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On the uniqueness of probability measure solutions to Liouville's equation of Hamiltonian PDEs

Quentin Liard*

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Abstract

In this paper, we give a uniqueness result to a transport equation fulfilled by probability measure on a infinite dimensional Hilbert space. Main arguments are based on projective aspects and a probabilistic representation of the measure-valued solutions. It extends the work of Maniglia, which concerns the finite dimensional case and the work of Ammari and Nier, for a wider class of velocity field.

Keywords: Wigner measures ,continuity equation, characteristics theory

2010 Mathematics subject classification: 81S05, 81T10, 35Q55, 28A33

1 Introduction

Liouville's equation is a fundamental equation of statistical mechanics which describes the time evolution of phase-space distribution functions. Consider for instance a Hamiltonian system $H(p, q) = H(p_1, \dots, p_n, q_1, \dots, q_n)$ of finite degrees of freedom where $(q_1, \dots, q_n, p_1, \dots, p_n)$ are the position-momentum canonical coordinates. Then the time evolution of a probability density function $\varrho(p, q, t)$ describing the system at time t is governed by the Liouville equation,

$$\frac{\partial \varrho}{\partial t} + \{\varrho, H\} = 0, \quad (1)$$

with the Poisson bracket defined as follows,

$$\{\varrho, H\} = \sum_{i=1}^n \left[\frac{\partial H}{\partial p_i} \frac{\partial \varrho}{\partial q^i} - \frac{\partial H}{\partial q^i} \frac{\partial \varrho}{\partial p_i} \right].$$

By formally differentiating $\varrho(p_t, q_t, t)$ with respect to time, where (p_t, q_t) are solutions of the Hamiltonian equations, we recover the Liouville's theorem as stated by Gibbs "The distribution function is constant along any trajectory in phase space", i.e.,

$$\frac{d}{dt} \varrho(p_t, q_t, t) = 0.$$

In fact the characteristics method says that if the Hamiltonian is sufficiently smooth and generates a unique Hamiltonian flow Φ_t on the phase-space, then the density function

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$\varrho(p, q, t)$ is uniquely determined by its initial value $\varrho(p, q, 0)$ and it is given as the propagation along the characteristics, i.e.,

$$\varrho(p, q, t) = \varrho(\Phi_t^{-1}(p, q), 0).$$

It is known that Liouville's theorem holds in more general framework than the Hamiltonian systems. Consider a differential equation,

$$\frac{d}{dt}X = F(X), \quad X(t=0) = X_0, \quad (2)$$

with $X = (X_1, \dots, X_n) \in \mathbb{R}^n$ and $F = (F_1, \dots, F_n) : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a given smooth vector field such that a unique flow map $\Phi_t : \mathbb{R}^n \mapsto \mathbb{R}^n$ exists and solves the ODE (2). If the system (2) is at an initial statistical state described by a probability density function $\varrho(X, 0)$ at $t = 0$, then under the flow map Φ_t , the evolution of this state is described by a density $\varrho(X, t)$, which is the pull-back of the initial one,

$$\varrho(X, t) = \varrho(\Phi_t^{-1}(X), 0). \quad (3)$$

If the vector field F satisfies the Liouville's property, which is the following divergence-free condition,

$$\operatorname{div}(F) = \sum_{j=1}^n \frac{\partial F_j}{\partial X_j} = 0,$$

then the flow map Φ_t is volume preserving (or measure preserving) on the phase space and for all times the density $\varrho(X, t)$ verifies the Liouville equation,

$$\frac{\partial \varrho}{\partial t} + F \cdot \nabla_X \varrho = 0. \quad (4)$$

Again when the vector field is smooth the characteristics theory says that (3) is the unique solution of the Liouville equation (4) with the initial value $\varrho(X, 0)$. This enlightens the relationship between individual solutions of the ODE (2) and statistical (probability measure) solutions of the Liouville equation (4) and suggests that this is a general principle that could extend to non-smooth vector fields or to dynamical systems with infinite degrees of freedom. Actually the non-smooth framework has been carefully studied and uniqueness of probability measure solutions of Liouville's equation is established via a general superposition principle, see [4, 2, 15, 19, 30, 32] and also [13, 18]. The extension to dynamical systems with infinite degrees of freedom is less studied and the investigations are not oriented toward the study of classical PDEs, see [3, 27, 38], at the exception of the work [11, Appendix C] where the ideas of [4] was adapted to a rigged Hilbert space and applied to the nonlinear Hartree equation with singular potential.

Our aim in this article is to consider the above uniqueness property for Hamiltonian systems with infinite degrees of freedom related to some interesting nonlinear PDEs like the wave or Schrödinger equations. Beyond the fact that Liouville's equation is a natural ground for a statistical theory of Hamiltonian PDEs, we do have another motivation when addressing the previous uniqueness problem. In fact, when we study the relationship between quantum field theories and classical PDEs we encounter the above uniqueness problem, see [6, 11, 12]. Roughly speaking, the quantum counterpart of Liouville's equation is the Von Neumann equation which describes the time evolution of quantum states of Hamiltonian (linear) systems. If we attempt to carry on the classical limit, i.e. $\hbar \rightarrow 0$ where \hbar is an effective Planck constant which depends on the scaling of the system at

hand, then quantum states transform in the limit into probability measures satisfying a Liouville equation related to a nonlinear Hamiltonian PDE, see [8, 9, 10]. Therefore the uniqueness property for probability measure solutions of Liouville's equation will be a crucial step towards a rigorous justification of the classical limit or the so-called Bohr's correspondence principle.

It is not so obvious how to generalize the above considerations to Hamiltonian systems with infinite degrees of freedom [38]. One of the difficulties for instance is the lack of translation-invariant measures on infinite dimensional normed spaces. Nevertheless, the approach elaborated in [4] is well suited to a generalization for systems with infinite degrees of freedom. This was accomplished in [11] with the following Liouville's equation considered in a weak sense,

$$\partial_t \mu_t + \nabla^T (F \cdot \mu_t) = 0, \quad (5)$$

where $t \mapsto \mu_t$ are probability measure solutions and F is a non-autonomous vector field defined on a rigged Hilbert space $\mathcal{Z}_1 \subset \mathcal{Z}_0 \subset \mathcal{Z}'_1$. The result on uniqueness of probability measure solutions of Liouville's (5) proved in [11, Appendix C] uses a slightly strong assumption on the vector field F ,

$$\forall T > 0, \exists C > 0, \int_{-T}^T \left[\int_{\mathcal{Z}_1} \|F(t, z)\|_{\mathcal{Z}'_1}^2 d\mu_t(z) \right]^{\frac{1}{2}} dt \leq C. \quad (6)$$

This result was applied in [6, 7, 11, 29] to the mean-field theory and to the classical limit of quantum field theories. In this article we relax the above condition so that we require only the uniform estimate,

$$\forall T > 0, \exists C > 0, \int_{-T}^T \int_{\mathcal{Z}_1} \|F(t, z)\|_{\mathcal{Z}_0} d\mu_t(z) dt \leq C, \quad (7)$$

which fits better the energy method communally used to solve PDEs. To illustrate the difference between (6) and (7), we consider the following example. Let $\mathcal{Z}_0 = L^2(\mathbb{R})$, $\mathcal{Z}_1 = H^1(\mathbb{R})$ and consider the nonlinear Schrödinger (NLS) equation,

$$\begin{cases} i\partial_t z_t = -\Delta z_t + |z_t|^2 z_t \\ z_{|t=0} = z_0. \end{cases} \quad (8)$$

In the interaction representation the NLS equation is equivalent to the PDE,

$$\begin{cases} \partial_t \tilde{z}_t = F(t, \tilde{z}_t) := -ie^{-it\Delta} |e^{it\Delta} \tilde{z}_t|^2 (e^{it\Delta} \tilde{z}_t) \\ z_{|t=0} = z_0. \end{cases} \quad (9)$$

So, $F : H^1(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ is interpreted as non-autonomous vector field defined on the energy space $H^1(\mathbb{R})$. Sobolev's embedding gives the existence of a constant $C > 0$ such that

$$\|F(t, z)\|_{L^2(\mathbb{R})} \leq C \|z\|_{H^1(\mathbb{R})}^2 \|z\|_{L^2(\mathbb{R})}.$$

So, suppose that we have the following a priori information on the measures $(\mu_t)_{t \in \mathbb{R}}$,

$$\int_{H^1(\mathbb{R})} \|z\|_{H^1(\mathbb{R})}^2 \|z\|_{L^2(\mathbb{R})} d\mu_t \leq C \quad (10)$$

for some time-independent constant C , then the assumption (7) is satisfied. The requirement (10) says in some sense that μ_t has a finite energy and actually this can be proved a

priori, see [11]. However, if we examine (6) in this case, we see that $\|F(t, z)\|_{H^1(\mathbb{R})}$ is bounded by

$$\|F(t, z)\|_{H^1(\mathbb{R})} \leq C \|z\|_{H^1(\mathbb{R})}^3,$$

and hence we need a stronger a priori estimate

$$\int_{H^1(\mathbb{R})} \|z\|_{H^1(\mathbb{R})}^6 d\mu_t \leq C,$$

which in contrast is difficult to prove a priori. In conclusion, the improvement provided in this article allows to show general and stronger results in the mean-field theory of quantum many-body dynamics, see [7]. The proof of our main Theorem 2.1 is based on the work of S. Magnilia [30], Z. Ammari and F. Nier [11] and L. Ambrosio, N. Gigli and G. Savaré [4].

2 Mains Results

Let \mathcal{Z}_0 be a separable Hilbert space. Denote \mathcal{Z}_1 a dense subset such that we have a rigged Hilbert space $\mathcal{Z}_1 \subset \mathcal{Z}_0 \subset \mathcal{Z}'_1$. In this sequel, we define the Liouville equation in a separable infinite dimensional Hilbert space \mathcal{Z}_0 . We denote $\mathfrak{P}(\mathcal{Z}_0)$ the set of Borel probability measures on \mathcal{Z}_0 .

Actually, we consider the equation (5) in a weak sense that we explain below. The extension of the characteristics theory to systems with infinite degrees of freedom is based on the integration of the equation (5) after testing by cylindrical test functions. Recall that a function $f : \mathcal{Z}_0 \rightarrow \mathbb{C}$ is said cylindrical if there exists a orthogonal projection \mathfrak{p} with finite rank and a function g on $\mathfrak{p}\mathcal{Z}_0$ such that $f(z) = g(\mathfrak{p}z)$ for all $z \in \mathcal{Z}_0$. The set of \mathcal{C}_0^∞ -cylindrical functions on \mathcal{Z}_0 is denoted $\mathcal{C}_{0,cyl}^\infty(\mathcal{Z}_0)$. So, we can define properly a weak Liouville equation by integrating against test functions on the space $\mathcal{C}_{0,cyl}^\infty(\mathbb{R} \times \mathcal{Z}_0)$. In this context, the velocity fields are singular and the characteristics theory cannot be applied directly. Moreover, the lack of compactness on balls of \mathcal{Z}_0 induces the choice of a topology on \mathcal{Z}_0 which is weaker than the strong one. Introduce $(e_n)_{n \in \mathbb{N}}$ a Hilbert basis on \mathcal{Z}_0 and define the topology $(\mathcal{Z}_0, d_{w,\mathcal{Z}_0})$ induced by the following distance

$$d_{w,\mathcal{Z}_0}(z_1, z_2) = \sqrt{\sum_{n \in \mathbb{N}} \frac{|\langle z_1 - z_2, e_n \rangle_{\mathcal{Z}}|^2}{1 + n^2}}, \quad z_1, z_2 \in \mathcal{Z}_0.$$

We will consider Borel probability measures, solution of the PDE (12), that are narrowly continuous for this weak topology. In the sequel, we talk about 'weak narrowly continuous' solution on \mathcal{Z}_0 of (12) (referring to the narrow convergence for continuous bounded test functions on $(\mathcal{Z}_0, d_{w,\mathcal{Z}_0})$).

Denote $\text{Prob}(\mathcal{Z}_0)$, the set of Borel probability measures μ on \mathcal{Z}_0 such that $\int_{\mathcal{Z}_0} \|z\|_{\mathcal{Z}_0} d\mu(z) < +\infty$. On $\text{Prob}(\mathcal{Z}_0)$, the Wasserstein distance is defined by

$$W_1(\mu_1, \mu_2) := \min \left\{ \int_{\mathcal{Z}_0 \times \mathcal{Z}_0} \|z_1 - z_2\|_{\mathcal{Z}_0} d\mu(z_1, z_2); \Pi_{j,*}\mu = \mu_j \right\},$$

where $\Pi_j : \mathcal{Z}_0^2 \rightarrow \mathcal{Z}_0$ is the natural projection, $j = 1, 2$. Actually, the narrow convergence of a sequence $(\mu_n)_{n \in \mathbb{N}}$ satisfying the uniform control

$$\exists C > 0, \forall n \in \mathbb{N}, \int_{\mathcal{Z}_0} \|z\| d\mu_n \leq C,$$

is equivalent to the W_1 convergence on $\text{Prob}(\mathcal{Z}_0)$.

We also denote for $T > 0$, $\Gamma_T(\mathcal{Z}_0)$ the space of continuous maps from $[0, T]$ into \mathcal{Z}_0

equipped with the 'sup' norm. Denote also $AC([0, T], \mathcal{Z}_0)$ the space of absolutely continuous curves on \mathcal{Z}_0 with $L^1([0, T]; \mathcal{Z}_0)$ derivative. We shall consider the Cauchy problem on \mathcal{Z}_1

$$\partial_t \gamma(t) = v_t(\gamma(t)), \gamma(0) = x, \quad (11)$$

for a suitable Borel velocity field $v_t : \mathcal{Z}_1 \rightarrow \mathcal{Z}_0$ and the following theorem provides the link between the Liouville equation (5) satisfied by the velocity field $v_t(\cdot)$ and the Cauchy problem (11) on a infinite dimensional separable Hilbert space. We give two theorems that completes the characteristics theory for singular velocity field in a infinite dimensional Hilbert spaces:

- Theorem 2.1 holds true when the Cauchy problem (11) admits a unique globally well-posed solution on \mathcal{Z}_1 for small initial data by assuming the charge conservation.
- Theorem 2.2 holds true when the Cauchy problem (11) admits a unique globally well-posed flow on \mathcal{Z}_1 for every initial data on \mathcal{Z}_1 .

Denote $B_{\mathcal{Z}_0}(0, R)$, the ball in \mathcal{Z}_0 of radius R centered at 0.

Theorem 2.1. ODE GWP for small initial data.

Let $\mu_t : \mathbb{R} \mapsto \mathfrak{P}(\mathcal{Z}_0)$ be a weakly narrowly continuous on \mathcal{Z}_0 , solution of the equation:

$$\partial_t \mu_t + \nabla^T(v_t \mu_t) = 0,$$

in the weak sense

$$\int_{\mathbb{R}} \int_{\mathcal{Z}_0} \partial_t \phi(t, z) + \operatorname{Re} \langle v_t(z), \nabla_z \phi(t, z) \rangle_{\mathcal{Z}_0} d\mu_t(z) dt = 0, \forall \phi \in C_{0, cyl}^{\infty}(\mathbb{R} \times \mathcal{Z}_0), \quad (12)$$

for a suitable Borel velocity field $v(t, z) = v_t(z) : \mathcal{Z}_1 \rightarrow \mathcal{Z}_0$ such that

$$\int_0^T \int_{\mathcal{Z}_0} \|v_t(z)\|_{\mathcal{Z}_0} d\mu_t(z) dt < \infty, \forall T > 0. \quad (13)$$

Furthermore, the Cauchy problem (11) is globally well posed for small initial data on \mathcal{Z}_1 , i.e.:

i) $\exists \lambda > 0, \forall x \in B_{\mathcal{Z}_0}(0, \lambda) \cap \mathcal{Z}_1$, there exists a unique solution

$$z(t) \in C^0(\mathbb{R}, B_{\mathcal{Z}_0}(0, \lambda) \cap \mathcal{Z}_1) \cap C^1(\mathbb{R}, \mathcal{Z}'_1),$$

ii) and the charge conservation holds true.

Assume additionally that

- 1) the family $\{\mu_t, t \in \mathbb{R}\}$ is weakly tight on \mathcal{Z}_0 ,
- 2) the time dependant measure μ_t is carried on \mathcal{Z}_1 and is a probability measure on \mathcal{Z}_1 ,
- 3) $\exists R \in (0, \lambda)$, such that $\mu_0(\overline{B_{\mathcal{Z}_0}(0, R)}) = 1$.

Then the measure μ_t is the push-forward of the initial measure μ_0 by the globally well defined flow $\Phi(t, 0) : B_{\mathcal{Z}_0}(0, \lambda) \cap \mathcal{Z}_1 \rightarrow B_{\mathcal{Z}_0}(0, \lambda) \cap \mathcal{Z}_1$ of the Cauchy problem (11), i.e. for all $t \in \mathbb{R}$

$$\mu_t = \Phi(t, 0)_* \mu_0. \quad (14)$$

In the following Theorem, we assume that the solution of the Cauchy problem (11) is globally well-posed on \mathcal{Z}_1 for every initial data on \mathcal{Z}_1 .

Theorem 2.2. ODE GWP for every initial data with finite energy.

Let $\mu_t : \mathbb{R} \mapsto \mathfrak{P}(\mathcal{Z}_0)$ be a weakly narrowly continuous on \mathcal{Z}_0 , solution of the equation:

$$\partial_t \mu_t + \nabla^T(v_t \mu_t) = 0,$$

in the weak sense

$$\int_{\mathbb{R}} \int_{\mathcal{Z}_0} \partial_t \phi(t, z) + \operatorname{Re} \langle v_t(z), \nabla_z \phi(t, z) \rangle_{\mathcal{Z}_0} d\mu_t(z) dt = 0, \forall \phi \in C_{0, \text{cyl}}^{\infty}(\mathbb{R} \times \mathcal{Z}_0), \quad (15)$$

for a suitable Borel velocity field $v(t, z) = v_t(z) : \mathcal{Z}_1 \rightarrow \mathcal{Z}_0$ such that

$$\int_0^T \int_{\mathcal{Z}_0} \|v_t(z)\|_{\mathcal{Z}_0} d\mu_t(z) dt < \infty, \forall T > 0. \quad (16)$$

Furthermore, the Cauchy problem (11) is globally well posed **for every initial data** on \mathcal{Z}_1 , i.e.: for every initial data $x \in \mathcal{Z}_1$, there exists a unique solution

$$z(t) \in C^0(\mathbb{R}, \mathcal{Z}_1) \cap C^1(\mathbb{R}, \mathcal{Z}'_1),$$

Assume additionally that

- 1) the family $\{\mu_t, t \in \mathbb{R}\}$ is weakly tight on \mathcal{Z}_0 ,
- 2) the time dependant measure μ_t is carried on \mathcal{Z}_1 and is a probability measure on \mathcal{Z}_1 ,

Then the measure μ_t is the push-forward of the initial measure μ_0 by the well defined flow $\Phi(t, 0) : \mathcal{Z}_1 \rightarrow \mathcal{Z}_1$ of the Cauchy problem (11), i.e. for all $t \in \mathbb{R}$

$$\mu_t = \Phi(t, 0)_* \mu_0. \quad (17)$$

Preliminaries and notations.

For the sake of simplicity, we introduce the vocabulary of triple solutions of the equation (5) in a general infinite dimensional separable Hilbert space \mathcal{Z} .

Definition 2.3. We say that a triple $(\mu_t, v_t, \mathcal{Z})$ is a solution of **the weak Liouville equation** if the equation (12) holds true, where $v_t(z) : \mathcal{Z} \rightarrow \mathcal{Z}$ is a velocity vector field associated to (11) and satisfying the estimate (13).

Remarks 1. Let $(\mu_t, v_t, \mathcal{Z}_0)$ be a triple solution of the **weak liouville equation**. Assume that the measure μ_t is carried on the dense subset $\mathcal{Z}_1 \subset \mathcal{Z}_0$. Without a loss of generality we can always assume that $v_t : \mathcal{Z}_0 \rightarrow \mathcal{Z}_0$. In fact for a Borel velocity field $v_t : \mathcal{Z}_1 \rightarrow \mathcal{Z}_0$ we set $\hat{v}_t := v_t$ on \mathcal{Z}_1 and $\hat{v}_t := 0$ on $\mathcal{Z}_0 \setminus \mathcal{Z}_1$. Therefore since $\forall t \in \mathbb{R}, \mu_t(\mathcal{Z}_1) = 1$, the triple $(\mu_t, \hat{v}_t, \mathcal{Z}_0)$ is also solution of the **weak Liouville equation**.

Definition 2.4. We introduce a new condition, called **(CP)** if a triple $(\eta, v_t, \mathcal{Z} \times \Gamma_T(\mathcal{Z}))$ satisfies

(CP): the measure $\eta \in \mathfrak{P}(\mathcal{Z} \times \Gamma_T(\mathcal{Z}))$ is concentrated on the set of (x, γ) with $\gamma \in AC([0, T]; \mathcal{Z})$ that are solutions of

$$\partial_t \gamma(t) = v_t(\gamma(t)), \gamma(0) = x, \quad (18)$$

with a Borel velocity vector field $v_t : \mathcal{Z} \rightarrow \mathcal{Z}$.

The set $\Gamma_T(\mathcal{Z})$ can be equipped with a weak topology $(\Gamma_T(\mathcal{Z}), d)$ induced by the distance d_w , i.e.

$$d(\gamma_1, \gamma_2) = \max_{t \in [0, T]} d_w(\gamma_1(t), \gamma_2(t)).$$

Additionally, denote the time dependent Borel probability measure $\mu_t^\eta \in \mathfrak{P}(\mathcal{Z})$ defined by

$$\int_{\mathcal{Z}} \varphi d\mu_t^\eta = \int_{\mathcal{Z} \times \Gamma_T(\mathcal{Z})} \varphi(\gamma(t)) d\eta(x, \gamma), \quad \forall \varphi \in C_{b, cyl}^0(\mathcal{Z}), \quad t \in [0, T], \quad (19)$$

where $C_{b, cyl}^0(\mathcal{Z})$ is the space of cylindrical bounded continuous functions on \mathcal{Z} . The measure μ_t^η is the push-forward of η by the evaluation map

$$e_t : (x, \gamma) \in \mathcal{Z} \times \Gamma_T(\mathcal{Z}) \mapsto \gamma(t) \in \mathcal{Z}, \quad \text{for } t \in [0, T].$$

Recall that we set a rigged Hilbert space $\mathcal{Z}_1 \subset \mathcal{Z}_0 \subset \mathcal{Z}'_1$ where \mathcal{Z}_1 is a dense subset of \mathcal{Z}_0 .

Proof of Theorem 2.1. *The plan is to project the equation (12) on a finite dimensional space to get the existence of a measure, which we denote, in the sequel, by μ_t^d belonging to $\mathfrak{P}(\mathbb{R}^d)$, and a velocity field $v_t^d : \mathbb{R}^d \rightarrow \mathbb{R}^d$ such that for $t \in [0, T]$ the triple $(\mu_t^d, v_t^d, \mathbb{R}^d)$ is a solution of the **weak Liouville equation**.*

i) Projection onto \mathbb{R}^d :

Consider the Hilbert basis $(e_n)_{n \in \mathbb{N}}$ of \mathcal{Z}_0 and the following diagram

$$\begin{array}{ccc} \mathcal{Z}_0 & \xrightarrow{\pi^d} & \mathbb{R}^d \\ & \searrow \hat{\pi}^d & \downarrow \pi^{d, T} \\ & & \mathcal{Z}_0 \end{array}$$

with $\pi^d(x) = (\langle e_1, x \rangle, \dots, \langle e_d, x \rangle)$, $\pi^{d, T}(y_1, y_2, \dots, y_d) = \sum_{j=1}^d y_j e_j$ and $\hat{\pi}^d = \pi^{d, T} \circ \pi^d$. Hence we define the measure μ_t^d as the push-forward of the measure μ_t by the projection π^d , i.e.:

$$\mu_t^d := \pi_*^d \mu_t. \quad (20)$$

Therefore, by the disintegration theorem (see Appendix A.1 or for a more general presentation Chapter V in [4]), there exists a family of measures $\{\mu_{t, y}, y \in \mathbb{R}^d\}$ such that we can define two velocity fields v_t^d and \hat{v}_t^d :

$$v_t^d(y) = \int_{(\pi^d)^{-1}(y)} \pi^d v_t(x) d\mu_{t, y}(x), \quad y \in \mathbb{R}^d \quad (21)$$

$$\hat{v}_t^d(y) = \int_{(\hat{\pi}^d)^{-1}(\hat{\pi}^d y)} \hat{\pi}^d v_t(x) d\mu_{t, \hat{\pi}^d y}(x), \quad y \in \mathcal{Z}_0. \quad (22)$$

Similarly, we also define the measure

$$\hat{\mu}_t^d := \hat{\pi}_*^d \mu_t. \quad (23)$$

Since $\pi^d, \hat{\pi}^d$ are projections with finite rank, the two maps

$$t \in [0, T] \mapsto \mu_t^d \in \mathfrak{P}(\mathbb{R}^d); \quad t \in [0, T] \mapsto \hat{\mu}_t^d \in \mathfrak{P}(\mathcal{Z}_0)$$

*are weakly narrowly continuous. By using Lemma A.3, the triples $(\mu_t^d, v_t^d, \mathbb{R}^d)$ and $(\hat{\mu}_t^d, \hat{v}_t^d, \mathcal{Z}_0)$ are solutions of the **weak Liouville equations**.*

ii) **Result in finite dimension:**

By using [30, Theorem 4.1] to the triple $(\mu_t^d, v_t^d, \mathbb{R}^d)$, there exists a triple $(\eta^d, v_t^d, \mathbb{R}^d \times \Gamma_T(\mathbb{R}^d))$ satisfying the condition **(CP)** where the measure η^d belongs to $\mathfrak{P}(\mathbb{R}^d \times \Gamma_T(\mathbb{R}^d))$. Moreover we can define a measure $\hat{\eta}^d \in \mathfrak{P}(\mathcal{Z}_0 \times \Gamma_T(\mathcal{Z}_0))$ by the following equality $\hat{\eta}^d := (\pi^{d,T} \times \pi^d)_* \eta^d$, i.e.

$$\int_{\mathcal{Z}_0 \times \Gamma_T(\mathcal{Z}_0)} \varphi(x, \gamma) d\hat{\eta}^d(x, \gamma) = \int_{\mathbb{R}^d \times \Gamma_T(\mathbb{R}^d)} f(\pi^{d,T} x, \pi^d \gamma) d\eta^d(x, \gamma),$$

for every function $\varphi \in \mathcal{C}_{b, \text{cyl}}^0(\mathcal{Z}_0 \times \Gamma_T(\mathcal{Z}_0))$. As a consequence of [30, Theorem 4.1], we have the equality for any $\varphi \in \mathcal{C}_b^0(\mathbb{R}^d)$, $t \in [0, T]$,

$$\int_{\mathcal{Z}_0} \varphi \circ \pi^d d\hat{\mu}_t^d = \int_{\mathbb{R}^d} \varphi d\mu_t^d = \int_{\mathbb{R}^d \times \Gamma_T(\mathbb{R}^d)} \varphi(\gamma(t)) d\eta^d = \int_{\mathcal{Z}_0 \times \Gamma_T(\mathcal{Z}_0)} \varphi \circ \pi^d(\gamma(t)) d\hat{\eta}^d.$$

iii) **Weak tightness:**

Let us show the weak tightness on $\mathcal{Z}_0 \times \Gamma_T(\mathcal{Z}_0)$ of the family $\{\hat{\eta}^d\}_{d \in \mathbb{N}}$. We will use two criterions recalled in Appendix A.2. Choose the maps r^1 and r^2 defined on $\mathcal{Z}_0 \times \Gamma_T(\mathcal{Z}_0)$ as

$$r^1 : (x, \gamma) \mapsto x \in \mathcal{Z}_0$$

and

$$r^2 : (x, \gamma) \mapsto \gamma - x \in \Gamma_T(\mathcal{Z}_0),$$

hence notice that the map $r = r^1 \times r^2 : \mathcal{Z}_0 \times \Gamma_T(\mathcal{Z}_0)$ is proper. The family $\{r_*^1 \hat{\eta}^d\}_{d \in \mathbb{N}}$ is given by the first marginal $\{\hat{\mu}_0^d\}_{d \in \mathbb{N}}$. Besides, the family $\{\hat{\mu}_0^d\}_{d \in \mathbb{N}}$ is weakly tight on \mathcal{Z}_0 , since the family $\{\mu_t\}_{t \in \mathbb{R}}$ is weakly tight on \mathcal{Z}_0 . For the family $\{r_*^2 \hat{\eta}^d\}_{d \in \mathbb{N}}$, since the functional

$$g \mapsto \int_0^T |\dot{g}(t)| dt,$$

defined on $\{g \in \Gamma_T(\mathcal{Z}_0), g(0) = 0\}$ and set to $+\infty$ if $g \neq AC([0, T], \mathcal{Z}_0)$, has compact sublevel sets in $\Gamma_T(\mathcal{Z}_0)$ but is not coercive in $\Gamma_T(\mathcal{Z}_0)$, the proof of the weak tightness differs from [11, Proposition C.2].

By using Lemma A.2, there exists a convex superlinear function $\Psi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that the functional

$$\gamma \mapsto \int_0^T \Psi(|\dot{\gamma}(t)|) dt$$

is coercive in $\{g \in \Gamma_T(\mathcal{Z}_0), g(0) = 0\}$. By using Lemma [30, 3.10], we get the existence of a regularized triple $\{(\mu_{t,\varepsilon}^d, v_{t,\varepsilon}^d, \mathbb{R}^d)\}_{\varepsilon > 0}$ that is a solution of the **weak Liouville equation** such that for $t \in [0, T]$

(a) there exists a unique maximal solution $X_\varepsilon(t, s, \pi^d x)$ of $\frac{d}{dt} X_\varepsilon(t, s, \cdot) = v_{t,\varepsilon}^d(X_\varepsilon(t, s, \cdot))$ with $X_\varepsilon(s, s, \pi^d x) = \pi^d x$,

(b) $\mu_{t,\varepsilon}^d = X(t, 0, \cdot)_* \mu_{0,\varepsilon}^d$,

(c) there exist two families of measures $\{\eta_\varepsilon^d\}_{\varepsilon > 0}$, $\{\hat{\eta}_\varepsilon^d\}_{\varepsilon > 0}$ given by the following equalities

$$\begin{aligned} \eta_\varepsilon^d &= (\text{Id}(x) \times X_\varepsilon(t, 0, x))_* \mu_{0,\varepsilon}^d \\ \hat{\eta}_\varepsilon^d &:= (\pi^{d,T} \times \pi^d)_* \eta_\varepsilon^d; \end{aligned} \tag{24}$$

(d) for any $T > 0$

$$\int_0^T \int_{\mathbb{R}^d} \Psi(|v_{t,\varepsilon}^d(x)|) d\mu_{t,\varepsilon}^d(x) dt \leq \int_0^T \int_{\mathbb{R}^d} \Psi(|v_t^d(x)|) d\mu_t^d(x) dt. \quad (25)$$

(e) For $d \in \mathbb{N}$ the family $\{\hat{\eta}_\varepsilon^d\}_{\varepsilon>0}$ is weakly tight.

Therefore, we compute the quantity

$$\begin{aligned} \int_{\Gamma_T(\mathcal{Z}_0)} \int_0^T \Psi(|\dot{\gamma}(t)|) dt d((r_2)_* \hat{\eta}_\varepsilon^d)(\gamma) &= \int_{\mathbb{R}^d} \int_0^T \Psi(|\dot{X}_\varepsilon(t, 0, x)|) dt d\mu_{0,\varepsilon}^d(x) \\ &\leq \int_0^T \int_{\mathbb{R}^d} \Psi(|v_{t,\varepsilon}^d(x)|) d\mu_{t,\varepsilon}^d(x) dt \\ &\leq \int_0^T \int_{\mathbb{R}^d} \Psi(|v_t^d(x)|) d\mu_t^d(x) dt \\ &\leq \int_0^T \int_{\mathcal{Z}_0} \Psi(\|v_t(x)\|_{\mathcal{Z}_0}) d\mu_t(x) dt < \infty, \end{aligned}$$

the last step coming from the estimate (45) in Lemma A.3. Since the family $\{\hat{\eta}_\varepsilon^d\}_{\varepsilon>0}$ is weakly tight, take the limit in the l.h.s when $\varepsilon \rightarrow 0$ to get

$$\int_{\Gamma_T(\mathcal{Z}_0)} \int_0^T \Psi(|\dot{\gamma}(t)|) dt d((r_2)_* \hat{\eta}^d)(\gamma) < +\infty. \quad (26)$$

Then, the family $\{(r_2)_* \hat{\eta}^d\}_{d \in \mathbb{N}}$ is weakly tight.

Let us sketch the rest of the proof by using diagrams for triples introduced in the previous steps. By Lemma A.3 we deduce the following diagram

$$\begin{array}{ccc} (\mu_t, v_t, \mathcal{Z}_0) & \longrightarrow & (\mu_t^d, v_t^d, \mathbb{R}^d) \\ & \searrow & \downarrow \\ & & (\hat{\mu}_t^d, \hat{v}_t^d, \mathcal{Z}_0) \end{array}$$

where each triple is a solution of the **weak Liouville equation**. By using the **weak tightness** step to the sequence $(\hat{\eta}^d)_{d \in \mathbb{N}}$, it gives rise to a probability measure $\eta \in \mathfrak{P}(\mathcal{Z}_0 \times \Gamma_T(\mathcal{Z}_0))$. Similarly as in the two previous diagrams we shall complete the following diagram for triples satisfying the **(CP)** condition:

$$\begin{array}{ccc} (\eta, v_t, \mathcal{Z}_0 \times \Gamma_T(\mathcal{Z}_0)) & \longleftarrow & (\eta^d, v_t^d, \mathbb{R}^d \times \Gamma_T(\mathbb{R}^d)) \\ & \swarrow & \downarrow \\ & & (\hat{\eta}^d, \hat{v}_t^d, \mathcal{Z}_0 \times \Gamma_T(\mathcal{Z}_0)) \end{array}$$

Therefore in the following step we shall prove the existence of the triple $(\eta, v_t, \mathcal{Z}_0 \times \Gamma_T(\mathcal{Z}_0))$ satisfying the **(CP)** condition.

iv) **Existence the measure concentrated on the solutions of the ODE:**

We have constructed two triples of probability measures $(\eta^d, v_t^d, \mathbb{R}^d \times \Gamma_T(\mathbb{R}^d))$ and $(\hat{\eta}^d, \hat{v}_t^d, \mathcal{Z}_0 \times \Gamma_T(\mathcal{Z}_0))$ such that the sequence $(\hat{\eta}^d)_{d \in \mathbb{N}}$ defined in [ii\)](#) is weakly tight. Therefore set η as a weak narrow limit point of $\hat{\eta}^d$. Assume that the test function

φ in *ii*) depends only on d' coordinates with $d' \leq d$. Hence taking the limit when $d \rightarrow +\infty$ gives

$$\int_{\mathcal{Z}_0} \varphi \circ \pi^{d'} d\mu_t = \int_{\mathcal{Z}_0 \times \Gamma_T(\mathcal{Z}_0)} (\varphi \circ \pi^{d'}) (\gamma(t)) d\eta(x, \gamma),$$

for all $\varphi \in \mathcal{C}_b^0(\mathbb{R}^{d'})$ and $t \in [0, T]$, where $\varphi \circ \pi^{d'}$ can be replaced by any cylindrical function or Borel bounded function on \mathcal{Z}_0 .

v) **The concentration condition, (CP):**

We shall prove the following equality for $t \in [0, T]$

$$\int_{\mathcal{Z}_0 \times \Gamma_T(\mathcal{Z}_0)} \|\gamma(t) - x - \int_0^t v_s(\gamma(s)) ds\|_{\mathcal{Z}_0} d\eta(x, \gamma) = 0. \quad (27)$$

In the finite dimensional case, by a regularization process like in Step 5 of Theorem [30, 4.1], there exists a sequence $(v_{t,n}^d)_{n \in \mathbb{N}}$ of uniformly continuous function in $\mathcal{C}_b^0([0, T] \times \mathcal{Z}_0; \mathcal{Z}_0)$ such that $\|v_t^d - v_{t,n}^d\|_{L^1(\mathbb{R}^d, d\mu_t^d)} \rightarrow 0$ and

$$\int_{\mathbb{R}^d \times \Gamma_T(\mathbb{R}^d)} |\gamma(t) - x - \int_0^t v_{s,n}^d(\gamma(s)) ds| d\eta^d(x, \gamma) \leq \int_0^T \int_{\mathbb{R}^d} |v_s^d - v_{s,n}^d| d\mu_s^d ds. \quad (28)$$

The equality (27) can be deduced from the finite dimensional case. Indeed set the function w_t belonging to $\mathcal{C}_b^0([0, T] \times \mathbb{R}^{d'}; \mathbb{R}^{d'})$ with $d' \leq d$ fixed and, by setting $\hat{w}_t = \pi^{d', T} \circ w_t \circ \pi^{d'} \in \mathcal{C}_b^0([0, T] \times \mathcal{Z}_0; \mathcal{Z}_0)$, we get

$$\begin{aligned} & \int_{\mathcal{Z}_0 \times \Gamma_T(\mathcal{Z}_0)} \|\gamma(t) - x - \int_0^t v_s(\gamma(s)) ds\|_{\mathcal{Z}_0} d\eta(x, \gamma) \leq \\ & \underbrace{\int_{\mathcal{Z}_0 \times \Gamma_T(\mathcal{Z}_0)} \|\gamma(t) - x - \int_0^t \hat{w}_s(\gamma(s)) ds\|_{\mathcal{Z}_0} d\eta(x, \gamma)}_{\mathcal{A}(\hat{w})} \\ & + \underbrace{\int_{\mathcal{Z}_0 \times \Gamma_T(\mathcal{Z}_0)} \int_0^t \|\hat{w}_s(\gamma(s)) - v_s(\gamma(s))\|_{\mathcal{Z}_0} ds d\eta(x, \gamma)}_{\mathcal{B}(\hat{w})}. \end{aligned}$$

The first term on the r.h.s, $\mathcal{A}(\hat{w})$ can be estimated owing to finite dimensional estimate (28). Indeed it follows by the same regularization process used in *iii*)c, the next estimate holds true

$$\mathcal{A}(\hat{w}) \leq \limsup_{d \rightarrow +\infty} \int_0^T \int_{\mathcal{Z}_0} \|\hat{v}_s^d - \hat{w}_s\|_{\mathcal{Z}_0} d\hat{\mu}_s^d ds \leq \int_0^T \|v_s - \hat{w}_s\|_{L^1(\mathcal{Z}_0, d\mu_s)} ds,$$

by using the estimate (44) in Lemma A.3 to the function $\hat{v}_s^d - \hat{w}_s$. We conclude the proof by noticing that there exists a sequence of cylindrical, uniformly bounded continuous bounded function $(\hat{w}_s^n)_{n \in \mathbb{N}}$ such that for any $t \in [0, T]$ $\|v_t - \hat{w}_t^n\|_{L^1(\mathcal{Z}_0, d\mu_t)} \rightarrow 0$. Therefore $\mathcal{A}(\hat{w}^n), \mathcal{B}(\hat{w}^n) \rightarrow 0$ and the equality (27) is proved.

vi) **End of the proof:** The relation $\mu_t = \mu_t^\eta$ defined according to *(ii)*) extends to any bounded Borel function φ on \mathcal{Z}_0

$$\int_{\mathcal{Z}_0} \varphi d\mu_t = \int_{\mathcal{Z}_1} \varphi d\mu_t = \int_{\mathcal{Z}_1 \times \Gamma_T(\mathcal{Z}_1)} \varphi(\gamma(t)) d\eta(x, \gamma), \quad (29)$$

since the measure μ_t is carried on \mathcal{Z}_1 . In particular this relation is true when $t = 0$, with a function $\varphi \in C_0^\infty(\mathcal{Z}_0)$ such that $\text{supp}(\varphi) \subset \mathcal{Z}_0 \setminus \mathcal{Z}_1$

$$\int_{\mathcal{Z}_1} \varphi d\mu_0 = 0 = \int_{(B_{\mathcal{Z}_0}(0,\lambda) \cap \mathcal{Z}_1) \times \Gamma_T(\mathcal{Z}_1)} \varphi(\gamma(0)) d\eta(x, \gamma). \quad (30)$$

Hence

$$\eta(\{(x, \gamma) \mid t \in \mathbb{R}, \partial_t \gamma(t) = v_t(\gamma(t)), \gamma(0) = x, x \notin B_{\mathcal{Z}_0}(0, \lambda) \cap \mathcal{Z}_1\}) = 0,$$

and by using the Cauchy problem uniqueness

$$\eta(\{(x, \gamma) \mid t \in \mathbb{R}, x \in B_{\mathcal{Z}_0}(0, \lambda) \cap \mathcal{Z}_1, \gamma(t)(x) = \Phi(t, 0)x\}) = 1,$$

where $\Phi(t, 0) : B_{\mathcal{Z}_0}(0, \lambda) \cap \mathcal{Z}_1 \rightarrow B_{\mathcal{Z}_0}(0, \lambda) \cap \mathcal{Z}_1$ is a Borel flow associated to the ODE (11), we deduce for any $t \in \mathbb{R}$

$$\int_{\mathcal{Z}_0} \varphi d\mu_t = \int_{(B_{\mathcal{Z}_0}(0,\lambda) \cap \mathcal{Z}_1) \times \Gamma_T(\mathcal{Z}_1)} [\varphi \circ \Phi(t, 0)](\gamma(0)) d\eta(x, \gamma) = \int_{B_{\mathcal{Z}_0}(0,\lambda) \cap \mathcal{Z}_1} [\varphi \circ \Phi(t, 0)] d\mu_0,$$

which ends the proof.

Proof of Theorem 2.2. Steps *i), ii), iii), iv), v)* follow similarly as above. It remains to prove the last step

End of the proof:

The relation $\mu_t = \mu_t^\eta$ gives also rise to assertion (29) and similarly we get for a function $\varphi \in C_0^\infty(\mathcal{Z}_0)$ such that $\text{supp}(\varphi) \subset \mathcal{Z}_0 \setminus \mathcal{Z}_1$

$$\int_{\mathcal{Z}_1} \varphi d\mu_0 = 0 = \int_{\mathcal{Z}_1 \times \Gamma_T(\mathcal{Z}_1)} \varphi(\gamma(0)) d\eta(x, \gamma). \quad (31)$$

Therefore,

$$\eta(\{(x, \gamma) \mid t \in \mathbb{R}, \partial_t \gamma(t) = v_t(\gamma(t)), \gamma(0) = x, x \notin \mathcal{Z}_1\}) = 0,$$

and by using the Cauchy problem uniqueness

$$\eta(\{(x, \gamma) \mid t \in \mathbb{R}, x \in \mathcal{Z}_1, \gamma(t)(x) = \Phi(t, 0)x\}) = 1,$$

where $\Phi(t, 0) : \mathcal{Z}_1 \rightarrow \mathcal{Z}_1$ is a Borel flow associated to the ODE (11), we deduce for any $t \in \mathbb{R}$

$$\int_{\mathcal{Z}_0} \varphi d\mu_t = \int_{\mathcal{Z}_1 \times \Gamma_T(\mathcal{Z}_1)} [\varphi \circ \Phi(t, 0)](\gamma(0)) d\eta(x, \gamma) = \int_{\mathcal{Z}_1} [\varphi \circ \Phi(t, 0)] d\mu_0,$$

which ends the proof.

3 Examples

We illustrate our main Theorem 2.1 with few examples. Consider a semi-linear Hamiltonian PDEs with a (real-valued) energy functional having the form,

$$h(z, \bar{z}) = \langle z, Az \rangle_{\mathcal{Z}_0} + h_I(z, \bar{z}),$$

where \mathcal{Z}_0 is a complex Hilbert space, A is a non-negative self-adjoint operator, $h_I(z, \bar{z})$ is a nonlinear functional and (z, \bar{z}) are the complex classical fields of the Hamiltonian theory. So that the related PDE (or equation of motion) is,

$$i\partial_t u = Au + \partial_{\bar{z}} h_I(u, \bar{u}). \quad (32)$$

By differentiating $\tilde{u} := e^{itA}u$ with respect to time, we equivalently express the above equation in the interaction representation, i.e.,

$$\partial_t \tilde{u} = -ie^{itA} \partial_{\bar{z}} h_I(e^{-itA} \tilde{u}, e^{-itA} \overline{\tilde{u}}).$$

Hence the original PDE (32) can be reformulated as an ODE,

$$\frac{d}{dt} u = v(t, u),$$

with a non-autonomous vector field $v(t, \cdot)$ given by

$$v(t, z) := -ie^{itA} \partial_{\bar{z}} h_I(e^{-itA} z, e^{-itA} \overline{z}).$$

The natural energy space is $Q(A) := D(A^{\frac{1}{2}})$, the form domain of A equipped with the graph norm,

$$\|z\|_{Q(A)}^2 = \langle z, (A+1)z \rangle_{\mathcal{Z}_0}.$$

In all the examples considered below we have that $v(\cdot, \cdot) : \mathbb{R} \times Q(A) \rightarrow \mathcal{Z}_0$ is a continuous map satisfying the following estimate (or a similar one),

$$\|v(t, z)\|_{\mathcal{Z}_0} \leq C \|z\|_{Q(A)}^2 \|z\|_{\mathcal{Z}_0},$$

for some time-independent constant $C > 0$. Moreover, we have that the energy $h(z, \bar{z})$ makes sense on the space $Q(A)$ and the Cauchy-problem (32) is globally well-posed on $Q(A)$ in the sense of existence and uniqueness of a global strong solution $t \mapsto z(t) \in C^0(\mathbb{R}, Q(A)) \cap C^1(\mathbb{R}, Q'(A))$ for each $z_0 \in Q(A)$ and continuous dependence on initial data.

Example 1 (The nonlinear Schrödinger equation). *The energy functional of the NLS equation in dimension $d = 1$ is,*

$$h(z, \bar{z}) = \langle z, -\Delta_x + V(x)z \rangle_{L^2(\mathbb{R}^d)} + \frac{\lambda}{2} \int_{\mathbb{R}} |z(x)|^4 dx, \quad (33)$$

where V is a real-valued potential which splits into a positive and negative part $V = V_+ + V_-$ such that $V_+ \in L^1_{loc}(\mathbb{R})$ and V_- is $-\Delta$ -form bounded with a relative bound less than one. So the quadratic form $A = -\Delta + V$ defines a self-adjoint operator semi-bounded from below and its natural domain $Q(A)$ is a Hilbert space when equipped with the graph norm,

$$\|u\|_{Q(A)}^2 = \langle u, (A + V_+ + 1)u \rangle.$$

The vector field in this case is $v(t, z) = 2\lambda |e^{-itA} z|^2 e^{-itA} z : Q(A) \rightarrow L^2(\mathbb{R})$ and satisfies the following inequalities for all $z \in Q(A)$,

$$\|v(t, z)\|_{L^2(\mathbb{R})} \leq C \|z\|_{H^1(\mathbb{R})}^2 \|z\|_{L^2(\mathbb{R})} \leq C \|z\|_{Q(A)}^2 \|z\|_{L^2(\mathbb{R})}, \quad (34)$$

since the inclusion $Q(A) \subset H^1(\mathbb{R}) \subset L^\infty(\mathbb{R})$ holds by Sobolev's embedding and the fact that $Q(A) = \{u \in L^2(\mathbb{R}), u' \in L^2(\mathbb{R}), V_+^{\frac{1}{2}} u \in L^2(\mathbb{R})\}$. Moreover, it is known that the NLS equation

$$\begin{cases} i\partial_t z = -\Delta z + Vz + \lambda |z|^2 z \\ z|_{t=0} = z_0, \end{cases} \quad (\text{NLS})$$

is globally well-posed on $Q(A)$ with energy and charge conservation. Therefore, Theorem 2.2 applies here. The derivation of such equation from quantum many-body dynamics is established for instance in [1, 5].

Example 2 (Non-relativistic Hartree equation). *The energy functional of the Hartree equation is*

$$h(z, \bar{z}) = \langle z, -\Delta_x + V(x) z \rangle_{L^2(\mathbb{R}^d)} + \iint_{\mathbb{R}^d \times \mathbb{R}^d} |z(x)z(y)|^2 W(x-y) dx dy, \quad (35)$$

where $W : \mathbb{R}^d \rightarrow \mathbb{R}$ is an even measurable function and V is a real-valued potential both satisfying the following assumptions for some p and q ,

$$\begin{aligned} V &\in L^p(\mathbb{R}^d) + L^\infty(\mathbb{R}^d), \quad p \geq 1, \quad p > \frac{d}{2}, \\ W &\in L^q(\mathbb{R}^d) + L^\infty(\mathbb{R}^d), \quad q \geq 1, \quad q \geq \frac{d}{2} \text{ (and } q > 1 \text{ if } d = 2). \end{aligned}$$

The vector field $v(t, z) := W * |z|^2 z : Q(A) \rightarrow L^2(\mathbb{R}^d)$ verifies the estimate,

$$\|W * |z|^2 z\|_{L^2(\mathbb{R}^d)} \leq \|(-\Delta + 1)^{-\frac{1}{2}} W (-\Delta + 1)^{-\frac{1}{2}}\| \|z\|_{H^1(\mathbb{R}^d)}^2 \|z\|_{L^2(\mathbb{R}^d)}. \quad (36)$$

The global well-posedness on $Q(A)$, conservation of energy and charge of the Hartree equation

$$\begin{cases} i\partial_t z = -\Delta z + Vz + W * |z|^2 z \\ z_{t=0} = z_0, \end{cases}$$

are proved in [16] Corollary 4.3.3 and Corollary 6.1.2. Therefore, Theorem 2.2 applies here.

We remark that the assumption on W are satisfied by the Coulomb type potentials $\frac{\lambda}{|x|^\alpha}$ when $\alpha < 2$, $\lambda \in \mathbb{R}$ and $d = 3$. The derivation of such equation from quantum many-body dynamics is extensively investigated, see for instance [11, 14, 21, 22, 23, 24, 25, 26, 37].

Example 3 (Semi-relativistic Hartree equation). *The semi-relativistic Hartree equation has the energy functional*

$$h(z, \bar{z}) = \langle z, \sqrt{-\Delta_x + m^2} + V(x) z \rangle_{L^2(\mathbb{R}^3)} + \kappa \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{|z(x)z(y)|^2}{|x-y|} dx dy,$$

with $-\kappa_c < \kappa$, with $\kappa_{cr}^{-1} := 2 \lim_{\alpha \rightarrow \infty} \|\frac{1}{|x|} (-\Delta + \alpha)^{-\frac{1}{2}}\|$, $m \geq 0$ and V is real-valued measurable function which splits into a positive and negative part $V = V_+ + V_-$ satisfying,

$$\begin{aligned} V_+ &\in L_{loc}^1(\mathbb{R}^3), \\ V_- &\text{ is } \sqrt{-\Delta} - \text{form bounded with a relative bound less than 1.} \end{aligned}$$

The quadratic form

$$\begin{aligned} A[u, u] &= \langle u, \sqrt{-\Delta + m^2} u \rangle + \langle u, Vu \rangle, \\ Q(A) &= \{u \in L^2(\mathbb{R}^3), (-\Delta + m^2)^{\frac{1}{4}} u \in L^2(\mathbb{R}^3), V_+^{\frac{1}{2}} u \in L^2(\mathbb{R}^3)\}, \end{aligned}$$

is semi-bounded from below and closed. So it defines a unique self-adjoint operator denoted by A . Thanks to a Hardy type inequality (see for instance [11, Proposition D.3]), we have the bound,

$$\|\frac{1}{|x|} * |z|^2 z\|_{L^2(\mathbb{R}^3)} \leq C \|z\|_{H^{1/2}(\mathbb{R}^3)}^2 \|z\|_{L^2(\mathbb{R}^3)}.$$

The global well-posedness in $Q(A)$, conservation of energy and charge of the semi-relativistic Hartree equation

$$\begin{cases} i\partial_t z = \sqrt{-\Delta + m^2} z + V(x)z + \frac{\kappa}{|x|} * |z|^2 z \\ z|_{t=0} = z_0. \end{cases}$$

are proved in [28, Theorem 4] for all $\kappa \geq 0$. Therefore when $\kappa \geq 0$, Theorem 2.2 can be applied here. In the case $-\kappa_c < \kappa < 0$, Theorem 2.1 holds true, since the mean field dynamics is globally well-posed $Q(A)$ for small initial data (see also [28, Theorem 4]).

Example 4 (The Klein-Gordon equation). The classical energy functional formally associated with the quantum field theory $P(\varphi)_2$ is given by

$$h(z, \bar{z}) := \langle z, Az \rangle_{L^2(\mathbb{R})} + G(z),$$

where A is a multiplication operator by the function $\omega(k) = \sqrt{m_0^2 + k^2}$, $m_0 > 0$, and G is a polynomial interaction defined as follows, see [35, 36]. Consider a bounded from below real polynomial

$$P(x) = \sum_{j=0}^{2n} \alpha_j x^j, \quad (\alpha_{2n} > 0).$$

Let $\varphi(x)$ be the scalar-field of mass $m_0 > 0$, i.e.:

$$\varphi(x) := \int_{\mathbb{R}} e^{-ikx} [\bar{z}(k) + z(-k)] \frac{dk}{\sqrt{\omega(k)}},$$

where (z, \bar{z}) are scalar complex fields. Let g a non-negative function in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ such that $g(x) = g(-x)$. The nonlinear term G is defined as the following real-valued polynomial

$$G(z) := \int_{\mathbb{R}} g(x) P\left(2 \operatorname{Re}\left\langle z, \frac{e^{-ikx}}{\sqrt{\omega(k)}} \right\rangle_{L^2(\mathbb{R})}\right) dx.$$

So that we have at hand the nonlinear Klein-Gordon equation with a non-local nonlinearity,

$$i\partial_t \varphi = \omega \varphi + \partial_{\bar{z}} G(\varphi). \quad (37)$$

The local Cauchy problem is studied for instance in [33, Theorem 1] and [34, Theorem X.72]. Actually, one can prove the energy conservation and hence global well-posedness holds true in this specific case. Moreover, the vector field $v(t, z) = -ie^{itA} \partial_{\bar{z}} G(e^{-itA} z) : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ is continuous and satisfies,

$$\|v(t, z)\|_{L^2(\mathbb{R})} \leq C(1 + \|z\|_{L^2(\mathbb{R})}^{2n-1}).$$

The derivation of such PDE from $P(\varphi)_2$ quantum field theory is established in [12, 20, 25].

Example 5 (The Schrödinger-Klein-Gordon system). The Schrödinger-Klein-Gordon system with Yukawa interaction is defined by:

$$\begin{cases} i\partial_t u = -\frac{\Delta}{2M} u + Au \\ (\square + m^2)A = -|u|^2 \end{cases}; \quad (\text{S-KG})$$

where (u, A) are the unknowns and $M, m > 0$ are real parameters. If we introduce the complex field α , defined by

$$A(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{\mathbb{R}^3} \frac{1}{\sqrt{2\omega(k)}} (\bar{\alpha}(k)e^{-ik \cdot x} + \alpha(k)e^{ik \cdot x}) dk, \quad \omega(k) = \sqrt{k^2 + m^2}, \quad (38)$$

we can rewrite (S-KG) as the equivalent system:

$$\begin{cases} i\partial_t u = -\frac{\Delta}{2M}u + Au \\ i\partial_t \alpha = \omega\alpha + \frac{1}{\sqrt{2\omega}}\mathcal{F}(|u|^2) \end{cases}, \quad (\text{S-KG}_\alpha)$$

where \mathcal{F} denotes the Fourier transform. It is known that the Cauchy problem for the Schrödinger-Klein-Gordon system (S-KG) is globally well posed on the energy space, see for instance [17, 31] and references therein. In particular, there is a unique Hamiltonian flow Φ_t for (S-KG $_\alpha$) on the energy space $H^1(\mathbb{R}^3) \oplus \mathcal{F}H^{\frac{1}{2}}(\mathbb{R}^3)$ where $\mathcal{F}H^s(\mathbb{R}^d)$ denotes the Fourier Sobolev space,

$$\mathcal{F}H^s(\mathbb{R}^d) = \left\{ f, \mathcal{F}^{-1}f \in H^s(\mathbb{R}^d) \right\}.$$

Moreover, the vector field $v(t, \cdot) : H^1(\mathbb{R}^3) \oplus \mathcal{F}H^{\frac{1}{2}}(\mathbb{R}^3) \rightarrow L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)$ satisfies by Gagliardo-Nirenberg's inequality,

$$\|v(t, u \oplus \alpha)\|_{L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)} \leq C(\|u\|_{H^1(\mathbb{R}^3)}^2 + \|\alpha\|_{\mathcal{F}H^{\frac{1}{2}}(\mathbb{R}^3)}^2).$$

The derivation of such equation from quantum field theory is studied in [6].

A Projective tools

Let us begin this section by recalling the disintegration theorem.

A.1 The disintegration theorem

Let E, F be separable metric spaces and let $x \in E \mapsto \eta_x \in \mathfrak{P}(F)$ be a measure-valued map. We say that η is a Borel map if $x \in E \mapsto \eta_x(B)$ is a Borel map for any Borel set $B \subset F$. Then we get by the monotone class theorem

$$x \mapsto \int_F f(x, y) d\eta_x(y), \quad (39)$$

is Borel for every bounded function $f : E \times F \rightarrow \mathbb{R}$. Hence by using (39) the formula

$$\eta(f) = \int_E \int_F f(x, y) d\eta_x(y) d\nu,$$

defines for any $\nu \in \mathfrak{P}(E)$ a unique measure $\eta \in \mathfrak{P}(E \times F)$, that will be denoted $\int_E \eta_x d\nu(x)$. Actually the disintegration theorem below implies for any $\eta \in \mathfrak{P}(E \times F)$ whose first marginal is ν can be represented in this way.

Proposition A.1. *Let E, F be complete separable metric spaces, $\eta \in \mathfrak{P}(E \times F)$, let $\pi : E \times F \rightarrow E$ be a Borel map and let $\mu_\pi^\eta = \pi_*\eta \in \mathfrak{P}(E)$. Then there exists a μ_π^η -a.e. uniquely determined Borel family of probability measures $\{\eta_x^\pi\}_{x \in E} \subset \mathfrak{P}(F)$ such that*

$$\eta = \int_E \eta_x^\pi d\mu_\pi^\eta(x), \quad (40)$$

i.e.

$$\int_{E \times F} f(x, y) d\eta(x, y) = \int_E \left(\int_F f(x, y) d\eta_x^\pi(y) \right) d\mu_\pi^\eta(x), \quad (41)$$

for every Borel map $f : E \times F \rightarrow [0, +\infty]$.

A.2 Tightness

Denote \mathcal{Z}_0 a infinite dimensional separable Hilbert space. Recall that we introduce the space $(\mathcal{Z}_0, d_{w, \mathcal{Z}_0})$ induced by the following distance

$$d_{w, \mathcal{Z}_0}(z_1, z_2) = \sqrt{\sum_{n \in \mathbb{N}} \frac{|\langle z_1 - z_2, e_n \rangle_{\mathcal{Z}_0}|^2}{1 + n^2}}, \quad z_1, z_2 \in \mathcal{Z}_0.$$

One of the main arguments in the proof of Theorems 2.1-2.2 is the weak tightness of a family of measures (see the weak tightness step in the proof). Therefore we recall below the definition and the criterions used in this proof. We say that a set $\mathcal{K} \subset \mathfrak{P}(\mathcal{Z}_0)$ is *tight* if,

$$\forall \lambda > 0, \exists K_\lambda \text{ compact in } (\mathcal{Z}_0, \|\cdot\|_{\mathcal{Z}_0}) \text{ such that } |\mu(\mathcal{Z}_0 \setminus K_\lambda)| \leq \lambda, \quad (\mathbf{Tightness})$$

and *weakly tight* if,

$$\forall \lambda > 0, \exists K_\lambda \text{ compact in } (\mathcal{Z}_0, d_{w, \mathcal{Z}_0}) \text{ such that } |\mu(\mathcal{Z}_0 \setminus K_\lambda)| \leq \lambda \quad (\mathbf{Weak tightness})$$

A useful characterisation is given here (for more details see Chapter V in [4]). The tightness (resp. weak tightness) condition for a subspace \mathcal{K} is equivalent to an integral condition, i.e there exists a function $\varphi : \mathcal{Z}_0 \rightarrow [0, +\infty]$, whose sublevels $\{x \in \mathcal{Z}_0 / \varphi(x) \leq c\}$ are compact in $(\mathcal{Z}_0, \|\cdot\|_{\mathcal{Z}_0})$ (resp. $(\mathcal{Z}_0, d_{w, \mathcal{Z}_0})$), such that

$$\sup_{\mu \in \mathcal{K}} \int_{\mathcal{Z}_0} \varphi(x) d\mu(x) < +\infty.$$

We also use the following tightness criterion:

Let X, X_1, X_2 be separable metric spaces and let $r^i : X \rightarrow X_i$ be continuous maps such that the product map

$$r := r^1 \times r^2 : X \rightarrow X_1 \times X_2$$

is proper. Let $\mathcal{K} \subset \mathfrak{P}(X)$ be such that $\mathcal{K}_i := r_*^i(\mathcal{K})$ is tight in $\mathfrak{P}(X_i)$ for $i = 1, 2$. Then also \mathcal{K} is tight in $\mathfrak{P}(X)$.

The following Lemma is a fundamental step in the proof of the main Theorem 2.1

Lemma A.2. *Let \mathcal{Z}_0 be a separable Hilbert space and $\{\mu_t\}_{t \in \mathbb{R}}$ be a family of weakly narrowly continuous Borel probability measures. Assume that a Borel velocity field $v_t : \mathcal{Z}_0 \rightarrow \mathcal{Z}_0$ satisfies the following uniform control*

$$\forall T > 0, \int_0^T \int_{\mathcal{Z}_0} \|v_t(z)\|_{\mathcal{Z}_0} d\mu_t(z) dt < +\infty. \quad (42)$$

Then there exists a convex, superlinear and non-decreasing function $\Psi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$, such that

$$\forall T > 0, \int_0^T \int_{\mathcal{Z}_0} \Psi(\|v_t(z)\|_{\mathcal{Z}_0}) d\mu_t(z) dt < +\infty. \quad (43)$$

Proof. Without loss of generality, we can always assume that

$$\forall T > 0, \int_0^T \int_{\mathcal{Z}_0} \|v_t(z)\|_{\mathcal{Z}_0} = 1.$$

By setting the measure μ by the following formula: for any $\alpha, \beta \in [0, T]$, for every Borel subset E in \mathcal{Z}_0

$$\mu := \int_\alpha^\beta \mu_t(E) dt,$$

we have

$$\int \|v_t(z)\|_{\mathcal{Z}_0} d\mu = \int_0^{+\infty} \mu(\|v_t(z)\|_{\mathcal{Z}_0} \geq t) dt.$$

Moreover, there exists a subsequence $(t_n)_{n \in \mathbb{N}}$, such that $t_0 = 0$, satisfying

$$\int_{t_n}^{t_{n+1}} \mu(\|v_t(z)\|_{\mathcal{Z}_0} \geq t) dt = \frac{1}{2^{n+1}}.$$

Then, by setting the superlinear convex function $\Psi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$

$$\forall t \in \mathbb{R}, \Psi(t) = \int_0^t n \mathbb{1}_{[t_n, t_{n+1})}(s) ds,$$

we have for any $T > 0$

$$\int_0^T \Psi(\|v_t(z)\|_{\mathcal{Z}_0}) d\mu_t(z) dt = \int \Psi(\|v_t(z)\|_{\mathcal{Z}_0}) d\mu = \int_0^{+\infty} \Psi'(t) \mu(\|v_t(z)\|_{\mathcal{Z}_0} \geq t) dt = \sum_{n \geq 0} \frac{n}{2^{n+1}} < +\infty.$$

□

A.3 Projection onto a finite-dimensional space.

Theorems 2.1-2.2 are obtained in a infinite dimensional separable Hilbert space \mathcal{Z}_0 . The proof is based on the projection $\pi^d, \hat{\pi}^d, \pi^{d,T}$ introduced in the diagram i) and definitions (21)-(22) of the projected velocity vector fields v_t^d and \hat{v}_t^d . Then the following Lemma is fundamental in our approach.

Lemma A.3. *Let $t \mapsto \mu_t : [0, T] \mapsto \mathfrak{P}(\mathcal{Z}_0)$ be a weakly narrowly continuous map on \mathcal{Z}_0 such that the triple $(\mu_t, v_t, \mathcal{Z}_0)$ is solution of the **weak Liouville equation** (see Definition 2.3). Then the following assertions hold true:*

- i) *The triples $(\mu_t^d, v_t^d, \mathcal{Z}_0)$ and $(\hat{\mu}_t^d, \hat{v}_t^d, \mathcal{Z}_0)$ are solutions of the **weak Liouville equations**.*
- ii) *The velocity field v_t^d and \hat{v}_t^d satisfy the following inequality*

$$\|v_t^d\|_{L^1(\mathbb{R}^d, d\mu_t^d)} = \|\hat{v}_t^d\|_{L^1(\mathcal{Z}_0, d\hat{\mu}_t^d)} \leq \|v_t\|_{L^1(\mathcal{Z}_0, d\mu_t)} < +\infty. \quad (44)$$

- iii) *For every non-decreasing convex function $\Psi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ the following estimate holds true*

$$\int_0^T \int_{\mathbb{R}^d} \Psi(|v_t^d(x)|) d\mu_t^d(x) dt \leq \int_0^T \int_{\mathcal{Z}_0} \Psi(\|v_t(z)\|_{\mathcal{Z}_0}) d\mu_t(z). \quad (45)$$

Proof. In order to prove the estimates (44) we introduce a new norm on \mathbb{R}^d given by $\|u\|_{\mathbb{R}^d} = \|\pi^{d,T}u\|_{\mathcal{Z}_0}$ and we compute

$$\begin{aligned} \|v_t^d\|_{L^1(\mathbb{R}^d, d\mu_t^d)} &= \int_{\mathbb{R}^d} \|\pi^{d,T} \int_{(\pi^d)^{-1}(y)} \pi^d v_t(x) d\mu_{t,y}(x)\|_{\mathbb{R}^d} d\mu_t^d(y) \\ &= \int_{\mathbb{R}^d} \|\hat{\pi}^d \int_{(\pi^d)^{-1}(y)} v_t(x) d\mu_{t,y}(x)\|_{\mathbb{R}^d} d\mu_t^d(y) = \|\hat{v}_t^d\|_{L^1(\mathcal{Z}_0, d\hat{\mu}_t^d)} \\ &\leq \int_{\mathbb{R}^d} \int_{(\pi^d)^{-1}(y)} \|\hat{\pi}^d v_t(x)\|_{\mathbb{R}^d} d\mu_{t,y}(x) d\mu_t^d(y) \\ &\leq \int_{\mathcal{Z}_0} \|\hat{\pi}^d v_t(x)\|_{\mathbb{R}^d} d\mu_t(x) \leq \|v_t\|_{L^1(\mathcal{Z}_0, d\mu_t)} < +\infty. \end{aligned}$$

The estimate (45) is a generalization of [30, Lemma 3.9] in the infinite dimensional case by using the disintegration theorem and the Jensen inequality. Consider a regular test function $\varphi = \psi \circ \pi^d$ in (12), with $\nabla\varphi = (\pi^d)^* \circ \nabla\psi \circ \pi^d$ gives

$$\begin{aligned} \frac{d}{dt} \int_{\mathcal{Z}_0} \varphi d\mu_t(x) &= \int_{\mathcal{Z}_0} \langle \pi^d(v_t), \nabla\psi \circ \pi^d \rangle d\mu_t(x) \\ &= \int_{\mathbb{R}^d} \int_{(\pi^d)^{-1}(y)} \langle \pi^d(v_t), \nabla\psi \circ \pi^d \rangle d\mu_{t,y}(x) d\mu_t^d(y) \\ \implies \frac{d}{dt} \int_{\mathbb{R}^d} \psi d\mu_t^d(x) &= \int_{\mathbb{R}^d} \langle v_t^d(y), \nabla\psi(y) \rangle d\mu_t^d(y). \end{aligned}$$

Therefore for $t \in [0, T]$, the triple $(\mu_t^d, v_t^d, \mathbb{R}^d)$ is a solution of the **weak Liouville equation**. It follows similarly that the triple $(\hat{\mu}_t, \hat{v}_t^d, \mathcal{Z}_0)$ is also a solution of the **weak Liouville equation**. \square

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