{I}}$  admits a finite subcover  $\bar{I} = \cup_{i=1}^n I(t_i)$ . By relabelling the  $t_i$ , one can assume that  $s \in I(t_1)$  and  $I(t_1) \cap I(t_2) \neq \emptyset$ . Then, using (45), we have the bound

$$(46) \quad \sup_{\bar{I}(t_1)} \|\gamma_1(t) - \gamma_2(t)\|_{\mathcal{X}_0} < \sup_{\bar{I}(t_1)} \|\gamma_1(t) - \gamma_2(t)\|_{\mathcal{X}_0}.$$

So, the two curves  $\gamma_1$  and  $\gamma_2$  coincide on the subinterval  $\bar{I}(t_1)$  and the following inequality holds true for any  $t \in \bar{I}$  and any  $r \in I(t_1) \cap I(t_2)$ ,

$$\|\gamma_1(t) - \gamma_2(t)\|_{\mathcal{X}_0} \leq C(M, \bar{I}) \int_r^t (\|\gamma_1(\tau)\|_{\mathcal{X}_1}^2 + \|\gamma_2(\tau)\|_{\mathcal{X}_1}^2) \|\gamma_1(\tau) - \gamma_2(\tau)\|_{\mathcal{X}_0} d\tau.$$

Hence, a similar inequality as (46) holds true with  $\bar{I}(t_2)$  instead of  $\bar{I}(t_1)$ . Iterating the same argument one proves that  $\gamma_1 = \gamma_2$  on  $\bar{I}$ . In particular, this proves Definition 2.2-(i).

The assumption (ii) of Thm. 2.4 implies that the measure  $\eta$  is concentrated on the set  $B \cap \mathcal{X}_1 \times \Gamma_I(\mathcal{X}'_{1,w})$ . Moreover, the assumption (iii) with the above uniqueness property imply

that for each  $x \in \mathcal{Z}_1 \cap B$  there exists a unique curve  $\gamma \in L^2(I, \mathcal{Z}_1) \cap W^{1,1}(I, \mathcal{Z}_0)$  satisfying the initial value problem (10) with  $\gamma(s) = x$ . This means that the measure  $\eta$  is concentrated on the set  $\{(x, \gamma) : x \in \mathcal{Z}_1 \cap B, \gamma(t) = \Phi(t, s)(x), \forall t \in \bar{I}\}$ , where  $\Phi(t, s)$  is the local flow of (10). Let  $A = \mathcal{Z}_1 \cap B$ , then one can prove that  $A$  is  $\Phi(t, s)$ -invariant modulo  $\mu_s$ . In fact, using the properties of  $\eta$  one shows

$$\mu_s(A \Delta \Phi(t, s)^{-1}(A)) \leq \mu_s(\Phi(t, s)^{-1}(A)^c) = \int_A 1_{A^c}(\Phi(t, s)(x)) d\mu_s = \int_{\bar{x}} 1_{A^c}(\gamma(t)) d\eta.$$

Hence, the invariance follows since  $\mu_t(A^c) = 0$ . Now, repeating the same argument as in (44), we obtain the claimed result in Thm. 2.4.  $\square$

## Appendix: Measure theoretical tools

We briefly review some tools in measure theory that have been used throughout the text.

### A Borel sets

Let  $\mathcal{Z}_1 \subset \mathcal{Z}_0 \subset \mathcal{Z}'_1$  a rigged Hilbert space such that  $(\mathcal{Z}_1, \mathcal{Z}_0)$  is a pair of complex *separable* Hilbert spaces. Notice that  $\mathcal{Z}'_1$  is automatically separable. Let  $\mathcal{I} : \mathcal{Z}_1 \rightarrow \mathcal{Z}_0$  denotes the continuous embedding of  $\mathcal{Z}_1$  into  $\mathcal{Z}_0$ , i.e.  $\mathcal{I}(x) = x$  for all  $x \in \mathcal{Z}_1$ . Taking the adjoint of  $\mathcal{I}$  and identifying the spaces  $(\mathcal{Z}_1, \mathcal{Z}_0)$  with their topological duals, one obtains a continuous linear map  $\mathcal{I}^* : \mathcal{Z}_0 \rightarrow \mathcal{Z}_1$ . Moreover, the operator  $B = \mathcal{I}\mathcal{I}^* : \mathcal{Z}_0 \rightarrow \mathcal{Z}_0$ , is bounded self-adjoint and nonnegative. Hence, we can define its square root  $\sqrt{B} : \mathcal{Z}_0 \rightarrow \mathcal{Z}_1$ . Remark that we have

$$\langle \sqrt{B}x, \sqrt{B}y \rangle_{\mathcal{Z}_0} = \langle \mathcal{I}^*x, \mathcal{I}^*y \rangle_{\mathcal{Z}_1} = \langle x, y \rangle_{\mathcal{Z}'_1}, \quad \text{and} \quad \langle \sqrt{B}x, \sqrt{B}y \rangle_{\mathcal{Z}_1} = \langle \mathcal{I}^*x, y \rangle_{\mathcal{Z}_1} = \langle x, y \rangle_{\mathcal{Z}_0}.$$

One can also prove that  $\text{Ran}(\sqrt{B}) = \mathcal{Z}_1$ . Hence, the linear operator  $A = (\sqrt{B})^{-1} : \mathcal{Z}_1 \subset \mathcal{Z}_0 \rightarrow \mathcal{Z}_0$ , with domain  $\mathcal{Z}_1$ , is clearly self-adjoint and nonnegative. Furthermore, we have for any  $x \in \mathcal{D}(A) = \mathcal{Z}_1$ ,

$$\|x\|_{\mathcal{Z}'_1} = \|A^{-1}x\|_{\mathcal{Z}_0}, \quad \text{and} \quad \|x\|_{\mathcal{Z}_1} = \|Ax\|_{\mathcal{Z}_0}.$$

So, the pair of Hilbert spaces  $(\mathcal{Z}_1, \mathcal{Z}'_1)$  identifies with  $((\mathcal{D}(A), \|A \cdot\|_{\mathcal{Z}_0}), (\overline{\mathcal{D}(A^{-1})}^{comp}, \|A^{-1} \cdot\|_{\mathcal{Z}_0}))$  where the latter notation stands for the completion of  $\mathcal{D}(A^{-1})$  with respect to the norm  $\|A^{-1} \cdot\|_{\mathcal{Z}_0}$ .

As consequence of the above remarks,  $\mathcal{Z}_1$  is a Borel subset of  $\mathcal{Z}_0$  and  $\mathcal{Z}_0$  is a Borel subset of  $\mathcal{Z}'_1$ . Indeed, consider the sequence of continuous functions  $f_n : \mathcal{Z}_0 \rightarrow \mathbb{R}$ ,

$$f_n(x) = \|(1 + \frac{A}{n})^{-1}x\|_{\mathcal{Z}'_1}.$$

Hence,  $f_n$  converges pointwise to a measurable function  $f(x) = \|x\|_{\mathcal{Z}'_1}$ , if  $x \in \mathcal{Z}_1$ ,  $f(x) = +\infty$  otherwise and  $\mathcal{Z}_1 = f^{-1}(\mathbb{R})$  is a Borel subset of  $\mathcal{Z}_0$ . A similar argument works for  $\mathcal{Z}'_1$  and we have the following inclusions of Borel  $\sigma$ -algebras

$$\mathcal{B}(\mathcal{Z}_1) \subset \mathcal{B}(\mathcal{Z}_0) \subset \mathcal{B}(\mathcal{Z}'_1).$$

In particular, a Borel probability measure  $\mu \in \mathfrak{P}(\mathcal{Z}'_1)$  that concentrates on  $\mathcal{Z}_1$ , i.e.  $\mu(\mathcal{Z}_1) = 1$ , is a Borel probability measure in  $\mathfrak{P}(\mathcal{Z}_1)$ .

## B Radon spaces, Tightness

*Radon spaces:* Let  $X$  be a Hausdorff topological space and  $\mathcal{B}(X)$  denotes the  $\sigma$ -algebra of Borel sets. Recall that a Borel measure  $\mu$  is a Radon measure if it is locally finite and inner regular. A topological space is called a *Radon space* if every finite Borel measure is a Radon measure. In particular, it is known that any Polish and more generally any Suslin space is Radon (see e.g. [43]). Recall that a Polish space is a topological space homeomorphic to a separable complete metric space and a Suslin space is the image of a Polish space under a continuous mapping.

*Tightness:* Let  $X$  be a separable metric space and  $\mathfrak{P}(X)$  the set of Borel probability measures on  $X$ . The narrow convergence topology in  $\mathfrak{P}(X)$  is given by the neighborhood basis

$$N(\mu, \delta, \varphi_1, \dots, \varphi_n) = \{\nu \in \mathfrak{P}(X) : \max_{i=1, \dots, n} |\nu(\varphi_i) - \mu(\varphi_i)| < \delta\},$$

with  $\mu \in \mathfrak{P}(X)$ ,  $\delta > 0$  and  $\varphi_1, \dots, \varphi_n \in C_b(X, \mathbb{R})$ . It is known, in this case, that the narrow topology on  $\mathfrak{P}(X)$  is a separable metric topology. We say that a set  $\mathcal{K} \subset \mathfrak{P}(X)$  is *tight* if,

$$(47) \quad \forall \varepsilon > 0, \exists K_\varepsilon \text{ compact in } X \text{ such that } \mu(X \setminus K_\varepsilon) \leq \varepsilon, \forall \mu \in \mathcal{K}.$$

Prokhorov's theorem says that any tight set  $\mathcal{K} \subset \mathfrak{P}(X)$  is relatively (sequentially) compact in  $\mathfrak{P}(X)$  in the narrow topology. A useful characterization is given below (see [5, Remark 5.15]).

**Lemma B.1.** *A set  $\mathcal{K} \subset \mathfrak{P}(X)$  is tight if and only if there exists a function  $\varphi : X \rightarrow [0, +\infty]$ , whose sublevels  $\{x \in X : \varphi(x) \leq c\}$  are relatively compact in  $X$  and satisfying*

$$\sup_{\mu \in \mathcal{K}} \int_X \varphi(x) d\mu(x) < +\infty.$$

In particular, in a metric separable Radon space  $X$  every narrowly converging sequence is tight. We also use the following tightness criterion from [5, Lemma 5.2.2].

**Lemma B.2.** *Let  $X, X_1, X_2$  be separable metric spaces and let  $r^i : X \rightarrow X_i$  be continuous maps such that the product map*

$$r := r^1 \times r^2 : X \rightarrow X_1 \times X_2$$

*is proper. Let  $\mathcal{K} \subset \mathfrak{P}(X)$  be such that  $\mathcal{K}_i := r_{\#}^i(\mathcal{K})$  is tight in  $\mathfrak{P}(X_i)$  for  $i = 1, 2$ . Then also  $\mathcal{K}$  is tight in  $\mathfrak{P}(X)$ .*

## C Dense subsets in $L^p$ spaces

Several interesting classes of functions are known to be dense in  $L^p$  spaces for  $p \geq 1$ . Unfortunately, those type of results are scattered throughout the literature with various degrees of generality. So, we prefer to recall some useful statements here that hold for metric spaces and finite Radon measures. Since the proofs are simple and elegant, we provide them for reader's convenience.

**Proposition C.1.** *Let  $X$  be a metric space,  $\mu$  a finite Radon measure on  $X$  and  $f : X \rightarrow \mathbb{R}$  a Borel function. Then for every  $\varepsilon > 0$  there exists a bounded uniformly continuous function  $f_\varepsilon : X \rightarrow \mathbb{R}$  such that*

$$\mu(\{x \in X : f(x) \neq f_\varepsilon(x)\}) < \varepsilon.$$

*Furthermore,  $\sup_{x \in X} |f_\varepsilon(x)| \leq \sup_{x \in X} |f(x)|$ .*

*Proof.* Decompose the function  $f$  into a Borel positive and negative part,  $f = f^+ - f^-$ ,  $f^\pm \geq 0$  with  $f^+ = 1_{\{x: f(x) \geq 0\}} f$  and  $f^- = -1_{\{x: f(x) < 0\}} f$ . Since  $\mu$  is a Radon finite measure, one can chose a compact set  $K \subset X$  such that  $\mu(X \setminus K) < \varepsilon/2$ . Since the functions  $f|_K^\pm : K \rightarrow \mathbb{R}$  are

Borel, then by Lusin's theorem (see [47, Theorem (3.1.1) 2]) there exist continuous functions  $g_\varepsilon^\pm : K \rightarrow \mathbb{R}$  such that

$$(48) \quad \mu(\{x \in K : f^\pm(x) \neq g_\varepsilon^\pm(x)\}) < \varepsilon/4,$$

with  $\sup_{x \in K} |g_\varepsilon^\pm(x)| \leq \sup_{x \in K} |f^\pm(x)|$ . Remark that the bound (48) is still true if we replace  $g_\varepsilon^\pm$  with  $|g_\varepsilon^\pm|$ . So, we can take  $g_\varepsilon^\pm$  to be nonnegative. Notice also that  $g_\varepsilon^\pm$  are bounded uniformly continuous functions on  $K$ . The Tietze extension theorem in [39], says that any real-valued bounded uniformly continuous function on a subset of  $X$  can be extended to a bounded uniformly continuous function on the whole space. In fact, the following extensions

$$f_\varepsilon^\pm(x) = \begin{cases} g_\varepsilon^\pm(x), & \text{if } x \in K, \\ \inf_{y \in K} g_\varepsilon^\pm(y) \frac{d(x, y)}{d(x, K)} & \text{if } x \notin K. \end{cases}$$

are uniformly continuous on  $X$ . Moreover, we have  $\sup_{x \in X} f_\varepsilon^\pm(x) \leq \sup_{x \in X} f^\pm(x)$ , since a simple estimate yields

$$0 \leq \inf_{y \in K} g_\varepsilon^\pm(y) \frac{d(x, y)}{d(x, K)} \leq \sup_{y \in K} g_\varepsilon^\pm(y) \leq \sup_{y \in K} f^\pm(y).$$

Taking  $f_\varepsilon = f_\varepsilon^+ - f_\varepsilon^-$ , then one checks that

$$\mu(\{x \in K : f(x) \neq f_\varepsilon(x)\}) < \varepsilon/2,$$

and moreover  $\sup_{x \in X} |f_\varepsilon(x)| \leq \sup_{x \in X} |f(x)|$ .  $\square$

Let  $C_{b,u}(X)$  and  $\text{Lip}_b(X)$  denote respectively the spaces of bounded uniformly continuous functions and bounded Lipschitz continuous functions on  $X$ .

**Corollary C.2.** *Let  $X$  be a metric space,  $\mu$  a finite Radon measure on  $X$ . Then the space  $C_{b,u}(X)$  is dense in  $L^p(X, \mu)$  for  $p \geq 1$ .*

*Proof.* The space  $L^\infty(X, \mu)$  is dense in  $L^p(X, \mu)$ . For any  $\varepsilon > 0$  there exists  $f_\varepsilon$  satisfying the conclusion of Proposition C.1. Hence,

$$\begin{aligned} \|f - f_\varepsilon\|_{L^p}^p &= \int_{\{x: f(x) \neq f_\varepsilon(x)\}} |f(x) - f_\varepsilon(x)|^p d\mu \\ &\leq 2^p \mu(\{x : f(x) \neq f_\varepsilon(x)\}) \sup_{x \in X} \text{ess}|f(x)|^p \\ &\leq 2^p \varepsilon \sup_{x \in X} \text{ess}|f(x)|^p. \end{aligned}$$

$\square$

**Lemma C.3.** *If  $\mathcal{F}$  is a family of  $k$ -Lipschitz functions on a metric space  $X$ , then*

$$F(x) = \inf\{f(x), f \in \mathcal{F}\},$$

*is  $k$ -Lipschitz on the set of points where it is finite.*

*Proof.* Let  $x_1, x_2 \in X$ , then for any  $\varepsilon > 0$  there exists  $f_\varepsilon \in \mathcal{F}$  such that  $F(x_2) \geq f_\varepsilon(x_2) - \varepsilon$ . Hence,

$$F(x_1) - F(x_2) \leq f_\varepsilon(x_1) - f_\varepsilon(x_2) + \varepsilon \leq k d(x_1, x_2) + \varepsilon.$$

$\square$

**Corollary C.4.** *Let  $X$  be a metric space,  $\mu$  a finite Radon measure on  $X$ . Then the space of bounded Lipschitz functions  $\text{Lip}_b(X)$  is dense in  $L^p(X, \mu)$  for  $p \geq 1$ .*

*Proof.* Let  $f \in C_{b,u}(X)$  decomposes into a positive and negative part, i.e.:  $f = f^+ - f^-$ ,  $f^\pm \geq 0$  with  $f^+ = \frac{|f|+f}{2}$  and  $f^- = \frac{|f|-f}{2}$ . Let for any  $k \in \mathbb{N}$ ,

$$f_k^\pm(x) = \inf_{y \in X} f^\pm(y) + kd(x, y).$$

One easily checks that  $f_k^\pm$  are  $k$ -Lipschitz by Lemma C.3, nondecreasing sequences satisfying for any  $x \in X$ ,

$$0 \leq f_k^\pm(x) \leq f^\pm(x) \quad \text{and} \quad \lim_{k \rightarrow \infty} f_k^\pm(x) = f^\pm(x).$$

So, by dominated convergence and triangle inequality  $f_k := f_k^+ - f_k^- \in \text{Lip}_b(X)$  converges to  $f$  in the  $L^p$  norm.  $\square$

## D Equi-integrability and Dunford-Pettis theorem

Let  $(X, \Sigma)$  be a measurable space and  $\mu$  a finite measure on  $(X, \Sigma)$ . We say that a family  $\mathcal{F} \subset L^1(X, \mu)$  is *equi-integrable* if for any  $\varepsilon > 0$  there exists  $\delta > 0$  such that:

$$\forall B \in \Sigma, \mu(B) < \delta \Rightarrow \sup_{f \in \mathcal{F}} \int_B |f| d\mu < \varepsilon.$$

An interesting characterization of equi-integrability is given below (see e.g. [4, Proposition 1.27]).

**Lemma D.1.** *A bounded set  $\mathcal{F}$  in  $L^1(X, \mu)$  is equi-integrable if and only if*

$$\mathcal{F} \subset \{f \in L^1(X, \mu) : \int_X \theta(|f|) d\mu \leq 1\},$$

for some nondecreasing convex continuous function  $\theta : \mathbb{R}^+ \rightarrow [0, \infty]$  satisfying  $\theta(t)/t \rightarrow \infty$  when  $t \rightarrow \infty$  or equivalently if and only if

$$\lim_{M \rightarrow \infty} \sup_{f \in \mathcal{F}} \int_{\{|f| > M\}} |f| d\mu = 0.$$

**Theorem D.2** (Dunford-Pettis). *A bounded set  $\mathcal{F}$  in  $L^1(X, \mu)$  is relatively sequentially compact for the weak topology  $\sigma(L^1, L^\infty)$  if and only if  $\mathcal{F}$  is equi-integrable.*

## E Disintegration theorem

Let  $E, F$  be Radon separable metric spaces. We say that a measure-valued map  $x \in F \rightarrow \mu_x \in \mathfrak{P}(E)$  is Borel if  $x \in F \rightarrow \mu_x(B)$  is a Borel map for any Borel set  $B$  of  $E$ . We recall below the disintegration theorem (see e.g. [5, Theorem 5.3.1]).

**Theorem E.1.** *Let  $E, F$  be Radon separable metric spaces and  $\mu \in \mathfrak{P}(E)$ . Let  $\pi : E \rightarrow F$  be a Borel map and  $\nu = \pi_* \mu \in \mathfrak{P}(F)$ . Then there exists a  $\nu$ -a.e. uniquely determined Borel family of probability measures  $\{\mu_x\}_{x \in F} \subset \mathfrak{P}(E)$  such that  $\mu_x(X \setminus \pi^{-1}(x)) = 0$  for  $\nu$ -a.e.  $x \in E$ , and*

$$(49) \quad \int_E f(x) d\mu(x) = \int_F \left( \int_{\pi^{-1}(x)} f(y) d\mu_x(y) \right) d\nu(x),$$

for every Borel map  $f : E \rightarrow [0, +\infty]$ .

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