



**HAL**  
open science

# The type problem for a class of inhomogeneous random walks: a series criterion by a probabilistic argument

Basile de Loynes

► **To cite this version:**

Basile de Loynes. The type problem for a class of inhomogeneous random walks: a series criterion by a probabilistic argument. 2016. hal-01274873v4

**HAL Id: hal-01274873**

**<https://hal.science/hal-01274873v4>**

Preprint submitted on 18 Jan 2018 (v4), last revised 22 Jun 2022 (v6)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



Distributed under a Creative Commons Attribution - NonCommercial - NoDerivatives 4.0 International License

# The type problem for a class of inhomogeneous random walks : a series criterion by a probabilistic argument

Basile de Loynes<sup>1</sup>

<sup>1</sup>Ensaï - Campus de Ker-Lann, Rue Blaise Pascal , BP 37 203 , 35 712 Bruz Cedex, France ,  
basile.deloyne@ensai.fr

---

**Abstract** In the classical framework, a random walk on a group is a Markov chain with independent and identically distributed increments. In some sense, random walks are time and space homogeneous. In this paper, a criterion for the recurrence or transience of Markov additive processes is given. This criterion is deduced using Fourier analysis and a perturbation argument of a Markov operator. The latter extends the results of the literature since it does not involve a quasi-compactness condition on the operator. Finally, this criterion is applied to a new family of model of strongly drifted random walks on the lattice.

**Key words** Weakly inhomogeneous random walks . Recurrence . Transience . Random Walks with Random Transition Probabilities . Markov additive processes . Fourier analysis . Operator perturbations

**Mathematics Subject Classification (2000)** 60F15 . 60J05 . 60J10 . 60J15 . 60J45 . 60J55 . 60K15

---

## Contents

<b>1</b>	<b>The type problem for Markov additive processes</b>	<b>3</b>
1.1	Markov additive processes . . . . .	4
1.2	Basic assumptions . . . . .	5
1.3	The type problem : a series criterion . . . . .	8
1.4	Few comments on the main theorem . . . . .	9
<b>2</b>	<b>Applications : Markov additive processes with strong local drift</b>	<b>9</b>
<b>3</b>	<b>Proof of the main theorem</b>	<b>14</b>
3.1	Estimates far from the origin . . . . .	14
3.2	Estimates in the neighborhood of the origin . . . . .	14

## Introduction

In the classical framework, a random walk on a group  $\mathbf{G}$  is a discrete time stochastic process  $(Z_n)_{n \geq 0}$  defined as the product of independent and identically distributed (*i.i.d.*) random

variables  $(\xi_n)_{n \geq 1}$ . More precisely, for any  $g \in \mathbf{G}$ , we set  $Z_0 = g$  and

$$Z_n = g\xi_1 \cdots \xi_n.$$

Random walks on groups are obviously Markov chains that are adapted to the group structure in the sense that the underlying Markov operator is invariant under the group action of  $\mathbf{G}$  on itself. Thus, this homogeneity naturally gives rise to deep connections between stochastic properties of the random walk and algebraic properties of the group. Starting with the seminal paper of Pòlya, [37], a large part of the literature is devoted to the study of these connections in this homogeneous case (for instance [29, 31, 18, 21, 43, 40, 23, 22, 2] and references therein).

In this paper, we aim at investigating “weakly” inhomogeneous random walks. It turns out that there are at least two ways to introduce inhomogeneity. First, we can consider “spatial” inhomogeneity by weakening the group structure, replacing it, for instance, by a directed graph or semigroupoid structure (see [4, 16, 17, 35, 14]). Secondly, we can study “temporal” inhomogeneous random walks by introducing a notion of memory as in the model of reinforced [34, 42], excited [38, 3], self-interacting [11, 36] or also persistent random walks [8, 7, 9]. All these models belong to the larger class of stochastic processes with long range dependency.

The strategy in the sequel consists of making use of the connection between such stochastic processes and Random Walk with Random Transition Probabilities — or for short, **RWRTP** — the terminology comes from [20]. Roughly speaking, a **RWRTP** consists of a dynamical system  $(\Omega, T)$ , endowed with a quasi-invariant (preimages by  $T$  of null measure sets have null measure) probability measure  $\lambda$ , together with a family  $\{\mu^\omega\}_{\omega \in \Omega}$  on a group  $\mathbf{G}$ . The dynamics dictate the way the measures of  $\{\mu^\omega\}_{\omega \in \Omega}$  can be convoluted<sup>1</sup>. More precisely, choosing at random (with respect to the probability measure  $\lambda$ ) a trajectory  $\omega \in \Omega$ , we are interested in the asymptotic properties of the sequence of convoluted measures  $\mu^\omega * \mu^{T\omega} * \mu^{T^2\omega} * \cdots * \mu^{T^n\omega}$ . As a matter of facts, along a trajectory of the dynamical system, the resulting stochastic process has independent but, in general, not stationary increments. If in addition, the dynamical system admits a stationary probability measure, in mean, the increments are no longer independent, however, they are in some sense stationary.

As it is noticed in [20], this model is actually a generalization of other well known models such as Random Walk With Internal Degree of Freedom — see [27] — or also in the modern formulation Markov additive processes (or semi-Markov processes, Markov random walks), for short **MAP** — see [26, 19, 32, 30, 1, 41, 24]. In this context, many results have been proved. Though, the basic assumptions are generally too much restrictive to encompass the class of very inhomogeneous random walks.

Generally speaking, in the context of Markov additive processes, we may introduce the Fourier transform operator, denoted by  $\mathcal{F}_t$ ,  $t \in \mathbb{R}^d$ , which is a continuous perturbation of the Markov operator  $\mathring{P}$  defining the underlying dynamics. The rate of the return probability is then estimated, under mild conditions, by inverse Fourier transform. In the literature, the Markov operator  $\mathring{P}$  is usually supposed to admit an ergodic invariant probability measure and to be quasi-compact. In addition, by a perturbation argument the Fourier transform operator remains quasi-compact for all  $t$  in a neighborhood of the origin. It allows, under suitable moment conditions on the system of probability measures, to derive a Taylor expansion at the second order of the perturbed dominating eigenvalue  $\gamma(t)$  (whose the coefficients are given roughly by the mean and the variance operators). Finally, under an assumption on the spectrum of the Fourier transform operator outside a neighborhood of the origin, it can be concluded that all the needed stochastic information is actually contained in the nature of the singularity at zero of  $(1 - \gamma(t))^{-1}$  (note  $\gamma(0) = 1$ ). This has to be compared to the classical context of random

---

<sup>1</sup>In this sense, it is comparable to Random Dynamical Systems: instead of randomly composing transformations, we randomly compose random (continuous) group homomorphisms.

walks for which there is an integral test criterion involving a singularity of this kind — see for instance [10] and [39].

In this paper, the quasi-compactness condition of  $\mathring{P}$  is dropped. It is only assumed that  $\mathring{P}$  is irreducible, recurrent and aperiodic. The condition on the spectrum of  $\mathcal{F}_t$  for large perturbations remains similar but the nature of the singularity at the origin is analyzed *via* probabilistic methods (there is also a kind of Chung and Fuchs integral test involving  $(I - \mathcal{F}_t)^{-1}$  but, without a strong knowledge on the spectrum of  $\mathring{P}$ , it seems impracticable, see Section 1.4). These estimates give rise to a criterion for the type problem in terms of the summability of a series. More precisely, introduce the following quantities for  $n \geq 1$  and  $\omega \in \Omega$

$$\Sigma_n := \Sigma_n(\omega) = -\frac{1}{n} \sum_{k=0}^{n-1} \left[ \nabla^* \widehat{\nabla \mu^{T^k \omega}}(0) - (\widehat{\nabla \mu^{T^k \omega}}(0))^* (\widehat{\nabla \mu^{T^k \omega}}(0)) \right],$$

$$\Delta_n := \Delta_n(\omega) = -i \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} \widehat{\nabla \mu^{T^k \omega}}(0), \quad (1)$$

and the (almost surely finite) random variable  $\tau$  which is the first instant such that the rank  $\Sigma_n$  is maximal. The main theorem of this paper states (under the assumptions 1, 2, 3 and 4 given in Section 1) that the Markov additive process is transient or recurrent (see Definition 1.6) accordingly as

$$\sum_{n \geq 1} \frac{1}{n^{d/2}} \int_{\Omega} \mathbf{1}_{n \geq \tau} \exp \left( -\frac{1}{2} \Delta_n^* \Sigma_n^{-1} \Delta_n \right) d\lambda$$

is finite or infinite provided  $\tau$  has a finite expectation.

Thus, the main term of the series involves the expectation of a functional of two random variables. The recurrence or transience behavior is interpreted as the result of the competition of this two random quantities. More precisely, the process is recurrent if locally it is sufficiently diffusive in order to balance the instantaneous trend. It turns out that  $\Delta^* \Sigma^{-1} \Delta$  is in some circumstances a self-normalized martingale (see [13] for instance).

Inspired by models introduced in [4], the main theorem is applied to a new class of examples in Section 2. In facts, in these examples, depending on the moves of a simple random walk on  $\mathbb{Z}$  and an (deterministic or random) environment  $\{\varepsilon_x\}_{x \in \mathbb{Z}} \in \{-1, 1\}^{\mathbb{Z}}$ , the random walk admits a strong drift taking values in one of the quarter plane of  $\mathbb{Z}^2$ .

In [4] are studied in particular the simple random walks on graphs represented on the figure 1 below. For the graph (a) of Figure 1, denoted by  $\mathbb{L}$ , a random walker choose at random one of the nearest neighbour among North, South or West if the ordinate of its current position is odd, and the nearest neighbour among North, South and East if it is even. It is shown in [4] that this random walk is recurrent. For the graph (b) of Figure 1, denoted by  $\mathbb{H}$ , the possible movements for a random walker are toward the North, South and West on the upper half-plane and North, South and East otherwise. The resulting random walk is shown to be transient. Beyond their own interests in the classical probability theory, it is worth noting the study of these models was initially motivated by natural questions arising in quantum theory as described in [5].

For modified random walks studied in Section 2, very similar results are proved. In particular, it is shown that recurrence is rather exceptional whereas transience is typical.

## 1 The type problem for Markov additive processes

In this section, we shall consider the restricted class of Markov additive processes, for short **MAP**, that are stochastic processes whose jumps, taking values in an Abelian group  $\mathbf{G}$ , are

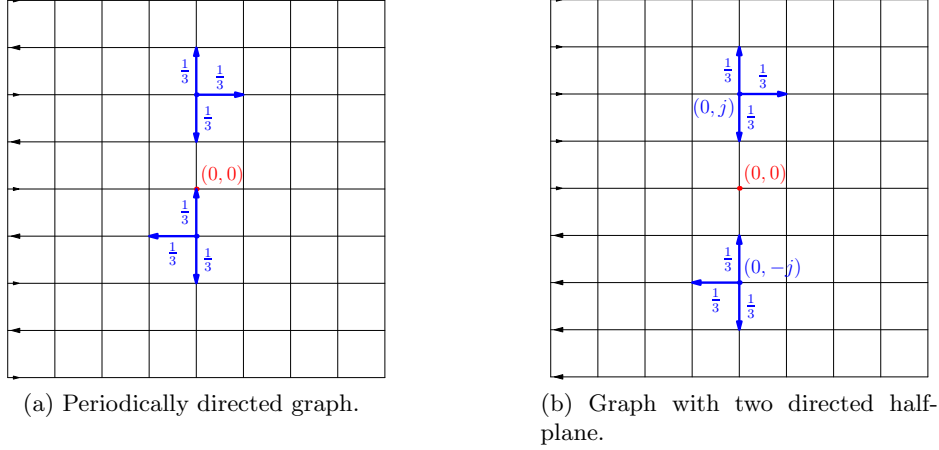


Figure 1: Directed graphs  $\mathbb{L}$  et  $\mathbb{H}$  de [4].

independent but with a distribution depending on a Markovian dynamics. We shall be even more restrictive by setting  $\mathbf{G} = \mathbb{Z}^d$ . For a non discrete Abelian group, say  $\mathbf{G} = \mathbb{R}^d$  or  $\mathbf{G}$  compact, the Fourier analysis of continuous measure is sensibly different but remains irrelevant for the scope of this paper.

### 1.1 Markov additive processes

In the sequel, we refer to [33] for the notion of Markov chain on general spaces. We shall consider a Markov kernel  $\dot{P}$  relatively to a measurable space  $(\mathbb{X}, \mathcal{X})$  for which the  $\sigma$ -algebra is separable. The space  $(\mathbb{X}, \mathcal{X})$  can be endowed with a  $\sigma$ -finite measure  $m$  dominating  $m\dot{P}$ . Without loss of generality, we may assume that the Markov chain induced by  $\dot{P}$  is  $m$ -irreducible and aperiodic. A stronger assumption is to suppose the Markov chain  $m$ -recurrent. In this case, the time shift on  $\mathbb{X}^{\mathbb{N}}$  preserves the possibly infinite, but  $\sigma$ -finite, Markov measure  $\mathbf{P}^m$ . The probability measures  $\mu^\omega$ ,  $\omega \in \mathbb{X}^{\mathbb{N}}$  are supposed to depend only on the two first coordinates<sup>2</sup> of  $\omega \in \mathbb{X}^{\mathbb{N}}$  and shall be alternatively denoted by  $\mu^{x,y}$ ,  $x, y \in \mathbb{X}$ .

A Markov additive process with internal Markov chain  $\dot{P}$  together with the system of probability measures  $\mu$  is the Markov chain on the space  $\mathbb{X} \times \mathbb{Z}^d$  whose Markov operator is given, for  $f \in \mathbf{L}_m^\infty(\mathbb{X} \times \mathbb{Z}^d)$ , by

$$\bar{P}f(x, \mathbf{u}) = \int_{\mathbb{X} \times \mathbb{Z}^d} \dot{P}(x, dy) (\delta_y \otimes \mu^{x,y}) (dzd\mathbf{v}) f(z, \mathbf{u} + \mathbf{v}), \quad \text{for } (x, \mathbf{u}) \in \mathbb{X} \times \mathbb{Z}^d.$$

The Markov operator  $\bar{P}$  is invariant by translations of the kind  $\mathbb{X} \times \mathbb{Z}^d \ni (x, \mathbf{u}) \rightarrow (x, \mathbf{u} + \mathbf{v}) \in \mathbb{X} \times \mathbb{Z}^d$ .

In this context, we may introduce the so called Fourier transform operator, denoted by  $\mathcal{F}_t$ , for  $t \in \mathbb{R}^d$ , acting as a contraction on  $\mathbf{L}_m^\infty(\mathbb{X})$  and defined for  $f \in \mathbf{L}_m^\infty(\mathbb{X})$  by

$$\mathcal{F}_t f(x) = \int_{\mathbb{X}} \dot{P}(x, dy) \widehat{\mu^{x,y}}(t) f(y), \quad \text{for } m - \text{a.e. } x \in \mathbb{X}, \quad (2)$$

<sup>2</sup>Actually, we use here the traditional definition, though it is completely equivalent to suppose that the probabilities  $\mu^\omega$  depends only on the first coordinate providing we consider the 2-order Markov chain induced by  $\dot{P}$ .

where  $\widehat{\mu^{x,y}}$  is the standard Fourier transform of the probability measure  $\mu^{x,y}$  defined, for  $t \in \mathbb{R}^d$ , by

$$\widehat{\mu^{x,y}}(t) = \sum_{\mathbf{v} \in \mathbb{Z}^d} \mu^{x,y}(\mathbf{v}) e^{i\langle t, \mathbf{v} \rangle},$$

$\langle \cdot, \cdot \rangle$  standing for the standard inner product.

## 1.2 Basic assumptions

Let  $\sigma$  be a bounded map from  $\mathbb{X}$  to  $\mathbb{Z}^d$  and define

$$\overline{P}_\sigma f(x, \mathbf{u}) = \int_{\mathbb{X} \times \mathbb{Z}^d} \dot{P}(x, dy) (\delta_y \otimes \mu^{x,y}) (dz d\mathbf{v}) f(z, \mathbf{u} + \mathbf{v} - \sigma(y) + \sigma(x)), \quad \text{for } (x, \mathbf{u}) \in \mathbb{X} \times \mathbb{Z}^d.$$

Intuitively, it corresponds to a change of origin of each fiber parametrized by  $\mathbb{X}$ . Such a function  $\sigma$  is called a change of section. The translated probability shall be denoted  $\mu_\sigma^{x,y} = \mu^{x,y} * \delta_{\sigma(y) - \sigma(x)}$ , or simply  $\mu^{x,y}$  if no ambiguity. Because of the invariance of  $\overline{P}$  under translations of  $\mathbb{Z}^d$ , the changes of section have no fundamental importance in the study of the recurrent or transient behavior of the Markov additive process.

To keep notations light and readable, we shall adopt alternatively the ones related to dynamical system (such as  $\mu^\omega$ ,  $\mathbf{P}^m$ -a.e.,...) or those ones proper to Markov additive processes ( $\mu^{x,y}$ ,  $m$ -a.e.,...).

**Definition 1.1** (Adaptation, aperiodicity, irreducibility). *A MAP is respectively adapted, aperiodic and irreducible if for any change of section  $\sigma$*

1. *there is no proper subgroup  $H \subset \mathbb{Z}^d$  such that  $\mathbf{P}^m$ -a.e.  $\mu^\omega(H) = 1$  ;*
2. *there is no proper subgroup  $H \subset \mathbb{Z}^d$  and no  $a \in \mathbb{R}^d$  such that  $\mathbf{P}^m$ -a.e.  $\mu^\omega(a + H) = 1$ .*
3. *there is no half-space  $H \subset \mathbb{R}^d$  such that  $\mathbf{P}^m$ -a.e.  $\mu^\omega(H) = 1$ .*

The properties of adaptation and aperiodicity can be read on the spectrum of the Fourier transform  $\mathcal{F}_t$  as shown in Corollary 1.3. Below  $\mathbf{W}_d$  denotes the set  $[-\pi, \pi]^d$  and  $\mathbf{W}_d^*$  the set  $\mathbf{W}_d \setminus \{0\}$ .

**Proposition 1.2.** *A MAP is adapted if and only if  $\widehat{\mu_\sigma^\omega}(t) \neq 1$ ,  $\mathbf{P}^m$ -a.e. for any change of section  $\sigma : \mathbb{X} \rightarrow \mathbb{Z}^d$  and any  $t \in \mathbf{W}_d^*$ . A MAP is aperiodic if and only if  $\widehat{\mu_\sigma^\omega}(t) \neq e^{i\theta}$ ,  $\mathbf{P}^m$ -a.e., for any  $\theta \in \mathbb{R}$ , any change of section  $\sigma$  and any  $t \in \mathbf{W}_d^*$ .*

*Proof.* If a MAP is not aperiodic then there exist a change of section, a proper subgroup  $H$  and  $a \in \mathbb{R}^d$  such that  $\mu_\sigma^\omega(a + H) = 1$ ,  $\mathbf{P}^m$ -a.e.. It is well known that there exist integers  $n_1, \dots, n_r$ ,  $1 \leq r \leq d$ , such that the subgroup  $H$  is generated by a set  $\{n_1 e_{i_1}, \dots, n_r e_{i_r}\}$  for some indices  $1 \leq i_1 \leq \dots \leq i_r \leq d$ , where  $e_i$  is the  $i^{\text{th}}$  vector of the canonical basis of  $\mathbb{Z}^d$ . Suppose  $r < d$ , then for any  $t \in \mathbf{W}_d^* \cap (\text{span } H)^\perp$  it follows that  $\widehat{\mu_\sigma^\omega}(t) = e^{i\langle t, a \rangle}$ . If  $r = d$ , set  $t = (\pi/n_1, \dots, \pi/n_d)$  then again  $\widehat{\mu_\sigma^\omega}(t) = e^{i\langle t, a \rangle}$  and  $t$  obviously belongs to  $\mathbf{W}_d^*$  unless  $n_1 = \dots = n_d = 1$ , i.e.  $H = \mathbb{Z}^d$ .

Conversely, suppose there exists a change of section  $\sigma$  and  $t \in \mathbf{W}_d^*$  such that  $\widehat{\mu_\sigma^\omega}(t) = e^{i\theta}$ ,  $\mathbf{P}^m$ -a.e., for some  $\theta \in \mathbb{R}$ . It means that  $\mu_\sigma^\omega(t)$  is an extremal convex combination and since  $\mu_\sigma^\omega$  is supported by a subset of  $\mathbb{Z}^d$ , one can choose  $a \in \mathbb{Z}^d$  such that  $\theta = \langle t, a \rangle$ . Then, for any  $n \geq 1$

$$e^{i\langle t, na \rangle} = \prod_{k=0}^{n-1} \widehat{\mu_\sigma^{T^k \omega}}(t) = \sum_{\mathbf{w} \in \mathbb{Z}^d} \mu^\omega * \dots * \mu^{T^{n-1} \omega}(\mathbf{w}) e^{i\langle t, \mathbf{w} \rangle}, \quad \mathbf{P}^m - \text{a.e.},$$

or equivalently,

$$1 = \sum_{\mathbf{w} \in \mathbb{Z}^d} \{\mu_\sigma^\omega * \delta_{-a}\} * \cdots * \{\mu_\sigma^{T^{n-1}\omega} * \delta_{-a}\}(\mathbf{w}) e^{i\langle t, \mathbf{w} \rangle}, \quad \mathbf{P}^m - a.e..$$

Thus, the convex combination of points on the unit circle on the right hand side is extremal so that each  $\mathbf{w} \in \mathcal{S}_n^\omega = \text{supp} \{\mu_\sigma^\omega * \delta_{-a}\} * \cdots * \{\mu_\sigma^{T^{n-1}\omega} * \delta_{-a}\}$ ,  $n \geq 1$ , satisfies  $\langle t, \mathbf{w} \rangle = 0$  modulo  $2\pi$ . Setting  $N = \{\omega \in \mathbb{X}^\mathbb{N} : \widehat{\mu_\sigma^\omega}(t) \neq e^{i\langle t, a \rangle}\}$  and defining  $H$  as the smallest subgroups containing the sets  $\mathcal{S}_n^\omega$ , for all  $n \geq 1$  and  $\omega \in N$ , it follows that,  $\mathbf{P}^m$ -a.e.,  $(\mu_\sigma^\omega * \delta_{-a})(H) = 1$ , i.e.  $\mu_\sigma^\omega(a + H) = 1$ . If the **MAP** was aperiodic, the group  $H$  should not be a strict subgroup of  $\mathbb{Z}^d$  so that  $\langle t, e_i \rangle = 0$  modulo  $2\pi$  for  $1 \leq i \leq d$ . Contradiction with the fact  $t \in \mathbf{W}_d^*$ .

The statement involving the adaptation property follows exactly the same lines by setting  $a = 0$  and  $\theta = 0$ .  $\square$

**Corollary 1.3.** *A **MAP** is adapted if and only if, for any  $t \in \mathbf{W}_d^*$  there exists a closed  $\mathcal{F}_t$ -invariant subspace  $E$  containing the constants such that one is not an eigenvalue of the operator  $\mathcal{F}_t$  acting on  $E$ . It is aperiodic if and only if, for all  $t \in \mathbf{W}_d^*$  there exists a closed  $\mathcal{F}_t$ -invariant subspace  $E$  containing the constants such that the operator  $\mathcal{F}_t$ , acting on  $E$ , has no eigenvalue of modulus one.*

**Remark 1.1.** *The proof of this corollary requires that the bounded superharmonic functions are constant. A natural assumption to ensure this property is the  $m$ -recurrence of the Markov operator  $\mathring{P}$  (see for instance [33, Proposition 3.13, p. 44]). In the terminology of [28], one might suppose  $\mathring{P}$  conservative and ergodic.*

*Proof.* Let  $t \in \mathbf{W}_d^*$  and suppose there exists  $f \in E$ , where  $E$  is any closed  $\mathcal{F}_t$ -invariant subspace containing the constants, such that

$$\mathcal{F}_t f(x) = \int_{\mathbb{X}} \mathring{P}(x, dy) \widehat{\mu_\sigma^{x,y}}(t) f(y) = e^{i\theta} f(x), \quad \text{for some } \theta \in \mathbb{R}.$$

By Jensen inequality and the fact that  $|\widehat{\mu_\sigma^{x,y}}(t)| \leq 1$ , it follows that  $|f| \leq \mathring{P}|f|$ . Thus, the function  $\|f\|_\infty - |f|$  is superharmonic and hence constant  $m$ -a.e. since  $\mathring{P}$  is supposed  $m$ -recurrent. As a consequence,

$$1 = \int_{\mathbb{X}} |\widehat{\mu_\sigma^{x,y}}(t)| \mathring{P}(x, dy), \quad m - a.e.,$$

and  $\widehat{\mu_\sigma^{x,y}}(t)$  is of modulus one,  $\mathbf{P}^m$ -a.e.. By Proposition 1.2, the **MAP** can not be aperiodic. Conversely, if the **MAP** is periodic, the same proposition implies  $e^{i\langle t, a \rangle}$  is an eigenvalue.

The proof of the statement involving the adaptation property follows exactly the same lines.  $\square$

**Remark 1.2.** *Since the probability measures  $\mu^\omega$  are supposed to be supported by  $\mathbb{Z}^d$ ,  $\mathbf{P}^m$ -a.e., it follows that the operator valued map  $t \rightarrow \mathcal{F}_t$  is  $2\pi$ -periodic along the directions given by the vectors of the standard basis of  $\mathbb{R}^d$ . In fact, the aperiodicity means that it can not be periodic with shorter periods.*

**Definition 1.4.** *A **MAP** is said to satisfy condition (S) if for any  $t \in \mathbf{W}_d^*$  there exists a closed  $\mathcal{F}_t$ -invariant subspace  $E$ , containing the constants, for which the set of spectral values of modulus 1 of  $\mathcal{F}_t$ , operating on  $E$ , consists of eigenvalues.*



**Remark 1.3.** *It is worth noting that the invariant space  $E$  appearing in the definition of condition (S) may vary with respect to  $t \in \mathbf{W}_d^*$ . Though, it is not clear if this flexibility is actually necessary. In a word, may we find an example of perturbations  $\mathcal{F}_t$  for which the condition (S) is satisfied but there is no common invariant space  $E$ , shared for every  $t \in \mathbf{W}_d^*$ , such that the spectral values of modulus one are eigenvalues ?*

**Remark 1.4.** *If the Markov operator  $\dot{P}$  is quasi-compact (as in most of the litterature) then the Fourier transform operator, as a continuous perturbation, is also quasi-compact for every  $t$  in a neighborhood of the origin. It is known that the set of spectral values of modulus one consists of isolated eigenvalues. Somehow, the condition (S) extends this property to large perturbations. Nonetheless, condition (S) can be satisfied without  $\dot{P}$  being quasi-compact which is the main motivation of this paper. In examples of Section 2, the Markov operator  $\dot{P}$  is that one of a symmetric nearest neighbor random walk which is not quasi-compact (otherwise, it is well known, for instance see [28], that  $\dot{P}$  would admit an invariant probability measure).*

For any closed subspace  $E$ , and any bounded operator  $Q$  we denote by  $\|Q\|_E$  the subordinated norm restricted to  $E$  defined as  $\|Q\|_E = \sup_{f \in E: \|f\|=1} \|Qf\|$ . Let  $t \in \mathbf{W}_d$ , we denoted by  $E_t$  the intersection of all closed subspaces  $E$  such that  $\mathcal{F}_t E \subset E$  and  $\mathbf{1} \in E$ . Obviously, the subspace  $E_t$  is itself closed, invariant and contains the constant. Also, we may define the pseudo spectral radius  $\tilde{r}(t)$  of  $\mathcal{F}_t$  by

$$\tilde{r}(t) = \lim_{n \rightarrow \infty} \|\mathcal{F}_t^n\|_{E_t}^{1/n}.$$

**Lemma 1.5.** *If a MAP is aperiodic and satisfy condition (S), then the pseudo spectral radius  $\tilde{r}(t)$  of  $\mathcal{F}_t$  is strictly smaller than one for  $t \in \mathbf{W}_d^*$ .*

*Proof.* As a matter of fact, the pseudo spectral radius can be defined alternatively as follows

$$\tilde{r}(t) = \inf \left\{ \|\mathcal{F}_t^n\|_E^{1/n}, n \geq 1, E \text{ closed subspace satisfying } \mathcal{F}_t E \subset E \text{ and } \mathbf{1} \in E \right\}.$$

Moreover, let  $t_0 \in \mathbf{W}_d$  and  $E$  be a closed subspace such that  $\mathcal{F}_{t_0} E \subset E$  and  $\mathbf{1} \in E$ . Then, by reverse triangle inequality,

$$\left| \|\mathcal{F}_{t_0}^n\|_E - \|\mathcal{F}_t^n\|_E \right| \leq \|\mathcal{F}_t^n - \mathcal{F}_{t_0}^n\|_E \leq \|\mathcal{F}_t^n - \mathcal{F}_{t_0}^n\|$$

which can be made arbitrarily small since  $t \rightarrow \mathcal{F}_t$  is continuous. It follows that  $\tilde{r}$  is the point-wise infimum of continuous functions, thus  $\tilde{r}$  is upper semi-continuous. Furthermore, the pseudo spectral radius reaches its maximum on compact  $K \subset \mathbf{W}_d$ . Aperiodicity together with condition (S) imply that  $\max_{t \in K} \tilde{r}(t) < 1$  excepted when  $0 \in K$ .  $\square$

From now on, we shall assume the following.

**Assumptions 1.** *The MAP is adapted, aperiodic and irreducible.*

**Assumptions 2.** *The condition (S) is fulfilled.*

Additionally, we make the following assumption on the system of probability measure  $\mu$  where  $|\cdot|$  stands for the Euclidean norm.

**Assumptions 3.** *Assume that the system of probability measures  $\mu$  admits a uniform third order moment, that is*

$$\left\| \sum_{\mathbf{u} \in \mathbb{Z}^d} |\mathbf{u}|^3 \mu(\mathbf{u}) \right\|_{\infty} < \infty.$$



**Remark 1.5.** Recall that a change of section is a bounded function  $\sigma : \mathbb{X} \rightarrow \mathbb{Z}^d$ , thus Assumption 3 needs not to be stated relatively to any change of section.

For  $n \geq 1$  and  $\mathbf{u}, \mathbf{v} \in \mathbb{Z}^d$ , denote respectively the  $n$ -step transition probability and its Fourier transform by

$$P_{0,n}^\omega(\mathbf{u}, \mathbf{v}) := \mu^\omega * \cdots * \mu^{T^{n-1}\omega}(\mathbf{v} - \mathbf{u}) \quad \text{and} \quad \phi_{0,n}^\omega(t) = \prod_{k=0}^{n-1} \widehat{\mu^{T^k\omega}}(t). \quad (3)$$

Introduce the quantities

$$\Sigma_n(\omega) := -\frac{1}{n} \sum_{k=0}^{n-1} \left[ \nabla^* \nabla \widehat{\mu^{T^k\omega}}(0) - (\nabla \widehat{\mu^{T^k\omega}}(0))^* (\nabla \widehat{\mu^{T^k\omega}}(0)) \right]$$

$$\text{and} \quad \Delta_n(\omega) := -i \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} \nabla \widehat{\mu^{T^k\omega}}(0). \quad (4)$$

**Assumptions 4.** There exists  $\alpha > 0$  such that for all  $t \in \mathbb{R}^d$ ,  $\liminf_{n \rightarrow \infty} t^* \Sigma_n t \geq \alpha |t|^2$ ,  $\mathbf{P}^m$ -a.e..

### 1.3 The type problem : a series criterion

Let  $A \subset \mathbb{X}$  and  $K \subset \mathbb{Z}^d$  be measurable subsets. We are interested in the mean time spent by the Markov additive process — that is nothing but a Markov chain on  $\mathbb{X} \times \mathbb{Z}^d$  — in the product set  $A \times K$ . In fact, it is well known that this quantity, starting from  $(x, \mathbf{u}) \in \mathbb{X} \times \mathbb{Z}^d$ , is actually given by the Green operator defined by

$$G\mathbf{1}_{A \times K}(x, \mathbf{u}) = \sum_{n \geq 0} \bar{P}^n \mathbf{1}_{A \times K}(x, \mathbf{u}).$$

In the special case of  $A = \mathbb{X}$ , using notation of Equation (3), the equation above rewrites

$$G\mathbf{1}_{\mathbb{X} \times K}(x, \mathbf{u}) = \mathbf{1}_{\mathbb{X} \times K}(x, \mathbf{u}) + \sum_{\mathbf{v} \in \mathbb{Z}^d} \sum_{n \geq 1} \int_{\mathbb{X}^{\mathbb{N}}} P_{0,n}(u, v) \mathbf{1}_K(v) d\mathbf{P}^x.$$

**Definition 1.6** (Recurrence and Transience). A Markov additive process is said to be recurrent (resp. transient) if, for any bounded change of section  $\sigma$ ,  $G\mathbf{1}_{\mathbb{X} \times \{0\}}(x, 0) = \infty$ ,  $m$ -a.e. (resp.  $G\mathbf{1}_{\mathbb{X} \times \{0\}}(x, 0) < \infty$ ,  $m$ -a.e.).

**Remark 1.6.** Since the changes of section are supposed bounded, a MAP is simultaneously recurrent or transient for every change of section.

A simple computation gives rise to the identities

$$\begin{aligned} G\mathbf{1}_{\mathbb{X} \times \{0\}}(x, 0) &= 1 + \sum_{n \geq 1} \int_{\mathbb{X}^{\mathbb{N}}} P_{0,n}(0, 0) d\mathbf{P}^x = 1 + \lim_{r \uparrow 1} \sum_{n \geq 1} r^{n-1} \int_{\mathbb{X}^{\mathbb{N}}} P_{0,n}(0, 0) d\mathbf{P}^x \\ &= 1 + \lim_{r \uparrow 1} \frac{1}{(2\pi)^d} \int_{\mathbf{W}_d} \operatorname{Re} \sum_{n \geq 1} r^{n-1} \mathcal{F}_t^n \mathbf{1}(x) dt \end{aligned} \quad (5)$$

where the latter equality is obtained by inverse Fourier transform and Fubini's theorem.

**Theorem 1.7** (Series criterion). *Let  $d \geq 2$  and suppose that the assumptions 1, 2, 3 and 4 are satisfied. In addition, let  $\tau$  be the stopping time defined by  $\tau = \inf\{n \geq 1 : \text{rk } \Sigma_n = d\}$  and assume  $\mathbf{E}^x(\tau) < \infty$  for  $m$ -a.e.  $x \in \mathbb{X}$ . Then,  $G\mathbf{1}_{\mathbb{X} \times \{0\}}(x, 0)$  is finite (resp. infinite) if and only if*

$$\sum_{n \geq 1} \frac{1}{n^{d/2}} \int_{\mathbb{X}^{\mathbb{N}}} \mathbf{1}_{n \geq \tau} \exp\left(-\frac{1}{2} \Delta_n^* \Sigma_n^{-1} \Delta_n\right) d\mathbf{P}^x < \infty \text{ (resp. } = \infty \text{),}$$

and the **MAP** is transient or recurrent accordingly.

The proof of this theorem is postponed to Section 3. This section is ended with few comments.

#### 1.4 Few comments on the main theorem

In Theorem 1.7, there is a rather technical assumption on the integrability of the stopping time  $\tau$ , the first time at which the matrix  $\Sigma_n$  is invertible so that the quantities  $I_{1,2}(n)$  and  $I_{1,3}(n)$  in Section 3 are well-defined. Due to Assumption 1 and Proposition 3.3, this stopping time  $\tau$  is  $\mathbf{P}^x$ -a.s. finite for  $m$ -a.e.  $x \in \mathbb{X}$ . The integrability of  $\tau$  is required to deduce estimates in mean from almost-sure estimates. For instance, for the simple random walk on a comb (a subgraph of  $\mathbb{Z}^2$  with two parallel infinite half-lines joined by an edge at the origine), the related **MAP** should not enjoy this integrability condition.

The formulation of the criterion may seem non conventional and, inspired by [10], one may think about a criterion stating a **MAP** is recurrent or transient depending on

$$\lim_{r \uparrow 1} \frac{1}{(2\pi)^{d/2}} \text{Re} \int_{\mathbf{W}_d} (I - r\mathcal{F}_t)^{-1} \mathbf{1}(x) dt$$

is finite or infinite (and even without the  $r$ -limit adapting the proof of [39] to our context at the cost of a long and rather technical proof involving potential theory arguments). The main concern is related to the analysis of the invert of  $(I - \mathcal{F}_t)$  without strong conditions giving informations about the spectrum in the neighborhood of the origin.

Finally, in the equality (5), an  $r$ -limit is introduced allowing several exchanges of limits, integrals and summations in the computations. At some point in the proof of Theorem 1.7, the  $r$ -limit can be dropped using monotone convergence so that the final criterion obtained here is stated without this  $r$ -limit. In some circumstances, one may be interested in dropping this limit inside the computations in order to obtain other estimates that might be relevant for other problematics beyond the type problem.

## 2 Applications : Markov additive processes with strong local drift

In this section, the main result of this paper is applied to some examples inspired by [4]. More precisely, we consider the simple random walk  $\{S_n\}_{n \geq 0}$  with *i.i.d.* increments whose common distribution is given by the probability  $(\delta_{-1} + \delta_1)/2$ . As this random walk is null recurrent, the corresponding Markov operator can not be quasi-compact (see [28]). Let  $\varepsilon = \{\varepsilon_x\}_{x \in \mathbb{Z}}$  be a random or deterministic sequence taking value in  $\{-1, 1\}$ .

There is obviously many ways to define a **MAP** on  $\mathbb{Z}^2$ . Here, we consider simple examples for which the probability measures  $\mu^{x,y}$  are of the form  $\mu^{x,y} = \mu^{\varepsilon_x, \text{sgn}(y-x)}$ ,  $x, y \in \mathbb{Z}$  for some fixed measures  $\{\mu^{p,q}\}_{p,q \in \{-1,1\}}$ . In order to exhibit transient behavior, we follow the ideas in [4] by supposing that  $\mu^{p,q}$  is supported in the quarter plane  $p\mathbb{N} \times q\mathbb{N}$ . Thus, on one hand,

the vertical component of the Markov additive process will merely follow the moves of the underlying random walk in the sense they go north or south simultaneously. On the other hand, the horizontal component moves in the direction dictated by  $\varepsilon$ .

From now on, it is supposed, for all  $p, q \in \{-1, 1\}$ , that

1. the measures  $\mu^{p,q}$  admit a third moment,
2.  $\mu^{p,q}$  are aperiodic in the sense  $|\widehat{\mu^{p,q}}(t)| = 1$  on  $\mathbf{W}_d$  if and only if  $t = 0$ ,
3. The map  $x \rightarrow \varepsilon_x \in \{-1, 1\}$  is surjective.

These three assumptions implies the related **MAP** satisfied assumptions 1, 3 and 4. Besides, remark the random time  $\tau$  of the theorem is deterministic equal to one. Finally, as it follows from the proposition below, the condition (S) is satisfied.

**Proposition 2.1.** *The condition (S) is fulfilled.*

*Proof.* The quantity

$$s(t) = \max \left\{ |\widehat{\mu^{p,q}}(t)|, (p, q) \in \{-1, 1\}^2 \right\}, \quad t \in \mathbf{W}_d$$

is continuous and  $s(t) = 1$  if and only if  $t = 0$  in  $\mathbf{W}_d$  since  $\mu^{p,q}$ ,  $p, q \in \{-1, 1\}$ , are aperiodic. Besides, the Fourier transform operator acts as a contraction on  $\ell^\infty(\mathbb{Z})$ , moreover, for all  $x \in \mathbb{Z}$  and  $f \in \ell^\infty$  :

$$\mathcal{F}_t f(x) = \frac{1}{2} \widehat{\mu^{\varepsilon_x, 1}}(t) f(x+1) + \frac{1}{2} \widehat{\mu^{\varepsilon_x, -1}}(t) f(x-1).$$

As a matter of fact,  $\|\mathcal{F}_t\|_\infty \leq s(t)$  so that  $r(\mathcal{F}_t) \leq s(t)$ . Thus for  $t \in \mathbf{W}_d^*$ , the operator  $\mathcal{F}_t$  has no spectral values of modulus 1. In particular, it satisfies condition (S).  $\square$

Let  $T_{p,q}$  be a random variable with distribution  $\mu^{p,q}$ . For the sake of simplicity, we impose some symmetries to the problem : assume for all  $p, q \in \{-1, 1\}$  that  $T_{p,q} \stackrel{d}{=} (pT_h, qT_v)$  for some (non necessarily uncorrelated) random variables  $T_h$  and  $T_v$  taking values in  $\mathbb{N}^2$ . Then, the expectation and the variance matrix of  $T_{p,q}$  are respectively given by

$$\mathbf{E}(T_{p,q}) = \begin{pmatrix} p\mathbf{E}(T_h) \\ q\mathbf{E}(T_v) \end{pmatrix} \quad \text{and} \quad \mathbf{V}(T_{p,q}) = \begin{pmatrix} \mathbf{V}(T_h) & pq\text{Cov}(T_h, T_v) \\ pq\text{Cov}(T_h, T_v) & \mathbf{V}(T_v) \end{pmatrix},$$

where  $\text{Cov}(T_h, T_v)$  denotes the covariance between  $T_h$  and  $T_v$ . If we assume in addition that  $T_h$  and  $T_v$  are uncorrelated, then the off diagonal coefficients of the variance matrix are null and the resulting Markov additive process is then recurrent or transient accordingly as

$$\sum_{n \geq 1} \frac{1}{n} \int_{\mathbb{Z}^{\mathbb{N}}} \exp \left[ -\frac{\mathbf{E}(T_h)^2}{2n\mathbf{V}(T_h)} \left( \sum_{k=0}^{n-1} \varepsilon_{S_k} \right)^2 \right] \exp \left[ -\frac{\mathbf{E}(T_v)^2}{2n\mathbf{V}(T_v)} (S_n - S_0)^2 \right] d\mathbf{P}^x \quad (6)$$

is finite or infinite for some (any)  $x \in \mathbb{Z}$ .

**Proposition 2.2.** *Set  $\varepsilon_x = (-1)^x$  for all  $x \in \mathbb{Z}$ . Then, the resulting Markov additive process is recurrent.*

*Proof.* First, remark that

$$\left( \sum_{k=0}^{n-1} \varepsilon_{S_k} \right)^2 = \frac{1 + (-1)^{n-1}}{2}, \quad n \geq 1.$$

Thus,

$$\lim_{n \rightarrow \infty} \exp \left[ -\frac{\mathbf{E}(T_h)^2}{2n\mathbf{V}(T_h)} \left( \sum_{k=0}^{n-1} \varepsilon_{S_k} \right)^2 \right] = 1.$$

Secondly, note that  $(S_n - S_0)^2/n$  converges in distribution to  $N^2$  where  $N$  is a standard gaussian random variable. Since  $x \rightarrow \exp(-x)$  is continuous and bounded on  $\mathbb{R}_+$ , it follows that

$$\lim_{n \rightarrow \infty} \int_{\mathbb{Z}^{\mathbb{N}}} \exp \left[ -\frac{\mathbf{E}(T_v)^2}{2n\mathbf{V}(T_v)} (S_n - S_0)^2 \right] d\mathbf{P}^x \rightarrow \int_{\mathbb{R}} \exp \left[ -\frac{\mathbf{E}(T_v)^2}{2\mathbf{V}(T_v)} y^2 \right] \exp -\frac{y^2}{2} \frac{dy}{2\pi},$$

and the series in (6) is infinite.  $\square$

**Proposition 2.3.** *If, for all  $x \in \mathbb{Z}$ ,  $\varepsilon_x = \text{sgn } x$ , then the resulting Markov additive process is transient.*

In the sequel, it is useful to make the following transformation

$$\sum_{k=0}^{n-1} \varepsilon_{S_k} = \sum_{x \in \mathbb{Z}} N_n(x) \varepsilon_x \quad (7)$$

where  $N_n(x)$ ,  $x \in \mathbb{Z}$  and  $n \geq 1$  is the local time of  $\{S_n\}_{n \geq 0}$  up to time  $n - 1$  that is  $N_n(x) = \sum_{k=0}^{n-1} \mathbf{1}_{\{x\}}(S_k)$ .

*Proof.* As a matter of facts, an obvious majoration in (6) implies it suffices to prove that

$$\sum_{n \geq 1} \frac{1}{n} \int_{\mathbb{Z}^2} \exp \left[ -\frac{\mathbf{E}(T_h)^2}{2n\mathbf{V}(T_h)} \left( \sum_{k=0}^{n-1} \varepsilon_{S_k} \right)^2 \right] d\mathbf{P}^x \quad (8)$$

is finite for all  $x \in \mathbb{Z}$ . By Markov property and  $\mathbb{Z}^2$ -invariance on the Markov additive process, one can suppose without loss of generality  $x = 0$  and forget the exponent in  $\mathbf{P}^x$ .

Using the transformation (7) and remarking that  $\varepsilon = \text{sgn}$ , it follows that

$$\sum_{k=0}^{n-1} \varepsilon_x = 2N_n(\mathbb{Z}_+) - n.$$

Now, estimate the summand in (8)

$$\begin{aligned} \mathbf{E} \left[ \exp \left\{ -\frac{\mathbf{E}(T_h)^2}{2n\mathbf{V}(T_h)} (2N_n(\mathbb{Z}_+) - n)^2 \right\} \right] &\leq \left| \mathbf{E} \left[ \exp \left\{ -n \frac{\mathbf{E}(T_h)^2}{2\mathbf{V}(T_h)} \left( 2 \frac{N_n(\mathbb{Z}_+)}{n} - 1 \right)^2 \right\} \right] \right| \\ &\quad - \mathbf{E} \left[ \exp \left\{ -n \frac{\mathbf{E}(T_h)^2}{2\mathbf{V}(T_h)} (2\Gamma - 1)^2 \right\} \right] \\ &\quad + \mathbf{E} \left[ \exp \left\{ -n \frac{\mathbf{E}(T_h)^2}{2\mathbf{V}(T_h)} (2\Gamma - 1)^2 \right\} \right], \end{aligned}$$

where  $\Gamma$  is distributed as an arcsine law supported by  $[0, 1]$ . It turns out that for all  $n \geq 1$ , the function

$$[0, \infty) \ni x \rightarrow \exp\left(-\frac{n\mathbf{E}(T_h)^2}{2\mathbf{V}(T_h)}(2x-1)^2\right) \quad (9)$$

is  $k_n$ -Lipschitz with constant  $k_n = \sqrt{\frac{n\mathbf{E}(T_h)^2}{e\mathbf{V}(T_h)}}$ . Consequently, it follows a majoration of the first term on the right handside of (9) by  $k_n d_{\mathcal{W}}(\mathcal{L}(N_n(\mathbb{Z}_+)/n), \mathcal{L}(\Gamma))$  where  $d_{\mathcal{W}}$  denotes the Wasserstein metric on probability measure. Namely, for any measure  $\mu$  and  $\nu$  on  $\mathbb{R}$ , the Wasserstein metric is defined as

$$d_{\mathcal{W}}(\mu, \nu) = \sup_{h \in \mathcal{H}} \left| \int h d\mu - \int h d\nu \right|,$$

where  $\mathcal{H}$  stands for the set of 1-Lipschitz continuous functions. From [15], we deduce that this term is a  $O(n^{-1/2})$  for all  $n \geq 0$  large enough.

Besides, the second term on the right handside vanishes at a sub-exponential rate which ends the proofs of the proposition.  $\square$

We finish this series of examples by considering now that  $\varepsilon = \{\varepsilon_y\}_{y \in \mathbb{Z}}$  is an *i.i.d.* sequence of  $\mathbb{Z}$ -valued centered random variables uniformly bounded. In addition, suppose its common distribution is not supported by any proper subgroup of  $\mathbb{Z}$ . In the sequel, we denote by  $\mathbf{Q}$  the distribution of  $\{\varepsilon_y\}_{y \in \mathbb{Z}}$ .

The notations used in the previous examples extends easily. In fact,  $\mu^{x,y}$ ,  $x, y \in \mathbb{Z}$ , is the distribution of the vector  $(\varepsilon_x T_h, \mathbf{sgn}(y-x)T_v)$  where  $T_h$  and  $T_v$  are  $\mathbb{N}$ -valued random variable independent with  $\varepsilon$ . If we assume in addition that  $T_h$  and  $T_v$  are uncorrelated, one may check that

$$\Sigma_n = -\frac{1}{n} \sum_{k=0}^{n-1} \begin{pmatrix} \varepsilon_{S_k}^2 \mathbf{V}(T_h) & 0 \\ 0 & \mathbf{V}(T_v) \end{pmatrix} \quad \text{and} \quad \Delta_n = -\frac{i}{\sqrt{n}} \sum_{k=0}^{n-1} \begin{pmatrix} \varepsilon_{S_k} \mathbf{E}(T_h) \\ \mathbf{sgn}(S_{k+1} - S_k) \mathbf{E}(T_v) \end{pmatrix}.$$

Thus, the problem reduces to the study of

$$\sum_{n \geq 1} \frac{1}{n} \int_{\mathbb{Z}^{\mathbb{N}}} \exp \left\{ -\frac{\mathbf{E}(T_h)^2 \left( \sum_{k=0}^{n-1} \varepsilon_{S_k} \right)^2}{2\mathbf{V}(T_h) \left( \sum_{k=0}^{n-1} \varepsilon_{S_k}^2 \right)} \right\} \exp \left\{ -\frac{\mathbf{E}(T_v)^2}{2n\mathbf{V}(T_v)} (S_n - S_0)^2 \right\} d\mathbf{P}^x. \quad (10)$$

**Proposition 2.4.** *Suppose that there exists  $M > 0$  such that for all  $x \in \mathbb{Z}$ ,  $|\varepsilon_x| \leq M$   $\mathbf{Q}$ -a.s.. For  $\mathbf{Q}$ -a.e. sequences  $\{\varepsilon_y\}_{y \in \mathbb{Z}}$ , the random walk  $M$  is transient.*

*Proof.* Using Markov inequality and an obvious upper bound of the second factor in (10), it suffices to prove that the following series is finite for any  $x \in \mathbb{Z}$

$$\sum_{n \geq 1} \frac{1}{n} \int_{\mathbb{Z}^{\mathbb{Z}} \times \mathbb{Z}^{\mathbb{N}}} \exp \left\{ -\frac{\mathbf{E}(T_h)^2 \left( \sum_{k=0}^{n-1} \varepsilon_{S_k} \right)^2}{2\mathbf{V}(T_h) \left( \sum_{k=0}^{n-1} \varepsilon_{S_k}^2 \right)} \right\} d\mathbf{Q} \otimes \mathbf{P}^x.$$

The proof follows four steps.

**Step 1:** As a matter of facts, by assumption, one get

$$\sum_{k=0}^{n-1} \varepsilon_{S_k}^2 \leq nM^2 \leq n^{1+\delta} M^2,$$

for any  $\delta > 0$ . Thus, it follows that

$$\exp \left\{ -\frac{\mathbf{E}(T_h)^2 \left( \sum_{k=0}^{n-1} \varepsilon_{S_k} \right)^2}{2\mathbf{V}(T_h) \left( \sum_{k=0}^{n-1} \varepsilon_{S_k}^2 \right)} \right\} \leq \exp \left\{ -\frac{\mathbf{E}(T_h)^2 \left( \sum_{k=0}^{n-1} \varepsilon_{S_k} \right)^2}{2\mathbf{V}(T_h) M^2 n^{1+\delta}} \right\} \quad (11)$$

Then, using (7), and setting for  $n \geq 1$  and  $x \in \mathbb{R}$

$$f_n(x) = \exp \left\{ -\frac{\mathbf{E}(T_h)^2 n^{1/2-\delta}}{2\mathbf{V}(T_h) M^2} x^2 \right\},$$

the right handside of (11) rewrites

$$f_n \left( n^{-3/4} \sum_{x \in \mathbb{Z}} N_n(x) \varepsilon_x \right).$$

**Step 2:** It turns out that for each  $n \geq 1$ , the function is  $k_n$ -Lipschitz with  $k_n = kM^{-1}n^{1/4-\delta/2}$  where  $k$  is a deterministic constant independent of  $n$ .

Now, from [25], it follows that

$$n^{-3/4} \sum_{x \in \mathbb{Z}} N_n(x) \varepsilon_x \xrightarrow{\mathcal{D}} \Delta_1,$$

where, for  $t \geq 0$ ,

$$\Delta_t = \int_{-\infty}^{\infty} L_t(x) dZ(x).$$

Here,  $L_t$  denotes the local time, up to time  $t$ , of a one dimensional Brownian motion independent of the bilateral one dimensional Brownian motion  $Z$ .

Decompose the right handside of (11) as follows

$$\mathbf{E}_{\mathbf{Q} \otimes \mathbf{P}^x} \left[ \exp \left\{ -\frac{\mathbf{E}(T_h)^2 \left( \sum_{k=0}^{n-1} \varepsilon_{S_k} \right)^2}{2\mathbf{V}(T_h) M^2 n^{1+\delta}} \right\} \right] \leq \left| \mathbf{E}_{\mathbf{Q} \otimes \mathbf{P}^x} \left[ f_n \left( n^{-3/4} \sum_{x \in \mathbb{Z}} N_n(x) \varepsilon_x \right) \right] - \mathbf{E}[f_n(\Delta_1)] \right| + \mathbf{E}[f_n(\Delta_1)]. \quad (12)$$

**Step 3:** Now, one may estimate the first term in the upper bound in (12). First remark that the stochastic process  $\{\Delta_t\}_{t \geq 0}$  is self-similar of index  $3/4$  (see [25]). That implies that for each  $n \geq 1$

$$\mathbf{E}[f_n(\Delta_1)] = \mathbf{E}[f_n(n^{-3/4} \Delta_n)].$$

Using the strong embedding given in [12], it turns there exists a coupling such that for any  $\eta > 0$

$$\sup_{0 \leq k \leq n} \left| \sum_{x \in \mathbb{Z}} N_k(x) \varepsilon_x - \Delta_k \right| = o(n^{5/8+\eta}).$$

In particular, using this coupling and renormalizing by  $n^{3/4}$ , we get the first term of the right handside (12) is equal to  $o(k_n n^{-1/8+\eta})$ . Now, choose  $\delta = 3/8$ ,  $\eta = 1/32$ , it gives the first term is a  $o(n^{-1/32})$ .

**Step 4:** Since  $\Delta_1$  is positive almost-surely (see [6] for an expression of the density), applying the theorem of dominated convergence, the second term vanishes at a higher rate than any polynomial. □

### 3 Proof of the main theorem

This section is devoted to the proof of Theorem 1.7. We shall be interested in the asymptotics as  $n$  goes to infinity of

$$\int_{\mathbb{X}^{\mathbb{N}}} P_{0,n}(\mathbf{u}, \mathbf{v}) d\mathbf{P}^x = \frac{1}{(2\pi)^d} \int_{\mathbf{W}_d} \mathcal{F}_t^n \mathbf{1}(x) e^{-i\langle t, \mathbf{v} - \mathbf{u} \rangle} dt$$

and more specifically when  $\mathbf{u} = \mathbf{v}$ .

Similarly to the context of classical random walks, let  $\delta > 0$  and split

$$\int_{\mathbf{W}_d} \mathcal{F}_t^n \mathbf{1}(x) e^{-i\langle t, \mathbf{v} - \mathbf{u} \rangle} dt = \underbrace{\int_{(-\delta, \delta)^d} \mathcal{F}_t^n \mathbf{1}(x) e^{-i\langle t, \mathbf{v} - \mathbf{u} \rangle} dt}_{I_1(n)} + \underbrace{\int_{\mathbf{W}_d \setminus (-\delta, \delta)^d} \mathcal{F}_t^n \mathbf{1}(x) e^{-i\langle t, \mathbf{v} - \mathbf{u} \rangle} dt}_{I_2(n)}. \quad (13)$$

#### 3.1 Estimates far from the origin

**Proposition 3.1.** *Under Assumptions 1 and 2, there exist constants  $\kappa \in (0, 1)$  and  $C > 0$  such that for all  $\mathbf{v}, \mathbf{u} \in \mathbb{Z}^d$  and all  $\delta > 0$  sufficiently small*

$$\left\| \int_{\mathbf{W}_d \setminus (-\delta, \delta)^d} \mathcal{F}_t^n \mathbf{1}(\cdot) e^{-i\langle t, \mathbf{v} - \mathbf{u} \rangle} dt \right\|_{\infty} \leq C\kappa^n.$$

*Proof.* By Lemma 1.5, under Assumptions 1 and 2, setting  $K := \mathbf{W}_d^* \setminus (-\delta, \delta)^d$ , it follows that  $M := \max\{\tilde{r}(t) : t \in K\} < 1$ . Set  $\kappa := (M + 1)/2$ . It is a matter of fact that  $\kappa^{-n} \|\mathcal{F}_t^n\|_{E_t}$  vanishes as  $n$  goes to infinity so it is for  $\kappa^{-n} \|\mathcal{F}_t^n \mathbf{1}\|_{\infty}$  and the result follows.  $\square$

**Remark 3.1.** *In the literature, it is usually considered that the whole family  $\{\mathcal{F}_t\}_{t \in K}$  shares the same invariant space  $E$ . Allowing different spaces for different points  $t \in K$  improves the estimate of Proposition 3.1 while the pseudo spectral radius  $\tilde{r}$  still satisfies the nice property of upper semi-continuity for continuous perturbations of operators.*

#### 3.2 Estimates in the neighborhood of the origin

At this level, it only remains to estimate the first integral term of Equation (13). Setting  $t = \frac{u}{\sqrt{n}}$ , it is given by

$$I_1(n) = \frac{1}{n^{d/2}} \int_{(-\delta\sqrt{n}, \delta\sqrt{n})^d} \mathcal{F}_{\frac{u}{\sqrt{n}}}^n \mathbf{1}(x) e^{-i\langle u/\sqrt{n}, \mathbf{v} - \mathbf{u} \rangle} du.$$

The expected estimate shall be obtained by integrating an almost-sure estimate exploiting the following expression

$$I_1(n) = \underbrace{\int_{\mathbb{X}^{\mathbb{N}}} n^{-d/2} \int_{|u| \leq \delta\sqrt{n}} \phi_{0,n}(u/\sqrt{n}) e^{-i\langle u/\sqrt{n}, \mathbf{v} - \mathbf{u} \rangle} du}_{I_1^{\omega}(n)} d\mathbf{P}^x. \quad (14)$$

**Proposition 3.2.** *Under Assumption 3, there exists a deterministic  $\delta > 0$  such that for  $|t| \leq \delta$ , the quantity  $\log \phi_{0,1}(t)$  is well defined. In addition, the following approximation formula holds  $\mathbf{P}^m$ -a.e.*

$$\log \phi_{0,1}(t) = \nabla \phi_{0,1}(0)t + \frac{1}{2} t^* [\nabla^* \nabla \phi_{0,1}(0) - \nabla \phi_{0,1}(0)^* \nabla \phi_{0,1}(0)] t + R(t), \quad (15)$$

where the remaining term  $R$  satisfies for  $|t| \leq \delta$  and for any  $\epsilon \in [0, 1)$  :

$$\|R(t)\|_{\infty} \leq \delta^{1-\epsilon} K |t|^{2+\epsilon} \text{ with } K \geq 0.$$



*Proof.* Under Assumption 3, the function  $\phi_{0,1}$  is  $\mathbf{P}^m$ -a.e. three times continuously differentiable. Therefore, the following majoration holds  $\mathbf{P}^m$ -a.e.

$$|\phi_{0,1}(t) - 1| \leq |t| \|\nabla \phi_{0,1}(0)\|_\infty.$$

Thus, there exists a deterministic  $\delta > 0$  such that for all  $|t| \leq \delta$  the function  $t \rightarrow \log \phi_{0,1}^\omega(t)$  is well defined. In addition, for  $|t| \leq \delta$ , the Taylor formula yields  $\mathbf{P}^m$ -a.e.

$$\log \phi_{0,1}(t) = \nabla \phi_{0,1}(0)t + \frac{1}{2}t^* [\nabla^* \nabla \phi_{0,1}(0) - \nabla \phi_{0,1}(0)^* \nabla \phi_{0,1}(0)] t + R(t),$$

where the remaining term  $R$  satisfies for  $|t| \leq \delta$  and any  $\epsilon \in [0, 1)$

$$\|R(t)\|_\infty \leq \delta^{1-\epsilon} K \|t\|^{2+\epsilon}, \quad \text{with } K = \left\| \sum_{\mathbf{u} \in \mathbb{Z}^d} \|\mathbf{u}\|^3 \mu(\mathbf{u}) \right\|_\infty.$$

□

This proposition implies that for  $\delta > 0$  sufficiently small and  $|u| \leq \delta\sqrt{n}$

$$\phi_{0,n}(t) = \exp \left\{ i \langle \Delta_n, t \rangle - \frac{1}{2} t^* \Sigma_n t + R_n(t) \right\},$$

with the notation

$$R_n(t) := R_n^\omega(t) = \sum_{k=0}^{n-1} R^{T^k \omega}(t/\sqrt{n}).$$

**Proposition 3.3.** *Under Assumption 3 the following properties hold*

1. for all  $n \geq 1$ , the matrix  $\Sigma_n$  is real positive symmetric  $\mathbf{P}^m$ -a.e.,
2. the sequence  $(\Sigma_n)_{n \geq 0}$  remains bounded in the following sense

$$\sup_{n \geq 0} \|\Sigma_n\|_\infty < \infty,$$

3.  $\mathbf{P}^m$ -a.e., the rank  $\text{rk } \Sigma_n$  is non decreasing with  $n \geq 1$ ,
4. in addition, under Assumption 1,  $\lim_{n \rightarrow \infty} \text{rk } \Sigma_n = d$ ,  $\mathbf{P}^m$ -a.e..

*Proof.* 1. Under Assumption 3,  $n\Sigma_n$  is a sum of covariance matrices so is positive semidefinite real symmetric.

2. As a matter of facts, the sequence of matrices  $\Sigma_n$  satisfies

$$\text{ess sup}_{\omega \in \Omega} \|\Sigma_n\| \leq \text{ess sup}_{\omega \in \Omega} \|\Sigma_1\| < \infty.$$

3. As a sum of positive semidefinite real symmetric matrices, the kernel of  $\Sigma_n$  is given by

$$\ker \Sigma_n = \bigcap_{k=0}^{n-1} \ker \left[ \nabla^* \nabla \phi_{0,1}^{T^k \omega}(0) - (\nabla \phi_{0,1}^{T^k \omega}(0))^* (\nabla \phi_{0,1}^{T^k \omega}(0)) \right],$$

and the result follows.

4. Since  $\text{rk } \Sigma_n$  is a non decreasing discrete bounded sequence  $\mathbf{P}^m$ -a.e., it suffices to show that

$$\mathbf{P}^x(\liminf\{\text{rk } \Sigma_n = d\}) = \mathbf{P}^x\left(\bigcup_{n \geq 1}\{\text{rk } \Sigma_n = d\}\right) = 1, \quad \text{for } m\text{-a.e. } x \in \mathbb{X}. \quad (16)$$

Moreover, the operator  $P$  is supposed aperiodic and recurrent so that we only need the asymptotic event in (16) holds with positive probability. Thus suppose on the contrary

$$\mathbf{P}^x(N) = 1 \quad \text{with} \quad N = \bigcap_{n \geq 1}\{\text{rk } \Sigma_n \leq d - 1\}.$$

Then the subgroup  $H$  of  $\mathbb{Z}^d$  generated by the supports  $\text{supp } \mu^\omega * \dots * \mu^{T^{n-1}\omega}$ ,  $n \geq 1$ ,  $\omega \in N$ , is  $\mathbf{P}^x$ -a.s. independent of  $\omega \in \mathbb{X}^{\mathbb{N}}$  and satisfies  $\mu^\omega(H) = 1$ ,  $\mathbf{P}^m$ -a.e.. Assumption 1 yields  $H = \mathbb{Z}^d$  which contradicts the non maximality of the asymptotic rank.  $\square$

Recall  $\tau = \inf\{n \geq 1 : \text{rk } \Sigma_n = d\}$  and remark it is finite  $\mathbf{P}^m$ -a.e. by Proposition 3.3. Rewrite  $I_1^\omega(n)$  as the sum of three terms  $I_{11}, I_{12}$  and  $I_{13}$  defined by

$$I_{11}^\omega(n) = n^{-d/2} \int_{|t| \leq \delta\sqrt{n}} e^{-\frac{1}{2}t^* \Sigma_n t} e^{i\langle t, \Delta_n \rangle - i\langle t, (\mathbf{v} - \mathbf{u})/\sqrt{n} \rangle} \left( e^{R_n(t/\sqrt{n})} - 1 \right) dt,$$

$$\text{for } n \geq \tau, \quad I_{12}^\omega(n) = n^{-d/2} \int_{\mathbb{R}^d} e^{-\frac{1}{2}t^* \Sigma_n t} e^{i\langle t, \Delta_n \rangle - i\langle t, (\mathbf{v} - \mathbf{u})/\sqrt{n} \rangle} dt,$$

and,

$$\text{for } n \geq \tau, \quad I_{13}^\omega(n) = -n^{-d/2} \int_{|t| > \delta\sqrt{n}} e^{-\frac{1}{2}t^* \Sigma_n t} e^{i\langle t, \Delta_n \rangle - i\langle t, (\mathbf{v} - \mathbf{u})/\sqrt{n} \rangle} dt.$$

**Proposition 3.4.** *Under Assumptions 1 and 3 for all  $n \geq \tau$  the following holds  $\mathbf{P}^m$ -a.e. for  $\mathbf{u}, \mathbf{v} \in \mathbb{Z}^d$ :*

$$1. \quad I_{12}(n) = \frac{(2\pi)^{d/2}}{n^{d/2} \det(\Sigma_n)^{1/2}} \exp \left\{ \left( -\frac{1}{2}[\Delta_n - (\mathbf{v} - \mathbf{u})/\sqrt{n}]^* \Sigma_n^{-1} [\Delta_n - (\mathbf{v} - \mathbf{u})/\sqrt{n}] \right) \right\}.$$

If additionally Assumption 4 is fulfilled, uniformly in  $\omega \in \Omega$ ,

$$2. \quad \text{there exists } \alpha > 0 \text{ such that } |I_{13}(n)| \leq \mathcal{O}\left(n^{-d/2} \exp\{-\frac{1}{2}\alpha\delta\sqrt{n}\}\right),$$

$$3. \quad \text{then there exists a deterministic } \delta > 0 \text{ such that } |I_{11}(n)| = \mathcal{O}\left(n^{-\frac{d+\epsilon}{2}}\right) \text{ for any } \epsilon \in [0, 1).$$

*Proof.* First, set  $\mathbf{v} = \mathbf{u}$ .

1. Under Assumptions 1 and 3, Proposition 3.3 implies  $\tau < \infty$   $\mathbf{P}^m$ -a.e. and for  $n \geq \tau$ , there exist orthogonal matrices  $P_n$  and diagonal matrices  $D_n$  such that

$$\Sigma_n = P_n D_n P_n^{-1} \quad \text{and} \quad D_n = \text{diag}(\alpha_1^2(n), \dots, \alpha_d^2(n))$$

with  $\alpha_i^2(n) > 0$  for all  $i = 1, \dots, d$  and  $n \geq \tau$ . Setting  $t = P_n u$ , we obtain as the Fourier transform of Gaussian vectors

$$\begin{aligned} n^{d/2} I_{12}(n) &= \int_{\mathbb{R}^d} e^{-\frac{1}{2}u^* D_n u} e^{i\langle u, P_n^* \Delta_n \rangle} du \\ &= \det(\Sigma_n)^{-1/2} (2\pi)^{d/2} e^{-\frac{1}{2} \Delta_n^* \Sigma_n^{-1} \Delta_n}. \end{aligned}$$

2. For the term  $I_{13}(n)$  we can proceed analogously and we get the following (not sharp) upper bound for  $n \geq \tau$  :

$$\begin{aligned} n^{d/2}|I_{13}(n)| &\leq \int_{|u|>\delta\sqrt{n}} e^{-\frac{1}{2}u^*D_nu} du \\ &= \mathcal{O}\left(\exp\left\{-\frac{1}{2}\alpha\delta\sqrt{n}\right\}\right), \end{aligned}$$

for  $\alpha > 0$  of Assumption 4.

3. Because of the point (2) of Proposition 3.3, the eigenvalues of  $\Sigma_n$  remain bounded uniformly for  $n \geq 0$ . Thus, with Assumption 4, we deduce the following bound for  $I_{11}(n)$

$$\begin{aligned} n^{d/2}|I_{11}(n)| &\leq \int_{|t|\leq\delta\sqrt{n}} e^{-\frac{1}{2}t^*\Sigma_n t} \left| \exp\{R_n(tn^{-1/2})\} - 1 \right| dt \\ &\leq \int_{|t|\leq\delta\sqrt{n}} e^{-\frac{1}{2}t^*\Sigma_n t} \left| R_n(tn^{-1/2}) \right| \exp\{|R_n(tn^{-1/2})|\} dt \\ &\leq \frac{K\delta^{1-\epsilon}}{n^{\epsilon/2}} \int_{|t|\leq\delta\sqrt{n}} e^{-\frac{1}{2}t^*\Sigma_n t} |t|^{2+\epsilon} e^{|t|^2\delta K} dt. \end{aligned}$$

The last estimates comes from Proposition 3.2 and holds for any  $\epsilon \in [0, 1)$ . We conclude by choosing  $\delta > 0$  such that  $\delta K \leq \alpha/4$  (where  $\alpha$  is given by Assumption 4). Consequently the integral is convergent and the whole term goes to zero at rate, up to a constant,  $n^{-\epsilon/2}$ .

Setting  $\tilde{\Delta}_n = \Delta_n - (\mathbf{v} - \mathbf{u})/\sqrt{n}$ , the result follows for the general case.  $\square$

*Proof of Theorem 1.7.* Let  $r \in (0, 1)$ , and compute

$$\begin{aligned} &\int_{\mathbf{W}_d} \operatorname{Re} \sum_{n \geq 1} r^{n-1} \mathcal{F}_t^n \mathbf{1}(x) dt - \sum_{n \geq 1} r^{n-1} \int_{\mathbb{X}^{\mathbb{N}}} \mathbf{1}_{\{n \geq \tau\}} I_{12}(n) d\mathbf{P}^x \\ &= \sum_{n \geq 1} r^{n-1} \int_{\mathbb{X}^{\mathbb{N}}} \mathbf{1}_{\{n \geq \tau\}} [I_{11}(n) + I_{13}(n)] d\mathbf{P}^x \\ &+ \sum_{n \geq 1} r^{n-1} I_2(n) \\ &+ \sum_{n \geq 1} r^{n-1} \int_{\mathbb{X}^{\mathbb{N}}} \mathbf{1}_{\{n < \tau\}} I_1(n) d\mathbf{P}^x \end{aligned}$$

Taking absolute values on both side, under Assumptions 1, 2, 3 and 4, using Proposition 3.1, Proposition 3.4 with a suitable  $\delta > 0$ , it follows that, for some  $K \geq 0$ ,  $C > 0$ ,  $\kappa \in (0, 1)$  and any  $\epsilon \in (0, 1)$ ,

$$\begin{aligned} &\left| \int_{\mathbf{W}_d} \operatorname{Re} \sum_{n \geq 1} r^{n-1} \mathcal{F}_t^n \mathbf{1}(x) dt - \sum_{n \geq 1} r^{n-1} \int_{\mathbb{X}^{\mathbb{N}}} \frac{\mathbf{1}_{\{n \geq \tau\}}}{\det(\Sigma_n)n^{d/2}} \exp\left(-\frac{1}{2}\Delta_n^* \Sigma_n^{-1} \Delta_n\right) d\mathbf{P}^x \right| \\ &\leq \sum_{n \geq 1} r^{n-1} \frac{K}{n^{(d+\epsilon)/2}} + \sum_{n \geq 1} r^{n-1} \frac{K}{n^{d/2}} \exp\left\{-\frac{1}{2}\alpha\delta\sqrt{n}\right\} \\ &+ C \sum_{n \geq 1} \kappa^{n-1} + (2\pi)^d \sum_{n \geq 1} r^{n-1} \mathbf{P}^x(\tau > n). \end{aligned}$$

The latter summation follows from the fact  $|I_1(n)| \leq (2\pi)^d$ ,  $\mathbf{P}^m$ -a.e.. Then, letting  $r \uparrow 1$ , the right handside remains bounded in  $\mathbf{L}_m^\infty(\mathbb{X})$  since  $d \geq 2$  and  $\tau$  is integrable with respect to  $\mathbf{P}^x$ ,  $m$ -a.e.  $x \in \mathbb{X}$ . The result follows by letting  $r \uparrow 1$  on the left hand-side and remarking that

$$\begin{aligned} \lim_{r \uparrow 1} \sum_{n \geq 1} r^{n-1} \int_{\mathbb{X}^{\mathbb{N}}} \mathbf{1}_{n \geq \tau} \frac{1}{\det(\Sigma_n) n^{d/2}} \exp\left(-\frac{1}{2} \Delta_n^* \Sigma_n^{-1} \Delta_n\right) d\mathbf{P}^x \\ = \sum_{n \geq 1} \int_{\mathbb{X}^{\mathbb{N}}} \frac{\mathbf{1}_{n \geq \tau}}{\det(\Sigma_n) n^{d/2}} \exp\left(-\frac{1}{2} \Delta_n^* \Sigma_n^{-1} \Delta_n\right) d\mathbf{P}^x \in [0, \infty] \end{aligned}$$

$m$ -a.e. by monotone convergence. □

### Acknowledgements

The author thanks the referee for its careful reading of the paper and its useful remarks improving the readability.

The author acknowledges the financial support from the Swiss National Foundation Grant FN 200021-138242/1. This work was mainly done while the author holds a position at the University of Burgundy, the University of Strasbourg and at the Ensai.

### References

- [1] M. Babilot. Théorie du renouvellement pour des chaînes semi-markoviennes transientes. *Ann. Inst. H. Poincaré Probab. Statist.*, 24(4):507–569, 1988. URL: [http://www.numdam.org/item?id=AIHPB\\_1988\\_\\_24\\_4\\_507\\_0](http://www.numdam.org/item?id=AIHPB_1988__24_4_507_0). 2
- [2] M. Babilot. An introduction to Poisson boundaries of Lie groups. In *Probability measures on groups: recent directions and trends*, pages 1–90. Tata Inst. Fund. Res., Mumbai, 2006. 2
- [3] Itai Benjamini, Gady Kozma, and Bruno Schapira. A balanced excited random walk. *C. R. Math. Acad. Sci. Paris*, 349(7-8):459–462, 2011. URL: <http://dx.doi.org/10.1016/j.crma.2011.02.018>, doi:10.1016/j.crma.2011.02.018. 2
- [4] M. Campanino and D. Petritis. Random walks on randomly oriented lattices. *Markov Process. Related Fields*, 9(3):391–412, 2003. 2, 3, 4, 9
- [5] Massimo Campanino and Dimitri Petritis. On the physical relevance of random walks: an example of random walks on a randomly oriented lattice. In *Random walks and geometry*, pages 393–411. Walter de Gruyter GmbH & Co. KG, Berlin, 2004. 3
- [6] Fabienne Castell, Nadine Guillotin-Plantard, Françoise Pène, and Bruno Schapira. A local limit theorem for random walks in random scenery and on randomly oriented lattices. *Ann. Probab.*, 39(6):2079–2118, 2011. URL: <http://dx.doi.org/10.1214/10-AOP606>, doi:10.1214/10-AOP606. 13
- [7] P. Cénac, B. Chauvin, S. Herrmann, and P. Vallois. Persistent random walks, variable length Markov chains and piecewise deterministic Markov processes. *Markov Process. Related Fields*, 19(1):1–50, 2013. 2
- [8] Peggy Cénac, Brigitte Chauvin, Frédéric Paccaut, and Nicolas Pouyanne. Context trees, variable length Markov chains and dynamical sources. In *Séminaire de Probabilités XLIV*, volume 2046 of *Lecture Notes in Math.*, pages 1–39. Springer, Heidelberg, 2012. URL: [http://dx.doi.org/10.1007/978-3-642-27461-9\\_1](http://dx.doi.org/10.1007/978-3-642-27461-9_1), doi:10.1007/978-3-642-27461-9\_1. 2

- [9] Peggy Cénac, Arnaud Le Ny, Basile de Loynes, and Yoann Offret. Persistent random walks. i. recurrence versus transience. *Journal of Theoretical Probability*, pages 1–12, 2016. URL: <http://dx.doi.org/10.1007/s10959-016-0714-4>, doi:10.1007/s10959-016-0714-4. 2
- [10] K. L. Chung and W. H. J. Fuchs. On the distribution of values of sums of random variables. *Mem. Amer. Math. Soc.*, No. 6:12, 1951. 3, 9
- [11] Francis Comets, Mikhail V. Menshikov, Stanislav Volkov, and Andrew R. Wade. Random walk with barycentric self-interaction. *J. Stat. Phys.*, 143(5):855–888, 2011. URL: <http://dx.doi.org/10.1007/s10955-011-0218-7>, doi:10.1007/s10955-011-0218-7. 2
- [12] Endre Csáki, Wolfgang König, and Zhan Shi. An embedding for the Kesten-Spitzer random walk in random scenery. *Stochastic Process. Appl.*, 82(2):283–292, 1999. URL: [http://dx.doi.org/10.1016/S0304-4149\(99\)00036-8](http://dx.doi.org/10.1016/S0304-4149(99)00036-8), doi:10.1016/S0304-4149(99)00036-8. 13
- [13] Victor H. de la Peña, Michael J. Klass, and Tze Leung Lai. Theory and applications of multivariate self-normalized processes. *Stochastic Process. Appl.*, 119(12):4210–4227, 2009. URL: <http://dx.doi.org/10.1016/j.spa.2009.10.003>, doi:10.1016/j.spa.2009.10.003. 3
- [14] Basile de Loynes. Marche aléatoire sur un di-graphe et frontière de Martin. *C. R. Math. Acad. Sci. Paris*, 350(1-2):87–90, 2012. URL: <http://dx.doi.org/10.1016/j.crma.2011.12.005>, doi:10.1016/j.crma.2011.12.005. 2
- [15] C. Döbler. A rate of convergence for the arcsine law by Stein’s method. *ArXiv e-prints*, July 2012. URL: <https://arxiv.org/abs/1207.2401v1>, arXiv:1207.2401. 12
- [16] N. Guillin-Plantard and A. Le Ny. Transient random walks on 2D-oriented lattices. *Teor. Veroyatn. Primen.*, 52(4):815–826, 2007. URL: <http://dx.doi.org/10.1137/S0040585X97983353>, doi:10.1137/S0040585X97983353. 2
- [17] Nadine Guillin-Plantard and Arnaud Le Ny. A functional limit theorem for a 2D-random walk with dependent marginals. *Electron. Commun. Probab.*, 13:337–351, 2008. 2
- [18] Yves Guivarc’h. Groupes de Lie à croissance polynomiale. *C. R. Acad. Sci. Paris Sér. A-B*, 271:A237–A239, 1970. 2
- [19] Yves Guivarc’h. Application d’un théorème limite local à la transience et à la récurrence de marches de Markov. In *Théorie du potentiel (Orsay, 1983)*, volume 1096 of *Lecture Notes in Math.*, pages 301–332. Springer, Berlin, 1984. URL: <http://dx.doi.org/10.1007/BFb0100117>, doi:10.1007/BFb0100117. 2
- [20] V. A. Kaimanovich, Y. Kifer, and B.-Z. Rubshtein. Boundaries and harmonic functions for random walks with random transition probabilities. *J. Theoret. Probab.*, 17(3):605–646, 2004. 2
- [21] V. A. Kaimanovich and A. M. Vershik. Random walks on discrete groups: boundary and entropy. *Ann. Probab.*, 11(3):457–490, 1983. 2
- [22] Vadim A. Kaimanovich. Amenability and the Liouville property. *Israel J. Math.*, 149:45–85, 2005. Probability in mathematics. 2
- [23] Vadim A. Kaimanovich and Wolfgang Woess. Boundary and entropy of space homogeneous Markov chains. *Ann. Probab.*, 30(1):323–363, 2002. 2

- [24] Takahiro Kazami and Kohei Uchiyama. Random walks on periodic graphs. *Trans. Amer. Math. Soc.*, 360(11):6065–6087, 2008. URL: <http://dx.doi.org/10.1090/S0002-9947-08-04451-6>, doi:10.1090/S0002-9947-08-04451-6. 2
- [25] H. Kesten and F. Spitzer. A limit theorem related to a new class of self-similar processes. *Z. Wahrsch. Verw. Gebiete*, 50(1):5–25, 1979. URL: <http://dx.doi.org/10.1007/BF00535672>, doi:10.1007/BF00535672. 13
- [26] Harry Kesten. Renewal theory for functionals of a Markov chain with general state space. *Ann. Probability*, 2:355–386, 1974. 2
- [27] András Krámli and Domokos Szász. Random walks with internal degrees of freedom. I. Local limit theorems. *Z. Wahrsch. Verw. Gebiete*, 63(1):85–95, 1983. URL: <http://dx.doi.org/10.1007/BF00534179>, doi:10.1007/BF00534179. 2
- [28] Michael Lin. On quasi-compact Markov operators. *Ann. Probability*, 2:464–475, 1974. 6, 7, 9
- [29] G. A. Margulis. Positive harmonic functions on nilpotent groups. *Soviet Math. Dokl.*, 7:241–244, 1966. 2
- [30] P. Ney and E. Nummelin. Markov additive processes II. Large deviations. *Ann. Probab.*, 15(2):593–609, 1987. URL: [http://links.jstor.org/sici?sici=0091-1798\(198704\)15:2<593:MAPILD>2.0.CO;2-0&origin=MSN](http://links.jstor.org/sici?sici=0091-1798(198704)15:2<593:MAPILD>2.0.CO;2-0&origin=MSN). 2
- [31] P. Ney and F. Spitzer. The Martin boundary for random walk. *Trans. Amer. Math. Soc.*, 121:116–132, 1966. 2
- [32] Peter Ney and Esa Nummelin. Markov additive processes: large deviations for the continuous time case. In *Probability theory and mathematical statistics, Vol. II (Vilnius, 1985)*, pages 377–389. VNU Sci. Press, Utrecht, 1987. 2
- [33] Esa Nummelin. *General irreducible Markov chains and nonnegative operators*, volume 83 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1984. URL: <http://dx.doi.org/10.1017/CB09780511526237>, doi:10.1017/CB09780511526237. 4, 6
- [34] Robin Pemantle. Vertex-reinforced random walk. *Probab. Theory Related Fields*, 92(1):117–136, 1992. URL: <http://dx.doi.org/10.1007/BF01205239>, doi:10.1007/BF01205239. 2
- [35] F. Pène. Transient random walk in with stationary orientations. *ESAIM: Probability and Statistics*, 13:417–436, 2009. 2
- [36] Yuval Peres, Serguei Popov, and Perla Sousi. On recurrence and transience of self-interacting random walks. *Bull. Braz. Math. Soc. (N.S.)*, 44(4):841–867, 2013. URL: <http://dx.doi.org/10.1007/s00574-013-0036-4>, doi:10.1007/s00574-013-0036-4. 2
- [37] Georg Pólya. Über eine Aufgabe der Wahrscheinlichkeitsrechnung betreffend die Irrfahrt im Straßennetz. *Math. Ann.*, 84(1-2):149–160, 1921. 2
- [38] Olivier Raimond and Bruno Schapira. Random walks with occasionally modified transition probabilities. *Illinois J. Math.*, 54(4):1213–1238 (2012), 2010. 2

- [39] Frank Spitzer. *Principles of random walks*. Springer-Verlag, New York, second edition, 1976. Graduate Texts in Mathematics, Vol. 34. [3](#), [9](#)
- [40] Kôhei Uchiyama. Green's functions for random walks on  $\mathbf{Z}^N$ . *Proc. London Math. Soc.* (3), 77(1):215–240, 1998. [2](#)
- [41] Kohei Uchiyama. Asymptotic estimates of the Green functions and transition probabilities for Markov additive processes. *Electron. J. Probab.*, 12:no. 6, 138–180, 2007. URL: <http://dx.doi.org/10.1214/EJP.v12-396>, doi:10.1214/EJP.v12-396. [2](#)
- [42] Stanislav Volkov. Vertex-reinforced random walk on arbitrary graphs. *Ann. Probab.*, 29(1):66–91, 2001. URL: <http://dx.doi.org/10.1214/aop/1008956322>, doi:10.1214/aop/1008956322. [2](#)
- [43] Wolfgang Woess. The Martin boundary for harmonic functions on groups of automorphisms of a homogeneous tree. *Monatsh. Math.*, 120(1):55–72, 1995. [2](#)