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# Local Limit Theorem for Markov Additive Processes with Null Recurrent Internal Markov Chain

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**Abstract** In the classical framework, a random walk on a group is a Markov chain with independent and identically distributed increments. In some sense, random walks are time and space homogeneous. This paper is devoted to a class of inhomogeneous random walks on  $\mathbb{Z}^d$  termed Markov Additive processes (also known as Markov Random Walks, Random Walks with Internal Degree of Freedom or semi-Markov Processes). In this model, the increments of the walk are still independent but their distributions are dictated by a Markov chain termed the internal Markov chain. Whereas this model is largely studied in the literature, most of the results involves internal Markov chains whose operators are quasi-compact. This paper extends two results for more general internal operator: a Local Limit Theorem and a sufficient criterion for their transience. These results are thereafter applied to a new family of models of drifted random walks on the lattice  $\mathbb{Z}^d$ . Incidentally, in the examples under consideration, we give a new bound on the rate of convergence in the Wasserstein metric of appropriate scaling of Random Walks in Random Scenery.

**Key words** Inhomogeneous random walks . Markov Additive processes . Semi-Markov processes . Markov Random Walks . Random Walks with Internal Degrees of Freedom . Local Limit Theorem . Recurrence . Transience . Fourier analysis

**Mathematics Subject Classification (2000)** 60F15 . 60J05 . 60J10 . 60J15 . 60J45 . 60J55 . 60K15

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## Introduction

In the classical framework, a random walk on a group  $\mathbf{G}$  is a discrete time stochastic process  $(Z_n)_{n \geq 0}$  defined as the product of independent and identically distributed (*i.i.d.*) random variables  $(\xi_n)_{n \geq 1}$ . Random walks on groups are Markov chains that are adapted to the group structure in the sense that the underlying Markov operator is invariant under the group action of  $\mathbf{G}$  on itself. Thus, this homogeneity naturally gives rise to deep connections between stochastic properties of the random walk and algebraic properties of the group. Starting with the seminal paper of Pòlya, [47], a large part of the literature is devoted to the study of these connections in this homogeneous case (among them [42, 24, 31, 54, 51, 33, 32, 2] and references therein).

In this paper, we aim at investigating inhomogeneous random walks. It turns out that there are at least two ways to introduce inhomogeneity. First, we can consider *spatial* inhomogeneity by weakening the group structure, replacing it, for instance, by a directed graph as in [4, 22, 23, 45, 15]. Secondly, we can study *temporal* inhomogeneous random walks by introducing a notion of memory as in the model of reinforced [44, 53], excited [48, 3], self-interacting [13, 46], persistent random walks [7, 6, 9, 10, 8] or also Markov Additive Processes (abbreviated by MAP) that shall be at the core of this paper. All these models belong to the larger class of stochastic processes with long range dependency.

Markov Additive Processes are also known as Random Walk With Internal Degree of Freedom — see [38] — semi-Markov processes or Markov random walks — see for instance [37, 25, 43, 41, 1, 52, 35]. Roughly speaking, a Markov Additive Process are discrete time  $\mathbb{Z}^d$ -valued (or  $\mathbb{R}^d$ ) processes whose increments are still independent but no longer stationary. The distribution of an increment is then driven by a Markov chain termed the *internal* Markov chain. Most of results in the context of standard random walks are generalized to Markov Additive Processes when the Markov operator of the internal chain is assumed to be quasi-compact on a suitable Banach space. Among them, a renewal theorem [37, 1, 20, 21], local limit theorem [38, 25, 26, 29, 17, 27], central limit theorem [29, 17], results on the recurrence set [30], large deviations [43, 41], asymptotic expansion of the Green function [52, 35], one-dimensional Berry-Essen theorem [29, 17, 28] with applications to  $M$ -estimation, first passage time [18].

However, assuming the internal operator to be quasi-compact is rather strong (see [39]). Actually, beyond the technical difficulties inherent to the infinite dimension, there is no real difference in nature with the finite dimension under this assumption. On the other hand, relax this assumption and the study of Markov Additive processes can be very challenging. Besides, it is worth noting that many (interesting) Markov Additive processes do not admit a quasi-compact internal operator as it will be illustrated by the examples considered in this paper.

In the context of Markov additive processes, classical Fourier analysis can be extended by introducing the Fourier transform operator which is a perturbation of the internal Markov operator in an appropriate Banach space. As in the classical context, the Fourier transform operator characterizes the distribution of the additive part of Markov Additive processes. By a continuous perturbation argument (see for instance [34]), when the internal Markov operator is

quasi-compact, the Fourier transform operator remains quasi-compact for all sufficiently small perturbations. It allows, under suitable moment conditions on the distribution of increments, to derive a Taylor expansion at the second order of the perturbed dominating eigenvalue  $\lambda(t)$  (whose coefficients are given roughly speaking by the mean and the variance operators). Finally, under an assumption on the spectrum of the Fourier transform operator for large perturbations, it can be concluded that all the needed stochastic information is actually contained in the nature of the singularity at zero of  $(1 - \lambda(t))^{-1}$  (note that  $\lambda(0) = 1$ ). For instance, an integral test criterion, similar to the Chung-Fuchs criterion (see [12] or [50]), involving a singularity of this kind is given in [8].

In this paper, the quasi-compactness condition is dropped and the internal Markov chain is only assumed to be irreducible recurrent. The condition on the spectrum of Fourier transform operator for large perturbations remains similar but the nature of the singularity at the origin is analyzed considering the Taylor expansion of the *quenched characteristic exponent* (in a wide sense) defined in Section 1.2. The terms of order 1,  $U_n$ , and 2,  $V_n$ , also defined in Section 1.2, are termed the *quenched drift* and the *quenched dispersion* respectively. These quantities are characteristics of the increments of the process and naturally appears in the Local Limit in Theorem 2.2 and the transience condition in Theorem 2.5.

The paper is organized as follows. In Section 1 are gathered the main notions involved in the statement of Theorems 2.2 and 2.5. Section 2 is devoted to the statement of these two theorems. In Section 3, Theorems 2.2 and 2.5 are applied on various families of Markov Additive processes that extends the models considered in [40] and [4]. Those models are simple random walk on directed graphs built upon  $\mathbb{Z}^2$ . Various phenomena are observed whether the directions are fixed periodically or not. Incidentally, the result of [36] is slightly extended by giving an approximate rate of the convergence in the Wasserstein metric. As it shall be clear for the model periodically directed, it is possible to factorize a Markov Additive process without changing the distribution of the additive part (see Section 3). It will be concluded that the simple random walk on the periodically directed graph is a Markov Additive process with a finite internal Markov chain. As such, the internal Markov operator is a matrix and is quasi-compact. For more general directions (random directions for instance), such a reduction is no longer possible. Finally, in Section 4 are gathered the proofs of the two Theorems 2.2 and 2.5.

## 1 Markov additive processes

### 1.1 Definitions

Let  $(\Omega, \mathcal{F}, \mathbf{P})$  be a probability space and  $\mathbb{X}$  a countable set. The set  $\mathbb{X}$  is naturally endowed with the  $\sigma$ -algebra consisting of all subsets of  $\mathbb{X}$ .

**Definition 1.1** (Markov additive process). *Let  $d \geq 1$  be an integer. A Markov Additive Process (MAP for short) is a Markov chain  $((X_n, Z_n))_{n \geq 0}$  taking values in  $\mathbb{X} \times \mathbb{Z}^d$  defined on  $(\Omega, \mathcal{F}, \mathbf{P})$  satisfying for all  $n \geq 0$  and all bounded functions  $f : \mathbb{X} \times \mathbb{Z}^d \rightarrow \mathbb{R}$*

$$\mathbf{E}[f(X_{n+1}, Z_{n+1} - Z_n) | ((X_k, Z_k))_{0 \leq k \leq n}] = \mathbf{E}[f(X_{n+1}, Z_{n+1} - Z_n) | X_n]. \quad (1)$$

From equality (1), it follows immediately that  $(X_n)_{n \geq 0}$  is a Markov chain on  $\mathbb{X}$ . The latter is termed *the internal Markov chain*. The corresponding Markov kernel shall be denoted by  $P$ , namely  $Pf(x) := \mathbf{E}[f(X_1) | X_0 = x]$  for any bounded measurable function  $f : \mathbb{X} \rightarrow \mathbb{R}$ . Generally speaking, there exists a  $\sigma$ -finite measure  $m$  dominating the family of probabilities  $(P(x, \cdot))_{x \in \mathbb{X}}$ , i.e.  $m(y) = 0$  implies  $P(x, y) = 0$  for all  $x \in \mathbb{X}$ . If the internal Markov chain is irreducible and recurrent, the invariant measure (unique up to a positive constant) is a natural choice for  $m$ .

The conditional distribution of  $Z_{n+1} - Z_n$  given  $(X_n, X_{n+1}) = (x, y)$  will be denoted by  $\mu^{x,y}$  and the Fourier transform of  $\mu^{x,y}$  by  $\widehat{\mu^{x,y}}$ . Then, for any  $t \in \mathbb{R}^d$ , the Fourier transform operator  $P_t$  acting on the bounded function  $f : \mathbb{X} \rightarrow \mathbb{C}$  is defined as follows:

$$P_t f(x) := \mathbf{E}[f(X_1)e^{i\langle t, Z_1 - Z_0 \rangle} | X_0 = x] = \sum_{y \in \mathbb{X}} P(x, y) f(y) \widehat{\mu^{x,y}}(t).$$

From Markov property, it follows for all  $n \geq 1$

$$P_t^n f(x) = \mathbf{E}[f(X_n)e^{i\langle t, Z_n - Z_0 \rangle} | X_0 = x].$$

Moreover,

$$P_t^n \mathbf{1}(x) = \sum_{z \in \mathbb{Z}^d} e^{i\langle t, z \rangle} \mathbf{P}(Z_n - Z_0 = z | X_0 = x). \quad (2)$$

Consequently, the function  $\mathbb{R}^d \ni t \rightarrow P_t^n \mathbf{1}(x) \in \mathbb{C}$  is the Fourier transform of the conditional distribution of  $Z_n - Z_0$  given  $X_0 = x$ .

## 1.2 Conditional characteristic exponent, conditional drift and conditional dispersion

**Proposition 1.2.** *For any MAP, the following identity holds for all  $n \geq 0$ :*

$$\mathbf{P}(Z_n - Z_0 = z | X_0 = x) = \frac{1}{(2\pi)^d} \mathbf{E} \left[ \int_{\mathbb{T}_d} e^{-i\langle t, z \rangle} \Pi_n(t) dt \middle| X_0 = x \right], \quad (3)$$

where

$$\Pi_n(t) = \prod_{k=0}^{n-1} \mathbf{E} \left[ e^{i\langle t, Z_{k+1} - Z_k \rangle} \middle| X_k, X_{k+1} \right]. \quad (4)$$

The proof of Proposition 1.2 relies on a lemma from [16] whose statement is recalled below.

**Lemma 1.3.** *For all  $m \geq n \geq p \geq 0$ ,*

$$\begin{aligned} \mathbf{E} \left[ e^{i\langle t, Z_n - Z_p \rangle} \middle| \sigma(X_j, p \leq j \leq n) \right] & \mathbf{E} \left[ e^{i\langle t, Z_m - Z_n \rangle} \middle| \sigma(X_j, n \leq j \leq m) \right] \\ & = \mathbf{E} \left[ e^{i\langle t, Z_m - Z_p \rangle} \middle| \sigma(X_j, p \leq j \leq m) \right]. \end{aligned} \quad (5)$$

*Proof of Proposition 1.2.* By inverse Fourier transform,

$$(2\pi)^d \mathbf{P}(Z_n - Z_0 = z | X_0 = x) = \int_{\mathbb{T}_d} P_t^n \mathbf{1}(x) e^{-i\langle t, z \rangle} dt = \mathbf{E} \left[ \int_{\mathbb{T}_d} e^{i\langle t, Z_n - Z_0 - z \rangle} dt \middle| X_0 = x \right].$$

Then, by Lemma 1.3, setting  $\mathcal{G}_n = \sigma(X_\ell, 0 \leq \ell \leq n)$ ,  $n \geq 1$ ,

$$\begin{aligned} \mathbf{P}(Z_n - Z_0 = z | X_0 = x) & = \frac{1}{(2\pi)^d} \mathbf{E} \left[ \int_{\mathbb{T}_d} e^{-i\langle t, z \rangle} \mathbf{E} \left[ \prod_{k=0}^{n-1} e^{i\langle t, Z_{k+1} - Z_k \rangle} \middle| \mathcal{G}_n \right] dt \middle| X_0 = x \right] \\ & = \frac{1}{(2\pi)^d} \mathbf{E} \left[ \int_{\mathbb{T}_d} e^{-i\langle t, z \rangle} \prod_{k=0}^{n-1} \mathbf{E} \left[ e^{i\langle t, Z_{k+1} - Z_k \rangle} \middle| X_k, X_{k+1} \right] dt \middle| X_0 = x \right] \end{aligned}$$

□

As a matter of fact,  $\Pi_n$  is a continuous  $\mathbb{C}$ -valued function of  $\mathbb{T}_d$  satisfying  $\Pi_n(0) = 1$ . Hence, in a neighborhood of the origin, the logarithm of  $\Pi_n$  is well defined. We may call  $\log \Pi_n$  the *quenched characteristic exponent* (with a slight abuse of terminology since  $\log \Pi_n$  is not defined on  $\mathbb{R}^d$  in general). Its second order Taylor expansion at  $t = 0$  then exhibits two important quantities:

$$U_n = \sum_{\ell=0}^{n-1} \mathbf{E}[Z_{\ell+1} - Z_\ell | X_\ell, X_{\ell+1}] \quad \text{and} \quad V_n = \frac{1}{n} \sum_{\ell=0}^{n-1} \mathbf{Cov}(Z_{\ell+1} - Z_\ell | X_\ell, X_{\ell+1}).$$

The latter are called respectively the *quenched drift* and the *quenched dispersion* of the additive component.

### 1.3 Spectral condition

For any closed linear subspace  $E \subset \ell^\infty(\mathbb{X})$  and any bounded operator  $Q$  on  $E$ , denote by  $\|Q\|_E$  the subordinated norm restricted to  $E$ :  $\|Q\|_E = \sup\{\|Qf\|_\infty, f \in E : \|f\|_\infty = 1\}$ . Now, introduce the following quantity for  $t \in \mathbb{T}_d$ :

$$r(t) = \inf \left\{ \|P_t^n\|_E^{1/n}, n \geq 1, E \text{ closed subspace satisfying } P_t E \subset E \text{ and } \mathbf{1} \in E \right\}.$$

**Definition 1.4** (Spectral condition). *A MAP is said to satisfy the spectral condition if, for any compact  $K \subset \mathbb{T}_d$  such that  $0 \notin K$ ,  $\sup_{t \in K} r(t) < 1$ .*

**Example 1.5.** *In the literature, the invariant subspace  $E$  does not depend on the parameter  $t \in \mathbb{T}_d$  which may be sometimes a restriction as illustrated by the following example.*

*Let  $P$  be the Markov operator acting on  $\ell^\infty(\mathbb{Z})$  defined as follows*

$$\forall f \in \ell^\infty(\mathbb{Z}), \quad \forall x \in \mathbb{Z}, \quad Pf(x) = \frac{1}{3}[f(x-1) + f(x) + f(x+1)].$$

*Let  $(\varepsilon_x)_{x \in \mathbb{Z}}$  be a sequence taking values in  $\{-1, 1\}$ . For all  $x, y \in \mathbb{Z}$ , define the probability measure on  $\mathbb{Z}^2$  by*

$$\mu^{x,y} \begin{cases} \delta_{(0,1)} & \text{if } y = x + 1 \\ \delta_{-(0,1)} & \text{if } y = x - 1 \\ \delta_{(1,0)} & \text{if } y = x, \varepsilon_x = 1 \\ \delta_{-(1,0)} & \text{if } y = x, \varepsilon_x = -1 \end{cases}$$

*Now, let us prove that the associated second order MAP satisfies the spectral condition. For  $f \in \ell^\infty(\mathbb{Z})$ ,  $x \in \mathbb{Z}$  and  $t = (t_1, t_2) \in \mathbb{R}^2$ ,*

$$\begin{aligned} P_t^2 f(x) &= \left( \frac{2}{9} + \frac{1}{9} e^{2it_1 \varepsilon_x} \right) f(x) + \frac{1}{9} e^{2it_2} f(x+2) + \frac{1}{9} e^{-2it_2} f(x-2) \\ &\quad + \frac{1}{9} e^{it_2} f(x+1) (e^{it_1 \varepsilon_x} + e^{it_1 \varepsilon_{x+1}}) + \frac{1}{9} e^{-it_2} f(x-1) (e^{it_1 \varepsilon_x} + e^{it_1 \varepsilon_{x-1}}). \end{aligned}$$

*First, suppose that  $t_1 \neq 0$ , it follows that*

$$\|P_t^2\|_{\ell^\infty(\mathbb{Z})} \leq \frac{2}{3} + \sup_{x \in \mathbb{Z}} \left| \frac{2}{9} + \frac{1}{9} e^{2it_1 \varepsilon_x} \right| \leq \frac{2}{3} + \frac{1}{9} \sqrt{5 + 4 \cos(2t_1)} < 1.$$

*Secondly, if  $t_1 = 0$ , then the expression of  $P_t^2$  simplifies as follows*

$$P_t^2 f(x) = \frac{1}{3} f(x) + \frac{2}{9} e^{it_2} f(x+1) + \frac{2}{9} e^{-it_2} f(x-1) + \frac{1}{9} e^{2it_2} f(x+2) + \frac{1}{9} e^{-2it_2} f(x-2).$$

As a matter of facts, the space  $E$  of  $\mathbb{C}$ -valued constant functions is closed and invariant under  $P_t^2$ . Moreover

$$\|P_t^2\|_E \leq \left( \frac{1}{3} + \frac{4}{9} \cos(t_2) + \frac{2}{9} \cos(2t_2) \right) = \frac{1}{9} (2 \cos(t_2) + 1)^2.$$

Summarizing, as soon as  $t \in \mathbb{T}_d^*$ , the spectral radius  $r(t)$  is strictly smaller than 1 and the spectral condition is satisfied.

Since the spectral condition is rather technical, the end of this section is devoted to alternative and more tractable sufficient conditions.

**Proposition 1.6.** *The Fourier operator satisfies, for all  $t \in \mathbb{T}_d$ ,*

$$\|P_t\|_{\ell^\infty(\mathbb{X})} \leq \sup_{x,y \in \mathbb{X}} |\widehat{\mu^{x,y}}(t)|.$$

In particular, if the family  $(\mu^{x,y})_{x,y \in \mathbb{X}}$  of probability measures is uniformly aperiodic in the following sense

$$\forall t \in \mathbb{T}_d \setminus \{0\}, \quad \sup_{x,y \in \mathbb{X}} |\widehat{\mu^{x,y}}(t)| < 1$$

then the MAP fulfills the spectral condition.

*Proof.* For each  $t \in \mathbb{T}_d$ , the Fourier transform operator  $P_t$  acts as a linear contraction on the complex Banach space  $\ell^\infty(\mathbb{X})$ . In addition, for all  $x \in \mathbb{X}$  and all  $f \in \ell^\infty(\mathbb{X})$ ,

$$|P_t f(x)| = \left| \sum_{y \in \mathbb{X}} P(x,y) \widehat{\mu^{x,y}}(t) f(y) \right| \leq \|f\|_{\ell^\infty(\mathbb{X})} \sup_{x,y \in \mathbb{X}} |\widehat{\mu^{x,y}}(t)|.$$

Therefore, setting

$$\forall t \in \mathbb{R}^d, \quad s(t) = \sup_{x,y \in \mathbb{X}} |\widehat{\mu^{x,y}}(t)|,$$

one deduce that  $\|P_t\|_{\ell^\infty(\mathbb{X})} \leq s(t)$ . Moreover,  $r(t) \leq s(t)$ ,  $t \in \mathbb{R}^d$ , so that the spectral condition is satisfied if the family  $(\mu^{x,y})_{x,y \in \mathbb{X}}$  is uniformly aperiodic.  $\square$

The uniform aperiodicity condition is far from being necessary. Below, we introduce the usual notion of aperiodic MAP.

**Definition 1.7** (Aperiodic Markov Additive process). *A MAP is said to be periodic if there exists  $z \in \mathbb{Z}^d$  and a proper subgroup  $\Gamma \subsetneq \mathbb{Z}^d$  such that, for all  $x, y \in \mathbb{X}$  satisfying  $m(x)P(x,y) > 0$ ,  $\mu^{x,y}(z + \Gamma) = 1$ . On the contrary, it is said to be aperiodic.*

**Remark 1.8.** *The MAP considered in Example 1.5 is 2-periodic, that is why we rather studied the second order MAP which is aperiodic. Similarly to the classical context of random walk, a periodic MAP admits a cyclic decomposition so that there is no loss of generality to assume aperiodicity.*

**Lemma 1.9.** *Suppose there exists  $t_0 \in \mathbb{T}_d \setminus \{0\}$  such that, for all  $x, y \in \mathbb{X}$  satisfying  $m(x)P(x,y) > 0$ ,  $|\widehat{\mu^{x,y}}(t_0)| = 1$ . Then, the corresponding Markov Additive Process is periodic.*

*Proof.* The assumption means that there exists  $z \in \mathbb{R}^d$  such that  $\widehat{\mu^{x,y}}(t_0) = e^{i\langle t, z \rangle}$  for all  $x, y \in \mathbb{X}$  satisfying  $m(x)P(x, y) > 0$ . In fact,  $\mu^{x,y}$  is supported by  $\mathbb{Z}^d$  implies that  $z \in \mathbb{Z}^d$ . For any  $n \geq 1$ , set

$$P_n = \{(x_0, \dots, x_n) \in \mathbb{X}^{n+1} : m(x_0) > 0, \quad \forall k = 0, \dots, n-1, \quad P(x_k, x_{k+1}) > 0\}.$$

Then, for any  $n \geq 1$  and any  $(x_0, \dots, x_n) \in P_n$

$$e^{i\langle t_0, nz \rangle} = \prod_{k=0}^{n-1} \widehat{\mu^{x_k, x_{k+1}}}(t_0) = \sum_{w \in \mathbb{Z}^d} \mu^{x_0, x_1} * \dots * \mu^{x_{n-1}, x_n}(w) e^{i\langle t_0, w \rangle},$$

or equivalently,

$$1 = \sum_{w \in \mathbb{Z}^d} (\mu^{x_0, x_1} * \delta_{-z}) * \dots * (\mu^{x_{n-1}, x_n} \delta_{-z})(w) e^{i\langle t_0, w \rangle}.$$

As an extremal convex combination, it follows that  $\langle t_0, w \rangle = 0$  modulo  $2\pi$  as soon as  $w$  is in the support  $S_n(x_0, \dots, x_n)$  of  $(\mu^{x_0, x_1} * \delta_{-z}) * \dots * (\mu^{x_{n-1}, x_n} \delta_{-z})$ . This property naturally extends to the group  $H$  generated by  $\cup_{n \geq 1} \cup_{(x_0, \dots, x_n) \in P_n} S_n(x_0, \dots, x_n)$ . In symbols,  $\langle t_0, w \rangle$  for all  $w \in H$ . Besides,  $H$  satisfies, for all  $(x, y) \in P_2$ ,  $(\mu^{x,y} * \delta_{-z})(H) = 1$  or equivalently  $\mu^{x,y}(z + H) = 1$ . Consequently, if the MAP is supposed aperiodic, the group  $H$  is not a strict subgroup of  $\mathbb{Z}^d$ . In particular,  $\langle t_0, e_i \rangle = 0$  modulo  $2\pi$ ,  $i = 1, \dots, d$ , where  $e_i$  stands for the  $i^{\text{th}}$  canonical vector of  $\mathbb{R}^d$ . This is a contradiction with  $t_0 \in \mathbb{T}_d \setminus \{0\}$ .  $\square$

**Proposition 1.10.** *Let  $(X, Z)$  be an aperiodic MAP and suppose that  $X$  is recurrent. Then for any  $t \in \mathbb{T}_d \setminus \{0\}$  there exists a closed  $P_t$ -invariant linear subspace  $E$  of  $\ell^\infty(\mathbb{X})$  containing the constants such that the operator  $P_t$ , acting on  $E$ , has no eigenvalue of modulus one.*

*Proof.* Suppose there exists  $t_0 \in \mathbb{T}_d \setminus \{0\}$  and  $f \in E$ , where  $E$  is any closed  $P_{t_0}$ -invariant subspace containing the constants, such that, for some  $\theta \in \mathbb{R}$ ,

$$P_{t_0} f(x) = \sum_{y \in \mathbb{X}} P(x, y) \widehat{\mu^{x,y}}(t_0) f(y) = e^{i\theta} f(x).$$

By Jensen's inequality and the fact that  $|\widehat{\mu^{x,y}}(t_0)| \leq 1$ , it follows that  $|f| \leq P|f|$ . Consequently,  $\|f\|_{\ell^\infty(\mathbb{X})} - |f|$  is a non-negative superharmonic function. Since  $P$  is recurrent, the function  $|f|$  is constant (see [55, Theorem 1.16, p.5] for instance). Hence, for all  $x \in \mathbb{X}$  such that  $m(x) > 0$ ,

$$1 = \left| \sum_{y \in \mathbb{X}} P(x, y) \widehat{\mu^{x,y}}(t_0) \right| \leq \sum_{y \in \mathbb{X}} P(x, y) |\widehat{\mu^{x,y}}(t_0)| \leq 1.$$

This means that, for all  $x, y \in \mathbb{X}$  with  $m(x)P(x, y) > 0$ ,  $|\widehat{\mu^{x,y}}(t_0)| = 1$ . By Lemma 1.9, the corresponding MAP is periodic.  $\square$

**Definition 1.11** (Condition (S)). *A MAP is said to satisfy condition (S) if for any  $t \in \mathbb{T}_d \setminus \{0\}$ , there exists a closed  $P_t$ -invariant subspace  $E$ , containing the constants, for which the spectral values of modulus one of  $P_t$ , operating on  $E$ , consists of eigenvalues.*

**Proposition 1.12.** *Let  $(X, Z)$  be an aperiodic MAP satisfying the condition (S) and suppose that  $X$  is recurrent. If in addition  $\mathbb{R}^d \ni t \longrightarrow P_t \in \ell^\infty(\mathbb{X})$  is continuous, then the MAP satisfies the spectral condition.*



*Proof.* Let  $t_0 \in \mathbb{T}_d$  and  $E$  be a closed subspace such that  $P_{t_0}E \subset E$  and  $\mathbf{1} \in E$ . Then, by reverse triangle inequality,

$$\forall t \in \mathbb{T}_d, \quad \left| \|P_{t_0}^n\|_E - \|P_t^n\|_E \right| \leq \|P_{t_0}^n - P_t^n\|_E \leq \|P_{t_0}^n - P_t^n\|_{\ell^\infty(\mathbb{Z})},$$

which can be made arbitrarily small by continuity assumption. It follows that  $r$  is the point-wise infimum of continuous functions and as such is upper semi-continuous. Therefore, the function  $r$  reaches its maximum on any compact  $K \subset \mathbb{T}_d$ . Aperiodicity together with condition (S) imply  $\max_{t \in K} r(t) < 1$  excepted when  $0 \in K$ .  $\square$

## 2 Main results

Let us first summarize the assumptions involved in the statement of the Local Limit Theorem and the transience sufficient criterion.

**Assumptions 1.** *The internal Markov chain  $P$  is irreducible and recurrent.*

**Assumptions 2.** *The family of probability measures  $(\mu^{x,y})_{x,y \in \mathbb{X}}$  admits a uniform third order moment:*

$$\sup_{x,y \in \mathbb{X}} \sum_{z \in \mathbb{Z}^d} \|z\|^3 \mu^{x,y}(z) < \infty \quad (6)$$

**Assumptions 3.** *The quenched dispersion is uniformly elliptic: there exists a (deterministic) constant  $\alpha > 0$  such that*

$$\forall t \in \mathbb{R}^d, \quad \forall n \geq 1, \quad \langle t, V_n t \rangle \geq \alpha \|t\|^2. \quad (7)$$

**Assumptions 4.** *The MAP satisfies the spectral condition.*

**Remark 2.1.** *Some of these assumptions might be slightly relaxed. In particular, it is only needed in Assumption 3 that the uniform ellipticity holds for all  $n \geq 1$  but finitely may. It is even possible to suppose it holds for all  $n \geq \tau$  for some stopping time  $\tau$ . In that case,  $\tau$  must be finite almost-surely in Theorem 2.2 and integrable in Theorem 2.5. On the other hand, Assumption 4 implies that  $\|P_t^n \mathbf{1}\|_{\ell^\infty(\mathbb{X})}$  decreases exponentially fast whereas it is only required to vanishes in Theorem 2.2 and to be summable in Theorem 2.5.*

### 2.1 A Local Limit Theorem

**Theorem 2.2** (Local Limit Theorem). *Under Assumptions 1, 2, 3 and 4, for all  $x \in \mathbb{X}$ ,*

$$\lim_{n \rightarrow \infty} \sup_{z \in \mathbb{Z}^d} \left| (2\pi n)^{d/2} \mathbf{P}(Z_n - Z_0 = z | X_0 = x) - \mathbf{E} \left[ \frac{1}{\sqrt{\det(V_n)}} \exp \left\{ -\frac{1}{2n} \langle U_n - z, V_n^{-1}(U_n - z) \rangle \right\} \middle| X_0 = x \right] \right| = 0. \quad (8)$$

### 2.2 A sufficient criterion for transience: some potential theory

This section is devoted to a sufficient criterion for the transience of the additive part of Markov Additive Processes.

**Definition 2.3** (Recurrence versus Transience). A MAP  $((X_n, Z_n)_{n \geq 0})$  is said to be recurrent if for any  $(x, z) \in \mathbb{X} \times \mathbb{Z}^d$ , there exists  $r > 0$  such that

$$\mathbf{P} \left[ \liminf_{n \rightarrow \infty} \|Z_n\| < r \mid X_0 = x, Z_0 = z \right] = 1.$$

It is said to be transient if for any  $(x, z) \in \mathbb{X} \times \mathbb{Z}^d$

$$\mathbf{P} \left[ \lim_{n \rightarrow \infty} \|Z_n\| = \infty \mid X_0 = x, Z_0 = z \right] = 1.$$

In [8], it is proved that a MAP is either recurrent or transient under Assumption 1. In particular, the recurrence or transience of such a MAP does not depend on the initial state  $(x, z) \in \mathbb{X} \times \mathbb{Z}^d$ .

For any positive function  $f$  on  $\mathbb{X} \times \mathbb{Z}^d$ , recall that the potential of the charge  $f$  is given by

$$Gf(x, z) := \mathbf{E} \left[ \sum_{n \geq 0} f(X_n, Z_n) \mid X_0 = x, Z_0 = z \right].$$

By analogy with the classical context of random walks, it is natural to look for a criterion for the recurrence or transience of a MAP that involves the mean sojourn time of the set  $\mathbb{X} \times \{z\}$ :

$$G\mathbf{1}_{\mathbb{X} \times \{z\}}(x, z) := \mathbf{E} \left[ \sum_{n \geq 0} \mathbf{1}_{\mathbb{X} \times \{z\}}(X_n, Z_n) \mid X_0 = x, Z_0 = z \right]. \quad (9)$$

Naturally, if the quantity in (9) is finite, the additive component of the MAP hits  $z \in \mathbb{Z}^d$  only finitely many times by applying the Markov inequality.

**Definition 2.4** (Irreducibility). A MAP is said to be irreducible if for any  $x \in \mathbb{X}$ , for any  $z, z' \in \mathbb{Z}^d$ , there exists  $n \geq 0$  such that  $\mathbf{P}[Z_n = z' \mid X_0 = x, Z_0 = z] > 0$ .

Therefore, if a MAP is irreducible and that the quantity in Equation (9) is finite for all  $x \in \mathbb{X}$  and for some (equivalently any)  $z \in \mathbb{Z}^d$  then the MAP is transient by the Markov property. Consequently, we obtain the following criterion as a Corollary of Theorem 2.2.

**Theorem 2.5** (Sufficient criterion for transience). Let  $d \geq 2$ . Suppose that the MAP is irreducible. Then, under Assumptions 1, 2, 3 and 4, if for any  $x \in \mathbb{X}$

$$\sum_{n \geq 1} \frac{1}{n^{d/2}} \mathbf{E} \left[ \frac{1}{\sqrt{\det(V_n)}} \exp \left\{ -\frac{1}{2n} \langle U_n, V_n^{-1} U_n \rangle \right\} \mid X_0 = x \right] < \infty \quad (10)$$

then, the MAP is transient.

Nonetheless, it is worth noting that if the converse holds true when the internal Markov chain is positive recurrent by [8, Proposition 2.2], it remains an open question for the null recurrent case.

**Remark 2.6.** Note that under Assumption 3,  $\det(V_n) \geq \alpha^d$  for all  $n \geq 1$  so that it does not play any role in the nature of the series (10). Intuitively, Assumption 3 means that the MAP remains genuinely  $d$ -dimensional and is not attracted by a sub-manifold.

### 3 Application: random walks with local drift

This section is dedicated to the illustration of Theorems 2.2 and 2.5. The examples presented below are based on the models introduced in [4].

Let  $(\xi_k)_{k \geq 1}$  be a sequence of *i.i.d.* random variables where  $\xi_1$  is uniformly distributed on  $\{-1, 1\}$ . Let  $S_0$  be a  $\mathbb{Z}$ -valued random variable and, for  $n \geq 1$ ,  $S_n = S_0 + \xi_1 + \dots + \xi_n$ . In the sequel, the random walk  $S = (S_n)_{n \geq 0}$  shall play the role of the internal Markov chain of the family of MAP, in particular  $\mathbb{X} = \mathbb{Z}$ . It is well known that  $S$  fulfills Assumption 1 and that the corresponding Markov operator can not be quasi-compact (see [39]).

Let  $T_h$  and  $T_v$  be  $\mathbb{N}$ -valued random variables. For  $p, q \in \{-1, 1\}$ , set  $T_{p,q} = (pT_h, qT_v)$  and denote by  $\mu^{p,q}$  its distribution on  $\mathbb{Z}^2$ . Furthermore, let  $\varepsilon = (\varepsilon_x)_{x \in \mathbb{Z}}$  be a random or deterministic sequence taking values in  $\{-1, 1\}$ . Then, the distribution of the additive component is defined through the identity  $\mu^{x,y} = \mu^{\varepsilon_x, \text{sgn}(y-x)}$ ,  $x, y \in \mathbb{Z}$ , where  $\text{sgn}$  stands for the sign function.

Intuitively, on one hand, the vertical component of the Markov additive process shall merely follow the moves of the underlying random walk in the sense they go north or south simultaneously. On the other hand, the horizontal component moves in the direction dictated by  $\varepsilon$ .

The expectation and the covariance matrix of  $T_{p,q}$  are respectively given by

$$\mathbf{E}(T_{p,q}) = \begin{pmatrix} p\mathbf{E}(T_h) \\ q\mathbf{E}(T_v) \end{pmatrix} \quad \text{and} \quad \mathbf{V}(T_{p,q}) = \begin{pmatrix} \mathbf{V}(T_h) & pq\mathbf{Cov}(T_h, T_v) \\ pq\mathbf{Cov}(T_h, T_v) & \mathbf{V}(T_v) \end{pmatrix},$$

where  $\mathbf{Cov}(T_h, T_v)$  denotes the covariance between  $T_h$  and  $T_v$ .

In the sequel, let us assume that:

1. the random variables  $T_h$  and  $T_v$  are uncorrelated;
2. for all  $p, q \in \{-1, 1\}$ 
  - (a) the measures  $\mu^{p,q}$  admit a third moment (*i.e.*  $T_v$  and  $T_h$  admit a third order moment),
  - (b)  $\mu^{p,q}$  are aperiodic in the sense  $|\widehat{\mu^{p,q}}(t)| = 1$  on  $\mathbb{T}_d$  if and only if  $t = 0$ ,

As a straightforward consequence, conditions 1 and 2 above ensures that the resulting MAP satisfies Assumptions 2 and 3. Finally, Assumption 4 is fulfilled by Proposition 1.6.

The main objective is now to compute and/or estimate the asymptotic of the following quantity of interest for any  $z = (z_1, z_2) \in \mathbb{Z}^2$ :

$$\frac{1}{\sqrt{\mathbf{V}(T_h)\mathbf{V}(T_v)}} \mathbf{E} \left[ \exp \left\{ -\frac{\mathbf{E}[T_h]^2}{2n\mathbf{V}[T_h]} \left( \sum_{\ell=0}^{n-1} \varepsilon_{S_\ell} - z_1 \right)^2 \right\} \exp \left\{ -\frac{\mathbf{E}[T_v]^2}{2n\mathbf{V}[T_v]} (S_n - S_0 - z_2)^2 \right\} \middle| S_0 = x \right] < \infty. \quad (11)$$

#### 3.1 Periodic environment

Suppose that  $\varepsilon_x = (-1)^x$  for  $x \in \mathbb{Z}$ . Then,

$$\forall n \geq 1, \quad \left( \sum_{k=0}^{n-1} \varepsilon_{S_k} - z_1 \right)^2 = \left( \varepsilon_{S_0} \frac{1 + (-1)^{n-1}}{2} - z_1 \right)^2.$$

Hence,

$$\lim_{n \rightarrow \infty} \exp \left[ -\frac{\mathbf{E}(T_h)^2}{2n\mathbf{V}(T_h)} \left( \sum_{k=0}^{n-1} \varepsilon_{S_k} - z_1 \right)^2 \right] = 1.$$

Beside, notice that  $(S_n - S_0 - z_2)^2/n$  converges in distribution to a chi-square random variable with 1 degree of freedom. Since  $x \rightarrow \exp(-x)$  is continuous and bounded on  $\mathbb{R}_+$ , it follows by Slutsky's lemma that

$$\begin{aligned} \lim_{n \rightarrow \infty} \mathbf{E} \left[ \exp \left\{ -\frac{\mathbf{E}[T_h]^2}{2n\mathbf{V}[T_h]} \left( \sum_{\ell=0}^{n-1} \varepsilon_{S_\ell} - z_1 \right)^2 \right\} \exp \left\{ -\frac{\mathbf{E}[T_v]^2}{2n\mathbf{V}[T_v]} (S_n - S_0 - z_2)^2 \right\} \middle| S_0 = x \right] \\ = \int_{\mathbb{R}} \exp \left[ -\frac{\mathbf{E}(T_v)^2 y^2}{2\mathbf{V}(T_v)} \right] \exp \left[ -\frac{y^2}{2} \right] \frac{dy}{\sqrt{2\pi}} = \left( 1 + \frac{E[T_v]^2}{V(T_v)} \right)^{-1/2}. \end{aligned}$$

This limit is not uniform in the variable  $z \in \mathbb{Z}^d$ . However, Theorem 2.2 does imply that uniformly for  $\frac{z}{\sqrt{n}}$  negligible (using for instance the estimate of [11, Corollary 4.2, p.66])

$$\mathbf{P}(Z_n - Z_0 = z | S_0 = x) \sim_{n \rightarrow \infty} \frac{1}{2\pi n} \sqrt{\frac{1}{\mathbf{V}(T_h)\mathbf{V}(T_v)}} \left( 1 + \frac{E[T_v]^2}{V(T_v)} \right)^{-1/2}.$$

It is worth noting that in this form Theorem 2.5 is inconclusive: we can not deduce the recurrence of the Markov Additive process from the fact that the series is infinite. However, in this specific example, it is worth noting that the process  $M = ((\varepsilon_{S_n}, \text{sgn}(S_{n+1} - S_n)))_{n \geq 0}$  is still a Markov chain taking values in the finite space  $\{-1, 1\}^2$ . The transition probabilities are given, for all  $x, x', y, y' \in \{-1, 1\}$ , as follows:

$$Q((x, y), (x', y')) = Q_1(x, x')Q_2(y, y') \quad \text{with} \quad Q_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad Q_2 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

This observation allows to define a new Markov Additive process  $(M, \tilde{Z})$  for which we know that the additive component is recurrent (by [25] for instance). The recurrence of the original Markov Additive process then follows from the equality of distributions

$$\mathcal{L}((Z_n)_{n \geq 0} | S_0 = x) = \mathcal{L}((\tilde{Z}_n)_{n \geq 0} | M_0 = \varepsilon_x).$$

In the two models considered below, such a reduction is no longer possible motivating the extension to null recurrent internal Markov chain.

### 3.2 Two directed half planes

For simplicity, it is supposed in the result below that  $T_h$  and  $T_v$  are fully supported. This assumption is sufficient to deduce irreducibility of the MAP but it can be drastically weakened.

**Proposition 3.1.** *Suppose that for all  $x \in \mathbb{Z}$ ,  $\varepsilon_x = \text{sgn}(x)$  and that  $T_h$  and  $T_v$  are fully supported on  $\mathbb{N}$ . Then the resulting MAP is transient.*

*Proof.* By irreducibility, it is only needed to estimate the quantity in (11) with  $z_1 = z_2 = 0$ . An immediate majoration in (11) implies it suffices to prove that

$$\sum_{n \geq 1} \frac{1}{n} \mathbf{E} \left[ \exp \left\{ -\frac{\mathbf{E}(T_h)^2}{2n\mathbf{V}(T_h)} \left( \sum_{k=0}^{n-1} \varepsilon_{S_k} \right)^2 \right\} \middle| S_0 = x \right]. \quad (12)$$

is finite for all  $x \in \mathbb{Z}$ .

Let us introduce the following useful transformation

$$\sum_{k=0}^{n-1} \varepsilon_{S_k} = \sum_{x \in \mathbb{Z}} N_n(x) \varepsilon_x \quad (13)$$

where  $N_n(x)$ ,  $x \in \mathbb{Z}$  and  $n \geq 1$  is the local time of  $(S_n)_{n \geq 0}$  up to time  $n - 1$  that is  $N_n(x) = \sum_{k=0}^{n-1} \mathbf{1}_{\{x\}}(S_k)$ . Remarking that  $\varepsilon = \text{sgn}$ , it follows that

$$\sum_{k=0}^{n-1} \varepsilon_x = 2N_n(\mathbb{Z}_+) - n \quad \text{with} \quad N_n(\mathbb{Z}_+) = \sum_{x \geq 0} N_n(x).$$

Now, estimate the summand in (12)

$$\begin{aligned} \mathbf{E} \left[ \exp \left\{ -\frac{\mathbf{E}(T_h)^2}{2n\mathbf{V}(T_h)} (2N_n(\mathbb{Z}_+) - n)^2 \right\} \right] &\leq \left| \mathbf{E} \left[ \exp \left\{ -n \frac{\mathbf{E}(T_h)^2}{2\mathbf{V}(T_h)} \left( 2\frac{N_n(\mathbb{Z}_+)}{n} - 1 \right)^2 \right\} \right] \right. \\ &\quad \left. - \mathbf{E} \left[ \exp \left\{ -n \frac{\mathbf{E}(T_h)^2}{2\mathbf{V}(T_h)} (2\Gamma - 1)^2 \right\} \right] \right| \\ &\quad + \mathbf{E} \left[ \exp \left\{ -n \frac{\mathbf{E}(T_h)^2}{2\mathbf{V}(T_h)} (2\Gamma - 1)^2 \right\} \right], \end{aligned} \quad (14)$$

where  $\Gamma$  is distributed as an arc-sine law supported by  $[0, 1]$ . It turns out that for all  $n \geq 1$ , the function

$$[0, \infty) \ni x \rightarrow \exp \left( -\frac{n\mathbf{E}(T_h)^2}{2\mathbf{V}(T_h)} (2x - 1)^2 \right)$$

is  $k_n$ -Lipschitz with constant  $k_n = e^{-\frac{1}{8}} \sqrt{\frac{n\mathbf{E}(T_h)^2}{\mathbf{V}(T_h)}}$ . Consequently, it follows a majoration of the first term on the right hand side of (14) by  $k_n d_{\mathcal{W}}(\mathcal{L}(N_n(\mathbb{Z}_+)/n), \mathcal{L}(\Gamma))$  where  $d_{\mathcal{W}}$  denotes the Wasserstein metric on probability measures. From [19, Theorem 1.2], we deduce that this term is a  $O(n^{-1/2})$ .

For the second term on the right hand side of (14), first observe that the quantity in (12) is finite as soon as the following is finite by Fubini theorem

$$\mathbf{E} \left[ \sum_{n \geq 1} \frac{1}{n} \exp \left\{ -n \frac{\mathbf{E}[T_h]^2}{2\mathbf{V}[T_h]} (2\Gamma - 1)^2 \right\} \right].$$

However, the power series inside the expectation can be computed explicitly so that the series in (12) is finite if and only if the following is finite

$$-\int_0^1 \log \left[ \exp \left\{ \frac{\mathbf{E}[T_h]^2}{2\mathbf{V}[T_h]} (2x - 1)^2 \right\} - 1 \right] \frac{dx}{\pi\sqrt{1-x^2}}.$$

The singularity at  $x = 1$  coming from the density function of the arc-sine distribution is integrable and so is the singularity at  $x = \frac{1}{2}$  as  $x \rightarrow \log x$  in the positive neighborhood of 0.  $\square$

### 3.3 Randomly directed random walks

In this paragraph, the sequence  $\varepsilon = (\varepsilon_x)_{x \in \mathbb{Z}}$  is supposed to be an *i.i.d.* sequence of  $\mathbb{Z}$ -valued. For the sake of simplicity, it is supposed that the common distribution, denoted by  $\pi$ , is aperiodic so that the resulting MAP is irreducible.

The notations used in the previous paragraph extends easily:  $\mu^{x,y}$ ,  $x, y \in \mathbb{Z}$ , is the distribution of the vector  $(\varepsilon_x T_h, \text{sgn}(y-x)T_v)$  where  $T_h$  and  $T_v$  are  $\mathbb{N}$ -valued random variables, independent with  $\varepsilon$ , satisfying the same properties as in the previous paragraph. In particular, the MAP is irreducible and  $T_h$  and  $T_v$  are uncorrelated. Therefore,

$$V_n = \frac{1}{n} \sum_{\ell=0}^{n-1} \begin{pmatrix} \varepsilon_{S_\ell}^2 \mathbf{V}(T_h) & 0 \\ 0 & \mathbf{V}(T_v) \end{pmatrix} \quad \text{and} \quad U_n = \sum_{\ell=0}^{n-1} \begin{pmatrix} \varepsilon_{S_\ell} \mathbf{E}(T_h) \\ \text{sgn}(S_{\ell+1} - S_\ell) \mathbf{E}(T_v) \end{pmatrix}.$$

Thus, the problem reduces to the study the quantity

$$\sum_{n \geq 1} \frac{1}{n} \mathbf{E} \left[ \exp \left\{ -\frac{\mathbf{E}(T_h)^2 \left( \sum_{k=0}^{n-1} \varepsilon_{S_k} \right)^2}{2\mathbf{V}(T_h) \left( \sum_{k=0}^{n-1} \varepsilon_{S_k}^2 \right)} \right\} \exp \left\{ -\frac{\mathbf{E}(T_v)^2}{2n\mathbf{V}(T_v)} (S_n - S_0)^2 \right\} \middle| S_0 = x, \varepsilon \right]. \quad (15)$$

**Proposition 3.2.** *Assume that  $\mathbf{E}[\varepsilon_0^2] < \infty$  and  $\mathbf{E}[\varepsilon_0] = 0$ . Then, for  $\pi^{\otimes \mathbb{Z}}$ -a.e. sequences  $(\varepsilon_x)_{x \in \mathbb{Z}}$ , the MAP in the environment  $\varepsilon$  is transient.*

*Proof.* Taking the expectation of (15) with respect to  $\pi^{\otimes \mathbb{Z}}$ , it is only needed by Markov inequality to prove that the following series is finite for any  $x \in \mathbb{Z}$

$$\sum_{n \geq 1} \frac{1}{n} \mathbf{E} \left[ \exp \left\{ -\frac{\mathbf{E}(T_h)^2 \left( \sum_{k=0}^{n-1} \varepsilon_{S_k} \right)^2}{2\mathbf{V}(T_h) \left( \sum_{k=0}^{n-1} \varepsilon_{S_k}^2 \right)} \right\} \middle| S_0 = x \right]. \quad (16)$$

The proof follows four steps.

**Step 1:** Let  $\delta > 0$ , set  $A_{n,\delta} = \{ \sum_{k=0}^{n-1} \varepsilon_{S_k} \leq n^{1+\delta} \}$ . Then,

$$\begin{aligned} & \sum_{n \geq 1} \frac{1}{n} \mathbf{E} \left[ \exp \left\{ -\frac{\mathbf{E}(T_h)^2 \left( \sum_{k=0}^{n-1} \varepsilon_{S_k} \right)^2}{2\mathbf{V}(T_h) \left( \sum_{k=0}^{n-1} \varepsilon_{S_k}^2 \right)} \right\} \middle| S_0 = x \right] \\ & \leq \sum_{n \geq 1} \frac{1}{n} \mathbf{E} \left[ \exp \left\{ -\frac{\mathbf{E}(T_h)^2 \left( \sum_{k=0}^{n-1} \varepsilon_{S_k} \right)^2}{2\mathbf{V}(T_h) n^{1+\delta}} \right\} \mathbf{1}_{A_{n,\delta}} \middle| S_0 = x \right] + \sum_{n \geq 1} \frac{1}{n} \mathbf{P}[A_{n,\delta}^c | S_0 = x]. \end{aligned}$$

For all  $\delta > 0$ , the second series is convergent applying the Markov inequality and observing that  $\mathbf{E}[\varepsilon_0^2] < \infty$ . It remains to estimate the expectation in the first series in which the immediate majoration  $\mathbf{1}_{A_{n,\delta}} \leq \mathbf{1}$  is made. Using the identity (13), and setting for  $n \geq 1$  and  $x \in \mathbb{R}$

$$f_n(x) = \exp \left\{ -\frac{\mathbf{E}(T_h)^2 n^{1/2-\delta}}{2\mathbf{V}(T_h) M^2} x^2 \right\},$$

this expectation rewrites as follows:

$$\mathbf{E} \left[ \exp \left\{ -\frac{\mathbf{E}(T_h)^2 \left( \sum_{k=0}^{n-1} \varepsilon_{S_k} \right)^2}{2\mathbf{V}(T_h) n^{1+\delta}} \right\} \middle| S_0 = x \right] = \mathbf{E} \left[ f_n \left( n^{-3/4} \sum_{x \in \mathbb{Z}} N_n(x) \varepsilon_x \right) \middle| S_0 = x \right].$$

**Step 2:** For each  $n \geq 1$ , the function is  $k_n$ -Lipschitz with  $k_n = \frac{e^{-1/2}}{M} \sqrt{\frac{\mathbf{E}[T_h]^2}{\mathbf{V}(T_h)}} n^{1/4-\delta/2}$ . Now, from [36], it follows that

$$n^{-3/4} \sum_{x \in \mathbb{Z}} N_n(x) \varepsilon_x \xrightarrow{\mathcal{D}} \Delta_1,$$

where, for  $t \geq 0$ ,

$$\Delta_t = \int_{-\infty}^{\infty} L_t(x) dZ(x).$$

Here,  $L_t$  denotes the jointly continuous version of the local time, up to time  $t$ , of a one dimensional Brownian motion  $B$  independent of the bilateral one dimensional Brownian motion  $Z$ .

Decompose the right hand side of (16) as follows

$$\begin{aligned} \mathbf{E} \left[ \exp \left\{ -\frac{\mathbf{E}(T_h)^2 \left( \sum_{k=0}^{n-1} \varepsilon_{S_k} \right)^2}{2\mathbf{V}(T_h)M^2n} \right\} \middle| S_0 = x \right] &\leq \left| \mathbf{E} \left[ f_n \left( n^{-3/4} \sum_{x \in \mathbb{Z}} N_n(x) \varepsilon_x \right) \middle| S_0 = x \right] \right. \\ &\quad \left. - \mathbf{E} \left[ f_n(n^{-3/4} \Delta_n) \right] \right| + \mathbf{E} \left[ f_n(n^{-3/4} \Delta_n) \right]. \end{aligned} \quad (17)$$

**Step 3:** At this level, it is needed to estimate the rate of convergence of:

$$d_{\mathcal{W}} \left( \mathcal{L} \left( n^{-3/4} \sum_{x \in \mathbb{Z}} N_n(x) \varepsilon_x \right), \mathcal{L}(n^{-3/4} \Delta_n) \right).$$

To this end, using the Kantorovich duality, it suffices to use the coupling exhibited in [14] and replace the growth rate in probability by the corresponding ones in  $\mathbf{L}^1$ . More precisely, it is shown in [14] that there exists a simple symmetric random walk  $\tilde{S}$ , distributed as the original random walk  $S$ , built from the linear Brownian motion  $B$  via the Skorokhod embedding. Moreover, there exists a scenery  $\tilde{\varepsilon}$ , distributed as  $\varepsilon$ , built from the bilateral Brownian motion  $Z$ . It is worth noting that  $W$  and  $Z$  are independent and so are the random variables  $(\tilde{S}, B)$  and  $(\tilde{\varepsilon}, W)$ . Denoting by  $(\tilde{N}_n(x))_{n \geq 0, x \in \mathbb{Z}}$  the local time process associated with  $\tilde{S}$  (and distributed as the original one  $(N_n(x))_{n \geq 0, x \in \mathbb{Z}}$ ), it remains to estimate the vanishing rates of:

$$\begin{aligned} \mathbf{E} \left| n^{-3/4} \sum_{x \in \mathbb{Z}} \tilde{N}_n(x) \tilde{\varepsilon}_x - n^{-3/4} \Delta_n \right| &\leq \mathbf{E} \left| n^{-3/4} \sum_{x \in \mathbb{Z}} \tilde{N}_n(x) \tilde{\varepsilon}_x - n^{-3/4} \sum_{x \in \mathbb{Z}} L_n^x \tilde{\varepsilon}_x \right| \\ &\quad + \mathbf{E} \left| n^{-3/4} \sum_{x \in \mathbb{Z}} L_n^x \tilde{\varepsilon}_x - n^{-3/4} \Delta_n \right|. \end{aligned} \quad (18)$$

*Coupling processes:* Let us set as in [14]

$$I(\infty, n) = \sum_{x \in \mathbb{Z}} \tilde{\varepsilon}_x (\tilde{N}_n(x) - L_n^x) \quad \text{and} \quad I(N, n) = \sum_{x=-N}^N \tilde{\varepsilon}_x (\tilde{N}_n(x) - L_n^x).$$

Denote by  $R_n$  the range, up to time  $n$ , of the random walk  $\tilde{S}$  (see [50, p. 35]). If  $x \in \mathbb{Z}$  is such that  $|x| > R_n$  then  $\tilde{N}_n(x) = L_n^x = 0$ . Moreover,  $R_n \leq n$ . Then, for all  $n \geq 1$ , the equality  $I(\infty, n) = I(R_n + 1, n)$  holds almost-surely. Besides,  $R_n$  is  $\sigma(\tilde{S}, B)$ -measurable and  $\tilde{\varepsilon}$  is independent of  $(\tilde{S}, B)$  so that the estimate (2.9) of [14] with  $p \geq 2$  rewrites, using the estimate

(2.10) of [14],

$$\begin{aligned}
\mathbf{E}[|I(R_n + 1, n)|]^p &\leq \mathbf{E} \left[ c_1 (2R_n + 3)^{p/2-1} \sum_{x=-R_n-1}^{R_n+1} \mathbf{E} \left[ |\tilde{\varepsilon}_x|^p |\tilde{N}_n(x) - L_n^x|^p | \sigma(\tilde{S}, B) \right] \right] \\
&= \mathbf{E}[|\tilde{\varepsilon}_0|^p] \mathbf{E} \left[ c_1 (2R_n + 3)^{p/2-1} \sum_{x=-R_n-1}^{R_n+1} |\tilde{N}_n(x) - L_n^x|^p \right] \\
&= c_1 \mathbf{E}[|\tilde{\varepsilon}_0|^p] \sum_{x \in \mathbb{Z}} \mathbf{E} \left[ (2R_n + 3)^{p/2-1} \mathbf{1}_{R_n+1 \geq |x|} |\tilde{N}_n(x) - L_n^x|^p \right] \\
&\leq c_1 \mathbf{E}[|\tilde{\varepsilon}_0|^p] \sum_{x=-n-1}^{n+1} \mathbf{E} \left[ (2R_n + 3)^{(p/2-1)q} |\tilde{N}_n(x) - L_n^x|^{pq} \right]^{1/q} \\
&\hspace{25em} \mathbf{P}[R_n + 1 \geq |x|]^{1/r} \\
&\leq c_1 \mathbf{E}[|\tilde{\varepsilon}_0|^p] \sum_{x=-n-1}^{n+1} \mathbf{E} \left[ (2R_n + 3)^{(p/2-1)qs} \right]^{1/qs} \\
&\hspace{15em} \mathbf{E} \left[ |\tilde{N}_n(x) - L_n^x|^{pqt} \right]^{1/qt} \mathbf{P}[R_n + 1 \geq |x|]^{1/r} \\
&\leq c_1 c_2 n^{p/4} \mathbf{E}[|\tilde{\varepsilon}_0|^p] (2n + 3)^{p/2-1} \sum_{x=-n-1}^{n+1} \mathbf{P}[R_n + 1 \geq |x|]^{1/r},
\end{aligned}$$

where  $q, r > 1$  (resp.  $s, t > 1$ ) are real conjugate indices and  $c_1, c_2 \geq 0$  are the constants given in [14].

Let us denote by  $\tau$  the time of first return to 0 of a simple random walk  $\tilde{S}$ . In symbols,  $\tau = \inf\{n \geq 1 : \tilde{S}_n = 0\}$ . It is shown in [50] that  $\mathbf{E}[R_n] = \sum_{k=1}^n \mathbf{P}[\tau > k]$ . Since  $\mathbf{P}[\tau > n] \sim_{n \rightarrow \infty} (\pi n)^{-1/2}$  (see [47] for instance),  $\mathbf{E}[R_n] = O(n^{1/2})$ . Then, the Markov inequality gives

$$\sum_{x=-n-1}^{n+1} \mathbf{P}[R_n + 1 \geq |x|]^{1/r} \leq \mathbf{E}[R_n + 1]^{1/r} \sum_{x=-n-1}^{n+1} |x|^{-1/r} \leq O \left( n^{1/2r} \frac{n^{1-1/r}}{1-1/r} \right) = O \left( \frac{n^{1-1/2r}}{1-1/r} \right).$$

Summarizing, choosing  $p = 2$ ,

$$\forall r > 1, \quad \mathbf{E}[|I(R_n + 1, n)|] \leq O \left( \frac{n^{3/4} n^{-1/4r}}{1-1/r} \right).$$

*Coupling environments:* First of all, the Cauchy-Schwartz's inequality implies

$$n^{-3/4} \mathbf{E} \left| \sum_{x \in \mathbb{Z}} L_n^x \tilde{\varepsilon}_x - n^{-3/4} \Delta_n \right| \leq n^{-3/4} \left[ \mathbf{E} \left| \sum_{x \in \mathbb{Z}} L_n^x \tilde{\varepsilon}_x - n^{-3/4} \Delta_n \right|^2 \right]^{1/2}.$$

Similarly to [14], let us set

$$J(\infty, n) = \int_0^\infty L_n^x dZ(x) - \sum_{j=1}^\infty \tilde{\varepsilon}_j L_n^j \quad \text{and} \quad J(N, n) = \int_0^{\rho(N)} L_n^x dZ(x) - \sum_{j=1}^N \tilde{\varepsilon}_j L_n^j.$$

Here again, for all  $n \geq 0$ ,  $J(\infty, n) = J(R_n + 1, n)$  almost-surely. Then, setting:

$$\forall x \in \mathbb{R}_+, \quad A_n(x) = \sum_{j=1}^{R_n+1} (L_n^x - L_n^j) \mathbf{1}_{(\rho(j-1), \rho(j)]}(x),$$



the quantity  $J(R_n + 1, n)$  rewrites

$$J(R_n + 1, n) = \int_0^\infty A_n(x) dW(x) = \sum_{j=1}^{R_n+1} \int_{\rho^{(j-1)}}^{\rho^{(j)}} A_n(x) dW(x).$$

Note that the process  $\left(\int_0^t A_n(x) dW(x)\right)_{t \geq 0}$  is a continuous local martingale. In addition, the random time  $\rho(R_n + 1)$  is a stopping time for this martingale. Therefore, by the Burkholder-Davis-Gundy's inequality (more precisely, [49, Corollary 4.2, Chapter IV, p.161]),

$$\mathbf{E} \left[ \left| \sum_{x \in \mathbb{Z}} L_n^x \tilde{\varepsilon}_x - n^{-3/4} \Delta_n \right|^2 \right] = \mathbf{E}[|J(\infty, n)|^2] = \mathbf{E}[|J(R_n + 1, n)|^2] \leq \mathbf{E} \int_0^{\rho(R_n+1)} A_n^2(x) dx.$$

Observing that  $R_n \leq n$ , the quantity above is majorized, by independence of the sequence  $\rho$  with  $(\tilde{S}, B)$ , as follows

$$\begin{aligned} \mathbf{E} \int_0^{\rho(R_n+1)} A_n^2(x) dx &\leq \mathbf{E} \sum_{j=1}^{R_n+1} \int_{\rho^{(j-1)}}^{\rho^{(j)}} A_n^2(x) dx = \sum_{j=1}^{n+1} \mathbf{E} \int_{\rho^{(j-1)}}^{\rho^{(j)}} A_n^2(x) \mathbf{1}_{R_n+1 \geq j} dx \\ &= \sum_{j=1}^{n+1} \mathbf{E} \int_{\rho^{(j-1)}}^{\rho^{(j)}} \mathbf{E}[A_n^2(x) \mathbf{1}_{R_n+1 \geq j}] dx. \end{aligned} \quad (19)$$

The Hölder's inequality yields

$$\mathbf{E}[A_n^2(x) \mathbf{1}_{R_n+1 \geq j}] \leq \mathbf{E}[A_n^{2q}(x)]^{1/q} \mathbf{P}[R_n + 1 \geq j]^{1/p}. \quad (20)$$

Let  $n \geq y > x > 0$  be real numbers and recall that  $B_0 = 0$  almost-surely. Then,

$$|L_n^x - L_n^y|^{2q} = \left[ (B_n - x)^+ - (B_n - y)^+ - \int_0^n \mathbf{1}_{\{B_s > x\}} dB_s + \int_0^n \mathbf{1}_{\{B_s > y\}} dB_s \right]^{2q} \quad (21)$$

$$\leq 2^{2q-1} \left[ (B_n - x)^+ - (B_n - y)^+ \right]^{2q} + \left[ - \int_0^n \mathbf{1}_{\{B_s > x\}} dB_s + \int_0^n \mathbf{1}_{\{B_s > y\}} dB_s \right]^{2q}. \quad (22)$$

Integrating this inequality, it yields by the Burkholder-Davis-Gundy's inequality

$$\begin{aligned} \mathbf{E} \left[ \left( - \int_0^n \mathbf{1}_{\{B_s > x\}} dB_s + \int_0^n \mathbf{1}_{\{B_s > y\}} dB_s \right)^{2q} \right] &\leq \mathbf{E} \left( \int_0^n \mathbf{1}_{(x,y]}(B_s) ds \right)^q \\ &\leq \int_0^n \mathbf{E} \mathbf{1}_{(x,y]}(B_s) ds \\ &\leq \int_0^n \int_x^y \frac{e^{-z^2/2s}}{\sqrt{2\pi s}} dz ds \\ &\leq |x - y| \int_0^n \frac{e^{-x^2/2s}}{\sqrt{2\pi s}} ds \\ &\leq |x - y| \sqrt{n}. \end{aligned}$$

Besides,

$$\begin{aligned}
\mathbf{E}[(B_n - x)^+ - (B_n - y)^+]^2 &= \int_{\mathbb{R}} ((z - x)^+ - (z - y)^+)^2 \frac{e^{-z^2/2n}}{\sqrt{2\pi n}} dz \\
&= \int_x^y (z - x)^2 \frac{e^{-z^2/2n}}{\sqrt{2\pi n}} dz + (x - y)^2 \int_y^\infty \frac{e^{-z^2/2n}}{\sqrt{2\pi n}} dz \\
&\leq (y - x)^2 \int_{\min(x,y)}^\infty \frac{e^{-z^2/2n}}{\sqrt{2\pi n}} dz. \quad (23)
\end{aligned}$$

Now, Equation (19) with the Hölder's inequality of (20) gives

$$\mathbf{E} \int_0^{\rho(n+1)} A_n^2(x) dx \leq \sum_{j=1}^{n+1} \mathbf{P}[R_n + 1 \geq j]^{1/p} \mathbf{E} \int_{\rho(j-1)}^{\rho(j)} \mathbf{E}[A_n^{2q}(x)]^{1/q} dx.$$

Thereafter, consider  $x \in (\rho(j-1), \rho(j)]$  and use the estimates of (21) and (23) to obtain

$$\mathbf{E}[A_n^{2q}(x)] \leq C_q \frac{j^2}{\sqrt{2\pi n}} |1 - x/j|^2 + C_q j \sqrt{n} |j - x|.$$

Since

$$x \in (\rho(j-1), \rho(j)] \implies |x - j| \leq \max(|j - \rho(j)|, |j - \rho(j-1)|) = jM_j,$$

it follows, for  $j \geq 1$ ,

$$\begin{aligned}
\mathbf{E} \int_{\rho(j-1)}^{\rho(j)} \mathbf{E}[A_n^{2q}(x)]^{1/q} dx &\leq C_q^{1/q} \mathbf{E} \left[ (\rho(j) - \rho(j-1)) (j^2 (2\pi n)^{-1/2} M_j^2 + j n^{1/2} M_j)^{1/q} \right] \\
&= O(j^{2/q} n^{1/2q} \mathbf{E}[(\rho(j) - \rho(j-1))(M_j^2 + M_j)^{1/q}])
\end{aligned}$$

Consequently, the Markov inequality yields

$$\mathbf{E} \int_0^{\rho(n+1)} A_n^2(x) dx \leq \mathbf{E}[R_n + 1]^{1/p} n^{1/2q} \sum_{j=1}^{n+1} j^{2/q-1/p} = O(n^{1/2p} n^{3/2q}) = O(n^{1/2+1/q}).$$

Summarizing,

$$n^{-3/4} \mathbf{E} \left| \sum_{x \in \mathbb{Z}} L_n^x \tilde{\varepsilon}_x - n^{-3/4} \Delta_n \right| = O(n^{1/4+1/2q-3/4}) = O(n^{-1/2+1/2q}).$$

All this implies that for any fixed  $q, r > 1$

$$d_{\mathcal{W}} \left( \mathcal{L} \left( n^{-3/4} \sum_{x \in \mathbb{Z}} N_n(x) \varepsilon_x \right), \mathcal{L}(n^{-3/4} \Delta_n) \right) = O(n^{-1/4r} + n^{-1/2+1/2q}).$$

Since the Lipschitz constant  $k_n = O(n^{1/4-\delta/2})$ , fix  $\delta = \frac{1}{2}$ ,  $r = 2$  and  $q = 2$ , it yields

$$\left| \mathbf{E} \left[ f_n \left( n^{-3/4} \sum_{x \in \mathbb{Z}} N_n(x) \varepsilon_x \right) \middle| S_0 = x \right] \right| = O(n^{-1/8}).$$

**Step 4:** It is shown in [5] that the distribution of the random variable  $\Delta_1$  is absolutely continuous with respect to the Lebesgue measure. More precisely, since the random variables  $\varepsilon_x$ ,  $x \in \mathbb{Z}$ , are centered, the density of  $\Delta_1$  is given, for some  $\sigma > 0$ , by

$$C(x) = \mathbf{E} \left[ \frac{\exp \left\{ -\frac{x^2}{2|L|_2^2 \sigma^2} \right\}}{|L|_2 \sigma \sqrt{2\pi}} \right] \quad \text{where} \quad |L|_2 = \left( \int_{\mathbb{R}} L_1(y)^2 \, dy \right)^{\frac{1}{2}}.$$

Therefore, making the change of variables  $x = y/n^{1/16}$ , it follows

$$\begin{aligned} n^{1/16} \mathbf{E}[f_n(\Delta_1)] &= n^{1/16} \int_{\mathbb{R}} \exp \left\{ -\frac{\mathbf{E}(T_h)^2 n^{1/8}}{2\mathbf{V}(T_h) M^2} x^2 \right\} C(x) \, dx \\ &= \int_{\mathbb{R}} \exp \left\{ -\frac{\mathbf{E}(T_h)^2}{2\mathbf{V}(T_h) M^2} y^2 \right\} C(y/n^{1/16}) \, dy. \end{aligned}$$

Then, the monotone convergence theorem implies

$$\lim_{n \uparrow \infty} \int_{\mathbb{R}} \exp \left\{ -\frac{\mathbf{E}(T_h)^2}{2\mathbf{V}(T_h) M^2} y^2 \right\} C(y/n^{1/16}) \, dy = \int_{\mathbb{R}} \exp \left\{ -\frac{\mathbf{E}(T_h)^2}{2\mathbf{V}(T_h) M^2} y^2 \right\} \, dy \frac{1}{\sqrt{2\pi}} \mathbf{E}[|L|_2^{-1}].$$

Finally, Jensen's inequality gives

$$|L|_2 = \left( \int_{\mathbb{R}} L_1(y)^2 \, dy \right)^{1/2} \geq \int_{\mathbb{R}} L_1(y) \, dy = 1.$$

Hence,  $\mathbf{E}[f_n(\Delta_1)] = O(n^{-1/16})$ . □

## 4 Proofs

### 4.1 Proof of the Local Limit Theorem

**Proposition 4.1.** *Under Assumption 2, for any sufficiently small  $\delta \in (0, 1)$  and any  $t \in \mathbb{T}^d$  such that  $\|t\| \leq \delta$ , there exists  $\Theta_n(t)$  such that*

$$\Pi_n(t) = \exp \left\{ i \langle t, U_n \rangle - \frac{1}{2} \langle t, nV_n t \rangle + \Theta_n(t) \right\}$$

Moreover, there exists a (deterministic) constant  $K$  such that  $|\Theta_n(t)| \leq nK\|t\|^3$ .

*Proof.* Fix  $\ell \geq 0$ . Under Assumption 2, there exists a constant  $\kappa > 0$  such that

$$\left| \mathbf{E} \left[ e^{i \langle t, Z_{\ell+1} - Z_\ell \rangle} \middle| X_\ell, X_{\ell+1} \right] - 1 \right| \leq \kappa \|t\|_\infty.$$

Then, for all  $\delta \in (0, \kappa/2) \cap (0, 1)$  and all  $t \in \mathbb{T}_d$  such that  $\|t\| \leq \delta$ , the  $\mathbb{C}$ -valued function

$$\pi_\ell : t \longrightarrow \log \mathbf{E} \left[ e^{i \langle t, Z_{\ell+1} - Z_\ell \rangle} \middle| X_\ell, X_{\ell+1} \right]$$

is three times continuously differentiable. Moreover,  $\pi_\ell(0) = 1$ ,

$$\partial_p \pi_\ell(t) = i \frac{\mathbf{E} \left[ e^{i \langle t, Z_{\ell+1} - Z_\ell \rangle} (Z_{\ell+1}^{(p)} - Z_\ell^{(p)}) \middle| X_\ell, X_{\ell+1} \right]}{\mathbf{E} \left[ e^{i \langle t, Z_{\ell+1} - Z_\ell \rangle} \middle| X_\ell, X_{\ell+1} \right]},$$

and

$$\begin{aligned} \partial_{p,q}^2 \pi_\ell(t) &= \mathbf{E} \left[ e^{i\langle t, Z_{\ell+1} - Z_\ell \rangle} \middle| X_\ell, X_{\ell+1} \right]^{-2} \\ &\left\{ -\mathbf{E} \left[ e^{i\langle t, Z_{\ell+1} - Z_\ell \rangle} (Z_{\ell+1}^{(p)} - Z_\ell^{(p)})(Z_{\ell+1}^{(q)} - Z_\ell^{(q)}) \middle| X_\ell, X_{\ell+1} \right] \mathbf{E} \left[ e^{i\langle t, Z_{\ell+1} - Z_\ell \rangle} \middle| X_\ell, X_{\ell+1} \right] \right. \\ &\quad \left. + \mathbf{E} \left[ e^{i\langle t, Z_{\ell+1} - Z_\ell \rangle} (Z_{\ell+1}^{(p)} - Z_\ell^{(p)}) \middle| X_\ell, X_{\ell+1} \right] \mathbf{E} \left[ e^{i\langle t, Z_{\ell+1} - Z_\ell \rangle} (Z_{\ell+1}^{(q)} - Z_\ell^{(q)}) \middle| X_\ell, X_{\ell+1} \right] \right\}. \end{aligned}$$

Hence, the Taylor expansion at  $t = 0$  yields, for all  $t \in \mathbb{T}_d$  such that  $\|t\| \leq \delta$ ,

$$\pi_\ell(t) = i\langle t, Z_{\ell+1} - Z_\ell \rangle - \frac{1}{2}\langle t, \mathbf{Cov}(Z_{\ell+1} - Z_\ell | X_\ell, X_{\ell+1})t \rangle + R_\ell(t)$$

where  $R_\ell$  satisfies

$$|R_\ell(t)| \leq K\|t\|^3, \quad \text{with } K = \sup_{x,y} \sum_{z \in \mathbb{Z}^d} \|z\|^3 \mu^{x,y}(z).$$

Now,

$$\Pi_n(t) = \prod_{\ell=0}^{n-1} \pi_\ell(t) = \exp \left\{ i\langle t, U_n \rangle - \frac{1}{2}\langle t, nV_n t \rangle + \Theta_n(t) \right\},$$

where

$$\Theta_n(t) = \sum_{\ell=0}^{n-1} R_\ell(t) \quad \text{and} \quad |\Theta_n(t)| \leq nK\|t\|^3.$$

□

*Proof of Theorem 2.2.* Let  $\delta \in (0, 1)$  such that Proposition 4.1 holds and  $a \in (0, \delta\sqrt{n})$ . Then,

$$(2\pi)^d \mathbf{P}(Z_n - Z_0 = z | X_0 = x) = \int_{\mathbb{T}_d} P_t^n \mathbf{1}(x) e^{-i\langle t, z \rangle} dt \quad (24)$$

$$= \int_{\mathbb{T}_d \setminus \delta\mathbb{T}} P_t^n \mathbf{1}(x) e^{-i\langle t, z \rangle} dt + \int_{\delta\mathbb{T}_d} P_t^n \mathbf{1}(x) e^{-i\langle t, z \rangle} dt \quad (25)$$

$$(26)$$

where  $\delta\mathbb{T}_d$  is the magnification of the hypercube  $\mathbb{T}_d$  by  $\delta$ . The change of variables  $u = t/\sqrt{n}$  in the second term yields

$$\int_{\delta\mathbb{T}_d} P_t^n \mathbf{1}(x) e^{-i\langle t, z \rangle} dt = \mathbf{E} \left[ \int_{\mathbb{T}_d} e^{-i\langle t, z \rangle} \Pi_n(t) dt \middle| X_0 = x \right] \quad (27)$$

$$= n^{-d/2} \mathbf{E} \left[ \int_{\delta\sqrt{n}\mathbb{T}_d} e^{-\frac{i}{\sqrt{n}}\langle t, z \rangle} \Pi_n(t/\sqrt{n}) dt \middle| X_0 = x \right]. \quad (28)$$

This latter quantity can be decomposed as follows:

$$\begin{aligned}
& n^{-d/2} \mathbf{E} \left[ \int_{\mathbb{R}^d} \exp \left\{ i \frac{1}{\sqrt{n}} \langle t, U_n - z \rangle - \frac{1}{2} \langle t, V_n t \rangle \right\} dt \middle| X_0 = x \right] \\
& \quad + n^{-d/2} \mathbf{E} \left[ \int_{\delta\sqrt{n}\mathbb{T} \setminus a\mathbb{T}} e^{-\frac{i}{\sqrt{n}} \langle t, z \rangle} \Pi_n(t/\sqrt{n}) dt \middle| X_0 = x \right] \\
& \quad + n^{-d/2} \mathbf{E} \left[ \int_{a\mathbb{T}} e^{-i \frac{1}{\sqrt{n}} \langle t, z \rangle} \left( \Pi_n(t/\sqrt{n}) - \exp \left\{ i \frac{1}{\sqrt{n}} \langle t, U_n \rangle - \frac{1}{2} \langle t, V_n t \rangle \right\} \right) dt \middle| X_0 = x \right] \\
& \quad - n^{-d/2} \mathbf{E} \left[ \int_{\mathbb{R}^d \setminus a\mathbb{T}} \exp \left\{ i \frac{1}{\sqrt{n}} \langle t, U_n - z \rangle - \frac{1}{2} \langle t, V_n t \rangle \right\} dt \middle| X_0 = x \right]
\end{aligned}$$

Finally, Equations (24) and (27) imply

$$\begin{aligned}
& n^{d/2} (2\pi)^d \mathbf{P}(Z_n - Z_0 = z | X_0 = x) - \mathbf{E} \left[ \int_{\mathbb{R}^d} \exp \left\{ i \frac{1}{\sqrt{n}} \langle t, U_n - z \rangle - \frac{1}{2} \langle t, V_n t \rangle \right\} dt \middle| X_0 = x \right] \\
& \quad = \left[ A_1(z, n, a) + A_2(z, n, a) + A_3(z, n, a, \delta) + A_4(z, n, \delta) \right]
\end{aligned} \tag{29}$$

$$\tag{30}$$

where

$$\begin{aligned}
A_1(z, n, a) &= \mathbf{E} \left[ \int_{a\mathbb{T}} \exp \left\{ i \frac{1}{\sqrt{n}} \langle t, U_n - z \rangle - \frac{1}{2} \langle t, V_n t \rangle \right\} \left( e^{\Theta_n(t/\sqrt{n})} - 1 \right) dt \middle| X_0 = x \right], \\
A_2(z, n, a) &= -\mathbf{E} \left[ \int_{\mathbb{R}^d \setminus a\mathbb{T}} \exp \left\{ i \frac{1}{\sqrt{n}} \langle t, U_n - z \rangle - \frac{1}{2} \langle t, V_n t \rangle \right\} dt \middle| X_0 = x \right], \\
A_3(z, n, a, \delta) &= \mathbf{E} \left[ \int_{\delta\sqrt{n}\mathbb{T} \setminus a\mathbb{T}} e^{-\frac{i}{\sqrt{n}} \langle t, z \rangle} \Pi_n(t/\sqrt{n}) dt \middle| X_0 = x \right], \\
A_4(z, n, \delta) &= n^{d/2} \int_{\mathbb{T} \setminus \delta\mathbb{T}} P_t^n \mathbf{1}(x) e^{-i \langle t, z \rangle} dt.
\end{aligned}$$

**Lemma 4.2.** *Under Assumption 2 and 3, for any  $a > 0$ ,*

$$\lim_{n \rightarrow \infty} \sup_{z \in \mathbb{Z}^d} A_1(z, n, a) = 0.$$

*Proof.* First remark that, under Assumptions 2 and 3, Proposition 4.1 implies for all  $t \in \mathbb{R}^d$ ,

$$\lim_{n \rightarrow \infty} \exp \left\{ i \frac{1}{\sqrt{n}} \langle t, U_n - z \rangle - \frac{1}{2} \langle t, V_n t \rangle \right\} \left( e^{\Theta_n(t/\sqrt{n})} - 1 \right) = 0.$$

Moreover, for  $t \in a\mathbb{T}_d$ ,

$$\begin{aligned}
\left| e^{i \frac{1}{\sqrt{n}} \langle t, U_n - z \rangle - \frac{1}{2} \langle t, V_n t \rangle} \left( e^{\Theta_n(t/\sqrt{n})} - 1 \right) \right| &\leq e^{-\frac{1}{2} \langle t, V_n t \rangle} |\Theta_n(t/\sqrt{n})| \exp |\Theta_n(t/\sqrt{n})| \\
&\leq e^{-\frac{1}{2} \langle t, V_n t \rangle} \frac{a^3 \pi n K}{n^{3/2}} \exp \frac{a^3 \pi n K}{n^{3/2}} = O(e^{-\frac{\alpha}{2} \|t\|^2}). \tag{31}
\end{aligned}$$

Therefore, we may apply the dominated convergence theorem which implies that, for any  $a > 0$ ,  $A_1(z, n, a)$  goes to 0 uniformly in  $z \in \mathbb{Z}^d$ .  $\square$

**Lemma 4.3.** *Under Assumption 3, there exists some constant  $K_1, K_2 > 0$  such that for any  $a > 0$ ,*

$$\limsup_{n \rightarrow \infty} \sup_{z \in \mathbb{Z}^d} |A_2(z, n, a)| \leq K_1 \exp\{-K_2 a^2\}.$$

*Proof.* Assumption 3 implies  $|A_2(z, n, a)| \leq K_1 e^{-K_2 a^2}$  for some constant  $K_1, K_2 > 0$ .  $\square$

**Lemma 4.4.** *Under Assumptions 2 and 3, for any  $a > 0$  and for any  $\delta > 0$  sufficiently small*

$$\limsup_{n \rightarrow \infty} \sup_{z \in \mathbb{Z}^d} |A_3(z, n, a, \delta)| \leq K_3 \exp\{-K_4 a^2\}.$$

*Proof.* As a matter of fact,

$$\begin{aligned} \left| \mathbf{E} \left[ \int_{\delta\sqrt{n}\mathbb{T} \setminus a\mathbb{T}} e^{-\frac{i}{\sqrt{n}}\langle t, z \rangle} \Pi_n(t/\sqrt{n}) dt \middle| X_0 = x \right] \right| &\leq \mathbf{E} \left[ \int_{\delta\sqrt{n}\mathbb{T} \setminus a\mathbb{T}} |\Pi_n(t/\sqrt{n})| dt \middle| X_0 = x \right] \\ &= \mathbf{E} \left[ \int_{\delta\sqrt{n}\mathbb{T} \setminus a\mathbb{T}} \left| \exp\left\{-\frac{i}{\sqrt{n}}\langle t, U_n \rangle\right\} + \Pi_n(t/\sqrt{n}) \right| dt \middle| X_0 = x \right] \end{aligned} \quad (32)$$

Under Assumptions 2 and 3, by Proposition 4.1, the integrand in Equation (32) satisfies

$$\left| \Pi_n(t/\sqrt{n}) \exp\left\{-\frac{i}{\sqrt{n}}\langle t, U_n \rangle\right\} \right| \leq \exp\left\{-\frac{1}{2}\langle t, V_n t \rangle + |\Theta_n(t/\sqrt{n})|\right\}.$$

Moreover, since  $t \in \delta\sqrt{n}\mathbb{T} \setminus a\mathbb{T}$

$$|\Theta_n(t/\sqrt{n})| \leq nK \|t/\sqrt{n}\|^2 \|t/\sqrt{n}\| = K \|t\|^2 \|t/\sqrt{n}\| \leq K\delta\pi\sqrt{8} \|t\|^2,$$

and  $-\frac{1}{2}\langle t, V_n t \rangle \leq -\frac{1}{2}\alpha \|t\|^2$ . Thus, choosing any  $\delta > 0$  so small that  $\delta K\pi\sqrt{8} \leq \alpha/4$  (and that Proposition 4.1 holds true), the integrand in Equation (32) is bounded above by  $\exp(-\beta \|t\|^2)$  for some  $\beta > 0$ . Consequently, there exists  $K_3, K_4 > 0$  such that  $|A_3(z, n, a, \delta)| \leq K_3 e^{-K_4 a^2}$ .  $\square$

**Lemma 4.5.** *Under Assumptions 4, for any  $\delta \in (0, 1)$*

$$\limsup_{n \rightarrow \infty} \sup_{z \in \mathbb{Z}^d} A_4(z, n, \delta) = 0.$$

*Proof.* Fix  $\delta \in (0, 1)$  and observe that the sequence  $(\|P_t^n\|_E)_{n \geq 1}$  is sub-multiplicative. Consequently, under Assumption 4, there exists  $\gamma > 0$  such that, for all  $t \in \mathbb{T}_d \setminus \delta\mathbb{T}_d$ ,  $\|P_t^n \mathbf{1}\|_\infty \leq (1 - \gamma)^n$ . The result follows immediately.  $\square$

The four lemmas above implies that

$$\begin{aligned} \limsup_{n \rightarrow \infty} \sup_{z \in \mathbb{Z}^d} \left| n^{d/2} (2\pi)^d \mathbf{P}(Z_n - Z_0 = z | X_0 = x) - \operatorname{Re} \mathbf{E} \left[ \int_{\mathbb{R}^d} e^{i\frac{1}{\sqrt{n}}\langle t, U_n - z \rangle - \frac{1}{2}\langle t, V_n t \rangle} dt \middle| X_0 = x \right] \right| \\ \leq K_1 \exp\{-K_2 a^2\} + K_3 \exp\{-K_4 a^2\}. \end{aligned}$$

The left hand side does not depend on  $a > 0$  which can be chosen arbitrarily large. Noting that  $V_n$  is invertible by Assumption 3, the proof of Theorem 2.2 ends with the help of the Fourier transform

$$\int_{\mathbb{R}^d} \exp\left\{i\frac{1}{\sqrt{n}}\langle t, U_n - z \rangle - \frac{1}{2}\langle t, V_n t \rangle\right\} dt = \frac{(2\pi)^{d/2}}{\sqrt{\det(V_n)}} \exp\left\{-\frac{1}{2n}\langle U_n - z, V_n^{-1}(U_n - z) \rangle\right\}. \quad (33)$$

$\square$

## 4.2 Proof of the transience criterion

The proof of Theorem 2.2 needs to be slightly adapted since it is required to determine the convergence rate of the limit.

*Proof of Theorem 2.5.* Let us rewrite the decomposition (29) as follows (using the same notations):

$$\mathbf{P}(Z_n - Z_0 = 0 | X_0 = x) = \frac{1}{n^{d/2}} \mathbf{E} \left[ \int_{\mathbb{R}^d} e^{i \frac{1}{\sqrt{n}} \langle t, U_n \rangle - \frac{1}{2} \langle t, V_n t \rangle} dt \middle| X_0 = x \right] \quad (34)$$

$$+ \frac{1}{n^{d/2}} \left[ A_1(0, n, a) + A_2(0, n, a) + A_3(0, n, a, \delta) \right] + A_4(0, n, \delta). \quad (35)$$

With the identity (33), it is only needed to prove that the second line is summable. The assertion of Theorem 2.5 then follows from the Markov inequality.

The idea now is to let the parameter  $a > 0$  to suitably depend on  $n \geq 1$  and  $\delta > 0$  with the condition  $a \in (0, \delta\sqrt{n})$ . Set  $a_n = n^{1/8}\delta/2$ . Then, the terms  $A_1, A_2$  and  $A_3$  are still well defined. Furthermore, the same arguments as in the proofs of Lemmas 4.3 and 4.4 implies that

$$|A_2(0, n, n^{1/8}\delta/2)| \leq K_1 \exp\{-K_2 n^{1/4} \delta^2/4\}$$

$$\text{and } |A_3(0, n, n^{1/8}\delta/2, \delta)| \leq K_3 \exp\{-K_4 n^{1/4} \delta^2/4\} \quad (36)$$

with the same positive constants  $K_1, K_2, K_3$  and  $K_4$ . As a consequence, the corresponding series are convergent.

In addition, the series corresponding to the terms  $A_4$  is convergent under Assumption 4 since for any sufficiently small  $\delta > 0$ , the sequence  $A_4(0, n, \delta)$  vanishes exponentially fast.

Finally, one can show that the series  $\sum_{n \geq 1} n^{-d/2} A_1(0, n, n^{1/8}\delta/2)$  is convergent thanks to the following Lemma:

**Lemma 4.6.** *Under Assumption 2 and 3,*

$$\lim_{n \rightarrow \infty} n^{1/8} A_1(0, n, n^{1/4}\delta/2) = 0. \quad (37)$$

*Proof.* Compare to the proof of Lemma 4.2, we observe that

$$\lim_{n \rightarrow \infty} n^{1/8} \exp \left\{ i \frac{1}{\sqrt{n}} \langle t, U_n \rangle - \frac{1}{2} \langle t, V_n t \rangle \right\} \left( e^{\Theta_n(t/\sqrt{n})} - 1 \right) = 0 \quad (38)$$

because  $|\Theta_n(t/\sqrt{n})| \leq nK \|n^{-3/8}\delta/2\|^3$  by Proposition 4.1 so that

$$\forall t \in \frac{\delta n^{1/8}}{2} \mathbb{T}, \quad n^{1/8} \left( e^{\Theta_n(t/\sqrt{n})} - 1 \right) \leq K \|\delta/2\|^3 n^{-1/8} \rightarrow 0. \quad (39)$$

We conclude applying the dominated convergence theorem with the domination (31) using Assumption 3 and noting that  $a^3 = n^{3/8}\delta^3/8$ .  $\square$

$\square$

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