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# SUMMABILITY CONDITION AND RIGIDITY FOR FINITE TYPE MAPS

MATTHIEU ASTORG

ABSTRACT. We extend a series of results due to Makienko, Dominguez and Sienra on the rigidity of some holomorphic dynamical systems with summable critical values to the setting of finite type maps. We also recover a shorter proof of a transversality theorem of Levin. Our methods are based on the deformation theory introduced by Epstein.

## 1. INTRODUCTION AND STATEMENT OF THE MAIN THEOREMS

Finite type maps are a class of analytic maps  $f : W \rightarrow X$  of complex 1-manifolds introduced and studied by A. Epstein in [Eps93]. More precisely:

**Definition 1.** *An analytic map  $f : W \rightarrow X$  of complex 1-manifolds is of finite type if*

- *$f$  is nowhere locally constant,*
- *$f$  has no isolated removable singularities,*
- *$X$  is a finite union of compact Riemann surfaces, and*
- *the singular set  $S(f)$  is finite.*

This class includes notably rational self-maps of  $\mathbb{P}^1$  and more generally ramified covers between compact Riemann surfaces, as well as entire functions of the complex plane with a finite singular set, such as the exponential family  $f_\lambda(z) = \lambda e^z$ . It also contains the so-called horn maps appearing in the theory of parabolic implosion, as was proved by Buff, Ecalle and Epstein (see [BÉE13] and also [Eps93]). When  $W \subset X$ , one can study the dynamics of the map  $f$ , that is the orbits  $(z, f(z), f^2(z) \dots)$ , for as long as  $f^n(z) \in W$ . If  $z \in W$  is such that for all  $n \in \mathbb{N}$ ,  $f^n(z) \in W$ , then we say that  $z$  is non-escaping. We may define the Fatou set  $\mathcal{F}(f)$  of  $f$  as the set of points  $z \in W$  such that there exists a neighborhood  $U$  of  $z$  in  $W$  such that either all points in  $U$  escape, or the family  $\{f|_U^n : W \rightarrow X, n \in \mathbb{N}\}$  is well-defined and normal. The Julia set is  $\mathcal{J}(f) = X - \mathcal{F}(f)$ .

Epstein proved several key results about the dynamics of these maps: they do not possess wandering domains, their Julia set is never empty, and as for rational maps, we have a classification theorem for periodic Fatou components. He also constructed so-called deformation spaces  $\text{Def}_A^B(f)$ , which are defined abstractly through Teichmüller theory but may be thought of as natural parameter spaces for  $f$ . These deformation spaces are finite-dimensional complex manifolds, and one can describe their cotangent bundle in terms of quadratic differentials.

Moreover, if  $f : W \rightarrow X$  is a finite type map, one can define the Teichmüller space of  $f$ , denoted by  $\text{Teich}(f)$ . This notion was first introduced by McMullen and Sullivan ([MS98]) in the context of algebraic correspondances, and was specifically described in the case of rational maps. Epstein studied in [Eps93] the case of the Teichmüller spaces

of finite type maps. Again, this is a finite-dimensional complex manifold, and it comes equipped with a natural holomorphic immersion into the deformation space (see [Ast14] for a detailed construction of  $\text{Teich}(f)$  from first principles in the case where  $f$  is a rational map, and the proof of the fact that  $\text{Teich}(f)$  immerses into the moduli space). Roughly speaking, the Teichmüller space of  $f$  represents the topological conjugacy class of  $f$ : the larger its dimension is, the more parameters of topological deformation exist for  $f$ . If  $\text{Teich}(f)$  is reduced to a point, then  $f$  is said to be rigid: this implies that for every quasiconformal homeomorphism  $\phi : X \rightarrow X'$  mapping  $W$  to  $W'$ , if there is a holomorphic map  $g$  making the following diagram commute:

$$\begin{array}{ccc} W & \xrightarrow{f} & X \\ \phi \downarrow & & \downarrow \phi \\ W' & \xrightarrow{g} & X' \end{array}$$

then  $X$  must be biholomorphic to  $X'$  and  $\phi$  must be the composition of a biholomorphism  $X \rightarrow X'$  and of a quasiconformal homeomorphism  $X \rightarrow X$  commuting with  $f$ . Thus  $g$  is a "trivial" deformation of  $f$ , as it is in fact biholomorphically conjugate to  $f$ . On the other hand, if  $\text{Teich}(f)$  has maximal dimension (that is, equal to the dimension of the moduli space), then every nearby parameter is quasiconformally conjugate to  $f$ ; we say that  $f$  is structurally stable.

The question of describing when a rational map  $f : \mathbb{P}^1 \rightarrow \mathbb{P}^1$  is structurally stable or rigid is equivalent to a central conjecture in holomorphic dynamics, which state that with the exception of one well-understood family (flexible Lattès maps), no rational map may have an invariant line field supported on its Julia set.

In the series of papers [Mak01], [Mak05], [DMS05], [MS06] and [Mak10], Makienko, Dominguez and Sienna proved that some sufficient expansion along at least one critical orbit was an obstruction to structural stability, and that expansion along *all* critical orbits implied in fact rigidity. This was proved first for rational maps (at first under unnecessary assumptions), then for the exponential family, and at last for some subset of the class of entire functions with only finitely many singular values. Avila gave a different proof in [Avi02] of this result in the case of rational maps. In particular, all of these maps are of finite type. More precisely, this is what we mean by sufficient expansion along a critical orbit:

**Definition 2.** *Let  $f : W \rightarrow X$  be a finite type map, with  $W \subset X$ , and let  $z \in W$  be a non-escaping point. We say that  $z$  is summable if there is a Hermitian metric on  $X$  such that the series*

$$\sum_{n \geq 0} \|Df^n(z)\|^{-1}$$

*is convergent.*

Note that by compactity of  $X$ , the choice of the metric does not matter. This type of condition was first introduced by Tsujii ([Tsu00]) for real quadratic polynomials. Stronger expansivity conditions (the so-called Collet-Eckmann condition) had previously been known to imply rigidity (see [PR99]).

We will also need the following definitions before we can state our main results:

**Definition 3.** A compact set  $K \subset X$  is called a  $C$ -compact if it satisfies the following property: any continuous function on  $K$  can be uniformly approximated by restrictions of functions that are holomorphic on a neighborhood of  $K$ .

This condition, though not always satisfied by the Julia sets of rational maps, is relatively mild; it is in particular always satisfied by Julia sets of polynomials (see [Lev14]).

**Definition 4.** Let  $f : W \rightarrow X$  be a finite type map, with  $W \subset X$ .

- Let  $p(f)$  denote the number of singular values with a periodic or preperiodic orbit
- Let  $s(f)$  denote the number of summable singular values with an infinite forward orbit, whose  $\omega$ -limit sets are  $C$ -compacts.

**Definition 5.** Let  $f : W \rightarrow X$  be a finite type analytic map. Then we say that  $f$  is exceptional if either  $f$  is an automorphism of  $X$ , or an endomorphism of a complex torus, or a flexible Lattès example.

The following is the main result, and generalizes the aforementioned results of Makienko, Dominguez and Sienna and Avila:

**Theorem A.** Let  $f : W \rightarrow X$  be a non-exceptional finite type analytic map, with  $W \subset X$ . We have:

$$\dim \text{Teich}(f) \leq \text{card } S(f) - p(f) - s(f).$$

In particular, if at least one singular value is summable with an  $\omega$ -limit set that is a  $C$ -compact, then  $f$  is not structurally stable, and if all singular values either are summable with  $C$ -compacts as  $\omega$ -limit sets, or have finite orbit, then  $f$  is rigid and therefore does not have any invariant line field.

Our second result is a simplified proof of a theorem of Levin ([Lev14]). Before we state it, let us introduce some notations.

**Definition 6.** Let  $\text{Rat}_d$  be the space of degree  $d$  rational maps, and let  $\text{rat}_d$  be the quotient of  $\text{Rat}_d$  by the group of Möbius transformation acting by conjugacy. We will call  $\text{Rat}_d$  the parameter space of degree  $d$  rational maps, and  $\text{rat}_d$  the moduli space of degree  $d$  rational maps.

The parameter space  $\text{Rat}_d$  is  $2d+1$  dimensional complex manifold, and  $\text{rat}_d$  is a  $2d-2$  complex orbifold.

Denote by  $\text{Crit}(f)$  the critical set of  $f$ , i.e. the set of points  $z$  where  $Df(z) = 0$ .

Let  $\Delta \ni \lambda \mapsto f_\lambda$  be a holomorphic curve in  $\text{Rat}_d$  passing through  $f$  at  $\lambda = 0$ . Denote by  $\dot{f}$  the section  $\frac{df_\lambda}{d\lambda}|_{\lambda=0}$  of the line bundle  $f^*T\mathbb{P}^1$ , and by  $\eta$  the meromorphic vector field

$$\eta = Df^{-1} \cdot (\dot{f}).$$

Note that  $\eta$  is holomorphic outside of  $\text{Crit}(f)$ , and that its poles have order at most the order of the corresponding critical points of  $f$ . Denote by  $T(f)$  the vector space of meromorphic vector fields on  $\mathbb{P}^1$  satisfying this property, i.e. if  $\eta \in T(f)$ , then all the poles of  $\eta$  are in  $\text{Crit}(f)$  and the pole at  $c \in \text{Crit}(f)$  of  $\eta$  has order at most the order of  $c$  as a critical point of  $f$ .

Then the map  $\dot{f} \mapsto \eta = Df^{-1} \cdot (\dot{f})$  induces a canonical isomorphism between  $T_f \text{Rat}_d$  and  $T(f)$  (indeed, this map is clearly injective and  $\dim T(f) = 2d+1 = \dim \text{Rat}_d$ ).

**Definition 7.** Let  $v$  be a summable critical value of a rational map  $f : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ . For any  $\eta \in T(f)$ , denote by:

$$\xi_\eta(v) := \sum_{k=0}^{\infty} (f^k)^* \eta(v) \in T_v \mathbb{P}^1.$$

**Theorem B** (see [Lev14]). Let  $f : \mathbb{P}^1 \rightarrow \mathbb{P}^1$  be a rational map with  $s$  summable critical values, that is not a Lattès map. Assume that either  $f$  has no invariant line field, or that the  $\omega$ -limit set of those  $s$  summable critical values are  $C$ -compact. Then the linear map

$$\begin{aligned} \mathcal{V} : T(f) &\rightarrow \bigoplus_{1 \leq i \leq s} T_{v_i} \mathbb{P}^1 \\ \eta &\mapsto (\xi_\eta(v_i))_{1 \leq i \leq s} \end{aligned}$$

has maximal rank, i.e. equal to  $s$ .

**Outline.** In section 2, we will recall some facts about the Teichmüller space and the deformation space of a finite type map, as well as a description of their cotangent bundle and the immersion of the Teichmüller space into the deformation space. In sections 3 and 4, we will prove some technical lemmas on quadratic differentials that will be useful to the proof of the main theorems. Finally, sections 5 and 6 are devoted to the proofs of Theorem A and Theorem B respectively.

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## 2. DEFORMATION SPACES

Recall the following objects from Teichmüller theory:

**Definition 8.** Let  $f : W \rightarrow X$  be a finite type map, and assume that  $W \subset X$ . Let us define:

- $\text{QC}(f)$  the group of quasiconformal homeomorphisms  $\phi$  of  $X$  such that  $\phi \circ f = f \circ \phi$  wherever this equation is defined
- $\text{QC}_0(f)$  the subgroup of  $\text{QC}(f)$  of those elements  $\phi$  such that there exists a uniformly quasiconformal isotopy relatively to the ideal boundary of  $X$   $\phi_t \in \text{QC}(f)$ ,  $0 \leq t \leq 1$ , with  $\phi_0 = \phi$  and  $\phi_1$  is the identity on  $X$
- $\text{Bel}(f)$  is the space of Beltrami forms on  $X$  that are invariant under  $f$  on  $W$ , and vanish on  $X - W$ .
- The Teichmüller space of  $f$ , denoted by  $\text{Teich}(f)$ , is defined as the quotient  $\text{Bel}(f)/\text{QC}_0(f)$
- $\text{Bel}(X)$  is the space of all Beltrami forms on  $X$
- $\text{bel}(f)$  is the space of Beltrami differentials on  $X$  invariant under  $f$  (a Beltrami differential is a Beltrami form for which we only assume that it has finite  $L^\infty$  norm instead of norm less than one)
- $\text{bel}(X)$  is the space of Beltrami differentials on  $X$ .

The reader unfamiliar with Teichmüller theory may find some background in [Hub06] and [GL00]. Notice that the Beltrami form that is identically zero gives a natural base-point in  $\text{Teich}(f)$ .

**Definition 9.** Let  $W$  be a complex 1-manifold, and let  $A \subset W$  be a finite set. We denote by  $Q(W)$  the space of integrable holomorphic quadratic differentials on  $W$ , and by  $Q(W, A)$  the space of integrable meromorphic quadratic differentials on  $W$ , with at worst simple poles, whose poles are in  $A$ .

Assume from now on that  $f : W \rightarrow X$  is a finite type map, with  $W \subset X$ , and that  $X$  is connected.

**Theorem 1** ([Eps93], Corollary 9 p. 137). *The space  $\text{Teich}(f)$  is a finite dimensional complex manifold.*

**Definition 10.** Let us denote by  $\Lambda_f$  the union of the Julia set of  $f$  and of the closure of the grand orbit of  $S(f)$  in  $X$ , and  $\Omega_f = X - \Lambda_f$ . Also denote by  $Q_f$  the space of integrable quadratic differentials on  $X$  that are holomorphic on  $\Omega_f$ .

Notice that  $\Omega_f \cap W$  is an open subset of the Fatou set of  $f$ ; also note that when  $\Lambda_f$  has zero Lebesgue measure,  $Q_f$  is canonically isomorphic to  $Q(\Omega_f)$ .

**Definition 11.** Let  $f : W \rightarrow X$  be a finite type map. Let  $A \subset W$  and  $B \subset X$  be two finite sets. We say that  $(A, B)$  is admissible for  $f$  if:

- $A \subset B$
- $f(A) \subset B$
- $S(f) \subset B$
- if  $X$  has genus 0, then  $\text{card } A \geq 3$ , and if  $X$  has genus 1, then  $\text{card } A \geq 1$ .

Let  $f : W \rightarrow X$  be a finite type analytic map, with  $W \subset X$ . If  $(A, B)$  is admissible for  $f$ , then  $A \subset B$ , so we have a natural forgetful map  $\varpi : \text{Teich}(X, B) \rightarrow \text{Teich}(X, A)$ .

Moreover, since  $f(A) \subset B$  and  $S(f) \subset B$ , we have a well-defined pullback map  $\sigma_f : \text{Teich}(X, B) \rightarrow \text{Teich}(X, A)$  obtained by pulling back Beltrami forms from  $X$  to  $W$  using  $f$ , and then extending them by 0 to the rest of  $X$ ; this operation on Beltrami forms descends to a holomorphic map from  $\text{Teich}(X, B)$  to  $\text{Teich}(X, A)$ .

**Definition 12** ([Eps93]). We define  $\text{Def}_A^B(f)$  by:

$$\text{Def}_A^B(f) = \{\tau \in \text{Teich}(X, B), \varpi(\tau) = \sigma_f(\tau)\}.$$

Note that once again, the zero Beltrami form on  $X$  induces a natural basepoint in  $\text{Def}_A^B(f)$ . From its definition,  $\text{Def}_A^B(f)$  is clearly an analytic set. But we can in fact say more:

**Theorem 2** ([Eps93]). *Let  $f : W \rightarrow X$  be a non-exceptional finite type map, and let  $(A, B)$  be admissible for  $f$ . Then  $\text{Def}_A^B(f)$  is a complex manifold of dimension  $\text{card}(B - A)$ , whose cotangent space at the basepoint canonically identifies with:*

$$Q(X, B)/\nabla_f Q(X, A).$$

Let  $\text{Bel}(X)$  denote the space of Beltrami forms on  $X$ . The identity map  $\text{Bel}(X) \rightarrow \text{Bel}(X)$  descends to a natural holomorphic map  $i : \text{Teich}(f) \rightarrow \text{Def}_A^B(f)$ , mapping basepoint to basepoint.

The next theorem has been proved in [Ast14] in the case of rational maps, through a new construction of the complex structure of  $\text{Teich}(f)$  that bypasses the use of certain sophisticated tools from Teichmüller theory. A. Epstein has a different (unpublished) proof.

**Theorem 3.** *The cotangent space at the basepoint of  $\text{Teich}(f)$  canonically identifies with  $Q_f/\overline{\nabla_f Q_f}$ . Moreover, if  $f$  is non-exceptionnal,  $(A, B)$  is admissible for  $f$  and  $B \subset \Lambda_f$ , then the natural map  $\Psi_T : \text{Teich}(f) \rightarrow \text{Def}_A^B(f)$  is an immersion, and the kernel of its codifferential at the basepoint is given by:*

$$\ker D\Psi_T(0)^* = (Q(X, B) \cap \overline{\nabla_f Q_f}) / \nabla_f Q(X, A).$$

The proof of the particular case where  $f$  is a rational map, available in [Ast14], can easily be adapted to the general case of a finite type map. For the convenience of the reader, we will include here a sketch of the proof. Note that just like in [Ast14], this approach may also lead to another construction of the complex structure on  $\text{Teich}(f)$ .

*Proof of Theorem 3.* Let  $(A, B)$  be admissible for  $f$ , with  $B \subset \Lambda_f$ . The only difference between the proof of Theorem 3 and that of the main theorem of [Ast14] is that we are going to replace the moduli space  $\text{rat}_d$  of degree  $d$  rational maps with the deformation space  $\text{Def}_A^B(f)$ .

The natural inclusion  $\text{Bel}(f) \rightarrow \text{Bel}(X)$  descends to a holomorphic map  $\Psi : \text{Bel}(f) \rightarrow \text{Def}_A^B(f)$ .

**Lemma 1.** *The kernel of  $D\Psi(0)$  is equal to*

$$\ker D\Psi(0) = \{\overline{\partial}\xi, \xi|_{\Lambda_f} = 0\}.$$

*Proof of Lemma 1.* The differential of this map at the basepoint is the restriction to  $\text{bel}(f)$  of the quotient map

$$\text{bel}(X) \rightarrow T_{[0]} \text{Teich}(X, B)$$

whose kernel is  $\{\overline{\partial}\xi, \xi \text{ quasiconformal vector field on } X \text{ s.t. } \xi|_B = 0\}$  (see for example [Hub06] for a description of  $T_{[0]} \text{Teich}(X, B)$ ). Since  $B \subset \Lambda_f$ , if  $\mu = \overline{\partial}\xi$  with  $\xi|_{\Lambda_f} = 0$ , then  $\mu \in \ker D\Psi(0)$ . Conversely, if for  $\overline{\partial}\xi \in \text{bel}(f)$  we let  $\eta = f^*\xi - \xi$  on  $W$ , then  $\eta$  is a meromorphic vector field on  $W$  (with poles at the critical points of  $f$ ). Indeed, we have  $\overline{\partial}\xi = f^*\overline{\partial}\xi$  so outside of the critical points of  $f$ ,  $\overline{\partial}\xi = \overline{\partial}(f^*\xi)$  so that  $\overline{\partial}\eta = 0$  outside of the critical points of  $f$ .

Now we claim that if  $\mu \in \ker D\Psi(0)$ , then  $\mu$  can be written as  $\overline{\partial}\xi$ , where  $\xi$  is a quasiconformal vector field invariant under pullback by  $f$  on  $W$ . Indeed, let  $Z$  be any union of repelling periodic points of  $f$  disjoint from  $B$ , of cardinal at least 3. Let  $A' = A \cup Z$ , and  $B' = B \cup Z$ ; then  $(A', B')$  is admissible for  $f$ . The forgetful map  $\text{Teich}(X, B') \rightarrow \text{Teich}(X, B)$  induces a natural holomorphic map  $\Phi : \text{Def}_{A'}^{B'}(f) \rightarrow \text{Def}_A^B(f)$ ; its codifferential at the basepoint is the natural map

$$D\Phi(0)^* : Q(B)/\nabla_f Q(A) \rightarrow Q(B')/\nabla_f Q(A')$$

induced by the inclusion  $Q(B) \rightarrow Q(B')$ . This map is injective: indeed, if  $q \in Q(B)$  is of the form  $q = \nabla_f \phi$  with  $\phi \in Q(A')$ , then  $\nabla_f \phi$  does not have any poles at any repelling points in  $Z$ . So this means that the polar part of  $\phi$  on  $Z$  is invariant, but since  $Z$  is a union of repelling points this implies that  $\phi$  does not have any pole on  $Z$  (see [Eps09]). So in fact  $q \in \nabla_f Q(A)$ , which means that  $D\Phi(0)^*$  is injective. Since

$$\dim Q(B)/\nabla_f Q(A) = \dim Q(B')/\nabla_f Q(A') = \text{card}(B - A),$$

$D\Phi(0)$  is also surjective, which implies that  $D\Phi(0)$  is injective. Now notice that if we let  $\Psi'$  be the natural map from  $\text{Bel}(f)$  to  $\text{Def}_A^{B'}(f)$ , then  $\Psi = \Phi \circ \Psi'$ , so that  $D\Psi(0) = D\Phi([0]) \circ D\Psi'(0)$ . Since  $D\Phi(0)$  is injective, we have therefore  $\ker D\Psi(0) = \ker D\Psi'(0)$ . Since this is true for any choice of  $Z$ , this means that for any finite union of repelling point  $Z$ , there is a quasiconformal vector field  $\xi_Z$  such that  $\bar{\partial}\xi_Z = \mu$  and  $\xi_Z = 0$  on  $Z$ . For every choice of  $Z_1, Z_2$  of cardinal ar least 3, the difference  $\xi_{Z_1} - \xi_{Z_2}$  is a holomorphic vector field on  $X$  vanishing on at least 3 points, so by the Riemann-Roch formula  $\xi_{Z_1} = \xi_{Z_2}$ . This means that every  $\mu \in \ker D\Psi(0)$  may be written as  $\mu = \bar{\partial}\xi$  with  $\xi$  vanishing on *every* repelling periodic point, so  $\eta = f^*\xi - \xi$  also vanishes on every repelling periodic point of  $f$ . Since  $\eta$  is meromorphic on  $W$ , we must have  $\eta = 0$  by the isolated zeroes theorem. So  $\xi = f^*\xi$  on  $W$ , which implies that  $\xi|_{\Lambda_f} = 0$ .  $\square$

Once we have this description of the kernel of  $D\Psi(0)$ , and using the classification of non-escaping Fatou components for finite type maps, the proof of [[Ast14], Th. 4.5] carries over verbatim, proving that the differential of  $\Psi$  has constant rank on  $\text{Bel}(f)$ . The map  $\Psi : \text{Bel}(f) \rightarrow \text{Def}_A^B(f)$  descends to the map  $\Psi_T : \text{Teich}(f) \rightarrow \text{Def}_A^B(f)$ . Applying the constant rank theorem to  $\Psi$ , we obtain local coordinates on  $\text{Bel}(f)$  and  $\text{Def}_A^B(f)$ , in which (following Section 5 of [Ast14]) the map  $\Psi_T$  can be written locally as a linear inclusion between two finite dimensional complex spaces. Therefore, the map  $\Psi_T$  is an immersion. Let  $\pi : \text{Bel}(f) \rightarrow \text{Teich}(f)$  be the quotient map; since  $D\Psi_T([0]) \circ D\pi(0) = D\Psi(0)$ , and since  $D\Psi_T(0)$  is injective, we must have  $\ker D\pi(0) = \ker D\Psi(0)$ . Therefore, using the preceding lemma, we have the following canonical identification:

$$T_{[0]} \text{Teich}(f) = \text{bel}(f) / \ker D\pi(0) = \text{bel}(f) / \{\bar{\partial}\xi, \xi|_{\Lambda_f} = 0\},$$

and by duality:

$$T_{[0]}^* \text{Teich}(f) = Q_f / \overline{\nabla_f Q_f}.$$

Finally, it just remains to prove that the kernel of  $D\Psi_T(0)^*$  is  $(Q(X, B) \cap \overline{\nabla_f Q_f}) / \nabla_f Q(X, A)$ . But since the differential  $D\Psi_T(0)$  is the natural map

$$\text{bel}(f) / \{\bar{\partial}\xi, \xi|_{\Lambda_f} = 0\} \rightarrow \text{bel}(X) / \{\bar{\partial}\xi, \xi|_B = 0\},$$

the codifferential  $D\Psi_T(0)^*$  is the natural map

$$Q(X, B) / \nabla_f Q(X, A) \rightarrow Q_f / \overline{\nabla_f Q_f},$$

whose kernel is clearly  $(Q(X, B) \cap \overline{\nabla_f Q_f}) / \nabla_f Q(X, A)$ .  $\square$

### 3. ACTION OF QUADRATIC DIFFERENTIALS ON VECTOR FIELDS

In this section,  $X$  will denote a compact Riemann surface of genus  $g$ . If we chose an arbitrary Hermitian metric on  $X$ , we get a topology on the space  $\Gamma(TX)$  of continuous vector fields on  $X$ , induced by the norm

$$\|\xi\|_\infty = \sup_{s \in X} \|\xi(s)\|.$$

This norm depends on the particular choice of the Hermitian metric, but not the topology it induces (by compacity of  $X$ ). We will refer to it as the *uniform topology* for continuous vector fields on  $X$ .



**Definition 13.** Denote by  $\Gamma(TX)^*$  the (topological) dual of the topological vector space of continuous vector fields on  $X$ , equipped the topology dual to the uniform topology.

Again, the choice of a Hermitian metric on  $X$  gives by duality a norm generating the topology on  $\Gamma(TX)^*$ , but that topology is independant from the choice of the norm. Depending on the genus  $g$  of  $X$ , it will be convenient to use different choices of metrics in the following proofs.

**Definition 14.** Let  $q$  be an integrable quadratic differential on  $X$ . Then  $q$  induces a linear form on the space of smooth vector fields in the following way:

$$\xi \mapsto \int_X q \cdot \bar{\partial}\xi.$$

If that linear form extends continuously to an element of  $\Gamma(TX)^*$ , we denote that extension by  $\bar{\partial}q$  and we say that  $q$  is regular.

Note that if  $q$  is written in local coordinates as  $q = h(z)dz^2$ , then  $q$  is regular if and only if  $\frac{\partial h}{\partial \bar{z}}$  (in the sense of distributions) is a complex Radon measure. It is in particular the case when  $q$  is meromorphic with at worst simple poles, in which case  $\bar{\partial}q$  has finite support.

An immediate consequence of Weyl's lemma is that if  $q$  is a regular quadratic differential such that  $\bar{\partial}q$  is supported in a compact  $K$ , then  $q$  is holomorphic outside of  $K$ .

**Proposition 1.** Let  $\mathcal{M}$  be the space of Radon measures on  $X$ , and  $A = C^0(X, \mathbb{C})$ . Let  $\Omega^{1,0}(X)$  denote the space of complex-valued continuous forms of bidegree  $(1,0)$  on  $X$ . The map

$$\begin{aligned} \mathcal{M} \otimes_A \Omega^{1,0}(X) &\rightarrow \Gamma(TX)^* \\ \mu \otimes \alpha &\mapsto \left( \xi \mapsto \int_X \alpha(\xi) d\mu \right) \end{aligned}$$

is an isomorphism of  $A$ -modules (and therefore of  $\mathbb{C}$ -vector spaces).

**Remark 1.** Since  $\Omega^{1,0}(X)$  is an  $A$ -module of rank 1, every element of  $\Omega^{1,0}(X) \otimes_A \mathcal{M}$  can be written as  $\alpha \otimes_A \mu$ , where  $\alpha \in \Omega^{1,0}(X)$  and  $\mu \in \mathcal{M}$ .

*Proof.* The considered map is clearly an injective morphism of  $A$ -modules. It is therefore enough to prove that it is surjective. Let  $u \in \Gamma(TX)^*$ . If the support of  $u$  is included in a local coordinate domain  $(U, z)$ , then it is a consequence of Riesz's representation theorem that  $u$  can be written as  $u = dz \otimes_A \mu$ , where  $\mu$  is a Radon measure of support included in  $U$ . We conclude easily using a partition of unity.  $\square$

We will therefore identify from now on  $\Gamma(TX)^*$  with  $\Omega^{1,0}(\mathbb{P}^1) \otimes_A \mathcal{M}$ .

**Definition 15.** Let  $f : W \rightarrow X$  be an open holomorphic map and let  $u = \alpha \otimes \mu \in \Gamma(TX)^*$  be such that  $\frac{\|\alpha\|}{\|Df\|} \in L^1(|\mu|)$  (for any continuous Hermitian metric on  $X$ ). We define the pushforward of  $u$ , denoted by  $f_*u$ , by :

$$\langle f_*u, \xi \rangle := \langle u, f^*\xi \rangle = \int_{\mathbb{P}^1} \alpha(f^*\xi) d\mu.$$

Note that  $f_*u \in \Gamma(TX)^*$ . In particular, if  $u \in \Gamma(TX)^*$  has support  $K$  that does not meet  $S(f)$ , then  $f_*u$  is well-defined and has a support included in  $f(K)$ .

**Lemma 2.** *Let  $Z \subset X$  be a subset of cardinal  $|3g - 3|$ . Let  $u \in \Gamma(TX)^*$  be supported in  $\{y\}$ , where  $y \in X$ . Then there is a unique meromorphic quadratic differential  $q$  on  $X$  with at worst simple poles such that:*

- if  $g = 0$ , then  $\bar{\partial}q - u$  is supported in  $Z$
- if  $g \geq 1$ ,  $\bar{\partial}q = u$  and for all  $z \in Z$ ,  $q(z) = 0$ .

Moreover, for any choice of Hermitian metric on  $X$ , there is a constant  $C > 0$  depending only on that metric and on  $Z$  such that  $\|q\|_{L^1} \leq C\|u\|_\infty$ .

*Proof.* We will treat separately the three following cases:  $g = 0$ ,  $g = 1$  and  $g \geq 2$ .

*The case of genus 0.* If  $X$  has genus 0, then  $X$  is isomorphic to the Riemann sphere  $\mathbb{P}^1$ . Note that a meromorphic quadratic differential  $q$  with at worst simple poles on  $X$  will satisfy the property that  $\bar{\partial}q - u$  if and only if  $q$  has at worst four simple poles, located in  $Z \cup \{y\}$ , and for all smooth vector fields  $\xi$  vanishing on  $Z$ ,

$$\int_X q \cdot \bar{\partial}\xi = \langle u, \xi \rangle.$$

If we work in affine coordinates in which  $Z = \{0, 1, \infty\}$ , then  $q$  has the form:

$$q(z) = \alpha \frac{y(y-1)}{z(z-1)(z-y)} dz^2$$

where  $\alpha$  is such that  $u = \delta_y \otimes (\alpha dz)$ ,  $\delta_y$  being the Dirac mass at  $y$ . Up to permuting the order of points in  $Z$ , we may assume that  $|y| < 1$ . Then it is easy to see that  $\|q\| \leq C|\alpha|$  for some constant  $C > 0$  (depending on the coordinates  $z$  and therefore on  $Z$ ) and that  $\|u\| = |\alpha| \sup_{|y| < 1} \|dz\|$ . Therefore there is a constant  $C_2 > 0$  depending only on the metric and on  $Z$  such that  $\|q\|_{L^1} \leq C_2\|u\|$ . This concludes the case of genus 0.

*The case of genus 1.* In this case,  $Z$  is empty, so we need to prove that there is a unique quadratic differential with at worst simple poles such that  $\bar{\partial}q = u$ . Any such quadratic differential must have at worst one simple pole, located at  $y$ . According to the Riemann-Roch formula, such quadratic differentials form a complex vector space of dimension one. Moreover, Stokes' theorem implies that for all smooth vector fields  $\xi$  on  $X$ ,

$$\langle \bar{\partial}q, \xi \rangle = \int_X q \cdot \bar{\partial}\xi = 2i\pi \text{Res}(q \cdot \xi, y).$$

Therefore, there is exactly one choice of polar part at  $y$  (and therefore exactly one choice of  $q$ ) such that  $\bar{\partial}q = u$ . Let us now prove the inequality. If  $g = 1$ , then  $X$  is a complex torus  $\mathbb{C}/\Lambda$ , and for any  $y \in X$ , there is a translation descending to an automorphism  $T_y$  of  $X$  mapping the basepoint  $[0]$  to  $y$ . The pullback map  $T_y^*$  induces an isometry for the  $L^1$  norm of integrable quadratic differentials, as well as for linear forms in  $\Gamma(TX)^*$  (endowed with the norm induced by the flat Hermitian metric on  $X$ ). In other words, we lose no generality in assuming that  $y = [0]$ . Then the desired inequality is trivial, since the map  $u \mapsto q$  is a complex linear map between finite dimensional normed vector spaces (in fact one-dimensional), therefore is continuous.

*The case of genus at least 2.* The existence and unicity is similar to the previous case: notice that if  $\bar{\partial}q = u$  and  $q$  has at worst simple poles, then  $q$  must have at worst a simple pole at  $y$  and must vanish on  $Z$ . According to the Riemann-Roch formula, such quadratic differentials form a vector space of complex dimension one, and once again, the choice of the right polar part at  $y$  uniquely determines  $q$ .

Now let us prove the desired inequality. Since  $g \geq 2$ ,  $X$  is hyperbolic, so we may pick its hyperbolic metric as a choice of Hermitian metric inducing a norm on  $\Gamma(TX)^*$ . We will work by duality. According to Theorem A in [Ast14], for any quasiconformal vector field  $\xi$  on  $X$ , we have  $\|\xi\| \leq 4\|\bar{\partial}\xi\|_\infty$  (here  $\|\xi\|$  is the supremum of the length of  $\xi$  is the hyperbolic metric on  $X$ , which is finite since  $X$  is compact). Therefore  $\|q\|_{L^1} \leq 4\|u\|$ .  $\square$

**Theorem 4.** *Let  $Z \subset X$  be a subset of cardinal  $|3g - 3|$ . Let  $u \in \Gamma(TX)^*$ . Then there is a unique regular quadratic differential  $q$  on  $X$  such that:*

- if  $g = 0$ , then  $\bar{\partial}q - u$  is supported in  $Z$
- if  $g \geq 1$ ,  $\bar{\partial}q = u$  and for all  $z \in Z$ ,  $q(z) = 0$ .

Moreover, for any choice of Hermitian metric on  $X$ , there is a constant  $C > 0$  depending only on that metric such that  $\|q\|_{L^1} \leq C\|u\|_\infty$ .

We will say that the quadratic differential  $q$  given by the above theorem is *the  $Z$ -normalized quadratic differential corresponding to  $u$* .

*Proof.* According to Proposition 1, we can write  $u = \mu \otimes \alpha$ . For any  $y \in X$ , let  $u_y = \delta_y \otimes \alpha$ , where  $\delta_y$  is the Dirac mass at  $y$ , and  $q_y$  be the corresponding quadratic differential given by the preceding lemma. Let  $r_y = \bar{\partial}q_y - u_y$ : by definition,  $r_y$  is supported in  $Z$ . The second part of that lemma implies that there is a constant  $C > 0$  depending only on the choice of Hermitian metric and on  $\alpha$  such that for all  $y \in X$ ,  $\|q_y\|_{L^1} \leq C$ . Let  $q = \int_X q_y \mu(y)$ . Note that  $q$  is integrable and  $\|q\|_{L^1} \leq C$ . We will prove that  $q$  satisfies the desired property. Let  $\xi$  be a smooth vector field on  $X$ . We have:

$$\begin{aligned} \int_X q \cdot \bar{\partial}\xi &= \int_X \left( \int q_y \mu(y) \right) \cdot \bar{\partial}\xi \\ &= \int \left( \int_X q_y \cdot \bar{\partial}\xi \right) \mu(y) \\ &= \int \langle u_y, \bar{\partial}\xi \rangle \mu(y) \\ &= \int \alpha_y(\xi) \mu(y) + \int \langle r_y, \xi \rangle \mu(y) \\ &= \langle u, \xi \rangle + \langle r, \xi \rangle \end{aligned}$$

where  $\langle r, \xi \rangle := \int \langle r_y, \xi \rangle \mu(y)$  is supported in  $Z$  if  $g = 0$ , and is identically zero otherwise. Thus the theorem is proved.  $\square$

**Proposition 2.** *Let  $q$  be a regular quadratic differential, and  $f : W \rightarrow X$  a finite type analytic map. Assume that  $f_*\bar{\partial}q$  and  $\bar{\partial}f_*q$  are well-defined as elements of  $\Gamma(TX)^*$ . Then*

$$\text{supp}(\bar{\partial}f_*q - f_*\bar{\partial}q) \subset S(f).$$

*Proof.* Let  $\xi$  be a quasiconformal vector field vanishing on a neighborhood of  $S(f)$ . Then  $f^*\xi$  is also quasiconformal (and it vanishes in the neighborhood of  $\text{Crit}(f)$ ), and  $\bar{\partial}f^*\xi = f^*\bar{\partial}\xi$ . Therefore:

$$\langle \bar{\partial}f_*q, \xi \rangle = \langle f_*\bar{\partial}q, \xi \rangle,$$

and so  $\langle f_*\bar{\partial}q - \bar{\partial}f_*q, \xi \rangle = 0$ . This exactly means that  $\text{supp}(\bar{\partial}f_*q - f_*\bar{\partial}q) \subset S(f)$ .  $\square$

#### 4. EXTENDED INFINITESIMAL THURSTON RIGIDITY

**Definition 16.** A compact set  $K \subset X$  is called a  $C$ -compact if it satisfies the following property: any continuous function on  $K$  can be uniformly approximated by restrictions of functions that are holomorphic on a neighborhood of  $K$ .

Note that a  $C$ -compact must have empty interior. The following proposition gives sufficient conditions for a compact to be a  $C$ -compact. The proof is adapted from [Mak10] to the case of a general Riemann surface.

**Remark 2.** In fact, it can be proved (see [BJ04, Th. 2]) that being a  $C$ -compact is a local property, in the sense that  $K$  is a  $C$ -compact if and only if for every point  $p \in K$ , there is a basis of neighborhoods  $(U_n)_{n \in \mathbb{N}}$  such that  $K \cap \overline{U_n}$  is a  $C$ -compact. Therefore we can replace functions by vector fields (or sections of any holomorphic line bundle) without changing the definition of  $C$ -compact.

**Proposition 3.** Let  $K$  be a compact subset of  $X$ . Each of the following properties imply that  $K$  is a  $C$ -compact :

- i)  $K$  has zero Lebesgue measure, or
- ii)  $K$  disconnects  $X$  into finitely many connected components.

*Proof.* Those conditions have been observed to imply that  $K$  is a  $C$ -compact in [Mak01] and [Lev14] in the case where  $X = \mathbb{P}^1$ . The following are immediate adaptations to the general case of a compact Riemann surface  $X$ .

- i) This follows from the local nature of being a  $C$ -compact (remark 2) and Vitushkin's theorem (see e.g. [Gam05]).
- ii) This follows from [Sch78], by taking  $M$  to be  $X$  with a closed disk removed from every connected component of  $X - K$ .

$\square$

**Theorem 5.** Let  $K \subset X$  be a  $C$ -compact, and let  $q$  be a regular quadratic differential supported in  $K$ . Then  $q = 0$  Lebesgue a.e.

*Proof.* Let  $q$  be a regular quadratic differential supported in a  $C$ -compact  $K$ . We shall prove that  $\bar{\partial}q = 0$  as an element of  $\Gamma(TX)^*$ . By definition of a  $C$ -compact and by Remark 2, any continuous vector field can be uniformly approximated on  $K$  by restrictions of vector fields that are holomorphic in the neighborhood of  $K$ , so it is enough to test  $\bar{\partial}q$  against such vector fields. Let  $\xi$  be a smooth vector field on  $X$  that is holomorphic on a neighborhood  $U$  of  $K$ . Then:

$$\begin{aligned} \langle \bar{\partial}q, \xi \rangle &= \int_X q \cdot \bar{\partial}\xi \\ \langle \bar{\partial}q, \xi \rangle &= \int_U q \cdot \bar{\partial}\xi + \int_{X-U} q \cdot \bar{\partial}\xi \end{aligned}$$

Since  $q$  is supported in  $K$ , we have  $\int_{X-U} q \cdot \bar{\partial}\xi = 0$ . Since  $\xi$  is holomorphic on  $U$ , we have  $\int_U q \cdot \bar{\partial}\xi = 0$ . Therefore  $\langle \bar{\partial}q, \xi \rangle = 0$  and so  $\bar{\partial}q = 0$ . So by Weyl's lemma,  $q$  is holomorphic on  $X$  (up to a set of Lebesgue measure zero), and vanishes on the (non-empty) open set  $X - K$ , so  $q = 0$  Lebesgue-a.e.  $\square$

Recall the following fundamental fact:

**Proposition 4** (see [Eps93], Corollary 8 p. 124). *Let  $f : W \rightarrow X$  be a non-exceptional finite type analytic map, with  $W \subset X$ . Let  $A \subset X$  be a finite set. Then if  $q \in Q(X, A)$  and  $q = f_*q$ , then  $q = 0$ .*

We will now investigate what happens if we relax the assumption that  $q$  is meromorphic. The following result will be needed:

**Theorem 6** ([Eps93]). *Let  $U$  be a non-escaping Fatou component for  $f$ . Then  $U$  is eventually mapped to a periodic component which is either an attracting basin, a parabolic basin, a Herman ring or a Siegel disk.*

**Definition 17.** *A rotation annulus for  $f$  is a connected component of  $\Omega_f$  which is an annulus of finite modulus and on which the dynamics of  $f$  is conjugate to an irrational rotation.*

*A cycle of rotation annuli for  $f$  of period  $p$  is a family of components  $(\mathcal{A}, \dots, f^{p-1}(\mathcal{A}))$  of  $\Omega_f$  which are all rotation annuli for  $f^p$ .*

To each rotation annulus  $\mathcal{A}$ , we may canonically associate a quadratic differential  $q_{\mathcal{A}}$  in the following way: let  $\phi : \mathcal{A} \rightarrow \mathbb{C}$  be a linearizing coordinate for  $f$  on  $\mathcal{A}$ , mapping  $\mathcal{A}$  to a straight annulus  $A(R) = \{1 < |z| < R\}$ . Let

$$(1) \quad q_{\mathcal{A}} = \phi^* \left( \frac{dz^2}{z^2} \right).$$

One can easily check using Laurent series that  $\frac{dz^2}{z^2}$  is up to scalar multiplication the only holomorphic quadratic differential on  $Q(A(R))$  that is rotation-invariant: in particular, there are no rotation invariant quadratic differential that are integrable near 0. Therefore  $q_{\mathcal{A}}$  is the only integrable holomorphic quadratic differential on  $Q(\mathcal{A})$  that is forward-invariant under  $f$ . We can extend it by zero outside of  $\mathcal{A}$  to obtain a forward invariant quadratic differential in  $Q(\Omega_f)$ , that we still denote by  $q_{\mathcal{A}}$ .

Similarly, if  $(A, \dots, f^{p-1}(A))$  is a cycle of rotation rings for  $f$ , then we get a quadratic differential  $\tilde{q}_{\mathcal{A}} \in Q(\mathcal{A})$  that is invariant under  $f^p$ . It is then easy to check that  $q_{\mathcal{A}} := \sum_{k=0}^{p-1} f_*^k \tilde{q}_{\mathcal{A}}$  is forward invariant under  $f$ , and it is (up to scalar multiplication) the only one on  $Q(\mathcal{A} \cup \dots \cup f^{p-1}(\mathcal{A}))$ .

**Proposition 5.** *Let  $f : W \rightarrow X$  be a finite type analytic map, with  $W \subset X$ . Then the only quadratic differentials on  $Q(\Omega_f)$  invariant by  $f_*$  are those described above.*

*Proof.* Let  $q \in Q(\Omega_f)$  be an invariant quadratic differential. Then  $|q|$  is an invariant measure on  $\Omega_f$ , that does not charge any escaping Fatou component.

Let  $U$  be a component of  $\Omega_f$  with positive mass for  $|q|$ , and let  $V$  be the non-escaping Fatou component containing  $U$ . According to Theorem 6,  $V$  is eventually mapped to an attracting basin, a parabolic basin, a Siegel disk or a Herman ring. If  $V$  is mapped to an

attracting or parabolic basin, then every point in  $U$  converges to the same finite cycle of points, so the grand orbit of  $V$  cannot support an invariant measure absolutely continuous with respect to the Lebesgue measure. Therefore  $V$  must be eventually mapped to either a (periodic) Siegel disk or Herman ring. Such a Fatou component can never be completely invariant, since it maps to itself with degree 1. So  $V$  must be in fact in the cycle: indeed, if it were not the case, then the preimages  $f^{-n}(V)$  would form a pairwise disjoint family of open sets, each having the same mass as  $V$  for  $|q|$ ; but this would contradict the fact that  $q$  is integrable.

Therefore  $U$  must be in a periodic rotation domain. But by the preceding discussion, the only invariant quadratic differentials on such domains are the quadratic differentials  $q_A$  associated to rotation annuli.  $\square$

## 5. PROOF OF THEOREM A

**Definition 18.** Let  $f : W \rightarrow X$  be a finite type map, with  $W \subset X$ .

- Let  $p(f)$  denote the number of singular values with a periodic or preperiodic orbit
- Let  $s(f)$  denote the number of summable singular values with an infinite forward orbit, whose  $\omega$ -limit sets are  $C$ -compact.

**Theorem A.** Let  $f : W \rightarrow X$  be a non-exceptional finite type analytic map, with  $W \subset X$ . We have:

$$\dim \text{Teich}(f) \leq \text{card } S(f) - p(f) - s(f).$$

*Proof.* Since repelling periodic points are dense in the Julia set of  $f$  (see [Eps93]), we may choose a finite union of repelling cycles of  $f$  of cardinal at least  $|3g - 3|$ , if  $g \neq 1$ , or of cardinal at least one if  $g = 1$ . Let  $Z$  denote such a set. Let  $S^0(f)$  denote the set of singular values of  $f$  which are periodic or preperiodic, and let  $A = \{f^n(s), n \in \mathbb{N} \text{ and } s \in S^0(f)\} \cup Z$ . Let  $B = S(f) \cup A$ . Then  $(A, B)$  are admissible, and therefore  $\text{Def}_A^B(f)$  is a complex manifold of dimension  $\text{card}(B - A) = \text{card}(S(f)) - p(f)$ .

Let  $s \in S(f) - S^0(f)$  be a summable singular value whose  $\omega$ -limit set is a  $C$ -compact. Let  $u \in \Gamma(TX)^*$  be a non-zero linear form with support equal to  $\{s\}$ . Let  $v_n = \sum_{k=0}^n f_*^k u$ . The pushforwards are well-defined, since by definition of summability, the orbit of  $s$  does not meet critical points. The fact that  $s$  is summable readily implies that the sequence  $(v_n)_{n \in \mathbb{N}}$  converges to some  $v_s \in \Gamma(TX)^*$ . Let  $q_s$  be the  $Z$ -normalized quadratic differential on  $X$  corresponding to  $v_s$  (see Theorem 4). From the definition of  $v_s$ , we have that  $v_s - f_* v_s = u$ ; moreover,  $v_s - \bar{\partial} q_s$  is supported in  $Z$  (if  $g = 0$ , or  $v_s = \bar{\partial} q_s$  if  $g > 0$ ). Therefore, in view of Proposition 2,  $q_s - f_* q_s$  is supported in  $Z \cup S(f)$  (and in fact, if  $g > 0$ , it is supported only in  $S(f)$ ). Let us denote by  $S^s(f)$  the set of summable singular values of  $f$  whose  $\omega$ -limit sets are  $C$ -compact.

**Lemma 3.** The quadratic differentials  $(\nabla_f q_s)_{s \in S^s(f)}$  are linearly independent.

*Proof of Lemma 3.* First note that the quadratic differentials  $(q_s)_{s \in S^s(f)}$  are linearly independent. Next, we prove that the vector space spanned by their restriction to  $\Omega_f$  is in direct sum with the kernel of the operator  $\nabla_f : Q(\Omega_f) \rightarrow Q(\Omega_f)$ . According to Proposition 5, we just need to prove that no non-trivial element of the vector space spanned by the  $q_s$ ,  $s \in S^s(f)$ , can be written as  $\lambda \frac{dz^2}{z^2}$  in linearizing coordinates in a rotation annulus of  $f$ , with  $\lambda \neq 0$  (we may assume that the rotation annulus is fixed, up to replacing  $f$  by

one of its iterates). So let  $q = \sum_{s \in S^s(f)} \lambda_s q_s$ , and let  $q_A = \frac{dz^2}{z^2}$ , where  $z$  is a linearizing coordinate for a rotation annulus. Notice that  $\bar{\partial}q = \sum_{s \in S^s(f)} \lambda_s v_s$  is of the form  $\mu \otimes \alpha$ , where  $\mu$  is a converging series of Dirac masses at points of the post-singular set of  $f$ . In particular,  $\mu$  is a measure of dimension 0. On the other hand, let us compute  $\bar{\partial}q_A$ . Let  $\xi$  be a smooth vector field on  $X$ . Let us denote by  $A$  the rotation annulus on which  $q_A$  is supported, and let  $\frac{1}{2\pi} \log R$  be its module. Then, by working in the  $z$ -coordinates, we get:

$$\begin{aligned} \int_X q_A \cdot \bar{\partial}\xi &= \int_A q_A \cdot \bar{\partial}\xi \\ &= \int_{1 < |z| < R} \frac{dz^2}{z^2} \cdot \bar{\partial}\xi(z). \end{aligned}$$

Let  $\epsilon > 0$ . Notice that Stokes' theorem implies that:

$$\int_{1+\epsilon < |z| < R-\epsilon} \frac{dz^2}{z^2} \cdot \bar{\partial}\xi(z) = \int_{|z|=R-\epsilon} \frac{dz^2}{z^2} \cdot \xi(z) - \int_{|z|=1+\epsilon} \frac{dz^2}{z^2} \cdot \xi(z).$$

Moreover, since  $\phi \cdot \bar{\partial}\xi$  is integrable, Lebesgue's dominated convergence theorem implies that

$$\lim_{\epsilon \rightarrow 0} \int_{1+\epsilon < |z| < R-\epsilon} \frac{dz^2}{z^2} \cdot \bar{\partial}\xi(z) = \int_{1 < |z| < R} \frac{dz^2}{z^2} \cdot \bar{\partial}\xi(z),$$

so that

$$\int_X \phi \cdot \bar{\partial}\xi = \lim_{\epsilon \rightarrow 0} \left( \int_{|z|=R-\epsilon} \frac{dz^2}{z^2} \cdot \xi(z) - \int_{|z|=1+\epsilon} \frac{dz^2}{z^2} \cdot \xi(z) \right).$$

Since the pullback of the Lebesgue measure on the circles  $|z| = 1 + \epsilon$  and  $|z| = R - \epsilon$  converge to the harmonic measure of the closure  $\bar{A}$  of  $A$  when  $\epsilon$  tends to 0, this means that  $q_A$  is regular in the sense of Definition 14, and that  $\bar{\partial}q_A$  is of the form  $\nu \otimes \beta$ , where  $\nu$  is measure that is absolutely continuous with respect to the harmonic measure of  $\bar{A}$ . Since the harmonic measure never has dimension 0,  $\bar{\partial}q$  can never be a non-zero multiple of  $\bar{\partial}\phi$ , which concludes the proof that  $q$  cannot be a non-zero multiple of  $\phi$ .

This means that if  $q$  is invariant under  $f$ , then  $q$  must vanish on  $\Omega_f$ . Moreover,  $q$  is holomorphic outside of the union of the closure of the orbit of  $S^s(f)$ , so in fact  $q$  vanishes outside of the union of the closure of the orbit of  $S^s(f)$ , which is a  $C$ -compact. Therefore, by Theorem 5,  $q = 0$ . This concludes the proof of the lemma.  $\square$

Let us now return to the proof of Theorem A. Recall that by construction, for any  $s \in S^s(f)$ ,  $\nabla_f q_s \in Q(X, B)$ . It is a consequence of the previous lemma that the classes  $([\nabla_f q_s])_{s \in S^s(f)}$  are linearly independent in  $Q(X, B)/\nabla_f Q(X, A)$ : indeed,  $\nabla_f$  is injective on the vector space spanned by the  $q_s, s \in S^s(f)$  and by  $Q(X, A)$ . Therefore no non-trivial linear combination of the  $\nabla_f q_s$  can be in  $\nabla_f Q(X, A)$ , since none of the  $q_s$  are in  $Q(X, A)$ . This means that  $\dim(Q(X, B) \cap \overline{\nabla_f Q_f})/\nabla_f Q(X, A) \geq s(f)$ . But by Theorem 3:

$$\dim T_{[0]}^* \text{Teich}(f) = \dim T_{[0]}^* \text{Def}_A^B(f) - \dim(Q(X, B) \cap \overline{\nabla_f Q_f})/\nabla_f Q(X, A)$$

so that

$$\dim \text{Teich}(f) \leq \text{card}(B - A) - s(f) = \text{card } S(f) - p(f) - s(f),$$

which is the desired inequality.  $\square$

## 6. PROOF OF THEOREM B

In this section, we will focus on the case where  $W = X = \mathbb{P}^1$ , so that  $f : W \rightarrow X$  is a rational map. We will recover from the work done in the previous sections a simpler proof of a result due to Levin (see [Lev14]). First let us introduce some notations.

If  $\lambda \mapsto f_\lambda$  is a holomorphic curve in  $\text{Def}_A^B(f)$  passing through the basepoint at  $\lambda = 0$ , then let  $(\phi_\lambda, \psi_\lambda, f_\lambda)$  be a corresponding holomorphic family of triples (recall that  $\phi_\lambda$  and  $\psi_\lambda$  are quasiconformal homeomorphisms, and that  $f_\lambda$  are rational maps of the form  $f_\lambda = \phi_\lambda \circ f \circ \psi_\lambda^{-1}$ ). Then

$$(2) \quad \eta = Df^{-1} \cdot \left( \frac{d}{d\lambda} \Big|_{\lambda=0} f_\lambda \right) = f^* \dot{\phi} - \dot{\psi}.$$

Let us also denote:

$$\eta_n = Df^{-n} \cdot \frac{d(f_\lambda^n)}{d\lambda} \Big|_{\lambda=0}.$$

Thus,  $\eta = \eta_1$ .

**Lemma 4.** *We have, for any  $n \in \mathbb{N}^*$ :*

$$\eta_n = \sum_{k=0}^{n-1} (f^k)^* \eta_0$$

*Proof.* Let us proceed by induction. Suppose that for some  $n \in \mathbb{N}$ :

$$\eta_n = \sum_{k=0}^{n-1} (f^k)^* \eta.$$

Then:

$$\begin{aligned} \eta_{n+1} &= Df^{-(n+1)} \cdot \frac{d}{d\lambda} \Big|_{\lambda=0} f_\lambda^{n+1} \\ \eta_{n+1} &= Df^{-(n+1)} \cdot \left( \dot{f} \circ f^n + Df \circ f^n \cdot (\dot{f}^n) \right) \\ \eta_{n+1} &= (f^n)^* \eta + Df^{-n} \cdot (\dot{f}^n) \\ \eta_{n+1} &= (f^n)^* \eta + \eta_n. \end{aligned}$$

This concludes the proof.  $\square$

As a consequence of the preceding lemma, if  $v$  is a summable critical value, then  $(\eta_n(v))_{n \in \mathbb{N}}$  converges.

Recall the following notation:

**Definition 19.** *Let  $v$  be a summable critical value and  $\eta \in T(f)$ . Denote by:*

$$\xi_\eta(v) := \sum_{k=0}^{\infty} (f^k)^* \eta(v) = \lim_{n \rightarrow \infty} \eta_n(v) \in T_v \mathbb{P}^1.$$



We now come to our second result:

**Theorem B.** *Let  $f$  be a rational map of degree  $d \geq 2$ , with  $s$  summable critical values, that is not a Lattès map. Assume that either  $f$  has no invariant line field, or that the  $\omega$ -limit set of those  $s$  summable critical values are  $C$ -compacts. Then the linear map*

$$\begin{aligned} \mathcal{V} : T(f) &\rightarrow \bigoplus_{1 \leq i \leq s} T_{v_i} \mathbb{P}^1 \\ \eta &\mapsto (\xi_\eta(v_i))_{1 \leq i \leq s} \end{aligned}$$

has maximal rank, i.e. equal to  $s$ .

*Proof.* Let  $A$  be a repelling cycle of cardinal at least 3 that does not contain any critical value, and let  $Z \subset A$  be a subset of cardinal exactly 3. Let  $B = A \cup S(f)$ : then  $\text{Def}_A^B(f)$  is a complex manifold of dimension  $\text{card } S(f)$ . Let  $\Phi : \text{Def}_A^B(f) \rightarrow \text{Rat}_d$  be the natural map from the deformation space to the moduli space, and let  $\Phi_Z : \text{Def}_A^B(f) \rightarrow \text{Rat}_d$  be its lift to the parameter space obtained by choosing quasiconformal homeomorphisms fixing  $Z$  pointwise. Then  $D\Phi_Z([0])$  takes values in  $T_Z(f)$ , the subspace of  $T(f)$  of vector fields vanishing on  $Z$ . For each summable critical value  $v_i$ , let  $u_i$  be a non-zero element of  $\Gamma(TX)^*$  supported in  $\{v_i\}$ , and let  $q_i$  be the quadratic differentials constructed in the proof of Theorem A (recall that  $q_i$  is the  $Z$ -normalized quadratic differential associated to  $\sum_{n=0}^{\infty} f_*^n u_i$ ).

Instead of proving that the linear map  $\mathcal{V}$  has rank  $s$ , we will prove the slightly stronger statement that the linear map:

$$\begin{aligned} \mathcal{U} : T_{[0]} \text{Def}_A^B(f) &\rightarrow \mathbb{C}^s \\ [\mu] &\mapsto (\langle u_i, \xi(v_i) \rangle)_{1 \leq i \leq s} \end{aligned}$$

has rank  $s$ . This will imply the Theorem, as  $D\Phi_Z$  is injective and  $(\xi(v_i))_{i \leq s} \mapsto (\langle u_i, \xi(v_i) \rangle)_{i \leq s}$  is invertible. Let  $\mathcal{U}_i : [\mu] \mapsto \langle u_i, \xi(v_i) \rangle$ , so that  $\mathcal{U} = (\mathcal{U}_i)_{i \leq s}$ . Let  $[\mu] \in T_{[0]} \text{Def}_A^B(f)$  and  $\eta = D\Phi_Z([0]) \cdot [\mu] \in T_Z(f)$ .

For  $n \in \mathbb{N}$ , denote by  $q_{i,n}$  the  $Z$ -normalized quadratic differential associated to  $f_*^n u_i$ : then  $q_i = \sum_{n=0}^{\infty} q_{i,n}$  and  $q_{i,n}$  is a quadratic differential with exactly four poles, all simple, which are in  $Z \cup \{y\}$ .

Note that

$$2i\pi \sum_{n \in \mathbb{N}} \text{Res}(q_{i,n} \cdot \eta, f^n(v_i)) = \mathcal{U}_i([\mu]).$$

Indeed, by definition:

$$\langle f_*^n u_i, \eta \rangle = 2i\pi \text{Res}(q_{i,n} \cdot (f^n)^* \eta, f^n(v_i))$$

and  $\xi(v_i) = \sum_{n \geq 0} ((f^n)^* \eta)(v_i)$ .

If  $\lambda \mapsto [\mu_\lambda]$  is a holomorphic curve in  $\text{Def}_A^B(f)$  tangent to  $[\mu] \in T_{[0]} \text{Def}_A^B(f)$  at the basepoint, we can lift  $\lambda \mapsto [\mu_\lambda]$  to a holomorphic curve of representatives  $\lambda \mapsto \mu_\lambda$ , and the normalized solutions  $\psi_\lambda$  of the associated Beltrami equation will satisfy the property that  $\bar{\partial} \psi = \bar{\partial} \frac{d}{d\lambda} \big|_{\lambda=0} \psi_\lambda$  is a representative of  $[\mu]$ .

The next lemma essentially says that the linear form  $\mathcal{U}_i$  is represented by  $\nabla_f q_i$  in  $T_{[0]}^* \text{Def}_A^B(f)$ :

**Lemma 5.** *Let  $\mu = \bar{\partial}\dot{\psi}$  be a representative of  $[\mu] \in T_0\text{Def}_A^B(f)$ . Then :*

$$\int_{\mathbb{P}^1} \nabla_f q_i \cdot \mu = 2i\pi \sum_{n \in \mathbb{N}} \text{Res}(q_{i,n} \cdot \eta, f^n(v_i)).$$

*Proof of lemma 5.* For all  $n \in \mathbb{N}$ , the differential form  $q_{i,n} \cdot \eta$  is meromorphic on  $\mathbb{P}^1$ , therefore the sum of its residues is null. The quadratic differential  $q_{i,n}$  has poles at  $Z \cup \{f^n(v_i)\}$ , and the vector field  $\eta$  has poles at  $\text{Crit}(f)$ . So:

$$\text{Res}(q_{i,n} \cdot \eta, f^n(v_i)) = - \sum_{z \in Z \cup \text{Crit}(f)} \text{Res}(q_{i,n} \cdot \eta, z).$$

Therefore:

$$\sum_{n \in \mathbb{N}} \text{Res}(q_{i,n} \cdot \eta, f^n(v_i)) = - \sum_{n \in \mathbb{N}} \sum_{z \in Z \cup C_f} \text{Res}(q_{i,n} \cdot \eta, z)$$

According to equation (2), we have:  $\eta = f^*\dot{\psi} - \dot{\phi}$ , and  $\dot{\phi} = \dot{\psi}$  on  $Z \subset A$ . Therefore for all  $z \in A$  and for all  $n \in \mathbb{N}$  :

$$\text{Res}(q_{i,n} \cdot \eta, z) = \text{Res}(q_{i,n} \cdot (f^*\dot{\psi} - \dot{\psi}), z) = \text{Res}(f_*q_{i,n} \cdot \dot{\psi}, f(z)) - \text{Res}(q_{i,n} \cdot \dot{\psi}, z).$$

Therefore:

$$- \sum_{n \in \mathbb{N}} \sum_{z \in A} \text{Res}(q_{i,n} \cdot \eta, z) = \sum_{z \in A} \text{Res}(\nabla_f q_i \cdot \dot{\psi}, z).$$

Moreover:

$$\begin{aligned} - \sum_{c \in \text{Crit}(f)} \text{Res}(q_{i,n} \cdot \eta, c) &= - \sum_{c \in \text{Crit}(f)} \text{Res}(q_{i,n} \cdot (f^*\dot{\psi} - \dot{\phi}), c) \\ &= - \sum_{c \in \text{Crit}(f)} \text{Res}(q_{i,n} \cdot f^*\dot{\psi}, c) \\ &= - \sum_{v \in V_f} \text{Res}(f_*q_{i,n} \cdot \dot{\psi}, v) \\ &= \sum_{v \in V_f} \text{Res}(\nabla_f q_{i,n} \cdot \dot{\psi}, v) \end{aligned}$$

and therefore:

$$- \sum_{n \in \mathbb{N}} \sum_{c \in \text{Crit}(f)} \text{Res}(q_{i,n} \cdot \eta, c) = \sum_{v \in V_f} \text{Res}(\nabla_f q_i \cdot \dot{\psi}, v).$$

To sum things up, we have:

$$\sum_{n \in \mathbb{N}} \text{Res}(q_{i,n} \cdot \eta, f^n(v_i)) = \sum_{z \in B} \text{Res}(\nabla_f q_i \cdot \dot{\psi}, z).$$

Finally, by Stokes' theorem,

$$2i\pi \sum_{z \in B} \text{Res}(\nabla_f q_i \cdot \dot{\psi}, z) = \int_{\mathbb{P}^1} \nabla_f q_i \cdot \bar{\partial}\dot{\psi},$$

which concludes the proof of the lemma.  $\square$

Let us now return to the proof of Theorem B. According to the preceding lemma, we have:

$$B_i^Z([\mu]) = \int_{\mathbb{P}^1} \nabla_f q_i \cdot [\mu]$$

for all  $[\mu] \in \text{Def}_A^B(f)$ . In other words, the class of the quadratic differential  $\nabla_f q_i$  in  $Q(\mathbb{P}^1, B)/\nabla_f Q(\mathbb{P}^1, A)$  represents the linear form  $\mathcal{U}_i$  in  $T_{[0]}^* \text{Def}_A^B(f)$ . Moreover, according to lemma 3, the quadratic differentials  $(\nabla_f q_i)_{i \leq s}$  are linearly independent, and as in the proof of Theorem A, this implies that the classes  $([\nabla_f q_i])_{i \leq s}$  are linearly independent in  $Q(\mathbb{P}^1, B)/\nabla_f Q(\mathbb{P}^1, A)$ . Therefore, the  $(\mathcal{U}_i)_{i \leq s}$  are linearly independent, which proves that  $\mathcal{U}$  has rank  $s$ .  $\square$

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