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Local Limit Theorem for Markov Additive Processes on $\mathbb{Z}^d$ with Null Recurrent Internal Markov Chain

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Abstract In the classical framework, a random walk on a group is a Markov chain with independent and identically distributed increments. In some sense, random walks are time and space homogeneous. This paper is devoted to a class of inhomogeneous random walks on $\mathbb{Z}^d$ termed Markov Additive processes (also known as Markov Random Walks, Random Walks with Internal Degree of Freedom or semi-Markov Processes). In this model, the increments of the walk are still independent but their distributions are dictated by a Markov chain termed the internal Markov chain. Whereas this model is largely studied in the literature, most of the results involve internal Markov chains whose operator is quasi-compact. This paper extends two results for more general internal operators: a Local Limit Theorem and a sufficient criterion for their transience. These results are thereafter applied to a new family of models of drifted random walks on the lattice $\mathbb{Z}^d$.

Key words Inhomogeneous random walks . Markov Additive processes . Semi-Markov processes . Markov Random Walks . Random Walks with Internal Degrees of Freedom . Local Limit Theorem . Recurrence . Transience . Fourier analysis

Mathematics Subject Classification (2020) 60F15 . 60J10 . 60J45 . 60J55 . 60K15

Contents

Introduction

In the classical framework, a random walk on a group $\mathbf{G}$ is a discrete time stochastic process $(Z_n)_{n\geq 0}$ defined as the product of independent and identically distributed (*i.i.d.*) random variables $(\xi_n)_{n\geq 1}$. Random walks on groups are Markov chains that are adapted to the group structure in the sense that the underlying Markov operator is invariant under the group action of $\mathbf{G}$ on itself. Thus, this homogeneity naturally gives rise to deep connections between stochastic properties of the random walk and algebraic properties of the group. Starting with the seminal paper of Pòlya, [56], many of these connections have been studied (see [64] and references therein) and this research area has remained in constant progress over the last two decades (without claiming to be exhaustive, see [59, 41, 55, 11, 4, 42, 6, 34, 49, 29, 3]).

In this paper, we aim at investigating inhomogeneous random walks. It turns out that there are at least two ways to introduce inhomogeneity. First, we can consider *spatial* inhomogeneity by weakening the group structure, replacing it, for instance, by a directed graph as in [12, 32, 33, 53, 23, 9, 10, 7]. Secondly, we can study *temporal* inhomogeneous random walks by introducing a notion of memory as in the model of reinforced [52, 63], excited [57, 5], self-interacting [22, 54], persistent random walks [17, 16, 19, 20, 18] or also Markov additive processes (abbreviated by MAP) that shall be at the core of this paper. All these models belong to the larger class of stochastic processes with long range dependency.

Markov additive processes are also known as Random Walk With Internal Degree of Freedom — see [46] — semi-Markov processes or Markov random walks — see for instance [45, 35, 15, 50, 51, 2, 61, 44]. Roughly speaking, Markov additive processes are discrete time $\mathbb{Z}^d$-valued (or $\mathbb{R}^d$) processes whose increments are still independent but no longer stationary. The distribution of an increment is then driven by a Markov chain termed the *internal* Markov chain. Most of results in the context of standard random walks are generalized to Markov additive processes when the Markov operator of the internal chain is assumed to be quasi-compact on a suitable Banach space: among them, a renewal theorem [45, 2, 30, 31], local limit theorem [46, 35, 36, 39, 26, 37], central limit theorem [39, 26], results on the recurrence set [58, 1, 40], large deviations [50, 51], asymptotic expansion of the Green function [61, 44], one-dimensional Berry-Essen theorem [39, 26, 38] with applications to M-estimation, first passage time [27].

However, assuming the internal operator to be quasi-compact is rather strong (see [47]). Actually, beyond the technical difficulties inherent to the infinite dimension, there is no real difference in nature with the finite dimension under this assumption. On the other hand, relax this assumption and the study of Markov Additive processes can be very challenging. Besides, it is worth noting that many (interesting) Markov Additive processes do not admit a quasi-compact internal operator as it will be illustrated by the examples considered in this paper.

In the context of Markov additive processes, classical Fourier analysis can be extended by introducing the Fourier transform operator which is a perturbation of the internal Markov operator in an appropriate Banach space. As in the classical context, the Fourier transform operator characterizes the distribution of the additive part of Markov Additive processes. By a continuous perturbation argument (see for instance [43]), when the internal Markov operator is quasi-compact, the Fourier transform operator remains quasi-compact for all sufficiently small perturbations. It allows, under suitable moment conditions on the distribution of increments,

to derive a Taylor expansion at the second order of the perturbed dominating eigenvalue, say $\lambda(t)$, $t \in [-\pi, \pi)^d$, whose coefficients are given, roughly speaking, by the mean and the variance operators. Finally, under an assumption on the spectrum of the Fourier transform operator for large perturbations, it can be concluded that all the needed stochastic information is actually contained in the nature of the singularity at zero of $(1 - \lambda(t))^{-1}$ (note that $\lambda(0) = 1$). For instance, an integral test criterion, similar to the Chung-Fuchs criterion (see [21] or [60]), involving a singularity of this kind is given in [18].

In this paper, the quasi-compacity condition is dropped and the internal Markov chain is only assumed to be irreducible recurrent. The condition on the spectrum of Fourier transform operator for large perturbations remains similar but the nature of the singularity at the origin is analyzed considering the Taylor expansion of the *quenched characteristic exponent* (in a wide sense) defined in Section 1.2. The terms of order 1 and 2 are termed the *quenched drift* and the *quenched dispersion* respectively. These quantities are characteristics of the increments of the process and naturally appears in the Local Limit in Theorem 2.1 and the transience condition in Corollary 2.4.

The paper is organized as follows. In Section 1 are gathered the main notions involved in the statement of Theorems 2.1 and its Corollary 2.4. Section 2 is devoted to the statement of these two results. In Section 3 is dedicated to the proof of the Theorem 2.1. In Section 4, Theorems 2.1 and Corollary 2.4 are applied on various families of Markov Additive processes that extends the models considered in [48, 12, 8]. Those models are simple random walks on directed graphs built upon $\mathbb{Z}^2$. Various phenomena are observed whether the directions are fixed periodically or not. As it shall be clear for the model periodically directed, it is possible to factorize a Markov Additive process without changing the distribution of the additive part (see Section 4). It will be concluded that the simple random walk on the periodically directed graph is a Markov Additive process with a finite internal Markov chain. As such, the internal Markov operator is a matrix and is quasi-compact. For more general directions (random directions for instance), such a reduction is no longer possible.

1 Markov additive processes

1.1 Definitions

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\mathbb{X}$ a countable set. The set $\mathbb{X}$ is naturally endowed with the σ-algebra consisting of all subsets of $\mathbb{X}$.

Definition 1.1 (Markov additive process)**.** *Let $d \geq 1$ be an integer. A Markov additive process (MAP for short) is a Markov chain $((X_n, Z_n))_{n\geq 0}$ taking values in $\mathbb{X} \times \mathbb{Z}^d$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$ satisfying for all $n \geq 0$ and all bounded functions $f : \mathbb{X} \times \mathbb{Z}^d \to \mathbb{R}$*

$$\mathbb{E}[f(X_{n+1}, Z_{n+1} - Z_n)|((X_k, Z_k))_{0\leq k\leq n}] = \mathbb{E}[f(X_{n+1}, Z_{n+1} - Z_n)|X_n]. \tag{1}$$

From equality (1), it follows immediately that $(X_n)_{n\geq 0}$ is a Markov chain on $\mathbb{X}$. The latter is termed *the internal Markov chain*. The corresponding Markov kernel shall be denoted by P, namely $Pf(x) := \mathbb{E}[f(X_1)|X_0 = x]$ for any bounded function $f : \mathbb{X} \to \mathbb{R}$ (in symbols, $f \in \ell^\infty(\mathbb{X})$). Generally speaking, there exists a σ-finite measure m dominating the family of probabilities $(P(x, \cdot))_{x\in\mathbb{X}}$, *i.e.* $m(y) = 0$ implies $P(x, y) = 0$ for all $x \in \mathbb{X}$. If the internal Markov chain is irreducible and recurrent, the invariant measure (unique up to a positive constant) is a natural choice for m.

The conditional distribution of $Z_{n+1} - Z_n$ given $(X_n, X_{n+1}) = (x, y)$ will be denoted by $\mu^{x,y}$ and the Fourier transform of $\mu^{x,y}$ by $\widehat{\mu^{x,y}}$. Then, for any $t \in \mathbb{R}^d$, the Fourier transform operator

P_t acting on the bounded function $f : \mathbb{X} \to \mathbb{C}$ is defined as follows:

$$P_t f(x) := \mathbb{E}[f(X_1)e^{i\langle t, Z_1 - Z_0\rangle}|X_0 = x] = \sum_{y\in\mathbb{X}} P(x,y)f(y)\widehat{\mu^{x,y}}(t).$$

From Markov property, it follows for all $n \geq 1$

$$P_t^n f(x) = \mathbb{E}[f(X_n)e^{i\langle t, Z_n - Z_0\rangle}|X_0 = x].$$

Moreover,

$$P_t^n \mathbf{1}(x) = \sum_{z\in\mathbb{Z}^d} e^{i\langle t,z\rangle}\mathbb{P}(Z_n - Z_0 = z|X_0 = x). \tag{2}$$

Consequently, the function $\mathbb{R}^d \ni t \to P_t^n\mathbf{1}(x) \in \mathbb{C}$ is the Fourier transform of the conditional distribution of $Z_n - Z_0$ given $X_0 = x$.

By periodicity, it is sufficient to consider the operator P_t for $t \in \mathbb{T}^d = [-\pi, \pi)^d$. In addition, $\mathbb{T}^d_*$ will stand for $\mathbb{T}^d \setminus \{0\}$.

1.2 Conditional characteristic exponent, conditional drift and conditional dispersion

Proposition 1.2. *For any MAP, the following identity holds for all $n \geq 0$:*

$$\mathbb{P}(Z_n - Z_0 = z|X_0 = x) = \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d} e^{-i\langle t,z\rangle} P_t^n\mathbf{1}(x)\ \mathrm{d}t \tag{3}$$

and $P_t^n\mathbf{1}(x) = \mathbb{E}[\Pi_n(t)|X_0 = x]$ where

$$\Pi_n(t) := \prod_{k=0}^{n-1} \mathbb{E}\left[e^{i\langle t, Z_{k+1} - Z_k\rangle}\middle| X_k, X_{k+1}\right]. \tag{4}$$

The proof of Proposition 1.2 relies on a result from [25, Equation (8)] whose statement is recalled below.

Lemma 1.3. *For all $m \geq n \geq p \geq 0$,*

$$\mathbb{E}\left[e^{i\langle t, Z_n - Z_p\rangle}\middle|\sigma(X_j, p \leq j \leq n)\right]\mathbb{E}\left[e^{i\langle t, Z_m - Z_n\rangle}\middle|\sigma(X_j, n \leq j \leq m)\right] = \mathbb{E}\left[e^{i\langle t, Z_m - Z_p\rangle}\middle|\sigma(X_j, p \leq j \leq m)\right]. \tag{5}$$

Proof of Proposition 1.2. By inverse Fourier transform,

$$(2\pi)^d\mathbb{P}(Z_n - Z_0 = z|X_0 = x) = \int_{\mathbb{T}^d} P_t^n\mathbf{1}(x)e^{-i\langle t,z\rangle}\ \mathrm{d}t.$$

Then, by Lemma 1.3, setting $\mathcal{G}_n = \sigma(X_\ell, 0 \leq \ell \leq n)$, $n \geq 1$,

$$P_t^n\mathbf{1}(x) = \mathbb{E}\left[\mathbb{E}\left[\prod_{k=0}^{n-1} e^{i\langle t, Z_{k+1} - Z_k\rangle}\middle|\mathcal{G}_n\right]\middle| X_0 = x\right] = \mathbb{E}\left[\prod_{k=0}^{n-1}\mathbb{E}\left[e^{i\langle t, Z_{k+1} - Z_k\rangle}\middle| X_k, X_{k+1}\right]\middle| X_0 = x\right]$$

□

As a matter of fact, Π_n is a almost-surely continuous $\mathbb{C}$-valued function of $\mathbb{T}^d$ satisfying $\Pi_n(0) = 1$. Hence, in a neighborhood of the origin, the logarithm of Π_n is well defined. We may call $\log \Pi_n$ the *quenched characteristic exponent* (with a slight abuse of terminology since $\log \Pi_n$ is not defined on $\mathbb{R}^d$ in general). Its second order Taylor expansion at $t = 0$ then exhibits two important quantities:

$$U_n := \sum_{\ell=0}^{n-1} \mathbb{E}[Z_{\ell+1} - Z_\ell | X_\ell, X_{\ell+1}] \quad \text{and} \quad V_n := \frac{1}{n} \sum_{\ell=0}^{n-1} \mathbf{Cov}(Z_{\ell+1} - Z_\ell | X_\ell, X_{\ell+1}).$$

The latter are called respectively the *quenched drift* and *the quenched dispersion* of the additive component.

Remark 1.4. *The normalization factor for the quenched dispersion is chosen as the approximate growth rate of the sum. In particular, its eigenvalues remain bounded and away from zero almost-surely as illustrated in Section 4, Equation (16).*

1.3 Spectral condition

Definition 1.5 (Spectral condition)**.** *A MAP is said to satisfy the spectral condition if, for any compact $K \subset \mathbb{T}^d$ with $0 \notin K$ there exist constants $C > 0$ and $\gamma \in (0, 1)$ such that,*

$$\forall n \geq 1, \qquad \|P_t^n \mathbf{1}\|_{\ell^\infty(\mathbb{X})} \leq C\gamma^n.$$

Proposition 1.6. *Assume that the family $(\mu^{x,y})_{x,y\in\mathbb{X}}$ of probability measures is uniformly aperiodic* i.e*, for all $t \in \mathbb{T}_*^d$, $\sup_{x,y\in\mathbb{X}} |\widehat{\mu^{x,y}}(t)| < 1$, then the MAP fulfills the spectral condition.*

Proof. For all $x \in \mathbb{X}$ and all $f \in \ell^\infty(\mathbb{X})$,

$$|P_t f(x)| = \left| \sum_{y\in\mathbb{X}} P(x, y)\widehat{\mu^{x,y}}(t) f(y) \right| \leq \|f\|_{\ell^\infty(\mathbb{X})} \sup_{x,y\in\mathbb{X}} |\widehat{\mu^{x,y}}(t)|.$$

Therefore,

$$\|P_t^n \mathbf{1}\|_{\ell^\infty(\mathbb{X})} \leq \left[\sup_{x,y\in\mathbb{X}} |\widehat{\mu^{x,y}}(t)| \right]^n.$$

□

In the literature, it is usually assumed a stronger condition than the spectral condition of Definition 1.5. It involves the spectral radius of each P_t acting on some Banach subspace of $\ell^\infty(\mathbb{X})$. Such assumptions imply the spectral condition of Definition 1.5 as shown in Proposition 1.7 just below.

Proposition 1.7. *Let $(\mathcal{B}, \|\cdot\|_\mathcal{B})$ be a Banach subspace of $\ell^\infty(\mathbb{X})$ such that*

(C1) *$\mathbf{1} \in \mathcal{B}$ and the canonical injection $\mathcal{B} \hookrightarrow \ell^\infty(\mathbb{X})$ is continuous;*

(C2) *for each $t \in \mathbb{T}^d$, the operator P_t acts continuously on $\mathcal{B}$;*

(C3) *the map $\mathbb{T}^d \ni t \to P_t \in \mathcal{L}(\mathcal{B})$ is continuous for the subordinated operator norm induced by $\|\cdot\|_\mathcal{B}$.*

Suppose that the spectral radius of P_t defined, for all $t \in \mathbb{T}^d$, by

$$r_{\mathcal{B}}(t) = \inf\{\|P_t^n\|_{\mathcal{B}}^{1/n}, n \geq 1\},$$

satisfies $r_{\mathcal{B}}(t) < 1$ as soon as $t \in \mathbb{T}^d \setminus \{0\}$. Then, the resulting MAP satisfies the spectral condition of Definition 1.5.

Proof. Since $\mathbb{T}^d \ni t \to P_t \in \mathcal{L}(\mathcal{B})$ is continuous, the spectral radius $t \to r_{\mathcal{B}}(t)$ is upper semi-continuous as the infimum of continuous functions. Consequently, if $K \subset \mathbb{T}^d$ with $0 \notin K$ then there exists $t_K \in K$ such that $1 > r_{\mathcal{B}}(t_K) = \sup_{t\in K} r_{\mathcal{B}}(t)$.

Choose $\gamma \in (r_{\mathcal{B}}(t_K), 1)$ and denote by $\Gamma \subset \mathbb{C}$ the circle of the complex plan centered at 0 of radius γ. As a matter of fact, for any $(\lambda, t) \in \Gamma \times K$ the operator $\lambda - P_t$ is invertible. By [24, Theorem 10, p. 560], it follows

$$\|P_t^n\|_{\mathcal{B}} = \frac{\gamma^n}{2\pi} \oint_{\Gamma} \|(\lambda - P_t)^{-1}\|_{\mathcal{B}} \, \mathrm{d}\lambda.$$

Now, the map $\Gamma \times K \ni (\lambda, t) \to \lambda - P_t \in \mathcal{L}(\mathcal{B})$ is continuous. Besides, by [24, Lemma 1, p. 584], the map $A \to A^{-1}$ is a homeomorphism of the open subset of invertible operators in $\mathcal{L}(\mathcal{B})$. Consequently, the map $\Gamma \times \mathbb{K} \ni (\lambda, t) \to (\lambda - P_t)^{-1} \in \mathcal{L}(\mathcal{B})$ is continuous on the compact $\Gamma \times K$ hence

$$\sup_{(\lambda,t)\in\Gamma\times K} \|(\lambda - P_t)^{-1}\|_{\mathcal{B}} < \infty.$$

The spectral condition of Definition 1.5 follows immediately from the continuity of the canonical injection of $\mathcal{B}$ in $\ell^\infty(\mathbb{X})$. □

The uniform aperiodicity condition introduced in Proposition 1.6 is far from being necessary. Below, we introduce the usual notion of aperiodic MAP (we refer to [61] for instance).

Definition 1.8 (Aperiodic Markov Additive process)**.** *A MAP is said to be aperiodic if there exists no proper subgroup H of the additive group $\mathbb{Z}^d$ such that, for every positive integer n and every $x, y \in \mathbb{X}$ with $m(x)P^n(x, y) > 0$, there exists $a = a_n(x, y) \in \mathbb{Z}^d$ satisfying*

$$\mathbb{P}[Z_n - Z_0 \in a + H | X_0 = x, X_n = y] = 1.$$

Proposition 1.9. *Let (X, Z) be an aperiodic MAP for which the internal Markov chain X is irreducible and recurrent. Let $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ be a Banach subspace of $\ell^\infty(\mathbb{X})$ satisfying the assumptions (C1), (C2) and (C3) of Proposition 1.7. Then, for any $t \in \mathbb{T}^d \setminus \{0\}$, the operator $P_t \in \mathcal{L}(\mathcal{B})$ has no eigenvalues of modulus one.*

Proof. Suppose on the contrary that there exists $t_0 \in \mathbb{T}^d \setminus \{0\}$, $f \in \mathcal{B} \setminus \{0\}$ and $\theta \in \mathbb{R}$ such that

$$P_{t_0} f(x) = \sum_{y\in\mathbb{X}} P(x, y)\widehat{\mu^{x,y}}(t_0) f(y) = e^{i\theta} f(x), \quad x \in \mathbb{X}. \tag{6}$$

By Jensen's inequality and the fact that $|\widehat{\mu^{x,y}}(t_0)| \leq 1$, it follows that $|f| \leq P|f|$. Consequently, $\|f\|_{\ell^\infty(\mathbb{X})} - |f|$ is a non-negative superharmonic function. Since P is irreducible and recurrent, the function $|f|$ is constant (see [64, Theorem 1.16, p.5] for instance). Hence, for all $x \in \mathbb{X}$ such that $m(x) > 0$,

$$1 = \left| \sum_{y\in\mathbb{X}} P(x, y)\widehat{\mu^{x,y}}(t_0) \right| \leq \sum_{y\in\mathbb{X}} P(x, y)|\widehat{\mu^{x,y}(t_0)}| \leq 1, \quad x \in \mathbb{X}.$$

This means that, for all $x, y \in \mathbb{X}$ with $m(x)P(x,y) > 0$, $|\widehat{\mu^{x,y}}(t_0)| = 1$. Since $|f|$ is constant and $f \neq 0$, Equation (6) can be rewritten as follows

$$\sum_{y \in \mathbb{X}} P(x,y)\widehat{\mu^{x,y}}(t_0)\frac{f(y)}{f(x)} = e^{i\theta}, \quad x \in \mathbb{X}.$$

This convex combination is again extremal so that, for all $x, y \in \mathbb{X}$, $\widehat{\mu^{x,y}}(t_0)\frac{f(y)}{f(x)} = e^{i\theta}$. More generally,

$$\forall n \geq 1, \quad \forall (x_0, \ldots, x_n), \quad \prod_{k=0}^{n-1} \widehat{\mu^{x_k, x_{k+1}}}(t_0) = e^{in\theta}\frac{f(x_0)}{f(x_n)}. \tag{7}$$

Now, fix $n \geq 1$ and $x, y \in \mathbb{X}$ and choose once for all $a_n(x,y) \in \mathbb{R}^d$ such that $e^{in\theta}\frac{f(x)}{f(y)} = e^{i\langle t_0, a_n(x,y)\rangle}$. In fact, $a_n(x,y) \in \mathbb{Z}^d$ since left handside of Equation (7) is nothing but the Fourier transform of a measure supported by $\mathbb{Z}^d$. Thereafter, denote by $S_n(x,y)$ the support of the distribution of $Z_n - Z_0 - a_n(x,y)$ conditionally on the event $\{X_0 = x\} \cap \{X_n = y\}$ and set H the group generated by the family of sets $S_n(x,y)$, $n \geq 1$ and $x, y \in \mathbb{X}$. By construction of H, one has that

$$\forall n \geq 1, \quad \forall x, y \in \mathbb{X}, \quad \exists a_n(x,y) \in \mathbb{Z}^d : \quad \mathbb{P}[Z_n - Z_0 \in a_n(x,y) + H | X_0 = x, X_n = y] = 1.$$

By aperiodicity, $H = \mathbb{Z}^d$.

Finally, take the expectation on both sides of Equation (7) so that

$$e^{i\langle t_0, a_n(x,y)\rangle} = \mathbb{E}\left[e^{i\langle t_0, Z_n - Z_0\rangle} | X_0 = x, X_n = y\right].$$

Then, the extremality in the convex combination yields that for all $w \in S_n(x,y)$, $\langle t_0, w\rangle = 0$ modulo 2π. By linearity, this identity extends to the whole group $H = \mathbb{Z}^d$. Taking $w = e_i$, $i = 1, \ldots, d$, the vectors of the standard basis of $\mathbb{Z}^d$, it follows that $t_0 = 0$ leading to a contradiction. □

Let us point out that an arithmetic condition such as the aperiodicity fails to give information on spectral values that are not eigenvalues. Nonetheless, if in addition, it is known that the peripheral spectrum consists of eigenvalues then the spectral condition of Definition 1.5 is clearly fulfilled. Such an assumption on the peripheral spectrum which is rather technical can nonetheless be difficult to verify in practice. That is why, it can be often preferable to check directly the spectral condition of Definition 1.5.

2 Main results

Let us first summarize the assumptions involved in the statement of the Local Limit Theorem and the transience sufficient criterion.

Assumptions 1. *The internal Markov chain P is irreducible and recurrent.*

Assumptions 2. *The family of probability measures $(\mu^{x,y})_{x,y \in \mathbb{X}}$ admits a uniform third order moment:*

$$\sup_{x,y \in \mathbb{X}} \sum_{z \in \mathbb{Z}^d} \|z\|^3 \mu^{x,y}(z) < \infty$$

Assumptions 3. *The quenched dispersion is uniformly elliptic: there exists a (deterministic) constant $\alpha > 0$ such that*

$$\forall t \in \mathbb{R}^d, \quad \forall n \geq 1, \quad \langle t, V_n t\rangle \geq \alpha \|t\|^2, \quad a.s..$$

Assumptions 4. *The MAP satisfies the spectral condition.*

2.1 A Local Limit Theorem

Theorem 2.1 (Local Limit Theorem). *Under Assumptions 1, 2, 3 and 4, for all $x \in \mathbb{X}$,*

$$\forall \varepsilon \in (0, \frac{1}{2}), \quad \sup_{z \in \mathbb{Z}^d} \Big| (2\pi n)^{d/2} \mathbb{P}(Z_n - Z_0 = z | X_0 = x) - \mathbb{E}\left[\frac{1}{\sqrt{\det(V_n)}} \exp\left\{ -\frac{1}{2n} \langle U_n - z, V_n^{-1}(U_n - z) \rangle \right\} \Big| X_0 = x \right] \Big| = O(n^{-(\frac{1}{2}-\varepsilon)}).$$

2.2 A sufficient criterion for transience: some potential theory

This section is devoted to a sufficient criterion for the transience of the additive part of Markov additive processes. The recurrence or transience in this context is defined in [18] as follows.

Definition 2.2 (Recurrence *versus* Transience). *A MAP $((X_n, Z_n)_{n \geq 0}$ is said to be recurrent if for any $(x, z) \in \mathbb{X} \times \mathbb{Z}^d$, there exists $r > 0$ such that*

$$\mathbb{P}\left[\liminf_{n \to \infty} \|Z_n\| < r \Big| X_0 = x, Z_0 = z \right] = 1.$$

It is said to be transient if for any $(x, z) \in \mathbb{X} \times \mathbb{Z}^d$

$$\mathbb{P}\left[\lim_{n \to \infty} \|Z_n\| = \infty \Big| X_0 = x, Z_0 = z \right] = 1.$$

In [18], it is proved that a MAP is either recurrent or transient under Assumption 1. In particular, the recurrence or transience of such a MAP does not depend on the initial state $(x, z) \in \mathbb{X} \times \mathbb{Z}^d$.

For any positive function f on $\mathbb{X} \times \mathbb{Z}^d$, recall that the potential of the charge f is given by

$$Gf(x, z) := \mathbb{E}\left[\sum_{n \geq 0} f(X_n, Z_n) \Big| X_0 = x, Z_0 = z \right].$$

By analogy with the classical context of random walks, it is natural to look for a criterion for the recurrence or transience of a MAP that involves the mean sojourn time of the set $\mathbb{X} \times \{z\}$:

$$G\mathbf{1}_{\mathbb{X} \times \{z\}}(x, z) = \mathbb{E}\left[\sum_{n \geq 0} \mathbf{1}_{\mathbb{X} \times \{z\}}(X_n, Z_n) \Big| X_0 = x, Z_0 = z \right]. \tag{8}$$

Naturally, if the quantity in (8) is finite, the additive component of the MAP hits $z \in \mathbb{Z}^d$ only finitely many times almost surely by applying the Markov inequality.

Definition 2.3 (Irreducibility). *A MAP is said to be irreducible if for any $x \in \mathbb{X}$, for any $z, z' \in \mathbb{Z}^d$, there exists $n \geq 0$ such that $\mathbb{P}[Z_n = z' | X_0 = x, Z_0 = z] > 0$.*

Therefore, if a MAP is irreducible and if the quantity in Equation (8) is finite for all $x \in \mathbb{X}$ and for some (equivalently any) $z \in \mathbb{Z}^d$ then the MAP is transient by the Markov property. Consequently, we obtain the following criterion as a Corollary of Theorem 2.1.

Corollary 2.4 (Sufficient criterion for transience)**.** *Let $d \geq 2$. Suppose that the MAP is irreducible. Then, under Assumptions 1, 2, 3 and 4, if for any $x \in \mathbb{X}$*

$$\sum_{n\geq 1} \frac{1}{n^{d/2}} \mathbb{E}\left[\frac{1}{\sqrt{\det(V_n)}} \exp\left\{ -\frac{1}{2n} \langle U_n, V_n^{-1} U_n \rangle \right\} \middle| X_0 = x \right] < \infty \tag{9}$$

then, the MAP is transient.

Remark 2.5. *Under the assumptions of Corollary 2.4, the quantities in Equations (8) and (9) are simultaneously finite or infinite. When the internal Markov chain is positive recurrent, the MAP is recurrent as soon as the quantity in Equation (8) is infinite as shown in Proposition [18, Proposition 2.2] giving rise to a complete characterization of the transient or recurrent type. Such a characterization remains an open question in the case of a null recurrent internal Markov chain since the possibility of a transient MAP for which the quantity in Equation (8) remains infinite can not be ruled out.*

Remark 2.6. *Intuitively, Assumption 3 means that the MAP remains genuinely d-dimensional and is not attracted by a sub-manifold. In addition, it is worth noting that, under Assumption 3, $\det(V_n) \geq \alpha^d$ for all $n \geq 1$ and, under Assumption 2, $\sup_{n\geq 1} \|V_n\|_2 < \infty$ so that the factor $\det(V_n)$ does not play any role in the nature of the series (9).*

3 Proofs

Proposition 3.1. *Under Assumption 2, for any sufficiently small $\delta \in (0,1)$ and any $t \in \delta\mathbb{T}^d$, where $\delta\mathbb{T}^d$ is the magnification of the hypercube $\mathbb{T}^d$ by δ, there exist a (deterministic) constant K and $\Theta_n(t)$ with $|\Theta_n(t)| \leq nK\|t\|_\infty^3$ such that*

$$\Pi_n(t) = \exp\left\{ i\langle t, U_n\rangle - \frac{1}{2}\langle t, nV_n t\rangle + \Theta_n(t) \right\}.$$

Proof. Fix $\ell \geq 0$. Under Assumption 2, there exists a (deterministic) constant $\kappa > 0$ such that

$$\left| \mathbb{E}\left[e^{i\langle t, Z_{\ell+1} - Z_\ell\rangle} \middle| X_\ell, X_{\ell+1} \right] - 1 \right| \leq \kappa \|t\|_\infty. \tag{10}$$

Then, for all $\delta \in (0, 1/2\kappa) \cap (0,1)$ and all $t \in \delta\mathbb{T}^d$, the $\mathbb{C}$-valued function

$$\pi_\ell : t \longrightarrow \log \mathbb{E}\left[e^{i\langle t, Z_{\ell+1} - Z_\ell\rangle} \middle| X_\ell, X_{\ell+1} \right]$$

is three times continuously differentiable. Observe that $\pi_\ell(0) = 0$. Also, the partial derivative with respect to the p^{th} coordinate, $p \in \{1, \ldots, d\}$, is given by

$$\partial_p \pi_\ell(t) = i \frac{\mathbb{E}\left[e^{i\langle t, Z_{\ell+1} - Z_\ell\rangle} (Z_{\ell+1}^{(p)} - Z_\ell^{(p)}) \middle| X_\ell, X_{\ell+1} \right]}{\mathbb{E}\left[e^{i\langle t, Z_{\ell+1} - Z_\ell\rangle} \middle| X_\ell, X_{\ell+1} \right]}.$$

Finally, the partial derivative with respect to the p^{th} and q^{th} coordinates, $p, q \in \{1, \ldots, d\}$,

writes

$$\partial^2_{p,q}\pi_\ell(t) = \mathbb{E}\left[e^{i\langle t, Z_{\ell+1}-Z_\ell\rangle}\middle| X_\ell, X_{\ell+1}\right]^{-2}$$
$$\left\{ -\mathbb{E}\left[e^{i\langle t, Z_{\ell+1}-Z_\ell\rangle}(Z^{(p)}_{\ell+1}-Z^{(p)}_\ell)(Z^{(q)}_{\ell+1}-Z^{(q)}_\ell)\middle| X_\ell, X_{\ell+1}\right]\mathbb{E}\left[e^{i\langle t, Z_{\ell+1}-Z_\ell\rangle}\middle| X_\ell, X_{\ell+1}\right]\right.$$
$$\left. +\mathbb{E}\left[e^{i\langle t, Z_{\ell+1}-Z_\ell\rangle}(Z^{(p)}_{\ell+1}-Z^{(p)}_\ell)\middle| X_\ell, X_{\ell+1}\right]\mathbb{E}\left[e^{i\langle t, Z_{\ell+1}-Z_\ell\rangle}(Z^{(q)}_{\ell+1}-Z^{(q)}_\ell)\middle| X_\ell, X_{\ell+1}\right]\right\}.$$

Hence, the Taylor expansion at $t=0$ yields for all $t\in\delta\mathbb{T}^d$

$$\pi_\ell(t) = i\langle t, \mathbb{E}[Z_{\ell+1}-Z_\ell|X_\ell, X_{\ell+1}]\rangle - \frac{1}{2}\langle t, \mathbf{Cov}(Z_{\ell+1}-Z_\ell|X_\ell, X_{\ell+1})t\rangle + R_\ell(t)$$

where R_ℓ stands for the Lagrange's reminder in the Taylor expansion. Then, for any $t\in\delta\mathbb{T}^d$, almost surely,

$$|R_\ell(t)| \le \|t\|^3_\infty \sup_{t\in\delta\mathbb{T}^d}\sup\left\{|\partial^3_{p,q,r}\pi_\ell(t)|,\ p,q,r\in\{1,\dots,d\}\right\}.$$

In fact, Equation (10) implies that

$$\forall\delta\in(0,1/2k)\cap(0,1),\quad \forall t\in\delta\mathbb{T}^d,\quad \left|\mathbb{E}\left[e^{i\langle t, Z_{\ell+1}-Z_\ell\rangle}\middle| X_\ell, X_{\ell+1}\right]\right| \ge \frac{1}{2}.$$

Thus, computing explicitly the third order derivatives of π_ℓ, it follows, for any $p,q,r\in\{1,\dots,d\}$ and any $t\in\delta\mathbb{T}^d$, that

$$\begin{aligned}|\partial^3_{p,q,r}\pi_\ell(t)| &\le 2^4\left\{\mathbb{E}\left[\|Z_{\ell+1}-Z_\ell\|_1^3\middle| X_\ell, X_{\ell+1}\right] + 2\mathbb{E}\left[\|Z_{\ell+1}-Z_\ell\|_1\middle| X_\ell, X_{\ell+1}\right]^3\right.\\ &\qquad\left. + 5\mathbb{E}\left[\|Z_{\ell+1}-Z_\ell\|_1^2\middle| X_\ell, X_{\ell+1}\right]\mathbb{E}\left[\|Z_{\ell+1}-Z_\ell\|_1\middle| X_\ell, X_{\ell+1}\right]\right\}\\ &\le 2^7\mathbb{E}\left[\|Z_{\ell+1}-Z_\ell\|_1^3\middle| X_\ell, X_{\ell+1}\right].\end{aligned}$$

The existence of the deterministic constant K follows immediately from Assumption 2. Now,

$$\Pi_n(t) = \prod_{\ell=0}^{n-1}\pi_\ell(t) = \exp\left\{i\langle t, U_n\rangle - \frac{1}{2}\langle t, nV_nt\rangle + \Theta_n(t)\right\},$$

where

$$\Theta_n(t) = \sum_{\ell=0}^{n-1} R_\ell(t) \quad\text{and}\quad |\Theta_n(t)| \le nK\|t\|^3_\infty.$$

□

Proof of Theorem 2.1. Let $\delta\in(0,1)$ be such that Proposition 3.1 holds. Then, by Proposition 1.2,

$$\begin{aligned}(2\pi)^d\mathbb{P}(Z_n - Z_0 = z|X_0=x) &= \int_{\mathbb{T}^d} P^n_t\mathbf{1}(x)e^{-i\langle t,z\rangle}\,\mathrm{d}t\\ &= \int_{\mathbb{T}^d\setminus\delta\mathbb{T}^d} P^n_t\mathbf{1}(x)e^{-i\langle t,z\rangle}\,\mathrm{d}t + \int_{\delta\mathbb{T}^d} P^n_t\mathbf{1}(x)e^{-i\langle t,z\rangle}\,\mathrm{d}t.\end{aligned} \tag{11}$$

The change of variables $u = t/\sqrt{n}$ in the second term yields

$$\begin{aligned}\int_{\delta\mathbb{T}^d} P_t^n \mathbf{1}(x) e^{-i\langle t,z\rangle}\, \mathrm{d}t &= \mathbb{E}\left[\int_{\delta\mathbb{T}^d} e^{-i\langle t,z\rangle}\Pi_n(t)\mathrm{d}t \middle| X_0 = x\right] \\ &= n^{-d/2}\mathbb{E}\left[\int_{\delta\sqrt{n}\mathbb{T}^d} e^{-\frac{i}{\sqrt{n}}\langle t,z\rangle}\Pi_n(t/\sqrt{n})\mathrm{d}t \middle| X_0 = x\right].\end{aligned} \tag{12}$$

Let $a \in (0, \delta\sqrt{n})$. The quantity in Equation 12 can be decomposed as follows:

$$\begin{aligned}&n^{-d/2}\mathbb{E}\left[\int_{\mathbb{R}^d} \exp\left\{i\frac{1}{\sqrt{n}}\langle t, U_n - z\rangle - \frac{1}{2}\langle t, V_n t\rangle\right\}\mathrm{d}t \middle| X_0 = x\right] \\ &\qquad + n^{-d/2}\mathbb{E}\left[\int_{\delta\sqrt{n}\mathbb{T}^d\setminus a\mathbb{T}^d} e^{-\frac{i}{\sqrt{n}}\langle t,z\rangle}\Pi_n(t/\sqrt{n})\mathrm{d}t \middle| X_0 = x\right] \\ &+ n^{-d/2}\mathbb{E}\left[\int_{a\mathbb{T}^d} e^{-i\frac{1}{\sqrt{n}}\langle t,z\rangle}\left(\Pi_n(t/\sqrt{n}) - \exp\left\{i\frac{1}{\sqrt{n}}\langle t, U_n - z\rangle - \frac{1}{2}\langle t, V_n t\rangle\right\}\right)\mathrm{d}t \middle| X_0 = x\right] \\ &\qquad - n^{-d/2}\mathbb{E}\left[\int_{\mathbb{R}^d\setminus a\mathbb{T}^d} \exp\left\{i\frac{1}{\sqrt{n}}\langle t, U_n - z\rangle - \frac{1}{2}\langle t, V_n t\rangle\right\}\mathrm{d}t \middle| X_0 = x\right]\end{aligned}$$

Finally, remarking that $\|t/\sqrt{n}\|_\infty \le \delta$ as soon as $t \in a\mathbb{T}^d$ so that Proposition 3.1 holds, Equations (11) and (12) imply

$$\begin{aligned}&(2\pi\sqrt{n})^d\mathbb{P}(Z_n - Z_0 = z | X_0 = x) - \mathbb{E}\left[\int_{\mathbb{R}^d} \exp\left\{i\frac{1}{\sqrt{n}}\langle t, U_n - z\rangle - \frac{1}{2}\langle t, V_n t\rangle\right\}\mathrm{d}t \middle| X_0 = x\right] \\ &\qquad = A_1(z,n,a) + A_2(z,n,a) + A_3(z,n,a,\delta) + A_4(z,n,\delta)\end{aligned}$$

where

$$\begin{aligned}A_1(z,n,a) &:= \mathbb{E}\left[\int_{a\mathbb{T}^d} \exp\left\{i\frac{1}{\sqrt{n}}\langle t, U_n - z\rangle - \frac{1}{2}\langle t, V_n t\rangle\right\}\left(e^{\Theta_n(t/\sqrt{n})} - 1\right)\mathrm{d}t \middle| X_0 = x\right], \\ A_2(z,n,a) &:= -\mathbb{E}\left[\int_{\mathbb{R}^d\setminus a\mathbb{T}^d} \exp\left\{i\frac{1}{\sqrt{n}}\langle t, U_n - z\rangle - \frac{1}{2}\langle t, V_n t\rangle\right\}\mathrm{d}t \middle| X_0 = x\right], \\ A_3(z,n,a,\delta) &:= \mathbb{E}\left[\int_{\delta\sqrt{n}\mathbb{T}^d\setminus a\mathbb{T}^d} e^{-\frac{i}{\sqrt{n}}\langle t,z\rangle}\Pi_n(t/\sqrt{n})\mathrm{d}t \middle| X_0 = x\right], \\ A_4(z,n,\delta) &:= n^{d/2}\int_{\mathbb{T}^d\setminus\delta\mathbb{T}^d} P_t^n\mathbf{1}(x)e^{-i\langle t,z\rangle}\,\mathrm{d}t.\end{aligned}$$

Lemma 3.2. *Under Assumptions 2 and 3, it holds that*

$$\forall n \ge 1, \quad \forall a \in \left(0, \delta\sqrt{n}\right), \quad \sup_{z\in\mathbb{Z}^d} |A_1(z,n,a)| = O\left(\frac{a^3K}{\sqrt{n}}\exp\left(\frac{a^3K}{\sqrt{n}}\right)\right)$$

where K is defined in Proposition 3.1.

Proof. Let $t \in a\mathbb{T}^d$. By Proposition 3.1 under Assumptions 2 and 3,

$$\begin{aligned}\left|e^{i\frac{1}{\sqrt{n}}\langle t, U_n - z\rangle - \frac{1}{2}\langle t, V_n t\rangle}\left(e^{\Theta_n(t/\sqrt{n})} - 1\right)\right| &\le e^{-\frac{1}{2}\langle t, V_n t\rangle}\left|\Theta_n(t/\sqrt{n})\right|\exp\left|\Theta_n(t/\sqrt{n})\right| \\ &\le e^{-\frac{1}{2}\langle t, V_n t\rangle}\frac{a^3nK}{n^{3/2}}\exp\frac{a^3nK}{n^{3/2}} \\ &\le e^{-\frac{\alpha}{2}\|t\|_2^2}\frac{a^3K}{\sqrt{n}}\exp\frac{a^3K}{\sqrt{n}} \quad \text{a.s.}.\end{aligned}$$

The result follows by integrating with respect to $t \in a\mathbb{T}^d$. □

Lemma 3.3. *Under Assumption 3,*

$$\forall n \geq 1, \quad \forall a > 0, \quad \sup_{z \in \mathbb{Z}^d} |A_2(z, n, a)| \leq \frac{2^d e^{-\alpha d a^2}}{(\alpha d a)^d}.$$

Proof. Assumption 3 implies

$$\sup_{z \in \mathbb{Z}^d} |A_2(z, n, a)| \leq \int_{\mathbb{R}^d \setminus a\mathbb{T}^d} \exp(-\frac{1}{2}\alpha \|t\|_2^2) \, \mathrm{d}t \leq \int_{\|t\|_2 \geq a} \exp(-\frac{1}{2}\alpha \|t\|_2^2) \, \mathrm{d}t \leq \frac{2^d e^{-\alpha d a^2}}{(\alpha a)^d}.$$

□

Lemma 3.4. *Under Assumptions 2 and 3, for any $\delta > 0$ sufficiently small and any $a \in (0, \delta\sqrt{n})$*

$$\sup_{z \in \mathbb{Z}^d} |A_3(z, n, a, \delta)| \leq K_3 \exp\left\{-K_4 a^2\right\}.$$

Proof. As a matter of fact,

$$\left| \mathbb{E}\left[\int_{\delta\sqrt{n}\mathbb{T}^d \setminus a\mathbb{T}^d} e^{-\frac{i}{\sqrt{n}}\langle t, z\rangle} \Pi_n(t/\sqrt{n}) \mathrm{d}t \middle| X_0 = x \right] \right| \leq \mathbb{E}\left[\int_{\delta\sqrt{n}\mathbb{T}^d \setminus a\mathbb{T}^d} \left|\Pi_n(t/\sqrt{n})\right| \mathrm{d}t \middle| X_0 = x \right]. \quad (13)$$

Under Assumptions 2 and 3, by Proposition 3.1, the integrand in Equation (13) satisfies

$$\left|\Pi_n(t/\sqrt{n})\right| \leq \exp\left\{ -\frac{1}{2}\langle t, V_n t\rangle + \left|\Theta_n(t/\sqrt{n})\right| \right\}.$$

Moreover, since $t \in \delta\sqrt{n}\mathbb{T}^d \setminus a\mathbb{T}^d$

$$|\Theta_n(t/\sqrt{n})| \leq nK\|t/\sqrt{n}\|_\infty^2 \|t/\sqrt{n}\|_\infty = K\|t\|_\infty^2 \|t/\sqrt{n}\|_\infty \leq K\delta\pi\|t\|_\infty^2 \leq K\delta\pi d\|t\|_2^2,$$

and $-\frac{1}{2}\langle t, V_n t\rangle \leq -\frac{1}{2}\alpha\|t\|_2^2$. Thus, choosing any $\delta > 0$ so small that $K\delta\pi d \leq \alpha/4$ (and that Proposition 3.1 holds true), the integrand in Equation (13) is bounded above by $\exp(-\beta\|t\|_2^2)$ for some $\beta > 0$. Consequently, there exists $K_3, K_4 > 0$ such that $|A_3(z, n, a, \delta)| \leq K_3 e^{-K_4 a^2}$. □

Lemma 3.5. *Under Assumptions 4, for any $\delta \in (0, 1)$, there exist $\gamma \in (0, 1)$ and a constant $C > 0$ such that*

$$\sup_{z \in \mathbb{Z}^d} |A_4(z, n, \delta)| \leq (2\pi)^d C\gamma^n.$$

Proof. Fix $\delta \in (0, 1)$. Then, under Assumption 4, there exists $\gamma \in (0, 1)$ and a constant $C > 0$ such that, for all $t \in \mathbb{T}^d \setminus \delta\mathbb{T}^d$, $\|P_t^n \mathbf{1}\|_\infty \leq C\gamma^n$. The result follows immediately. □

The four lemmas above imply that

$$\begin{aligned}\sup_{z \in \mathbb{Z}^d} \left| n^{d/2}(2\pi)^d \mathbb{P}(Z_n - Z_0 = z | X_0 = x) - \mathbb{E}\left[\int_{\mathbb{R}^d} e^{i\frac{1}{\sqrt{n}}\langle t, U_n - z\rangle - \frac{1}{2}\langle t, V_n t\rangle} \mathrm{d}t \middle| X_0 = x \right] \right| \\ = O\left(\frac{a^3 K}{\sqrt{n}} e^{\frac{a^3 K}{\sqrt{n}}} + \frac{2^d}{(\alpha a)^d} e^{-\alpha d a^2/2} + K_3 e^{-K_4 a^2} + C\gamma^n \right) = O(n^{-(\frac{1}{2} - \varepsilon)}),\end{aligned}$$

by setting $a = \delta n^{\varepsilon/3}$ for any $\varepsilon \in (0, \frac{1}{2})$.

Finally, noting that V_n is invertible by Assumption 3, the proof of Theorem 2.1 ends with the help of the Fourier transform

$$\int_{\mathbb{R}^d} \exp\left\{ i\frac{1}{\sqrt{n}}\langle t, U_n - z\rangle - \frac{1}{2}\langle t, V_n t\rangle \right\} \mathrm{d}t = \frac{(2\pi)^{d/2}}{\sqrt{\det(V_n)}} \exp\left\{ -\frac{1}{2n}\langle U_n - z, V_n^{-1}(U_n - z)\rangle \right\}.$$

□

4 Application: random walks with local drift

This section is dedicated to the illustration of Theorems 2.1 and its Corollary 2.4. The examples presented below are largely inspired by those studied in [12], which in turn were previously introduced in [48]. The discrete time processes considered in [12] are basically simple random walks on directed graphs built upon $\mathbb{Z}^2$. More precisely, whereas the random walker can move freely toward North or South at each step, he can only move toward East or West depending on a prescribed environment. As shown in [12], different environments lead to different behaviors for the simple random walk explaining that a significant part of the literature considers these models (see for instance [32, 53, 14, 23, 40, 13, 8, 9, 10] or even more recently [7]).

In the model considered below, the random walker can move simultaneously vertically and horizontally in one of the quarter-plane North West, North East, South East or South West. Whereas the choice between North or South remains unrestricted, the choice between East and West is dictated by a prescribed environment. Thus, the case of a periodic environment considered in Section 4.2 has to be compared to the model $\mathbb{L}$ of [12]. The two-directed half planes of Section 4.3 is expected to be similar to the model $\mathbb{H}$ of [12]. Finally, the randomly directed case of Section 4.4 is reminiscent of the model $\mathbb{O}$.

4.1 The model of random walks with local drift

In this section the internal Markov chain X of the MAP (X, Z) shall be the simple random walk on $\mathbb{X} = \mathbb{Z}$. More precisely, let $(\xi_k)_{k\geq 1}$ be a sequence of *i.i.d.* random variables where ξ_1 is uniformly distributed on $\{-1, 1\}$. Let X_0 be a $\mathbb{Z}$-valued random variable and, for $n \geq 1$, $X_n := X_0 + \xi_1 + \cdots + \xi_n$.

Now, let us define the additive part Z of the MAP (X, Z). To this end, introduce a sequence $((T_{h,k}, T_{v,k}))_{k\geq 0}$ of independent copies of some $\mathbb{N}^2$-valued random vector (T_h, T_v) and a sequence $\varepsilon = (\varepsilon_x)_{x\in\mathbb{Z}}$ of $\mathbb{Z}$-valued random variables. Then, the additive part Z taking values in $\mathbb{Z}^2$ is defined as follows:

$$\forall n \geq 0, \quad Z_n - Z_0 := \left(\sum_{k=0}^{n-1} \varepsilon_{X_k} T_{h,k}, \sum_{k=0}^{n-1} \xi_{k+1} T_{v,k} \right). \tag{14}$$

Assumptions 5.
1. *The internal Markov chain X, the sequences $((T_{h,k}, T_{v,k}))_{k\geq 0}$ and ε are independent,*
2. *$\mathbb{E}[T_h] = m_h \in (0, \infty)$ and $\mathbb{E}[T_v] = m_v \in (0, \infty)$,*
3. *$\mathbb{V}[T_h] = \sigma_v^2 > 0$ and $\mathbb{V}[T_v] = \sigma_h^2 > 0$,*
4. *T_h and T_v are independent.*

Proposition 4.1. *Under Assumptions 5, conditionally on the environment ε, the quenched drift and quenched dispersion are respectively given by*

$$U_n(\varepsilon) = \begin{pmatrix} m_h \sum_{k=0}^{n-1} \varepsilon_{X_k} \\ m_v(X_n - X_0) \end{pmatrix} \quad \text{and} \quad V_n(\varepsilon) = \frac{1}{n} \begin{pmatrix} \sigma_h^2 \sum_{k=0}^{n-1} \varepsilon_{X_k}^2 & 0 \\ 0 & n\sigma_v^2 \end{pmatrix}.$$

Proof. Conditionally on the environment ε, the quenched drift and quench dispersion are respectively given by

$$U_n(\varepsilon) = \sum_{\ell=0}^{n-1} \mathbb{E}[Z_{\ell+1} - Z_\ell | X_\ell, X_{\ell+1}, \varepsilon] \quad \text{and} \quad V_n(\varepsilon) = \frac{1}{n} \sum_{\ell=0}^{n-1} \mathbf{Cov}(Z_{\ell+1} - Z_\ell | X_\ell, X_{\ell+1}, \varepsilon).$$

By independence, for all $\ell \geq 0$,

$$\mathbb{E}[Z_{\ell+1} - Z_\ell | X_\ell, X_{\ell+1}, \varepsilon] = \begin{pmatrix} \varepsilon_{X_\ell} \mathbb{E}[T_h] \\ \xi_{\ell+1} \mathbb{E}[T_v] \end{pmatrix}$$

$$\text{and} \quad \mathbf{Cov}(Z_{\ell+1} - Z_\ell | X_\ell, X_{\ell+1}, \varepsilon) = \frac{1}{n} \begin{pmatrix} \mathbb{V}(T_h)\varepsilon_{X_\ell}^2 & 0 \\ 0 & \mathbb{V}(T_v)\xi_{\ell+1}^2 \end{pmatrix}.$$

The result follows immediately. □

Proposition 4.2. *Under Assumptions 5, conditionally on the environment ε, the Fourier transform operator is given by*

$$\forall x \in \mathbb{Z}, \quad \forall t \in \mathbb{T}^2, \quad P_t f(x) = \sum_{y \in \mathbb{Z}} P(x,y) \widehat{\mu_\varepsilon^{x,y}}(t) f(y),$$

where P is the Markov operator associated with the simple random walk X and

$$\widehat{\mu_\varepsilon^{x,y}}(t) = \varphi_{T_h}(\varepsilon_x t_1) \varphi_{T_v}((y-x)t_2), \quad x, y \in \mathbb{Z} \text{ s.t. } P(x,y) > 0, \quad t = (t_1, t_2) \in \mathbb{T}^2,$$

where φ_{T_h} and φ_{T_h} are the characteristic functions of T_h and T_v respectively.

Proof. Let $t = (t_1, t_2) \in \mathbb{T}^2$. Then,

$$\begin{aligned} P_t f(x) &= \mathbb{E}[e^{it_1 \varepsilon_{X_0} T_{h,0}} e^{it_2 (X_1 - X_0) T_{v,0}} f(X_1) | X_0 = x, \varepsilon] \\ &= \sum_{y \in \mathbb{Z}} P(x,y) \mathbb{E}[e^{it_1 \varepsilon_{X_0} T_{h,0}} e^{it_2 (X_1 - X_0) T_{v,0}} | X_0 = x, X_1 = y, \varepsilon] f(y) \\ &= \sum_{y \in \mathbb{Z}} P(x,y) \varphi_{T_h}(\varepsilon_x t_1) \varphi_{T_h}((y-x)t_2) f(y). \end{aligned}$$

□

In order to apply Theorem 2.1 and Corollary 2.4, it is necessary to make further assumptions.

Assumptions 6.

(UB) *There exists $B > 0$ such that $\mathbb{P}[\sup_{x \in \mathbb{Z}} |\varepsilon_x| \geq B] = 0$;*

(ND) *For all $x \in \mathbb{Z}$, $\mathbb{P}[\varepsilon_x = 0] = 0$;*

(M) *$\mathbb{E}[T_h^3], \mathbb{E}[T_v^3]$ are finite;*

(AP) *the distribution of T_h and T_v are aperiodic.*

Proposition 4.3. *Under Assumptions 5 and 6, conditionally on the environment ε, Assumptions 1, 2, 3 and 4 are fulfilled.*

Proof. It is well known that the simple random walk X is irreducible recurrent (Assumption 1).

The uniform moment condition involved in Assumption 2 is equivalent to

$$\sup_{x \in \mathbb{Z}} \left(|\varepsilon_x| \mathbb{E}[T_h^3] + \mathbb{E}[T_v^3] \right) < \infty.$$

The latter is straightforward by (UB) and (M) of Assumption 6.

The uniform ellipticity condition of Assumption 3 follows immediately of (ND) of Assumption 6.

Finally, since T_h and T_v are aperiodic, for all $t \in [-\pi, \pi) \setminus \{0\}$, $|\varphi_{T_h}(t)| < 1$ and $|\varphi_{T_v}(t)| < 1$. Consequently, conditionally on the environment ε, the family of probability measure $(\mu_\varepsilon^{x,y})_{x,y \in \mathbb{Z}}$ is uniformly aperiodic. By Proposition 1.6, Assumption 4 follows. □

As a matter of fact, the quantity $\frac{1}{2n}\langle (U_n(\varepsilon)-z), V_n^{-1}(U_n-z)\rangle$, $n\geq 1$, $z=(z_1,z_2)\in\mathbb{Z}^2$, appearing in Theorem 2.1 is given by

$$\frac{1}{2n}\langle (U_n(\varepsilon)-z), V_n^{-1}(U_n-z)\rangle = \frac{1}{2}\left[\frac{\left(m_h\sum_{k=0}^{n-1}\varepsilon_{X_k}-z_1\right)^2}{\sigma_h^2\sum_{k=0}^{n-1}\varepsilon_{X_k}^2}+\frac{(m_v(X_n-X_0)-z_2)^2}{n\sigma_v^2}\right]. \quad (15)$$

Also, it is worth noting that, under Assumptions (UB) and (ND) of Assumption 6,

$$\forall n\geq 1,\quad \frac{\det(V_n)}{\sigma_h^2\sigma_v^2}=\frac{1}{n}\sum_{k=0}^{n-1}\varepsilon_{X_k}^2\in[1,B^2]. \quad (16)$$

Consequently, this factor will be omitted in the sequel when applying Corollary 2.4.

The end of this section is devoted to the application of Theorem 2.1 or Corollary 2.4 for three different environments ε. The first two are deterministic whereas the last one is random.

4.2 Periodic environment

In this example, the environment ε is supposed periodic. Namely, it is assumed that, for all $x\in\mathbb{Z}$, $\varepsilon_x=(-1)^x$.

Proposition 4.4. *Let $z\in\mathbb{Z}^2$. Then, under Assumptions 5 and 6, for all $x\in\mathbb{Z}$,*

$$\mathbb{P}[Z_n-Z_0=z|X_0=x]\sim_{n\to\infty}\frac{1}{2\pi\sigma_h\sigma_v n}\left(1+\frac{m_v^2}{\sigma_v^2}\right)^{-1/2}.$$

Proof. First observe that

$$\forall n\geq 1,\quad \sum_{k=0}^{n-1}\varepsilon_{X_k}=(-1)^{X_0}\frac{1-(-1)^n}{2}\quad\text{and}\quad \det(V_n)=\sigma_h^2\sigma_v^2.$$

Then, setting $z=(z_1,z_2)$,

$$\begin{aligned}&\sigma_h\sigma_v\mathbb{E}\left[\frac{1}{\sqrt{\det(V_n)}}\exp\left\{-\frac{1}{2n}\langle U_n-z,V_n^{-1}(U_n-z)\rangle\right\}\middle|X_0=x\right]\\&=\exp\left\{-\frac{1}{2\sigma_h^2 n}\left(m_h(-1)^x\frac{1-(-1)^n}{2}-z_1\right)^2\right\}\mathbb{E}\left[\exp\left\{-\frac{(m_v(X_n-X_0)-z_2)^2}{2n\sigma_v^2}\right\}\middle|X_0=x\right].\end{aligned}$$

As a matter of fact, the first exponential factor converges toward 1 as $n\to\infty$.

For the second factor, it suffices to remark that $n^{-1/2}[m_v(X_n-X_0)-z_2]$ converges in distribution to a centered standard Gaussian random variable. Since $t\to e^{-t^2/2\sigma_v^2}$ is continuous, it follows

$$\lim_{n\to\infty}E\left[\exp\left\{-\frac{(m_v(X_n-X_0)-z_2)^2}{2n\sigma_v^2}\right\}\middle|X_0=x\right]=\int_{\mathbb{R}}e^{-\frac{y^2}{2}\left[1+\frac{m_v^2}{\sigma_v^2}\right]}\frac{\mathrm{d}y}{\sqrt{2\pi}}=\left(1+\frac{m_v^2}{\sigma_v^2}\right)^{-1/2}.$$

By Theorem 2.1, the result of the proposition follows immediately. □

The estimate of Proposition 4.4 implies that the series in Corollary 2.4 if infinite. However, since the internal Markov chain is null recurrent, the recurrence of the MAP can not be deduced directly. Actually, it turns out that, taking advantage of the periodicity of the environment, the distribution of the additive component can be described with a simpler internal Markov chain.

Proposition 4.5. *Suppose the MAP (X, Z) is irreducible. Then, under Assumptions 5 and 6, the MAP Z is recurrent.*

Proof. The key idea consists in remarking that $M = ((\varepsilon_{X_n}, \xi_{n+1}))_{n\geq 0}$ is a Markov chain taking values in the finite space $\{-1, 1\}^2$. The transition probabilities are given, for all $x, x', y, y' \in \{-1, 1\}$, as follows:

$$Q((x, y), (x', y')) := Q_1(x, x')Q_2(y, y') \quad \text{with} \quad Q_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad Q_2 := \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

Now, let us set for all $n \geq 1$,

$$\widetilde{Z}_n - \widetilde{Z}_0 = \left(\sum_{k=0}^{n-1} M_k^{(1)} T_{h,k}, \sum_{k=0}^{n-1} M_k^{(2)} T_{v,k} \right),$$

where $M^{(1)}$ and $M^{(2)}$ are respectively the first and second component of M. From Equation (14), the following equality in distribution holds

$$\mathcal{L}((Z_n - Z_0)_{n\geq 0} | X_0 = x) = \mathcal{L}((\widetilde{Z}_n - \widetilde{Z}_0)_{n\geq 0} | M_0^{(1)} = \varepsilon_x).$$

From this equality in distribution, the MAP $(M, \widetilde{Z})$ inherits irreducibility from the MAP (M, Z). Moreover, applying [18], the MAP $(M, \widetilde{Z})$ is recurrent noting that M is positive recurrent. Consequently, (X, Z) is recurrent. □

4.3 Two directed half planes

The reduction made in the proof of Proposition 4.5 is naturally not always possible as illustrated by the example of this section. This obstruction was the main motivation of the extension to the null recurrent case.

The two directed half planes presented below corresponds to the environment ε defined, for all $x \in \mathbb{Z}$, by $\varepsilon_x = 2\mathbf{1}_{\mathbb{Z}_+}(x) - 1$ where $\mathbb{Z}_+$ stands for the set of non negative integers.

Proposition 4.6. *Assume that the MAP (X, Z) is irreducible. Then, under Assumptions 5 and 6, (X, Z) is transient.*

Proof. By irreducibility, Equation (15) with $z_1 = z_2 = 0$ and the fact that, for all $n \geq 1$, $\det(V_n) = \sigma_h^2 \sigma_v^2$, the transience of (X, Z) follows Corollary 2.4 by proving the series

$$\sum_{n\geq 1} \frac{1}{n} \mathbb{E}\left[\exp\left\{ -\frac{m_h^2}{2n\sigma_h^2} \left(\sum_{k=0}^{n-1} \varepsilon_{X_k} \right)^2 \right\} \middle| X_0 = x \right]. \tag{17}$$

is finite for all $x \in \mathbb{Z}$. Thereafter, a simple computation yields, for all $n \geq 1$, that

$$\sum_{k=0}^{n-1} \varepsilon_{X_k} = 2N_n(\mathbb{Z}_+) - n \quad \text{where} \quad N_n(\mathbb{Z}_+) = \sum_{k=0}^{n-1} \mathbf{1}_{\mathbb{Z}_+}(X_k).$$

Then, verifying that the function

$$[0, \infty) \ni t \to \exp\left(-\frac{nm_h^2}{2\sigma_h^2} (2t - 1)^2 \right)$$

is k_n-Lipschitz with constant $k_n = e^{-\frac{1}{8}}\sqrt{\frac{nm_h^2}{\sigma_h^2}}$, it follows that

$$\mathbb{E}\left[\exp\left\{-\frac{m_h^2}{2n\sigma_h^2}\left(2N_n(\mathbb{Z}_+)-n\right)^2\right\}\middle| X_0=x\right] \leq k_n d_{\mathcal{W}}(\mathcal{L}(N_n(\mathbb{Z}_+)/n),\mathcal{L}(\Gamma)) + \mathbb{E}\left[\exp\left\{-n\frac{m_h^2}{2\sigma_h^2}\left(2\Gamma-1\right)^2\right\}\middle| X_0=x\right] \quad (18)$$

where Γ is distributed as an arc-sine law supported by $[0,1]$ and $d_{\mathcal{W}}$ stands for the Kantorovich-Rubinstein metric on probability measures (see [62, Chapter 6, p.94]).

It is shown in [28, Theorem 1.2] that $d_{\mathcal{W}}(\mathcal{L}(N_n(\mathbb{Z}_+)/n),\mathcal{L}(\Gamma)) = O(1/n)$ so that the series in Equation (17) is finite if and only if, applying Fubini's Theorem,

$$\infty > \mathbb{E}\left[\sum_{n\geq 1}\frac{1}{n}\exp\left\{-n\frac{m_h^2}{2\sigma_h^2}(2\Gamma-1)^2\right\}\right] = -\int_0^1 \log\left[\exp\left\{\frac{m_h^2}{2\sigma_h^2}(2y-1)^2\right\}-1\right]\frac{\mathrm{d}y}{\pi\sqrt{1-y^2}}.$$

However, the singularity at $y=1$ coming from the density function of the arc-sine distribution is integrable and so is the singularity at $y=\frac{1}{2}$ since $y\to\log y$ is locally integrable in the positive neighborhood of 0. □

4.4 Randomly directed random walks

The last family of examples involve a random environment. More precisely, it will be assumed that ε is a sequence of *i.i.d.* random variables. In the sequel, the common marginal distribution will be denoted by π.

Proposition 4.7. *Additionally to Assumptions 5 and 6, suppose that $\varepsilon=(\varepsilon_x)_{x\in\mathbb{Z}}$ is a sequence of* i.i.d. *random variables such that $\mathbb{E}[\varepsilon_0]=0$. Then, for $\pi^{\otimes\mathbb{Z}}$-a.e. sequences ε, if the MAP (X,Z) in the environment ε is irreducible, then it is transient.*

Proof. Applying Markov inequality to the probability measure $\pi^{\otimes\mathbb{Z}}$, it is only needed to prove that

$$\sum_{n\geq 1}\frac{1}{n}\mathbb{E}\left[\exp\left\{-\frac{m_h^2\left(\sum_{k=0}^{n-1}\varepsilon_{X_k}\right)^2}{2\sigma_h^2\left(\sum_{k=0}^{n-1}\varepsilon_{X_k}^2\right)}\right\}\middle| X_0=x\right]. \quad (19)$$

Without loss of generality, one may suppose that $x=0$ since $\pi^{\otimes\mathbb{Z}}$ is shift invariant. Remarking that $n\leq\sum_{k=0}^{n-1}\varepsilon_{X_k}^2\leq nB^2$ by conditions (UB) and (ND) of Assumption 6, then, for any $\alpha\in(\frac{1}{2},\frac{3}{4})$,

$$\mathbb{E}\left[\exp\left\{-\frac{m_h^2\left(\sum_{k=0}^{n-1}\varepsilon_{X_k}\right)^2}{2\sigma_h^2\left(\sum_{k=0}^{n-1}\varepsilon_{X_k}^2\right)}\right\}\right] \leq \mathbb{P}\left[\left|\sum_{k=0}^{n-1}\varepsilon_{X_k}\right|\leq n^\alpha\right] + \exp\left\{e^{-\frac{m_h^2}{\sigma_h^2B^2}n^{2\alpha-1}}\right\}. \quad (20)$$

Now, set $D_n=\left\{x\in\mathbb{R}: n^{3/4}x\in[-n^\alpha,n^\alpha]\cap\mathbb{Z}\right\}$. By [14, Theorem 1], with the very same function C, the following estimate holds

$$\mathbb{P}\left[\left|\sum_{k=0}^{n-1}\varepsilon_{X_k}\right|\leq n^\alpha\right] = \sum_{x\in D_n}\left[\frac{C(x)}{n^{3/4}}+o(n^{-3/4})\right]$$

where the $o(n^{-3/4})$ is uniform in $x\in\mathbb{R}$. Noting that C is bounded (for instance [14, Lemma 4]), the probability above behaves like $O(n^{\alpha-3/4})$ which ends the proof of the proposition. □

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