



**HAL**  
open science

## Eberlein oligomorphic groups

Itai Ben Yaacov, Tomás Ibarlucía, Todor Tsankov

► **To cite this version:**

Itai Ben Yaacov, Tomás Ibarlucía, Todor Tsankov. Eberlein oligomorphic groups. 2015. hal-01275112v1

**HAL Id: hal-01275112**

**<https://hal.science/hal-01275112v1>**

Preprint submitted on 16 Feb 2016 (v1), last revised 20 Oct 2017 (v3)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# EBERLEIN OLIGOMORPHIC GROUPS

ITAI BEN YAACOV, TOMÁS IBARLUCÍA, AND TODOR TSANKOV

**ABSTRACT.** We study the Fourier–Stieltjes algebra of Roelcke precompact, non-archimedean, Polish groups and give a model-theoretic description of the Hilbert compactification of these groups. We characterize the family of such groups whose Fourier–Stieltjes algebra is dense in the algebra of weakly almost periodic functions: those are exactly the automorphism groups of  $\aleph_0$ -stable,  $\aleph_0$ -categorical structures. This analysis is then extended to all semitopological semigroup compactifications  $S$  of such a group:  $S$  is Hilbert-representable if and only if it is an inverse semigroup. We also show that every factor of the Hilbert compactification is Hilbert-representable.

## CONTENTS

Introduction	1
1. Preliminaries	3
1.1. Compactifications of topological groups	3
1.2. The Fourier–Stieltjes algebra and the WAP algebra of a topological group	4
1.3. Representations on Hilbert spaces	5
2. Semitopological semigroup compactifications	7
2.1. Definitions	7
2.2. Inverse semigroups	8
2.3. The WAP compactification of pro-oligomorphic groups	9
3. The Fourier–Stieltjes algebra of pro-oligomorphic groups	12
3.1. Examples of functions in $\text{Hilb}(G) \setminus B(G)$	12
3.2. A model-theoretic description of the Hilbert compactification	13
3.3. Characterization of Eberlein pro-oligomorphic groups	15
4. Hilbert-representable factors	17
References	20

## INTRODUCTION

It has long been recognized in model theory that the action of the automorphism group of an  $\aleph_0$ -categorical structure on the structure (and its powers) captures all model-theoretic information about it. Moreover, by a classical result of Ahlbrandt and Ziegler [AZ86], the automorphism group remembers the structure up to bi-interpretability. As most interesting model-theoretic properties are preserved by interpretations, it is reasonable to expect that those would correspond to natural properties of the automorphism group.

It turns out that many model-theoretic properties of the structure are reflected in the behavior of a certain universal dynamical system associated to the group that we proceed to describe. First, recall that automorphism groups of  $\aleph_0$ -categorical structures are Roelcke precompact in the following sense.

**Definition 0.1.** A topological group  $G$  is called *Roelcke precompact* if for every neighborhood  $U$  of the identity, there exists a finite set  $F$  such that  $UFU = G$ .

To each Roelcke precompact, Polish group  $G$  is naturally associated its *Roelcke compactification*  $R(G)$ , the completion of  $G$  with respect to its Roelcke (or lower) uniformity; see Section 2.3 for more details. The natural action  $G \curvearrowright R(G)$  renders it a topological dynamical system. From the

model-theoretic point of view, if we represent  $G$  as the automorphism group of an  $\aleph_0$ -categorical structure  $M$ ,  $R(G)$  can be considered as a suitable closed subspace of the type space  $S_\omega(M)$  in infinitely many variables over the model. Thus, there is a natural correspondence between formulas with parameters from the model, on the one hand, and continuous functions on  $R(G)$ , on the other. This allows building a dictionary between the model-theoretic and the dynamical setting. For example, *stable* formulas correspond to *weakly almost periodic* functions and *NIP* formulas correspond to *tame* functions.

Particularly relevant to us is the theory of Banach representations of dynamical systems as developed by Glasner and Megrelishvili in a series of papers (see [GM14] and the references therein). If  $G \curvearrowright X$  is a topological dynamical system and  $V$  is a Banach space, a *representation of  $X$  on  $V$*  is a pair of continuous maps  $\iota: X \rightarrow B$ ,  $\pi: G \rightarrow \text{Iso}(V)$ , where  $B$  is the unit ball of  $V^*$  equipped with the weak\* topology,  $\text{Iso}(V)$  is the group of linear isometries of  $V$ , equipped with the strong operator topology,  $\pi$  is a homomorphism, and

$$\langle v, \iota(gx) \rangle = \langle \pi(g)^{-1}v, \iota(x) \rangle,$$

for all  $x \in X$ ,  $v \in V$ ,  $g \in G$ . A representation is *faithful* if  $\iota$  is an embedding. If  $\mathcal{K}$  is a class of Banach spaces, we say that  $G \curvearrowright X$  is  *$\mathcal{K}$ -representable* if it admits a faithful representation on a Banach space in the class  $\mathcal{K}$ .

All dynamical systems are representable on some Banach space; however, if one restricts to some (well-chosen) class of Banach spaces  $\mathcal{K}$ , the  $\mathcal{K}$ -representable systems usually form an interesting family. Somewhat unexpectedly, in the  $\aleph_0$ -categorical setting, there are some precise connections between model-theoretic properties of the structure and the classes of Banach spaces  $R(G)$  can be represented on: for example,  $M$  is stable iff  $R(G)$  can be represented on a reflexive Banach space [BT14, §5][GM14, §5.1] and  $M$  is NIP iff  $R(G)$  can be represented on a Banach space that does not contain a copy of  $\ell^1$  [Iba14, §4][GM14, §8.1]. One of the main motivating questions for this paper was what the appropriate model-theoretic condition is for  $R(G)$  to be representable on a Hilbert space.

For some classes  $\mathcal{K}$  of Banach spaces, there are dynamical systems that are universal for the  $\mathcal{K}$ -representable ones. For example,  $W(G)$ , the *WAP compactification* of  $G$  is universal for reflexively representable systems and  $H(G)$ , the *Hilbert compactification*, is universal for Hilbert-representable systems. Both  $W(G)$  and  $H(G)$  carry the structure of a compact *semitopological semigroup* and  $H(G)$  is a factor of  $W(G)$ .

The main focus of the paper are the automorphism groups of  $\aleph_0$ -categorical *classical, discrete* (multi-sorted) structures, or, equivalently, Roelcke precompact, Polish, *non-archimedean* groups. (A group is *non-archimedean* if it admits an open basis at the identity consisting of open subgroups.) We make this assumption tacitly throughout the paper: when we say “ $\aleph_0$ -categorical structure”, we will always mean a classical one, as opposed to metric. A non-archimedean, Polish, Roelcke precompact group will be called *pro-oligomorphic*; it is *oligomorphic* if the structure can be chosen one-sorted.

For every non-archimedean group  $G$ , the compactification  $G \rightarrow H(G)$  is a topological embedding. Our first result is a concrete description of  $H(G)$  for pro-oligomorphic groups, in model-theoretic terms. More precisely, we have the following.

**Theorem 0.2.** *Let  $M$  be an  $\aleph_0$ -categorical structure and let  $G = \text{Aut}(M)$ . Then  $H(G)$  is isomorphic to the semigroup of partial elementary embeddings  $M^{\text{eq}} \rightarrow M^{\text{eq}}$  with algebraically closed domains.*

Using this description, we give two characterizations of pro-oligomorphic groups for which  $W(G) = H(G)$ : one model-theoretic, and one in terms of the semigroup  $W(G)$ . This is the main result of the paper.

**Theorem 0.3.** *Let  $M$  be an  $\aleph_0$ -categorical structure and let  $G = \text{Aut}(M)$ . The following are equivalent:*

- (1) *The idempotents of  $W(G)$  commute;*
- (2)  *$M$  is one-based for stable independence;*
- (3)  *$W(G) = H(G)$ .*

Using Theorem 0.3 and a classical, deep result in model theory, we can now give a satisfactory answer of our initial question.

**Corollary 0.4.** *Let  $M$  be an  $\aleph_0$ -categorical structure and let  $G = \text{Aut}(M)$ . Then the following are equivalent:*

- (1)  $M$  is  $\aleph_0$ -stable;
- (2)  $R(G)$  is Hilbert-representable.

Corollary 0.4 and a well-known example of an  $\aleph_0$ -categorical, stable, non- $\aleph_0$ -stable structure, due to Hrushovski, give us the following corollary (cf. Example 3.14) which answers a question of Glasner and Megrelishvili [GM14, Question 6.10].

**Corollary 0.5.** *There exists an oligomorphic group  $G$  that satisfies  $R(G) = W(G) \neq H(G)$ .*

While all factors of  $W(G)$  are known to be reflexively representable (or *reflexively approximable*, for a general topological group  $G$ ), it is an open question whether all factors of  $H(G)$  are Hilbert-representable [GM14, Question 5.12.3]. We can give a positive answer to this question in the case of pro-oligomorphic groups (cf. Theorem 4.8).

**Theorem 0.6.** *Let  $G$  be a pro-oligomorphic group. Then all factors of  $H(G)$  are Hilbert representable.*

The correspondence between model-theoretic properties of  $\aleph_0$ -categorical structures and dynamical properties of their automorphism groups is not restricted to the non-archimedean case. The correct model-theoretic setting for dealing with general Roelcke precompact, Polish groups is that of continuous logic and in both [BT14] and [Iba14], the results are proved in full generality. However, the two most important tools used in this paper are currently only available in the non-archimedean setting: namely, the classification of the unitary representations on the dynamical side and the notion of one-basedness on the model-theoretic side. For the moment, we do not even have a plausible conjecture of what the model-theoretic characterization of Hilbert-representable functions on a Roelcke precompact Polish group should be in general. Theorem 0.3 clearly fails in the continuous setting (for example, for the unitary group). While we do not have a counterexample to Corollary 0.4 for general separably categorical structures, we strongly suspect that it also fails.

As one of the goals of this paper is to provide a dictionary between model theory and abstract topological dynamics, we have tried to make the exposition fairly self-contained (apart from a couple of difficult model-theoretic results) and accessible to people working in both areas.

**Acknowledgements.** Part of the present work was carried out during the trimester program *Universality and Homogeneity* organized by the Hausdorff Research Institute for Mathematics in Bonn in 2013. We would like to thank the institute as well as the organizers of the program for the excellent conditions and stimulating atmosphere they provided. All three authors were partially supported by the ANR contract GrupoLoco (ANR-11-JS01-0008). T.I. was also partially supported by the ANR contract ValCoMo (ANR-13-BS01-0006). T.T. was also partially supported by the ANR contract GAMME (ANR-14-CE25-0004).

## 1. PRELIMINARIES

**1.1. Compactifications of topological groups.** Let  $G$  be a topological group. The algebra of complex-valued continuous bounded functions on  $G$  will be denoted by  $C(G)$ . This algebra carries always the uniform norm,  $\|f\| = \sup_{g \in G} |f(g)|$ . The group  $G$  admits a left and a right action on  $C(G)$ , given, respectively, by  $(gf)(h) = f(g^{-1}h)$  and  $(fg)(h) = f(hg^{-1})$  for every  $f \in C(G)$  and  $g, h \in G$ . These actions are isometric but in general not continuous.

When considering subalgebras of  $C(G)$ , we will always assume that these are unital and closed under complex conjugation. If we say that a subalgebra is *closed*, we mean closed with respect to the uniform norm. *Left-invariant*, *right-invariant* and *bi-invariant* refer to the actions of  $G$  defined above.

A *compactification* of  $G$  is a compact Hausdorff space  $X$  with a continuous left action of  $G$ , together with a continuous  $G$ -map  $\alpha: G \rightarrow X$  with dense image (where  $G$  carries the natural left action on itself). Given a compactification  $\alpha: G \rightarrow X$ , we denote by  $\mathcal{A}(\alpha) := C(X) \circ \alpha$  the subalgebra of  $C(G)$  consisting of those functions that factor through  $\alpha$ . We may also denote it by  $\mathcal{A}(X)$ , if no confusion arises. The algebra  $\mathcal{A}(\alpha)$  is always left-invariant, and the compactification will be called *bi-invariant* if  $\mathcal{A}(\alpha)$  is moreover right-invariant.

Given compactifications  $\alpha_X: G \rightarrow X$  and  $\alpha_Y: G \rightarrow Y$ , we say that  $\alpha_Y$  is a  $G$ -factor of  $\alpha_X$  (or shortly, that  $Y$  is a factor of  $X$ ) if there is a continuous surjective  $G$ -map  $X \rightarrow Y$  commuting with  $\alpha_X$  and  $\alpha_Y$ . In this case, this  $G$ -map is unique.

A function  $f \in C(G)$  is *right uniformly continuous* if the orbit map  $g \in G \mapsto gf \in C(G)$  is norm-continuous. Let  $\text{RUC}(G)$  denote the family of all right uniformly continuous, bounded functions on  $G$ , which is a bi-invariant closed subalgebra of  $C(G)$ . Any function  $f \in C(G)$  that factors through a compactification of  $G$  is in  $\text{RUC}(G)$ . In fact, there is a one-to-one correspondence between compactifications of  $G$  (up to isomorphism) and left-invariant closed subalgebras of  $\text{RUC}(G)$ . It takes a compactification  $\alpha$  to the algebra  $\mathcal{A}(\alpha)$ . Conversely, it takes a left-invariant closed subalgebra  $A \subset \text{RUC}(G)$  to the maximal ideal space  $G^A$  (with its usual compact Gelfand topology) together with the canonical  $G$ -map  $G \rightarrow G^A$ . Under this correspondence,  $A$  is a subalgebra of  $B$  if and only if  $G^B$  is a factor of  $G^A$ .

**1.2. The Fourier–Stieltjes algebra and the WAP algebra of a topological group.** We start by recalling the definition of positive definite functions and the GNS construction. See, for example, [BdlHV08], Theorem C.4.10.

**Definition 1.1.** A function  $f: G \rightarrow \mathbb{C}$  is *positive definite* if the following equivalent conditions hold.

- (1) For every  $g_1, \dots, g_n \in G$ , the matrix  $(f(g_j^{-1}g_i))_{ij}$  is positive semi-definite, i.e., for any  $c_1, \dots, c_n \in \mathbb{C}$  we have

$$\sum_{ij} c_i \bar{c}_j f(g_j^{-1}g_i) \geq 0.$$

- (2) There exists a unitary representation  $\pi: G \rightarrow U(\mathcal{H})$  and a vector  $v \in \mathcal{H}$  such that, for all  $g \in G$ ,

$$f(g) = \langle v, \pi(g)v \rangle.$$

In particular, every positive definite function is bounded:  $|f(g)| \leq f(1)$  for all  $g \in G$ . Besides, we can assume that  $v$  is a *cyclic vector* for the representation  $\pi: G \rightarrow U(\mathcal{H})$  (that is,  $\mathcal{H}$  is the closed subspace generated by  $\pi(G)v$ ); then  $\pi$  is continuous if and only if  $f \in C(G)$ .

The family of continuous positive definite functions on  $G$  is denoted by  $P(G)$ . The (not necessarily closed) subalgebra of  $C(G)$  generated by  $P(G)$  is the *Fourier–Stieltjes algebra* of  $G$ , and is denoted by  $B(G)$ .

**Definition 1.2.** A (*unitary*) *matrix coefficient* of a topological group  $G$  is a function  $f \in C(G)$  of the form

$$f(g) = \langle v, \pi(g)w \rangle$$

for a continuous unitary representation  $\pi: G \rightarrow U(\mathcal{H})$  and vectors  $v, w \in \mathcal{H}$ . We will use the notation  $f = m_{v,w}$ , or  $f = m_{v,w}^\pi$  if we wish to specify  $\pi$ .

**Fact 1.3.**  $B(G)$  is the family of matrix coefficients of  $G$ .

That matrix coefficients form an algebra follows from considering orthogonal sums, tensor products, and duals of representations.

In fact, every matrix coefficient is a linear combination of four positive definite functions,

$$4m_{v,w} = m_{v+w,v+w} - m_{v-w,v-w} + im_{v+iw,v+iw} - im_{v-iw,v-iw},$$

so  $B(G)$  coincides with the linear span of  $P(G)$ .

Next we recall weakly almost periodic functions, Grothendieck’s double limit criterion and the reflexive representation theorem of Megrelishvili (see [Meg03], Theorem 5.1).

**Definition 1.4.** A function  $f \in C(G)$  is *weakly almost periodic* if the following equivalent conditions hold:

- (1) The orbit  $Gf$  is precompact for the weak topology on  $C(G)$ .
- (2) For all sequences  $g_i, h_j \in G$ , we have

$$\lim_i \lim_j f(g_i h_j) = \lim_j \lim_i f(g_i h_j)$$

whenever both limits exist.

- (3) There exists a continuous, isometric representation  $\pi: G \rightarrow \text{Iso}(V)$  on a reflexive Banach space  $V$  and vectors  $v \in V, w \in V^*$  such that, for all  $g \in G$ ,

$$f(g) = \langle v, \pi(g)w \rangle.$$

It follows easily that the family  $\text{WAP}(G)$  of weakly almost periodic functions on  $G$  is a closed bi-invariant subalgebra of  $\text{RUC}(G)$  containing  $B(G)$ . On the other hand,  $B(G)$  is almost never closed in  $C(G)$  (see the beginning of Section 3). Following [GM14, §6], we will denote the closure  $\overline{B(G)}$  by  $\text{Hilb}(G)$ . It consists precisely of the continuous functions on  $G$  that factor through Hilbert-representable compactifications of  $G$  (see the next subsection). The algebra  $B(G)$  is bi-invariant, hence so is  $\text{Hilb}(G)$ .

Thus we have

$$\text{Hilb}(G) \subset \text{WAP}(G),$$

or equivalently: the *Hilbert compactification*  $G \rightarrow H(G)$  associated to the closed left-invariant algebra  $\text{Hilb}(G)$  is a  $G$ -factor of the *WAP compactification*  $G \rightarrow W(G)$  associated to  $\text{WAP}(G)$ . We will review the main properties of these compactifications in Section 2.

Finally, we recall that a function  $f: G \rightarrow \mathbb{C}$  is *Roelcke uniformly continuous* if the map  $(g, g') \in G \times G \mapsto gf g' \in C(G)$  is norm-continuous. The family of all Roelcke uniformly continuous functions on  $G$  is a closed bi-invariant subalgebra of  $\text{RUC}(G)$ , denoted by  $\text{UC}(G)$ . We always have  $\text{WAP}(G) \subset \text{UC}(G)$ .

**Definition 1.5.** Let  $G$  be a topological group.

- (i)  $G$  is *Eberlein* if  $\text{Hilb}(G) = \text{WAP}(G)$ .
- (ii)  $G$  is a *WAP group* if  $\text{WAP}(G) = \text{UC}(G)$ .
- (iii)  $G$  is *strongly Eberlein* if  $\text{Hilb}(G) = \text{UC}(G)$ .

In his fundamental work [Ebe49], Eberlein introduced weakly almost periodic functions (in the context of locally compact abelian groups) and proved the inclusion  $B(G) \subset \text{WAP}(G)$ . In fact, all his examples of WAP functions lied in the closure of  $B(G)$ . Rudin writes in [Rud59] that Eberlein asked him whether the closure of  $B(G)$  may in fact coincide with  $\text{WAP}(G)$ . Of course, by the Peter–Weyl theorem, this is the case for compact groups (indeed,  $\text{Hilb}(G) = C(G)$ ). However, Rudin showed in the referred paper that this is not true in general. As an example, he exhibited a concrete function  $f \in \text{WAP}(\mathbb{Z}) \setminus \text{Hilb}(\mathbb{Z})$ . Later, Chou proved that, more generally, the inclusion  $\text{Hilb}(G) \subset \text{WAP}(G)$  is strict for any non-compact locally compact nilpotent group [Cho82]. On the other hand, he remarked that equality does hold for some non-compact locally compact groups, and introduced the name *Eberlein* for this class.

The definitions of *WAP groups* and *strongly Eberlein groups* were introduced by Glasner and Megrelishvili in [GM14].

Examples of non-compact Eberlein groups include  $SL_n(\mathbb{R})$  (and any semisimple Lie group with finite center; see [Vee79]), the unitary group  $U(\ell^2)$  [Meg08], the group  $\text{Aut}(\mu)$  of measure preserving transformations of the unit interval [Gla12] and the symmetry group of a countable set,  $S_\infty$  [GM14]. The latter three are in fact strongly Eberlein. We will give some new examples in Section 3.3.

**1.3. Representations on Hilbert spaces.** Let  $X$  be a compactification of a Polish group  $G$ . We say that  $X$  is *Hilbert-representable* if there exist a Hilbert space  $\mathcal{H}$ , an embedding  $\iota: X \rightarrow \mathcal{H}$  (where  $\mathcal{H}$  carries the weak topology) and a unitary representation  $\pi: G \rightarrow U(\mathcal{H})$  such that  $\iota(gx) = \pi(g)\iota(x)$  for all  $x \in X$  and  $g \in G$ . This definition coincides, for the class of Hilbert spaces, with the notion of  $\mathcal{K}$ -representability given in the introduction.

Given a function  $f \in \text{RUC}(G)$ , let  $X_f$  be the compactification of  $G$  associated to the invariant closed subalgebra of  $\text{RUC}(G)$  generated by  $f$ . In [GW12, §2], it is observed that  $X_f$  is Hilbert-representable whenever  $f$  is positive definite (in the case  $G = \mathbb{Z}$ ); more generally, the following holds.

**Lemma 1.6.** *If  $f \in B(G)$ , then  $X_f$  is Hilbert-representable.*

*Proof.* Write  $f = m_{v_0, w_0}^{\pi_0}$  for some continuous unitary representation  $\pi_0: G \rightarrow U(\mathcal{H}_0)$ . Let  $\mathcal{H}_1$  be the closed linear span of  $\pi(G)w_0$  and let  $v = \Pi_{\mathcal{H}_1} v_0$  be the orthogonal projection of  $v_0$  to  $\mathcal{H}_1$ . Next let  $\mathcal{H}$  be the closed linear span of  $\pi(G)v$  and let  $w = \Pi_{\mathcal{H}} w_0$  be the orthogonal projection of  $w_0$  to  $\mathcal{H}$ . Consider the restriction  $\pi = \pi_0|_{\mathcal{H}}$ . Then  $f = m_{v, w}^{\pi}$ .

Let  $Z$  be the weak closure of  $\pi(G)w$  in  $\mathcal{H}$ , which is naturally a (Hilbert-representable) compactification of  $G$  via the map  $g \mapsto gw$ . Consider for each  $h \in G$  the function  $F_h \in C(Z)$ ,  $F_h(z) = \langle \pi(h)v, z \rangle$ , and remark that  $F_h(gw) = hf(g)$ . Since  $\mathcal{H}$  is generated by  $\pi(G)v$ , the functions  $F_h$  separate points of  $Z$ . Hence, by the Stone–Weierstrass theorem,  $\mathcal{A}(Z)$  is the closed algebra generated by  $Gf$ . In other words,  $Z = X_f$  up to isomorphism.  $\square$

In contrast, if instead of a matrix coefficient we take any  $f \in \text{Hilb}(G)$ , it is unknown whether  $X_f$  is necessarily Hilbert-representable; see Question 1.9 below.

**Proposition 1.7.** *Let  $\alpha: G \rightarrow X$  be a metrizable compactification of  $G$ . Then the following are equivalent:*

- (1)  $\alpha$  is Hilbert-representable.
- (2)  $\mathcal{A}(\alpha) = \overline{\mathcal{A}(\alpha) \cap B(G)}$ .

*Proof.* Suppose  $(\iota, \pi)$  is a representation of  $(X, G)$  on a Hilbert space  $\mathcal{H}$ . The functions  $F_v: w \mapsto \langle v, w \rangle$  separate points of  $\mathcal{H}$ , hence the algebra generated by  $\{F_v \iota \alpha\}_{v \in \mathcal{H}}$  is dense in  $\mathcal{A}(\alpha)$  and contained in  $B(G)$ .

Conversely, suppose that (2) holds. The metrizability assumption on  $X$  says that  $\mathcal{A}(\alpha)$  is separable. Thus, let  $B \subset \mathcal{A}(\alpha) \cap B(G)$  be a countable dense subset. By the previous lemma, for each  $f \in B$  there is a representation  $(\iota_f, \pi_f)$  of  $(X_f, G)$  on a Hilbert space  $\mathcal{H}_f$ . We consider

$$\mathcal{H} = \bigoplus_{f \in B} \mathcal{H}_f$$

and let  $\pi = \bigoplus_{f \in B} \pi_f: G \rightarrow U(\mathcal{H})$  be the orthogonal sum of the representations  $\pi_f$ . For each  $f \in B$  let  $w_f = \iota_f \alpha(1) \in \mathcal{H}_f$ . Since  $B$  is countable, by rescaling we may assume that  $w = (w_f)_{f \in B}$  is summable, i.e.,  $w \in \mathcal{H}$ . Now we define  $\alpha': G \rightarrow \mathcal{H}$  by  $\alpha'(g) = \pi(g)w$ , and let  $Z$  be the weak closure of  $\alpha'(G)$  in  $\mathcal{H}$ . Then the restriction  $\alpha': G \rightarrow Z$  is a Hilbert-representable compactification of  $G$ , which we claim is isomorphic to  $\alpha$ . Indeed,  $Z$  is homeomorphic to a subspace of the product  $\prod_{f \in B} X_f$ , so the projection maps  $Z \rightarrow X_f$  separate points of  $Z$ ; hence  $\mathcal{A}(\alpha')$  is the closed algebra generated by the algebras  $\mathcal{A}(X_f)$ ,  $f \in B$ . Since  $B$  is dense in  $\mathcal{A}(\alpha)$ , we deduce that  $\mathcal{A}(\alpha') = \mathcal{A}(\alpha)$ , which proves our claim.  $\square$

**Remark 1.8.** A basic consequence of the first implication of the above proposition (which does not use the metrizability assumption) is that all Hilbert-representable compactifications of  $G$  are factors of  $H(G)$ .

**Question 1.9** ([GM14], Question 5.12.3; [Meg07], Question 7.6). Are Hilbert-representable dynamical systems closed under factors; equivalently (for ambits), are all factors of  $H(G)$  Hilbert representable?

This question has also been investigated in [GW12]. In Section 4, we will see that the answer is positive for pro-oligomorphic groups.

We should note that *reflexively representable* dynamical systems are preserved under factors. In fact, the reflexively representable (or rather, when  $W(G)$  is not metrizable, *reflexively approximable*) compactifications of  $G$  are exactly the factors of  $W(G)$ . See [Meg08] and the references therein.

2. SEMITOPOLOGICAL SEMIGROUP COMPACTIFICATIONS

**2.1. Definitions.** A *semitopological semigroup* is a semigroup that carries a topological structure such that the product operation is separately continuous. That is to say, multiplying by an arbitrary fixed element to the left is continuous, and similarly to the right. We shall be interested in semitopological semigroups arising in the following manner.

**Definition 2.1.** A compactification  $\alpha: G \rightarrow S$  is a *semitopological semigroup compactification* if  $S$  admits a semitopological semigroup law that makes of  $\alpha$  a homomorphism.

**Remark 2.2.** Suppose  $\alpha: G \rightarrow S$  is a semitopological semigroup compactification.

- (1) Then  $S$  is in fact a monoid:  $\alpha(1)$  is an identity.
- (2) By Lawson's joint continuity theorem,  $\alpha(G)$  is a topological group ([Law74], Corollary 6.3).
- (3) The compactification is bi-invariant.

Both the Hilbert and WAP compactifications introduced before,  $H(G)$  and  $W(G)$ , are semitopological semigroup compactifications. Moreover,  $W(G)$  is universal among these, in the following sense.

**Fact 2.3.** Let  $S$  be a bi-invariant compactification of  $G$ . The following are equivalent.

- (1)  $S$  is a semitopological semigroup compactification.
- (2)  $S$  is a factor of  $W(G)$ .

*Proof.* See, for instance, [BJM78, Ch. III, §8], Corollary 8.5. □

Given a reflexive Banach space  $V$ , the semigroup  $\Theta(V)$  of linear contractions of  $V$ ,

$$\Theta(V) = \{T \in L(V) : \|T\| \leq 1\},$$

endowed with the weak operator topology, is compact and semitopological. It turns out that every compact semitopological semigroup can be seen as a closed subsemigroup of  $\Theta(V)$  for some reflexive Banach space  $V$  [Sht94, Meg01]. Thus, in this sense, every compact semitopological semigroup is *reflexively representable*. A stricter notion of representability of semigroups is the following.

**Definition 2.4.** A semitopological semigroup  $S$  is *Hilbert-representable* if it can be embedded in the compact semitopological semigroup  $\Theta(\mathcal{H})$  of linear contractions of a Hilbert space  $\mathcal{H}$ .

It is not difficult to see the following.

**Fact 2.5.** Let  $G$  be a topological group.

- (1)  $H(G)$  is Hilbert-representable as a semitopological semigroup.
- (2)  $G$  is Eberlein if and only if  $W(G)$  is Hilbert-representable as a semitopological semigroup.

In fact, the two notions of representability on Hilbert spaces discussed so far essentially coincide. See Lemma 4.5 in [Meg08].

**Fact 2.6.** Let  $\alpha: G \rightarrow S$  be a metrizable semitopological semigroup compactification of  $G$ . Then  $S$  is a Hilbert-representable semitopological semigroup if and only if  $\alpha$  is a Hilbert-representable compactification.

In the non-metrizable case, Definition 2.4 is the correct property to consider, while Hilbert-representability of dynamical systems has to be relaxed. However, the semigroups that we study in this paper are metrizable.

**Definition 2.7.** Let  $\alpha: G \rightarrow S$  be a semitopological semigroup compactification. We will say that  $\alpha$  is *\*-closed* or, equivalently, that  $\alpha$  is a *semitopological \*-semigroup compactification*, if the inverse operation on the group  $\alpha(G)$  extends to a continuous involution  $*$ :  $S \rightarrow S$ .

**Remark 2.8.**  $\alpha: G \rightarrow S$  is \*-closed if and only if, whenever  $f \in \mathcal{A}(\alpha)$ , the function  $g \mapsto f(g^{-1})$  is also in  $\mathcal{A}(\alpha)$ .

It follows readily that both  $H(G)$  and  $W(G)$  are \*-closed.



**Proposition 2.9.** *Every Hilbert-representable semitopological semigroup compactification is  $*$ -closed.*

*Proof.* Let  $\alpha: G \rightarrow S$  be a compactification with an embedding  $\beta: S \rightarrow \Theta(\mathcal{H})$ . It suffices to see that the image of  $\beta$  is closed under the adjoint operation  $*$ :  $\Theta(\mathcal{H}) \rightarrow \Theta(\mathcal{H})$ ; indeed, then we can define  $s^*$  as the preimage of  $\beta(s)^*$ , and this gives a continuous map  $*$ :  $S \rightarrow S$  that extends the inverse operation on  $\alpha(G)$ . Now, if  $s \in S$  is the limit of a net  $\alpha(g_i) \in \alpha(G)$ , then  $\beta\alpha(g_i^{-1})$  converges to  $\beta(s)^*$ ; by compactness we may assume that  $\alpha(g_i^{-1})$  converges to some  $s' \in S$ , so  $\beta(s') = \beta(s)^*$ . Hence  $\beta(s)^* \in \beta(S)$ .  $\square$

**2.2. Inverse semigroups.** In this short subsection, we review some general notions of the theory of semigroups, and some particular facts that hold for compact semitopological  $*$ -semigroups with a dense subgroup.

An element  $e$  in a semigroup  $S$  is an *idempotent* if  $e^2 = e$ . If  $S$  has an involution  $*$ , then  $e \in S$  is *self-adjoint* if  $e^* = e$ .

**Definition 2.10.** Let  $S$  be a semigroup.

- (i) An element  $p \in S$  is *regular* if there exists  $q \in S$  such that  $pqp = p$ .
- (ii)  $S$  is *regular* if every element is regular.
- (iii) An element  $q \in S$  is an *inverse* for  $p \in S$  if  $pqp = p$  and  $qpq = q$ .
- (iv)  $S$  is an *inverse semigroup* if every element has a unique inverse.

The canonical example of an inverse semigroup is the *symmetric inverse semigroup* of all partial bijections of a set, with composition where it is defined.

A proof of the following general characterization can be found in [How95], Theorem 5.1.1.

**Fact 2.11.** *The following are equivalent for a semigroup  $S$ .*

- (1)  $S$  is an inverse semigroup.
- (2)  $S$  is regular and the idempotents commute.

When a compact semitopological structure is available, and the semigroup contains a dense subgroup, much more is true. We formulate these additional properties in the case that we are interested in.

**Fact 2.12.** *Let  $G \rightarrow S$  be a semitopological  $*$ -semigroup compactification.*

- (1) For every  $p, q \in S$  we have  $Sq = Spq$  if and only if  $q = p^*pq$ .
- (2) Every idempotent is self-adjoint.
- (3) Let  $e, f \in S$  be idempotents. The following are equivalent:
  - (a)  $e$  and  $f$  commute.
  - (b)  $ef$  is also an idempotent.
- (4) Let  $p \in S$ . The following are equivalent:
  - (a)  $p$  is regular.
  - (b)  $pp^*p = p$ .
  - (c)  $p$  has a unique inverse.
- (5) In particular, the following are equivalent:
  - (a)  $S$  is an inverse semigroup.
  - (b)  $S$  is regular.

*Proof.*

- (1) See Lawson [Law84], Proposition 4.1.
- (2) This follows easily from (1).
- (3) One implication is immediate and the other is clear using that idempotents are self-adjoint.
- (4) Suppose  $p$  is regular, so  $pqp = q$  for some  $q$ . Hence  $Sq = Spq$ , so by (1) we have  $q = p^*pq$ , and then  $p = pqp = pp^*pqp = pp^*p$ .  
If now we suppose that  $p = pp^*p$ , then  $p^* = p^*pp^*$ , so  $p$  and  $p^*$  are inverses. If  $q$  is another inverse, then as before we have  $q = p^*pq$ , whence  $qp = p^*pqp = p^*p$ . Dually,  $pq = pp^*$ . Then  $q = qpq = qpp^* = p^*pp^* = p^*$ .
- (5) Clear.  $\square$

**2.3. The WAP compactification of pro-oligomorphic groups.** In this subsection we will recall the model-theoretic description of the WAP compactification given in [BT14] for Roelcke precompact Polish groups. Since the results of the present paper are concerned with pro-oligomorphic groups, our presentation here will be restricted to these, i.e., to automorphism groups of *classical*  $\aleph_0$ -categorical structures (as opposed to *metric*). Still, it will be convenient to consider formulas as real-valued functions, taking values in  $\{0, 1\}$ .

We refer to [TZ12] for the necessary background in model theory and for the basics of  $\aleph_0$ -categorical structures. Let us recall the definition of the family of groups we will study.

**Definition 2.13.** A group  $G$  is *oligomorphic* if it can be presented as a closed permutation group  $G \leq S(X)$  of a countable set  $X$  such that the orbit spaces  $X^n/G$  are finite for every  $n < \omega$ . *Equivalently*: if  $G$  is the automorphism group of an  $\aleph_0$ -categorical, one-sorted, first-order structure.

A Polish group  $G$  obtained as an inverse limit of oligomorphic groups will be called *pro-oligomorphic*. *Equivalently*:  $G$  can be presented as the automorphism group of an  $\aleph_0$ -categorical, multi-sorted, first-order structure. These are exactly the Roelcke precompact, non-archimedean, Polish groups: see [Tsa12], Theorem 2.4.

Throughout this paper, whenever  $G$  is a pro-oligomorphic group and we write  $G = \text{Aut}(M)$ , we understand that  $M$  is an  $\aleph_0$ -categorical structure and  $G$  is its automorphism group. By the homogeneity of  $\aleph_0$ -categorical structures, we have the following.

**Fact 2.14.** Let  $G = \text{Aut}(M)$  be a pro-oligomorphic group and  $\widehat{G}_L$  be the completion of  $G$  with respect to its left uniformity, which is a topological semigroup. Then  $\widehat{G}_L$  can be identified with the topological semigroup  $E(M)$  of elementary embeddings of  $M$  into itself with the topology of pointwise convergence.

*Proof.* Let  $\xi \in M^\omega$  be an enumeration of  $M$  and define the distance  $d_L$  on  $E(M)$  by  $d_L(x, y) = \sup_{i < \omega} 2^{-i} d(x(\xi_i), y(\xi_i))$ , where  $d$  is the discrete distance on  $M$ . It induces the topology of pointwise convergence on  $E(M)$ . The restriction of  $d_L$  to  $G$  is a compatible left-invariant metric, thus inducing the left uniformity of  $G$ . By homogeneity,  $G$  is dense on  $E(M)$  with respect to  $d_L$ . Since moreover  $E(M)$  is complete with respect to  $d_L$ , it is the left completion of  $G$ .  $\square$

Recall that if  $(X, d)$  is a metric space and  $G$  acts on  $X$  by isometries,

$$X // G = \{\overline{Gx} : x \in X\}$$

is a metric space with distance

$$d(\overline{Gx}, \overline{Gy}) = \inf\{d(g_1x, g_2y) : g_1, g_2 \in G\}.$$

The completion with respect to the *Roelcke uniformity* of a Polish group  $G$  (the infimum of the left and right uniformities) can be described as the space of orbit closures  $R(G) = (\widehat{G}_L \times \widehat{G}_L) // G$ , where  $G$  acts diagonally on  $\widehat{G}_L \times \widehat{G}_L$  by left translation. The group  $G$  is Roelcke precompact precisely when  $R(G)$  is compact. That is, when the completion  $R(G)$  coincides with the compactification of  $G$  associated to the algebra  $\text{UC}(G)$ . Note that the completion  $R(G)$  is metrizable; thus, for Roelcke precompact Polish groups, this is a metrizable compactification, and so are all its factors.

For the rest of this section, we fix a pro-oligomorphic group  $G = \text{Aut}(M)$ . By Fact 2.14, in this case we can write  $R(G) = (E(M) \times E(M)) // G$ . Given elements  $x, y \in E(M)$ , we denote the class of  $(x, y)$  in  $R(G)$  by  $[x, y]_R$ . Now, each  $x \in E(M)$  can be coded by a tuple  $\bar{x} \in M^\omega$ :  $\bar{x}_i = x(\xi_i)$  (if  $\xi \in M^\omega$  is a fixed enumeration of  $M$ ). Then, the element  $[x, y]_R \in R(G)$  can be seen as the type  $\text{tp}(\bar{x}, \bar{y}/\emptyset)$ . In other words,  $[x, y]_R$  is determined by the values

$$\varphi(x(a), y(b)),$$

where  $\varphi(u, v)$  varies over the formulas of  $M$  and  $a, b$  vary over tuples of  $M$  of the appropriate length. In this fashion, a sequence  $[x_n, y_n]_R$  in  $R(G)$  converges to  $[x, y]_R$  if the truth value  $\varphi(x_n(a), y_n(b))$  converges to  $\varphi(x(a), y(b))$  for all  $\varphi, a$ , and  $b$ .

Since  $\text{WAP}(G)$  is a subalgebra of  $\text{UC}(G)$ , the WAP compactification  $W(G)$  is a factor of  $R(G)$ . We will denote the image of  $[x, y]_R$  in  $W(G)$  simply by  $[x, y]$ . That is to say,  $[x, y]$  is determined

by the values  $\varphi(x(a), y(b))$ , as before, only that  $\varphi(u, v)$  ranges over the stable formulas of  $M$ . In particular,  $G$  is a WAP group if and only if the structure  $M$  is stable.

The canonical  $G$ -map  $G \rightarrow W(G)$  is given by

$$g \mapsto [1_G, g],$$

and the  $G$ -action by

$$g[x, y] = [xg^{-1}, y].$$

The involution  $*$ :  $W(G) \rightarrow W(G)$  extending the inverse on the image of  $G$  is given by

$$[x, y]^* = [y, x].$$

Moreover, the semitopological semigroup law of  $W(G)$  can be described in terms of the *stable independence relation* of  $M$ . In order to explain this, we first recall the definition of *imaginaries* and some notions from stability theory.

Let  $M$  be a structure. An *imaginary element* of  $M$  is the class of a definable equivalence relation on some finite power of  $M$ . In other words, if a formula  $\varphi(u, v)$  defines an equivalence relation on  $M^n$ , then each class  $[a]_\varphi \in M^n/\varphi$  is an imaginary of  $M$ .

A standard model-theoretic construction allows to consider all the imaginaries of  $M$  as actual elements in a larger (multi-sorted) first-order structure, denoted  $M^{\text{eq}}$ . See [TZ12, §8.4] for the details. This enlargement of  $M$  is in many senses innocuous; in particular, the natural restriction map  $\text{Aut}(M^{\text{eq}}) \rightarrow \text{Aut}(M)$  is an isomorphism between their automorphism groups. Thus, for many purposes, it is convenient to work directly with the structure  $M^{\text{eq}}$ .

Moreover, imaginary elements of  $\aleph_0$ -categorical structures are in correspondence with the open subgroups of its automorphism group. Indeed, a subgroup  $V \leq G$  is open if and only if it is the stabilizer of an imaginary element of  $M$ . That is to say, if and only if there is a definable equivalence relation  $\varphi(u, v)$  and a tuple  $c$  such that

$$V = \{g \in G : \varphi(c, gc)\} = \{g \in G : [c]_\varphi = g[c]_\varphi\}.$$

See, for example, [Tsa12, §5].

A special kind of imaginaries is given as follows. If  $\varphi(u, v)$  is any formula, we can define a formula  $E_\varphi(u, u')$  by

$$E_\varphi(u, u') := \forall v(\varphi(u, v) \leftrightarrow \varphi(u', v)).$$

Then  $E_\varphi$  defines an equivalence relation on  $M^{|u|}$ . An imaginary  $[c]_{E_\varphi} \in M^{|u|}/E_\varphi$  should be seen as representing the formula  $\varphi(c, v)$ ;  $[c]_{E_\varphi}$  (also denoted simply by  $[c]_\varphi$ ) is called the *canonical parameter* of  $\varphi(c, v)$ ,

Given a type  $t \in S_u(M)$  and a formula  $\varphi(u, v)$ , the  $\varphi$ -*definition* of  $t$  is the function  $d_t\varphi: M^{|v|} \rightarrow \{0, 1\}$  given by

$$d_t\varphi(b) := \varphi(u, b)^t,$$

where the right term denotes the value of  $\varphi(u, b)$  in the type  $t$ . Then, the formula  $\varphi(u, v)$  is *stable* if and only if  $d_t\varphi$  is an  $M$ -definable predicate for every  $t \in S_u(M)$ . In this case we can write  $d_t\varphi(v)$  in the form  $\psi(c, v)$ , and then consider the canonical parameter of this formula; we denote this canonical parameter by  $\text{Cb}_\varphi(t)$ . (The choice of the formula  $\psi$  can be done uniformly in  $t$ , that is,  $c$  depends on  $t$  but  $\psi(w, v)$  does not.) The tuple

$$\text{Cb}(t) = (\text{Cb}_\varphi(t))_{\varphi \text{ stable}}$$

is the *canonical base* of  $t$ .

Finally, an element  $d \in M^{\text{eq}}$  is in the *algebraic closure* of a set  $A \subset M^{\text{eq}}$  if, for some finite tuple  $a \subset A$ ,  $d$  has only finitely many conjugates by automorphisms fixing  $a$ . We denote the algebraic closure of  $A$  by  $\text{acl}(A)$  (which is always a subset of  $M^{\text{eq}}$ ). The set  $A$  is *algebraically closed* if  $A = \text{acl}(A)$ .

**Fact 2.15.** *Let  $a \in (M^{\text{eq}})^{|u|}$  be a tuple and  $B \subset M^{\text{eq}}$  be any subset. There is an extension of the type  $\text{tp}(a/\text{acl}(B))$  to a type  $t \in S_u(M)$  such that  $\text{Cb}(t) \subset \text{acl}(B)$ . Moreover,  $\text{Cb}(t)$  does not depend on the particular extension; in other words, if  $s \in S_u(M)$  is another such extension, then  $d_t\varphi = d_s\varphi$  for every stable formula  $\varphi(u, v)$ .*

**Definition 2.16.**

- (i) If  $a, B$  and  $t$  are as in the previous fact, we define  $\text{Cb}_\varphi(a/B) := \text{Cb}_\varphi(t)$ ,  $\text{Cb}(a/B) := \text{Cb}(t)$ .  
 (ii) Given any sets  $A, B, C \subset M^{\text{eq}}$ , we say that  $A$  is *stably independent from  $C$  over  $B$* , denoted

$$A \downarrow_B C,$$

if for any tuple  $a \in A^{|u|}$  we have  $\text{Cb}(a/B) = \text{Cb}(a/BC)$ .

- (iii) If  $a, c$  are tuples from  $M^{\text{eq}}$  and  $B$  is any subset, we write  $a \equiv_B^s c$  to mean that  $a$  and  $c$  have the same *stable type* over  $B$ , that is,  $\varphi(a, b) = \varphi(c, b)$  for any stable formula  $\varphi(u, v)$  and parameter  $b \in B^{|v|}$ . When  $B$  is empty we shall write simply  $a \equiv c$ , since  $a \equiv_\emptyset^s c$  is indeed equivalent to  $\text{tp}(a/\emptyset) = \text{tp}(c/\emptyset)$ .

Note that the natural identification of  $\text{Aut}(M)$  and  $\text{Aut}(M^{\text{eq}})$  extends to an identification of  $E(M)$  and  $E(M^{\text{eq}})$ .

**Convention 2.17.** We may consider the elements of  $E(M)$  as sets (notably, to apply the relations  $\downarrow$  and  $\equiv^s$  to them), and this shall be done in the following way: an element  $x \in E(M)$  is interpreted as the set  $x(M^{\text{eq}})$ , that is, the imaginary completion of the image of  $x$ . For instance,  $x \cap y$  will denote  $x(M^{\text{eq}}) \cap y(M^{\text{eq}})$ .

If appearing as arguments of the relation  $\equiv^s$ , the elements of  $E(M)$  will be considered as infinite tuples indexed by  $M$  (or by  $\omega$  via a fixed enumeration  $\xi$ , as before).

In these contexts, the juxtaposition  $xy$  will denote the juxtaposed tuple (or merely the union of sets).

The pair  $(\downarrow, \equiv^s)$  satisfies the following usual properties.

**Fact 2.18.** *Let  $x, y, z, w$  be any tuples from  $M^{\text{eq}}$ .*

- (1) (*Invariance*) If  $x \downarrow_y z$  and  $xyz \equiv x'y'z'$ , then  $x' \downarrow_{y'} z'$ . If  $x \downarrow_y z$  and  $x \equiv_{yz}^s x'$ , then  $x' \downarrow_y z$ .
- (2) (*Symmetry*)  $x \downarrow_y z$  if and only if  $z \downarrow_y x$ .
- (3) (*Transitivity*)  $x \downarrow_y zw$  if and only if  $x \downarrow_{yz} w$  and  $x \downarrow_y z$ .
- (4) (*Existence*) There exist  $x', y', z'$  such that  $x'y' \equiv xy$ ,  $y'z' \equiv yz$  and  $x' \downarrow_{y'} z'$ .
- (5) (*Stationarity*) Suppose  $y$  is algebraically closed. If  $x \equiv_y^s z$ ,  $x \downarrow_y w$  and  $z \downarrow_y w$ , then  $x \equiv_{yw}^s z$ .
- (6) (*Non-triviality*) If  $x \downarrow_y z$ , then  $\text{acl}(x) \cap \text{acl}(z) \subset \text{acl}(y)$ .

*Proof.* We refer the reader to [Pil96, Ch. 1, §2]. □

**Fact 2.19.** *The semigroup law in  $W(G)$  is given by*

$$[x, y][y, z] = [x, z] \text{ if } x \downarrow_y z.$$

*The properties of the independence relation stated above ensure that, for any  $p, q \in W(G)$ , we can always find  $x, y, z \in E(M)$  such that  $p = [x, y]$ ,  $q = [y, z]$  and  $x \downarrow_y z$ .*

The latter allows for a model-theoretic description of the idempotents of  $W(G)$ . This was given in [BT14, §5]. Let us end this section by recalling this description and giving a complete proof. Moreover, we complement it with a characterization of the regular elements of  $W(G)$ , which will be used in our main result.

For the definition and properties of the  $\varphi$ -rank see [Pil96, Ch. 1, §3].

**Lemma 2.20.** *Let  $p = [x, y] \in W(G)$ ,  $C = x \cap y$ .*

- (1) *The following are equivalent.*
  - (a)  $p$  is an idempotent (i.e.,  $pp = p$ ).
  - (b)  $x \equiv_C^s y$  and  $x \downarrow_C y$ .
- (2) *The following are equivalent.*
  - (a)  $p$  is regular (i.e.,  $pp^*p = p$ ).
  - (b)  $x \downarrow_C y$ .

*Proof.* (1) Suppose  $p$  is an idempotent. By replacing  $x, y$  by an equivalent pair if necessary, we can find  $z \in E(M)$  such that  $x \downarrow_y z$  and  $xy \equiv yz$ . Then  $[x, y] = [x, y][y, z] = [x, z]$ , so  $y \equiv_x^s z$ . It is easy to deduce that  $C = x \cap z = y \cap z$ , and further that  $x \equiv_C^s y$ .

Next we argue that  $x \downarrow_z y$ . This is equivalent to show that for every stable formula  $\varphi$  the  $\varphi$ -rank of  $x$  over  $yz$  equals the  $\varphi$ -rank of  $x$  over  $z$ :  $R_\varphi(x/yz) = R_\varphi(x/z)$ . Indeed, since  $x \downarrow_y z$  and  $xy \equiv xz$ , we have

$$R_\varphi(x/yz) = R_\varphi(x/y) = R_\varphi(x/z).$$

Thus  $x \downarrow_y z$  and  $x \downarrow_z y$ , so we have  $\text{Cb}(x/yz) \subset y \cap z$ . This implies that  $x \downarrow_C y$ .

Conversely, if  $x \equiv_C^s y$  and  $x \downarrow_C y$ , take  $z$  with  $x \downarrow_y z$  and  $xy \equiv yz$ . It follows that  $C = x \cap z$ ,  $x \equiv_C^s z$  and  $x \downarrow_C z$ . This implies  $y \equiv_x^s z$ , i.e.,  $[x, y] = [x, z]$ , which means that  $p$  is an idempotent.

(2) Suppose  $p$  is regular. Replacing  $x, y$  by an equivalent pair if necessary, we can find  $z, w \in E(M)$  with  $x \downarrow_y z$ ,  $xy \equiv yz$ ,  $x \downarrow_z w$  and  $xy \equiv zw$ . Then the condition  $p = pp^*p$  becomes

$$[x, y] = [x, y][y, z][z, w] = [x, w],$$

so  $y \equiv_x^s w$ . It is easy to check that  $C = x \cap z = x \cap w$ . Moreover, since  $e = pp^* = [x, z]$  is an idempotent, we have that  $x \downarrow_C z$ . From  $x \downarrow_z w$  we get then  $x \downarrow_C w$ . Since  $y \equiv_x^s w$ , we have  $x \downarrow_C y$ .

Suppose conversely that  $x \downarrow_C y$ , and find  $z, w$  as above, so that in particular  $pp^*p = [x, w]$ . From the hypothesis we get  $x \downarrow_C z$ , and then  $x \downarrow_C w$ . The condition  $xy \equiv zw$  implies  $y \equiv_C^s w$ , whence  $y \equiv_x^s w$ . That is,  $pp^*p = [x, w] = [x, y] = p$ .  $\square$

### 3. THE FOURIER-STIELTJES ALGEBRA OF PRO-OLIGOMORPHIC GROUPS

**3.1. Examples of functions in  $\text{Hilb}(G) \setminus B(G)$ .** As mentioned before, the Fourier–Stieltjes algebra  $B(G)$  is, as a general rule, strictly contained in its closure  $\text{Hilb}(G)$ . For example, if  $G$  is compact, then  $B(G)$  is not closed in  $C(G)$  unless  $G$  is finite (see for instance [HR70], Theorem 37.4). Let us begin this section with a model-theoretic argument showing that the same holds for pro-oligomorphic groups.

For locally compact groups, the algebra  $C_0(G)$  of functions vanishing at infinity is always contained in  $\text{Hilb}(G)$ . Indeed, the functions in  $C_0(G)$  factor through the one-point compactification of  $G$ , and the latter is a Hilbert-representable semitopological semigroup: it can be embedded into the contractions of  $L^2(G, \mu)$  by sending the point at infinity to the zero operator, and otherwise extending the regular representation of  $G$ .

Similarly, for closed subgroups of  $S_\infty$ , we have a simple way of producing functions in  $\text{Hilb}(G)$ . Recall that if a group  $G$  acts continuously on a discrete set  $X$ , then we have a natural unitary representation  $\pi: G \rightarrow U(\ell^2(X))$  defined (on the canonical basis of  $\ell^2(X)$ ) by  $\pi(g)e_x = e_{gx}$ .

**Lemma 3.1.** *Let  $M$  be a first-order structure,  $G = \text{Aut}(M)$ . Let  $F: M^n \rightarrow \mathbb{C}$  be a function vanishing at infinity and let  $a \in M^n$ . Then the function  $f \in C(G)$  given by  $f(g) = F(ga)$  belongs to  $\text{Hilb}(G)$ .*

*Proof.* We can assume that  $F$  is zero everywhere except on a finite set  $B \subset M^n$ , since the general case can be uniformly approximated by instances of this form. Take the natural representation  $\pi: G \rightarrow U(\ell^2(M^n))$  and the vectors  $v = \sum_{b \in B} F(b)e_b$ ,  $w = e_a$ . Then we have  $f(g) = \langle v, \pi(g)w \rangle$ , which shows that  $f \in B(G)$ .  $\square$

Next, we see that, in contrast, functions in  $B(G)$  must satisfy a non-trivial decay condition. Given an action by isometries  $G \curvearrowright X$  and a sequence  $(x_i)_{i < \omega} \subset X$ , let us say that  $(x_i)$  is *indiscernible* if for all indices  $i_1 < i_2 < \dots < i_k$  and  $j_1 < j_2 < \dots < j_k$  we have the equality

$$[x_{i_1}, x_{i_2}, \dots, x_{i_k}] = [x_{j_1}, x_{j_2}, \dots, x_{j_k}]$$

in  $X^k // G$ .

Suppose  $G$  is pro-oligomorphic, so that  $E(M) = \widehat{G}_L$ . Note then that every function  $f \in B(G)$ , being left uniformly continuous, extends to a function on  $E(M)$ .

**Proposition 3.2.** *Let  $G$  be a pro-oligomorphic group, say  $G = \text{Aut}(M)$ . Let  $F: M^n \rightarrow \mathbb{C}$  be a function vanishing at infinity,  $a \in M^n$ , and let  $f: E(M) \rightarrow \mathbb{C}$  be given by  $f(x) = F(x(a))$ .*

*Suppose  $(x_i)_{i < \omega} \subset E(M)$  is an indiscernible sequence such that  $(x_i(a))_{i < \omega}$  is non-constant. If  $f|_G \in B(G)$ , then*

$$\left| \frac{1}{n} \sum_{i < n} f(x_i) \right| = O\left(\frac{1}{\sqrt{n}}\right),$$

*the implicit constant depending only on  $f$ .*

*Proof.* Suppose we have a continuous unitary representation  $\pi: G \rightarrow U(\mathcal{H})$  such that  $f(g) = \langle v, \pi(g)w \rangle$  for all  $g \in G$ . Being a homomorphism,  $\pi$  is left uniformly continuous, so it extends to a representation  $\pi: E(M) \rightarrow E(\mathcal{H})$ . (Here,  $E(\mathcal{H})$  is the semigroup of isometric linear endomorphisms of  $\mathcal{H}$ , which is also the left completion of  $U(\mathcal{H})$ .) We have  $f(x) = \langle v, \pi(x)w \rangle$  for all  $x \in E(M)$ .

Since  $(x_i) \subset E(M)$  is indiscernible, so is  $(w_i) \subset \mathcal{H}$  for  $w_i = \pi(x_i)w$ . Now, as the reader can check, an indiscernible sequence  $(w_i)$  in a Hilbert space is always of the form  $w_i = w' + w'_i$ , where  $w'_i \perp w'$ ,  $\|w'_i\| = \|w'_j\|$  and  $w'_i \perp w'_j$  for  $i \neq j$ . In particular,  $w'$  is the weak limit of  $(w_i)$ . Since  $F$  vanishes at infinity and  $(x_i a)$  is indiscernible and non-constant, we have that  $f(x_i) \rightarrow 0$ . That is,  $\langle v, w' \rangle = 0$ . We deduce that

$$\left| \frac{1}{n} \sum_{i < n} f(x_i) \right| = \left| \langle v, \frac{1}{n} \sum_{i < n} w'_i \rangle \right| \leq \frac{\|v\| \cdot \|\sum_{i < n} w'_i\|}{n} = \frac{\|v\| \cdot \sqrt{\sum_{i < n} \|w'_i\|^2}}{n} = \frac{\|v\| \cdot \|w'_0\|}{\sqrt{n}} \leq \frac{\|v\| \cdot \|w\|}{\sqrt{n}}$$

□

**Corollary 3.3.** *Let  $G$  be pro-oligomorphic and infinite. Then  $B(G)$  is not closed in the uniform norm.*

*Proof.* Choose any non-constant indiscernible sequence  $(x_i) \subset E(M)$  (which always exists if  $M$  is  $\aleph_0$ -categorical) and an element  $a \in M$  such that  $(x_i(a))$  is non-constant. Then take  $F: M \rightarrow \mathbb{C}$  vanishing at infinity and such that  $F(x_i(a)) = 1/i^{1/3}$ . Then, by Lemma 3.1 and Proposition 3.2, we obtain that the function defined by  $f(g) = F(ga)$  is in  $\text{Hilb}(G)$  and not in  $B(G)$ . □

**3.2. A model-theoretic description of the Hilbert compactification.** As explained in Section 2.3, the WAP compactification of a pro-oligomorphic group  $G$  is the space of types of pairs of embeddings  $x, y \in E(M)$  restricted to stable formulas. Dually, this can be stated by saying that  $\text{WAP}(G)$  is the closed algebra generated by the functions of the form

$$g \mapsto \varphi(a, gb),$$

where  $\varphi(u, v)$  is a stable formula and  $a, b$  are tuples from  $M$ . See [BT14, §5] or [Iba14, §4]. Hence it is natural to ask which formulas  $\varphi(u, v)$  give rise, in the preceding way, to functions in the subalgebra  $\text{Hilb}(G)$ .

We start with the following basic observation.

**Lemma 3.4.** *Let  $M$  be a first-order structure,  $G = \text{Aut}(M)$ . Let  $\varphi(u, v)$  be a formula defining an equivalence relation on  $M^n$  and let  $a, b \in M^n$ . Then the function*

$$f(g) = \varphi(a, gb)$$

*(which takes the value 1 if the elements are related and 0 otherwise) is in  $B(G)$ .*

*Proof.* It suffices to consider the natural representation  $\pi: G \rightarrow U(\ell^2(M^n/\varphi))$ , then observe that  $f(g) = \langle e_{[a]_\varphi}, \pi(g)e_{[b]_\varphi} \rangle$ . □

The reader can also check that  $f$  belongs to  $B(G)$  under the weaker assumption that  $\varphi(x, b)$  defines a *weakly normal set*, that is to say, that the canonical parameter of  $\varphi(x, b)$  is in the algebraic closure of any tuple  $a$  that satisfies the formula.

We want to give a converse to the previous lemma, for  $\aleph_0$ -categorical structures. For this we invoke the classification theorem of unitary representations of pro-oligomorphic groups proved in [Tsa12].

**Fact 3.5** (Classification Theorem). *Let  $G$  be a pro-oligomorphic group.*

- (1) *Every continuous unitary representation of  $G$  is a direct sum of irreducible representations.*
- (2) *Every irreducible continuous unitary representation is a subrepresentation of the quasi-regular representation  $\pi_V : G \rightarrow U(\ell^2(G/V))$  for some open subgroup  $V \leq G$ .*

**Proposition 3.6.** *Let  $G$  be pro-oligomorphic,  $G = \text{Aut}(M)$ . Then  $\text{Hilb}(G)$  is the closed linear span of the functions of the form*

$$g \mapsto \varphi(a, gb)$$

where  $\varphi(u, v)$  is a definable equivalence relation on some power  $M^n$  and  $a, b$  are tuples in  $M^n$ .

*Proof.* It suffices to show that every  $f \in B(G)$  can be uniformly approximated by linear combinations of functions of this form. By the classification theorem, every continuous unitary representation is a subrepresentation of one of the form  $\pi : G \rightarrow U\left(\bigoplus_k \ell^2(G/V_k)\right)$ , where each  $V_k$  is an open subgroup of  $G$ . Now, every matrix coefficient of  $\pi$  can be uniformly approximated by a linear combination of *basic* matrix coefficients, that is, given by

$$g \mapsto \langle e_{g_0 V_k}, \pi(g) e_{g_1 V_k} \rangle$$

for vectors  $e_{g_0 V_k}, e_{g_1 V_k}$  from the canonical basis of  $\ell^2(G/V_k)$ . Fix an open subgroup  $V = V_k$ ; it is the stabilizer of some imaginary element  $[c]_\varphi \in M^{\text{eq}}$ . If we take  $a = g_0 c$  and  $b = g_1 c$ , we have that  $g_0 V = g g_1 V$  if and only if  $[a]_\varphi = g[b]_\varphi$ . In other words,

$$\langle e_{g_0 V}, \pi(g) e_{g_1 V} \rangle = \varphi(a, gb).$$

The proposition follows. □

Dually, this characterization of  $\text{Hilb}(G)$  will provide a nice model-theoretic description of the Hilbert compactification  $H(G)$ .

We fix a pro-oligomorphic group  $G = \text{Aut}(M)$ . We recalled earlier that the left completion of  $G$  can be identified with the topological semigroup  $E(M)$  of elementary embeddings  $M \rightarrow M$ , which is the pointwise closure of  $G$  inside  $M^M$ . A natural generalization is to consider *partial* elementary maps of  $M$ .

In fact, the correct framework will be  $M^{\text{eq}}$ . Let  $K = M^{\text{eq}} \cup \{\infty\}$  be the one-point compactification of  $M^{\text{eq}}$ , and let

$$\Xi = \{p \in K^K : p(\infty) = \infty \text{ and } p \text{ is injective on } p^{-1}(M^{\text{eq}})\}.$$

Then  $\Xi$ , equipped with composition and the product topology, is a compact semitopological inverse semigroup (in fact, isomorphic to the semigroup of partial bijections of  $M^{\text{eq}}$ ). Let  $P(M) = \overline{G} \subset \Xi$  be the closure of  $G$  in the product space  $K^K$  (where we set  $g(\infty) = \infty$  for every  $g \in G$ ). Then, if we think of an element  $p \in K^K$  as a partial map  $M^{\text{eq}} \rightarrow M^{\text{eq}}$  (undefined on  $a$  whenever  $p(a) = \infty$ ), we get the following.

**Proposition 3.7.** *The elements of  $P(M)$  are precisely the partial elementary maps of  $M$  with algebraically closed domain. Besides,  $P(M)$  is closed under composition, and with this operation it becomes a semitopological  $*$ -semigroup compactification of  $G$ , which is moreover an inverse semigroup.*

*Proof.* It is clear that any  $p \in P(G)$  is a partial elementary map of  $M$ , and also that its domain must be algebraically closed. Conversely, let  $p : A \rightarrow M^{\text{eq}}$  be an elementary map with  $A$  algebraically closed. Fix a finite tuple  $a$  from  $A$ , a finite tuple  $b$  disjoint from  $A$  and a finite subset  $C \subset M^{\text{eq}}$  (intended as the complement of a neighborhood of  $\infty$  in  $K$ ); denote  $a' = p(a)$ . Choose a tuple  $b'$  such that  $ab \equiv a'b'$ , then take  $b''$  satisfying  $b''a' \equiv b'a'$  and  $b'' \downarrow_{a'} C$ . Since  $b$  is disjoint from  $\text{acl}(a) \subset A$  we have that  $b''$  is disjoint from  $\text{acl}(a')$ , whence  $b''$  is disjoint from  $C$ . Now, by homogeneity there is  $g \in G$  such that  $ga = a'$  and  $gb = b''$ . This shows that  $p$  can be approximated by elements of  $G$  in the topology of  $K^K$ .

Finally,  $P(M)$ , being a closed subsemigroup of  $\Xi$  closed under inverses, is also a compact inverse semitopological semigroup. □

We remark that we have defined  $P(M)$  directly as a family of partial maps on  $M^{\text{eq}}$ , and not on  $M$ . Unlike the case of  $E(M)$ , which can be identified with  $E(M^{\text{eq}})$ , the previous construction applied to  $M$  would yield a smaller object (a factor of  $P(M)$ ), which may lose information. However, it is easy to check the following.

**Remark 3.8.** The structure  $M$  has *weak elimination of imaginaries* (see for instance [TZ12, §8.4]) if and only if  $P(M)$  coincides with its factor consisting of partial elementary maps of  $M$  with (relatively) algebraically closed domain.

We also observe that  $P(M)$  can be alternatively defined as the closure of the image of  $G$  inside  $\Theta(\ell^2(M^{\text{eq}}))$ , induced by the natural unitary representation  $G \rightarrow U(\ell^2(M^{\text{eq}}))$ . Indeed, by identifying  $\infty \in K$  with the zero of the Hilbert space, we have natural topological embeddings

$$G \subset \Xi \subset \Theta(\ell^2(M^{\text{eq}})).$$

In particular,  $P(M)$  is a factor of the Hilbert compactification.

**Theorem 3.9.** *Let  $G = \text{Aut}(M)$  be a pro-oligomorphic group. Then  $P(M)$  coincides with the Hilbert compactification  $H(G)$ .*

*Proof.* This follows from the previous observation and the fact, implied by the classification theorem, that every separable continuous unitary representation of  $G$  is a subrepresentation of  $G \rightarrow U(\ell^2(M^{\text{eq}}))$ . Nevertheless, let us give an explicit isomorphism  $H(G) \rightarrow P(M)$  based on the model-theoretic description of  $W(G)$ . Given endomorphisms  $x, y \in E(M^{\text{eq}})$ , let  $[x, y]_H$  denote the image of  $[x, y] \in W(G)$  under the canonical  $G$ -map  $W(G) \rightarrow H(G)$ . Proposition 3.6 says that  $[x, y]_H$  is the type determined by the values  $\varphi(x(a), y(b))$  for definable equivalence relations  $\varphi(u, v)$  and parameters  $a, b$  from  $M$ . Equivalently, it is the type determined by the values  $x(a) = y(b)$  for parameters  $a, b \in M^{\text{eq}}$ . We consider the map

$$[x, y]_H \in H(G) \mapsto x^{-1} \circ y \in \Xi,$$

where, on the right,  $x, y$  are seen as elements of  $\Xi$ . By our description of  $H(G)$ , this is well-defined and injective, and it is clearly a continuous  $G$ -map. Since  $H(G)$  is compact, its image is  $P(M)$ .  $\square$

**3.3. Characterization of Eberlein pro-oligomorphic groups.** A corollary of the previous results is that if a pro-oligomorphic group  $G$  is Eberlein (that is, if we have  $W(G) = H(G)$ ), then  $W(G)$  must be an inverse semigroup. As it turns out, this is a sufficient condition. Moreover, this property is related to the following model-theoretic notion.

**Definition 3.10.** Let  $M$  be a first-order structure. We will say that  $M$  is *one-based for stable independence* if for any algebraically closed sets  $A, B \subset M^{\text{eq}}$  we have

$$A \downarrow_{A \cap B} B.$$

Equivalently: if for any tuple  $a$  and set  $B$  we have  $\text{Cb}(a/B) \subset \text{acl}(a)$ .

**Theorem 3.11.** *Let  $G = \text{Aut}(M)$  be a pro-oligomorphic group. The following are equivalent.*

- (1)  $W(G)$  is an inverse semigroup.
- (2) The idempotents of  $W(G)$  commute.
- (3)  $M$  is one-based for stable independence.
- (4)  $G$  is Eberlein.

*Proof.* (1)  $\Rightarrow$  (2) is just a consequence of the general characterization referred in Fact 2.11.

(2)  $\Rightarrow$  (1): Let  $p \in W(G)$ , say  $p = [x, y]$  for  $x, y \in E(M)$ . We identify  $x, y$  with their images under the embedding  $E(M) \rightarrow W(G)$ . Then we can write  $p = [x, y] = [x, 1][1, y] = x^*y$ . Now, for any  $z \in E(M)$  we have  $z^*z = 1$ , so the element  $zz^*$  is an idempotent. If idempotents commute, we obtain  $pp^*p = x^*y y^* x x^* y = x^* x x^* y y^* y = x^* y = p$ .

(1)  $\Rightarrow$  (3): By hypothesis, every element is regular, so by Lemma 2.20 we have  $x \downarrow_{x \cap y} y$  for any  $x, y \in E(M)$ . Now take algebraically closed sets  $A, B \subset M^{\text{eq}}$ . By replacing  $AB$  by an equivalent copy if necessary, we can find  $x \in E(M)$  such that  $A \subset x$  and  $x \downarrow_A B$ . Again, by replacing  $xAB$  by



an equivalent copy, we can find  $y \in E(M)$  such that  $B \subset y$  and  $x \downarrow_B y$ . In particular,  $x \cap y = x \cap B = A \cap B$ . Since  $x \downarrow_{x \cap y} y$  and  $x \cap y = A \cap B \subset y$ , we have  $x \downarrow_{A \cap B} y$ . Hence  $A \downarrow_{A \cap B} B$ .

(3)  $\Rightarrow$  (4): We want to show that the canonical  $G$ -map  $W(G) \rightarrow H(G)$  is injective. Given  $p, q \in W(G)$ , we can always choose  $x, y, z \in E(M)$  such that  $p = [x, y]$  and  $q = [x, z]$ . If the images of  $p$  and  $q$  in  $H(G)$  coincide, then  $x \cap y = x \cap z =: C$ , and moreover  $y \equiv_C^s z$ . Since  $M$  is one-based for stable independence,  $y \downarrow_C x$  and  $z \downarrow_C x$ . By stationarity, we get  $y \equiv_x^s z$ , that is to say,  $p = q$ .

(4)  $\Rightarrow$  (1): Clear from the identification  $H(G) = P(M)$ .  $\square$

**Corollary 3.12.** *The following are equivalent.*

- (1)  $G$  is strongly Eberlein.
- (2)  $M$  is  $\aleph_0$ -stable (i.e., the space of types  $S_u(M)$ , in any finite variable  $u$ , is countable).
- (3) The intersection  $\bigcap_{x \in E(M)} E(M) \cdot x$  is non-empty.

Moreover, if the previous conditions hold, the action of  $G$  on  $S_u(M)$  is oligomorphic.

*Proof.* (1)  $\Leftrightarrow$  (2): As mentioned before,  $G$  is a WAP group if and only if  $M$  is stable. By the previous theorem,  $G$  is strongly Eberlein if and only if  $M$  is stable and one-based. A classical result of Zilber (see Theorem 5.12 in [Pil96, Ch. 2], and also [BBH14], Proposition 3.12) states that an  $\aleph_0$ -categorical stable structure is one-based if and only if it is  $\aleph_0$ -stable.

(2)  $\Leftrightarrow$  (3): This is just a topological reformulation. Indeed, it is easy to see that

$$y \in \bigcap_{x \in E(M)} E(M) \cdot x$$

if and only if every type over the image  $M' = y(M)$  (possibly in countably many variables) is realized in  $M$ . In turn, there exists a submodel  $M'$  of  $M$  with this property if and only if  $M$  is  $\aleph_0$ -stable.

For the moreover part, it suffices to show that  $S_u(M)/G$  is finite. We sketch the (standard) argument. To every indiscernible sequence  $(a_i)_{i < \omega} \subset M^{|u|}$  we assign its limit type  $p \in S_u(M)$ . This is a surjective  $G$ -map. Since  $M$  is one-based, the type of an indiscernible sequence  $(a_i)_{i < \omega}$  is determined by  $\text{tp}(a_0 a_1)$ . By  $\aleph_0$ -categoricity, there are only finitely many types  $\text{tp}(a_0 a_1)$ .  $\square$

**Example 3.13.** As mentioned before, the group  $S_\infty$  of permutations of a countable set  $X$  is (strongly) Eberlein; its Roelcke compactification is the semigroup of partial bijections of  $X$ . We can give some new examples. Consider the following oligomorphic groups:

- (1) the automorphism group of a dense linear order,  $\text{Aut}(\mathbb{Q}, <)$ ;
- (2) the homeomorphism group of the Cantor space (or, equivalently, the automorphism group of its algebra of clopen sets),  $\text{Homeo}(2^\omega)$ ;
- (3) the automorphism group of the random graph.

It follows from the results in [BT14, §6] (see also [Iba14, §4.2]) that for each of these groups (as well as for  $S_\infty$ ) the algebra  $\text{WAP}(G)$  is generated by the functions of the form

$$g \mapsto a = gb$$

for elements  $a, b$  in the respective structures. Since these are obviously in  $\text{Hilb}(G)$ , we deduce that these groups are Eberlein. In fact, fixed some  $G = \text{Aut}(M)$  from the above list, we deduce that the WAP compactification consists of the partial elementary maps  $M \rightarrow M$  with relatively algebraically closed domain. (In particular, as is well-known, these structures have weak elimination of imaginaries.)

**Example 3.14.** A famous conjecture of Zilber claimed that an  $\aleph_0$ -categorical stable structure should be  $\aleph_0$ -stable (equivalently, one-based, or still: not encoding a *pseudoplane*). This was refuted by Hrushovski, who constructed an  $\aleph_0$ -categorical stable pseudoplane. The details of the construction can be found in [Wag94]. It follows from Theorem 3.11 that the automorphism group of this pseudoplane is an oligomorphic WAP group that is not Eberlein. This answers Question 6.10 in [GM14].

**Example 3.15.** The previous example can be used to produce a countable compact dynamical system of finite Cantor–Bendixson rank that is faithfully representable on a reflexive Banach space, but not on a Hilbert space, in the sense of representability defined in the introduction (see [Meg08] for more background). Indeed, let  $M$  be Hrushovski’s stable pseudoplane,  $G = \text{Aut}(M)$ , and choose some formula  $\varphi(u, v)$  and parameters  $a, b$  such that  $f: g \mapsto \varphi(a, gb)$  is not in  $\text{Hilb}(G)$ . Now, the space  $S_\varphi(M)$  of  $\varphi$ -types in the variable  $v$ , with parameters from  $M$ , induces a compactification  $X$  of  $G$  via the map  $g \mapsto \text{tp}_\varphi(gb/M)$ . Since  $f$  belongs to the associated algebra, the dynamical system  $G \curvearrowright X$  is not Hilbert-representable; but it is reflexively representable, since  $\varphi$  is stable. Finally, as is well-known, the space of local types  $S_\varphi(M)$  of a stable formula over a countable structure is a countable compact zero-dimensional space of finite Cantor–Bendixson rank (see, for instance, [Pil96], Remark 2.3 and Lemma 3.1).

#### 4. HILBERT-REPRESENTABLE FACTORS

In this section we extend our analysis to the factors of  $H(G)$  and  $W(G)$ . We start by showing that all factors of  $H(G)$  are zero-dimensional.

We recall that if  $\pi: G \rightarrow U(\mathcal{H})$  is a continuous unitary representation, then  $\pi$  extends naturally to a homomorphism  $\pi: H(G) \rightarrow \Theta(\mathcal{H})$ .

**Lemma 4.1.** *Let  $\pi: G \rightarrow U(\mathcal{H})$  be a continuous unitary representation of a Roelcke precompact Polish group. Let  $\eta \in \mathcal{H}$  be a vector such that  $\pi(V)\eta = \eta$  for some open subgroup  $V \leq G$  (i.e.,  $\pi(v)\eta = \eta$  for all  $v \in V$ ). Then  $\pi(H(G))\eta$  is countable.*

*Proof.* Since  $G$  is Polish, we can assume  $\mathcal{H}$  is separable. As  $G$  is Roelcke precompact and  $V$  is open, the space of double cosets  $V \backslash G / V$  is finite. Since  $\eta$  is fixed by  $V$ , the function  $g \mapsto \langle \eta, \pi(g)\eta \rangle$  factors through  $V \backslash G / V$ , hence the set

$$\{ \langle \pi(g_1)\eta, \pi(g_2)\eta \rangle : g_1, g_2 \in G \}$$

is finite. By continuity,  $\{ \langle \pi(p_1)\eta, \pi(p_2)\eta \rangle : p_1, p_2 \in H(G) \}$  is equal to it, and therefore also finite. Now the separability of  $\mathcal{H}$  implies that  $\pi(H(G))\eta$  is countable.  $\square$

**Proposition 4.2.** *Let  $G$  be a Roelcke precompact Polish group and let  $f \in C(H(G))$  be a function such that  $Vf = f$  for some open subgroup  $V \leq G$ . Then  $f(H(G))$  is countable.*

*Proof.* Let  $f = \lim_n f_n$ , where  $f_n(g) = \langle \xi_n, \pi(g)\eta_n \rangle$  for some representation  $\pi$  and vectors  $\xi_n, \eta_n$ . First, we may assume that each  $\xi_n$  is fixed by  $\pi(V)$ . Indeed, let  $n$  be such that  $\|f - f_n\| \leq \epsilon$  and let  $\xi'_n$  be the element of minimal norm of  $\overline{\text{co}}(\pi(V)\xi_n)$ . Note that  $\xi'_n$  is fixed by  $\pi(V)$  and for every  $g \in G$  and  $v \in V$ ,

$$| \langle \xi_n, \pi(g)\eta_n \rangle - \langle \pi(v)\xi_n, \pi(g)\eta_n \rangle | = | f_n(g) - v f_n(g) | \leq 2\epsilon,$$

implying that

$$| \langle \xi_n, \pi(g)\eta_n \rangle - \langle \xi'_n, \pi(g)\eta_n \rangle | \leq 2\epsilon$$

and thus we can replace  $\xi_n$  by  $\xi'_n$  without losing much.

Next, by replacing  $\pi$  with a sum of infinitely many copies of itself and rescaling if necessary, we may assume that  $\xi_n = \xi$  for all  $n$ . Finally, apply Lemma 4.1 to obtain that  $\pi(H(G))\xi$  is countable and let  $E$  be the equivalence relation on  $H(G)$  given by  $p E q \iff \pi(p^*)\xi = \pi(q^*)\xi$  (so that  $E$  has countably many classes). Now all  $f_n$  and  $f$  factor through  $E$ , so, in particular, the image of  $f$  is countable.  $\square$

**Lemma 4.3.** *Suppose  $G$  is pro-oligomorphic and let  $A \subset \text{WAP}(G)$  be a closed subalgebra. Let  $A_0 \subset A$  be the subalgebra of functions  $f$  such that  $Vf = f$  for some open subgroup  $V \leq G$ , and let  $A_1 \subset A$  be the subalgebra of functions with finite range.*

- (1) *If  $A$  is invariant, then  $A_0$  is dense in  $A$ .*
- (2) *If  $A$  is bi-invariant, then  $A_1$  is dense in  $A$ .*

*Proof.* This follows almost verbatim from the proofs of Proposition 4.7 and Theorem 4.8 in [BT14].  $\square$

**Theorem 4.4.** *If  $G$  is a pro-oligomorphic group, then every factor of  $H(G)$  is zero-dimensional.*

*Proof.* Let  $S$  be a factor of  $H(G)$ . Since  $\mathcal{A}(S)$  is invariant, using Lemma 4.3.(1) and Proposition 4.2, we see that continuous functions on  $S$  with countable range separate points in  $S$ . This implies the conclusion of the theorem.  $\square$

**Question 4.5.** Is the same true for all factors of  $W(G)$ ?

The automorphism group of the dense, countable circular order acts minimally on the circle and this dynamical system is a quotient of the Roelcke compactification of the group. So certainly some hypothesis is necessary to obtain zero-dimensionality.

The previous theorem, restated as follows, is useful to show that Hilbert-representability is preserved under factors.

**Corollary 4.6.** *Let  $A \subset \text{Hilb}(G)$  be an invariant closed subalgebra, and let  $A_1 \subset A$  be the subalgebra of functions with finite range. Then  $A_1$  is dense in  $A$ .*

**Proposition 4.7.** *Let  $G$  be a pro-oligomorphic group. If  $f \in \text{Hilb}(G)$  has finite range, then  $f \in B(G)$ .*

*Proof.* Suppose first that  $f$  is  $\{0, 1\}$ -valued. By Proposition 3.6, we know that  $f$  can be approximated in norm by a linear combination of  $\{0, 1\}$ -valued matrix coefficients  $m_0, \dots, m_{n-1} \in B(G)$ , say

$$\left\| f - \sum_{i < n} \lambda_i m_i \right\| < 1/2.$$

Hence there is a boolean function  $b: \{0, 1\}^n \rightarrow \{0, 1\}$  such that  $b((m_i(g))_{i < n}) = f(g)$  for every  $g \in G$ . This implies that  $f$  can be written as a boolean combination of the matrix coefficients  $m_i$ . Now it is enough to note that, first, the negation of a  $\{0, 1\}$ -valued function  $m \in B(G)$  is again in  $B(G)$ , since we can write it as the difference  $\neg m = 1 - m$ , and, second, the conjunction of two  $\{0, 1\}$ -valued functions  $m_0, m_1 \in B(G)$  is again in  $B(G)$ , since it is simply the product  $m_0 \wedge m_1 = m_0 m_1$ . We conclude that  $f$  is a matrix coefficient.

Finally, it is clear that every  $f \in \text{Hilb}(G)$  of finite range is a linear combination of  $\{0, 1\}$ -valued functions in  $\text{Hilb}(G)$ . Hence  $f \in B(G)$ .  $\square$

We can finally give an answer to Question 1.9 for pro-oligomorphic groups.

**Theorem 4.8.** *Let  $G$  be a pro-oligomorphic group. Every factor of  $H(G)$  is Hilbert-representable.*

*Proof.* Let  $A$  be a subalgebra of  $\text{Hilb}(G)$  and let  $A_1 \subset A$  be the subalgebra of functions with finite range. By Corollary 4.6 and Proposition 4.7, we have that  $A_1$  is dense in  $A$  and contained in  $A \cap B(G)$ . Hence  $A = \overline{A \cap B(G)}$  and, by Proposition 1.7, the factor of  $H(G)$  corresponding to  $A$  is Hilbert-representable.  $\square$

**Corollary 4.9.** *Every semitopological semigroup factor of  $H(G)$  is  $*$ -closed and is an inverse semigroup.*

*Proof.* The first claim follows from the previous theorem and Proposition 2.9. Now, any such  $G$ -factor  $H(G) \rightarrow S$  must preserve the involution. It follows that  $pp^*p = p$  for every  $p \in S$ , hence  $S$  is an inverse semigroup.  $\square$

It turns out that the converse of the above corollary also holds. The following is a generalization of Theorem 3.11.

**Theorem 4.10.** *Let  $S$  be a semitopological  $*$ -semigroup compactification of a pro-oligomorphic group  $G$ . The following are equivalent:*

- (1)  $S$  is an inverse semigroup;
- (2) the idempotents of  $S$  commute;
- (3)  $S$  is Hilbert-representable.

*Proof.* The equivalence (1)  $\Leftrightarrow$  (2) is exactly as in Theorem 3.11. The implication (3)  $\Rightarrow$  (1) is clear, for example by the previous corollary.

(1)  $\Rightarrow$  (3): Let  $A \subset \text{WAP}(G)$  be the algebra generated by the union of  $\mathcal{A}(S)$  and  $\text{Hilb}(G)$ . By Theorem 4.8, to prove (3), it is enough to show that  $A = \text{Hilb}(G)$ .

Let  $S_H$  be the compactification of  $G$  associated to the subalgebra  $A \subset \text{WAP}(G)$ . It follows from Fact 2.3 and Remark 2.8 that  $S_H$  is a semitopological  $*$ -semigroup compactification. Since  $S$  and  $H(G)$  are inverse semigroups, so is  $S_H$ . Indeed, if  $q \in S_H$  and  $f \in \mathcal{A}(S)$  or  $f \in \text{Hilb}(G)$ , then  $f(qq^*q) = f(q)$ , for this holds in  $S$  and in  $H(G)$ ; hence this holds for any  $f \in A$ , which shows that  $qq^*q = q$ .

Let  $\phi_0: W(G) \rightarrow S_H$  and  $\phi_1: S_H \rightarrow H(G)$  be the canonical factor maps. We need to show that  $\phi_1$  is injective, so that  $S_H = H(G)$ . The proof of Theorem 3.11 shows that the canonical factor map  $\phi_1\phi_0: W(G) \rightarrow H(G)$  is injective on the set of regular elements of  $W(G)$ .

Let  $p \in W(G)$  be any element, and let  $P$  be the closed subsemigroup of  $W(G)$  generated by  $p^*p$ . Then  $P^* = P$ , so by [BT14], Lemma 3.6, there exists an idempotent  $e_p \in P$  such that the set  $W(G)e_p$  is contained in every set  $W(G)s$  for  $s \in P$ . Then  $W(G)e_p \subset W(G)p^*pe_p \subset W(G)pe_p \subset W(G)e_p$ , so  $W(G)pe_p = W(G)e_p$ . It follows from Fact 2.12.(1) that  $e_p = p^*pe_p$ . We set  $\sigma(p) = pe_p$ , and we remark that  $\sigma(p)$  is regular.

We claim that  $\phi_1(\sigma(p)) = \phi_1(p)$ . Since  $P$  is generated by  $p^*p$ , there is a sequence  $n_i < \omega$  such that  $(p^*p)^{n_i} \rightarrow e_p$ . By continuity,  $\phi_1(p(p^*p)^{n_i}) \rightarrow \phi_1(pe_p)$ , but we also have  $\phi_1(p(p^*p)^{n_i}) = \phi_1(p)$  since  $\phi_1$  is a homomorphism and  $S_H$  is an inverse semigroup.

Finally, let  $q, q' \in S_H$  and suppose  $\phi_1(q) = \phi_1(q')$ . Let  $p, p' \in W(G)$  be any elements with  $\phi_0(p) = q$  and  $\phi_0(p') = q'$ . The associated elements  $\sigma(p)$  and  $\sigma(p')$  are regular and have the same image in  $H(G)$ , hence  $\sigma(p) = \sigma(p')$ . It follows that  $q = q'$ .  $\square$

We point out that, even for the group of integers  $G = \mathbb{Z}$ , a Hilbert-representable semitopological semigroup compactification of  $G$  need not be an inverse semigroup. The semigroup considered in [BLM01] serves as a counterexample.

To conclude, we give a bound on the complexity of the countable factors of  $H(G)$ .

**Proposition 4.11.** *Let  $Z \subset \mathcal{H}$  be a countable, weakly compact subset of a Hilbert space  $\mathcal{H}$ . Denote  $D(Z) = \{\|\xi - \eta\| : \xi, \eta \in Z\}$  and assume that  $D(Z)$  is finite. Then  $D(Z') \subsetneq D(Z)$  (here  $Z'$  is the Cantor–Bendixson derivative of  $Z$ ). In particular, the Cantor–Bendixson rank of  $Z$  is bounded by  $|D(Z)|$ .*

*Proof.* Let  $\delta = \max D(Z)$ ; we show that  $\delta \notin D(Z')$ . Suppose, towards a contradiction, that  $\xi, \eta \in Z'$  are such that  $\|\xi - \eta\| = \delta$ . By translating  $Z$ , we can assume that  $\xi = 0$ . Let  $\eta_n \rightarrow^w \eta$  with  $\eta_n$  distinct elements of  $Z$ . As  $\delta$  is maximal,  $D(Z)$  is finite,  $\eta_n \rightarrow^w \eta$ , and the norm is lower semicontinuous in the weak topology, we must have that eventually  $\|\eta_n\| = \|\eta\|$ . This, in turn, implies that  $\eta_n \rightarrow \eta$  in norm, which means that  $\eta_n$  is eventually constant (again, since  $D(Z)$  is finite). This contradicts the choice of the sequence  $\eta_n$ .  $\square$

Recall that a  $G$ -ambit  $(X, x_0)$  is just a compactification of  $G$  where  $x_0$  is the image of 1. If  $X$  is countable then  $x_0$  must be isolated.

**Corollary 4.12.** *Let  $G$  be a Roelcke precompact group and let  $(X, x_0)$  be a Hilbert-representable  $G$ -ambit. Then  $X$  is countable if and only if the stabilizer  $G_{x_0}$  is open, and in this case we have*

$$\text{rank } X \leq |\{G_{x_0}gG_{x_0} : g \in G\}|.$$

*Proof.* Follows from Lemma 4.1 (and its proof) and Proposition 4.11.  $\square$

Note that the previous result is a generalization of the following model-theoretic fact, which follows from one-basedness: in an  $\aleph_0$ -categorical  $\aleph_0$ -stable theory, the Morley rank of any finite tuple  $a$  is bounded by the number of distinct types  $\text{tp}(b/a)$  with  $\text{tp}(b) = \text{tp}(a)$ .

**Question 4.13.** Do countable ambits of pro-oligomorphic groups necessarily have finite Cantor–Bendixson rank?

We remark that every countable compactification of a Roelcke precompact Polish group  $G$  is a factor of  $W(G)$ . Indeed, since countable compact systems are *Asplund-representable* (see, for instance, [GM06], Corollary 10.2), this follows from [Iba14], Theorem 2.9.

Akin and Glasner [AG14] construct countable WAP  $\mathbb{Z}$ -ambits of arbitrarily high rank. But of course, the group  $\mathbb{Z}$  is not pro-oligomorphic.

---

#### REFERENCES

- [AG14] Ethan Akin and Eli Glasner, *WAP Systems and Labeled Subshifts*, arXiv:1410.4753 [math.DS].
- [AZ86] Gisela Ahlbrandt and Martin Ziegler, *Quasi-finitely axiomatizable totally categorical theories*, Ann. Pure Appl. Logic **30** (1986), no. 1, 63–82, Stability in model theory (Trento, 1984). MR 831437 (87k:03026)
- [BBH14] Itai Ben Yaacov, Alexander Berenstein, and C. Ward Henson, *Almost indiscernible sequences and convergence of canonical bases*, J. Symb. Log. **79** (2014), no. 2, 460–484. MR 3224976
- [BdlHV08] Bachir Bekka, Pierre de la Harpe, and Alain Valette, *Kazhdan’s property (T)*, New Mathematical Monographs, vol. 11, Cambridge University Press, Cambridge, 2008. MR 2415834 (2009i:22001)
- [BJM78] J. F. Berglund, H. D. Junghenn, and P. Milnes, *Compact right topological semigroups and generalizations of almost periodicity*, Lecture Notes in Mathematics, vol. 663, Springer, Berlin, 1978. MR 513591 (80c:22003)
- [BLM01] A. Bouziad, M. Lemańczyk, and M. K. Mentzen, *A compact monothetic semitopological semigroup whose set of idempotents is not closed*, Semigroup Forum **62** (2001), no. 1, 98–102. MR 1832257
- [BT14] Itai Ben Yaacov and Todor Tsankov, *Weakly almost periodic functions, model-theoretic stability, and minimality of topological groups*, arXiv:1312.7757 [math.LO].
- [Cho82] Ching Chou, *Weakly almost periodic functions and Fourier-Stieltjes algebras of locally compact groups*, Trans. Amer. Math. Soc. **274** (1982), no. 1, 141–157. MR 670924 (84a:43008)
- [Ebe49] W. F. Eberlein, *Abstract ergodic theorems and weak almost periodic functions*, Trans. Amer. Math. Soc. **67** (1949), 217–240. MR 0036455 (12,112a)
- [Gla12] Eli Glasner, *The group  $\text{Aut}(\mu)$  is Roelcke precompact*, Canad. Math. Bull. **55** (2012), no. 2, 297–302. MR 2957245
- [GM06] E. Glasner and M. Megrelishvili, *Hereditarily non-sensitive dynamical systems and linear representations*, Colloq. Math. **104** (2006), no. 2, 223–283. MR 2197078 (2006m:37009)
- [GM14] Eli Glasner and Michael Megrelishvili, *Representations of dynamical systems on Banach spaces*, Recent progress in general topology. III, Atlantis Press, Paris, 2014, pp. 399–470. MR 3205489
- [GW12] Eli Glasner and Benjamin Weiss, *On Hilbert dynamical systems*, Ergodic Theory Dynam. Systems **32** (2012), no. 2, 629–642. MR 2901363
- [How95] John M. Howie, *Fundamentals of semigroup theory*, London Mathematical Society Monographs. New Series, vol. 12, The Clarendon Press, Oxford University Press, New York, 1995, Oxford Science Publications. MR 1455373 (98e:20059)
- [HR70] Edwin Hewitt and Kenneth A. Ross, *Abstract harmonic analysis. Vol. II: Structure and analysis for compact groups. Analysis on locally compact Abelian groups*, Die Grundlehren der mathematischen Wissenschaften, Band 152, Springer-Verlag, New York-Berlin, 1970. MR 0262773 (41 #7378)
- [Iba14] Tomás Ibarlucía, *The dynamical hierarchy for Roelcke precompact Polish groups*, to appear in Israel Journal of Mathematics; arXiv:1405.4613 [math.LO].
- [Law74] J. D. Lawson, *Joint continuity in semitopological semigroups*, Illinois J. Math. **18** (1974), 275–285. MR 0335674 (49 #454)
- [Law84] Jimmie D. Lawson, *Points of continuity for semigroup actions*, Trans. Amer. Math. Soc. **284** (1984), no. 1, 183–202. MR 742420 (85m:54042)
- [Meg01] Michael G. Megrelishvili, *Operator topologies and reflexive representability*, Nuclear groups and Lie groups (Madrid, 1999), Res. Exp. Math., vol. 24, Heldermann, Lemgo, 2001, pp. 197–208. MR 1858149
- [Meg03] Michael Megrelishvili, *Fragmentability and representations of flows*, Proceedings of the 17th Summer Conference on Topology and its Applications, vol. 27, 2003, pp. 497–544. MR 2077804 (2005h:37042)
- [Meg07] ———, *Topological transformation groups: selected topics*, survey paper in: Open Problems in Topology II, (Elliott Pearl, editor), Elsevier Science, 2007.
- [Meg08] ———, *Reflexively representable but not Hilbert representable compact flows and semitopological semigroups*, Colloq. Math. **110** (2008), no. 2, 383–407. MR 2353912 (2008k:54062)
- [Pil96] Anand Pillay, *Geometric stability theory*, Oxford Logic Guides, vol. 32, The Clarendon Press, Oxford University Press, New York, 1996, Oxford Science Publications. MR 1429864 (98a:03049)
- [Rud59] Walter Rudin, *Weak almost periodic functions and Fourier-Stieltjes transforms*, Duke Math. J. **26** (1959), 215–220. MR 0102705 (21 #1492)
- [Sht94] Alexander I. Shtern, *Compact semitopological semigroups and reflexive representability of topological groups*, Russian J. Math. Phys. **2** (1994), no. 1, 131–132. MR 1297947 (95j:22008)
- [Tsa12] Todor Tsankov, *Unitary representations of oligomorphic groups*, Geom. Funct. Anal. **22** (2012), no. 2, 528–555. MR 2929072
- [TZ12] Katrin Tent and Martin Ziegler, *A course in model theory*, Lecture Notes in Logic, vol. 40, Association for Symbolic Logic, La Jolla, CA; Cambridge University Press, Cambridge, 2012. MR 2908005

- [Vee79] William A. Veech, *Weakly almost periodic functions on semisimple Lie groups*, *Monatsh. Math.* **88** (1979), no. 1, 55–68. MR 550072 (81b:22012)
- [Wag94] Frank O. Wagner, *Relational structures and dimensions*, *Automorphisms of first-order structures*, Oxford Sci. Publ., Oxford Univ. Press, New York, 1994, pp. 153–180. MR 1325473

UNIVERSITÉ DE LYON, INSTITUT CAMILLE JORDAN, 43 BLVD. DU 11 NOVEMBRE 1918, 69622 VILLEURBANNE CEDEX, FRANCE

UNIVERSITÉ DE LYON, INSTITUT CAMILLE JORDAN, 43 BLVD. DU 11 NOVEMBRE 1918, 69622 VILLEURBANNE CEDEX, FRANCE

INSTITUT DE MATHÉMATIQUES DE JUSSIEU–PRG, UNIVERSITÉ PARIS 7, CASE 7012, 75205 PARIS CEDEX 13, FRANCE