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# The type problem for a class of inhomogeneous random walks : a series criterion by a probabilistic argument

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**Abstract** In the classical framework, a random walk on a group is a Markov chain with independent and identically distributed increments. In some sense, random walks are time and space homogeneous. In this paper, a class of weakly inhomogeneous random walks termed Random Walk with Random Transition Probabilities is investigated— *c.f.* [19] for the terminology. As an application, a criterion for the recurrence or transience of these processes in the discrete Abelian case is given. This criterion is deduced using Fourier analysis of Markov additive processes and a perturbation argument of a Markov operator. The latter extends the results of the literature since it does not involve a quasi-compactness condition on the operator. Finally, this criterion is applied to some well known examples of random walks on directed graphs embedded in  $\mathbb{Z}^2$ . Despite the type problem has been already solved for these examples, the analysis brought a new insight to this problematic.

**Key words** Weakly inhomogeneous random walks . Recurrence . Transience . Random Walks with Random Transition Probabilities . Markov additive processes . Fourier analysis . Operator perturbations

**Mathematics Subject Classification (2000)** 60F15 . 60J05 . 60J10 . 60J15 . 60J45 . 60J55 . 60K15

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## Introduction

In the classical framework, a random walk on a group  $\mathbf{G}$  is a discrete time stochastic process  $(Z_n)_{n \geq 0}$  defined as the product of independent and identically distributed (*i.i.d.*) random variables  $(\xi_n)_{n \geq 1}$ . More precisely, for any  $g \in \mathbf{G}$ , we set  $Z_0 = g$  and

$$Z_n = g\xi_1 \cdots \xi_n.$$

Random walks on groups are obviously Markov chains that are adapted to the group structure in the sense that the underlying Markov operator is invariant under the group action of  $\mathbf{G}$  on itself. Thus, this homogeneity naturally gives rise to deep connections between stochastic properties of the random walk and algebraic properties of the group. Starting with the seminal paper of Pòlya, [33], a large part of the literature is devoted to the study of these connections in this homogeneous case (for instance [25, 27, 17, 20, 39, 36, 22, 21, 2] and references therein).

In this paper, we aim at investigating “weakly” inhomogeneous random walks. It turns out that there are at least two ways to introduce inhomogeneity. First, we can consider “spatial” inhomogeneity by weakening the group structure, replacing it, for instance, by a directed graph or semigroupoid structure (see [5, 15, 16, 31, 13]). Secondly, we can study “temporal” inhomogeneous random walks by introducing a notion of memory as in the model of reinforced [30, 38], excited [34, 3], self-interacting [11, 32] or also persistent random walks [9, 8, 10]. All these models belong to the larger class of stochastic processes with long range dependency.

The strategy in the sequel consists of making use of the connection between such stochastic processes and Random Walk with Random Transition Probabilities — or for short, **RWRTP** — the terminology comes from [19]. Roughly speaking, a **RWRTP** consists of a dynamical system  $(\Omega, T)$ , endowed with a quasi-invariant (preimages by  $T$  of null measure sets have null measure) probability measure  $\lambda$ , together with a family  $\{\mu^\omega\}_{\omega \in \Omega}$  on a group  $\mathbf{G}$ . The dynamics dictate the way the measures of  $\{\mu^\omega\}_{\omega \in \Omega}$  can be convoluted<sup>1</sup>. More precisely, choosing at random (with respect to the probability measure  $\lambda$ ) a trajectory  $\omega \in \Omega$ , we are interested in the asymptotic properties of the sequence of convoluted measures  $\mu^\omega * \mu^{T\omega} * \mu^{T^2\omega} * \cdots * \mu^{T^n\omega}$ . As a matter of facts, along a trajectory of the dynamical system, the resulting stochastic process has independent but, in general, not stationary increments. If in addition, the dynamical system admits a stationary probability measure, in mean, the increments are no longer independent, however, they are in some sense stationary. We refer to Section 1 for the rigorous definition of **RWRTP**. In Proposition 1.2 of Section 1, it is shown that any inhomogeneous Markov chain taking values in a group  $\mathbf{G}$  is in particular a **RWRTP** while Proposition 1.3 is a useful tool to simplify, in some case, the internal dynamics. At the end of this section, an application of this two propositions for directed graphs embedded (in some sense) in a finitely generated group, is given.

As it is noticed in [19], this model is actually a generalization of other well known models such as Random Walk With Internal Degree of Freedom — see [24] — or also in the modern formulation Markov additive processes, for short **MAP** — see [18, 28, 26, 1, 37, 23]. In this context, many results have been proved. Though, the basic assumptions are generally too much

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<sup>1</sup>In this sense, it is comparable to Random Dynamical Systems: instead of randomly composing transformations, we randomly compose random (continuous) group homomorphisms.

restrictive to encompass the class of very inhomogeneous random walks (in particular, the model of random walk on directed graphs of Figure 1).

Generally speaking, in the context of Markov additive processes, we may introduce the Fourier transform operator, denoted by  $\mathcal{F}_t$ ,  $t \in \mathbb{R}^d$ , which is a continuous perturbation of the Markov operator  $\hat{P}$  defining the underlying dynamics. The rate of the return probability is then estimated, under mild conditions, by inverse Fourier transform. In the literature, the Markov operator  $\hat{P}$  is usually supposed to admit an ergodic invariant probability measure and to be quasi-compact. In addition, by a perturbation argument the Fourier transform operator remains quasi-compact for all  $t$  in a neighborhood of the origin. It allows, under suitable moment conditions on the system of probability measures, to derive a Taylor expansion at the second order of the perturbed dominating eigenvalue  $\gamma(t)$  (whose coefficients are given roughly by the mean and the variance operators). Finally, under an assumption on the spectrum of the Fourier transform operator outside a neighborhood of the origin, it can be concluded that all the needed stochastic information is actually contained in the nature of the singularity at zero of  $(1 - \gamma(t))^{-1}$  (note  $\gamma(0) = 1$ ).

In this paper, the quasi-compactness condition of  $\hat{P}$  is dropped. It is only assumed that  $\hat{P}$  is irreducible, recurrent and aperiodic. The condition on the spectrum of  $\mathcal{F}_t$  for large perturbations remains similar but the nature of the singularity at the origin is analyzed *via* probabilistic methods. These estimates give rise to a criterion for the type problem in terms of the summability of a series. More precisely, introduce the following quantities for  $n \geq 1$  and  $\omega \in \Omega$

$$\Sigma_n := \Sigma_n(\omega) = -\frac{1}{n} \sum_{k=0}^{n-1} \left[ \nabla^* \nabla \widehat{\mu^{T^k \omega}}(0) - (\nabla \widehat{\mu^{T^k \omega}}(0))^* (\nabla \widehat{\mu^{T^k \omega}}(0)) \right],$$

$$\Delta_n := \Delta_n(\omega) = -i \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} \nabla \widehat{\mu^{T^k \omega}}(0), \quad (1)$$

and the (almost surely finite) random variable  $\tau$  which is the first instant such that the rank  $\Sigma_n$  is maximal. The main theorem of this paper states (under the assumptions 1, 2, 3 and 4 given in Section 2) that the Markov additive process is transient or recurrent accordingly as

$$\sum_{n \geq 1} \frac{1}{n^{d/2}} \int_{\Omega} \mathbf{1}_{n \geq \tau} \exp \left( -\frac{1}{2} \Delta_n^* \Sigma_n^{-1} \Delta_n \right) d\lambda$$

is finite or infinite provided  $\tau$  has a finite expectation.

Thus, the main term of the series involves the expectation of a functional of two random variables. The recurrence or transience behavior is interpreted as the result of the competition of this two random quantities. More precisely, the process is recurrent if locally it is sufficiently diffusive in order to balance the instantaneous deviation. It turns out that  $\Delta^* \Sigma^{-1} \Delta$  is in some circumstances a self-normalized martingale (see [12] for instance).

In Section 3, applying this criterion to the model of random walks on directed graphs introduced in [5] and represented on the figure 1 below, we aim at illustrating that this theorem is effectively more general than those found in the literature and can be useful in practice.

For the graph (a) of Figure 1, denoted by  $\mathbb{L}$ , a random walker choose at random one of the nearest neighbour among North, South or West if the ordinate of its current position is odd, and the nearest neighbour among North, South and East if it is even. It is shown in [5] that this random walk is recurrent. For the graph (b) of Figure 1, denoted by  $\mathbb{H}$ , the possible movements for a random walker are toward the North, South and West on the upper half-plane and North, South and East otherwise. The resulting random walk is shown to be transient.

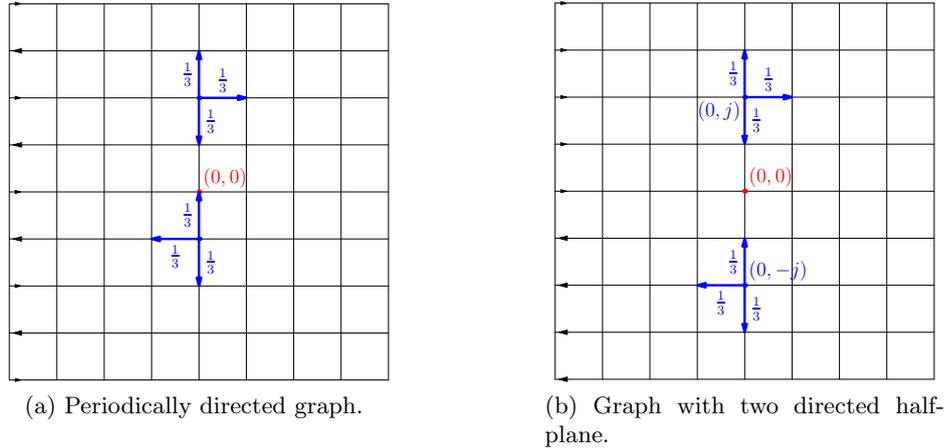


Figure 1: Directed graphs  $\mathbb{L}$  et  $\mathbb{H}$  de [5].

Using the theorem of factorization of Section 1, it follows easily that these random walks are Markov additive processes. The (minimal) internal Markov chains  $\mathring{P}$  are random walks on the group  $\mathbb{Z}/2\mathbb{Z}$  and  $\mathbb{Z}$  respectively. In the periodic case, the internal Markov chain admits a unique stationary probability measure, whereas for the graph with two directed half-plane it is null recurrent. The fundamental difference between the two random walks imposes to drop the strong quasi-compactness condition. Besides, beyond their own interests in the classical probability theory, it is worth noting the study of these models was initially motivated by natural questions arising in quantum theory as described [6].

## 1 The model of Random Walk with Random Transition Probabilities

### 1.1 Definitions and notations

Let  $(\mathbb{X}, \mathcal{X}, m)$  be a measured space. We denote by  $\mathbf{L}^\infty(m)$  the space of real essentially bounded function on  $\mathbb{X}$ . A linear operator  $P : \mathbf{L}^\infty(m) \rightarrow \mathbf{L}^\infty(m)$  is called Markov if  $P\mathbf{1} = \mathbf{1}$  and  $Pf \geq 0$  whenever  $f \geq 0$ . Let us denote by  $\mathbf{IMC}(\mathbb{X}, (P_n)_{n \geq 0}, \theta)$  the Inhomogeneous Markov Chain on  $\mathbb{X}$  where  $(P_n)_{n \geq 0}$  is a sequence of Markov operator on  $\mathbf{L}^\infty(m)$ ,  $\theta$  is absolutely continuous with respect to  $m$  — in symbols  $\theta \prec m$  — and stands for the initial probability measure. If the sequence of Markov operators  $(P_n)_{n \geq 0}$  is constant and equal to  $P$ , we denote by  $\mathbf{MC}(\mathbb{X}, P, \theta)$  the resulting time homogeneous Markov chain.

For a  $\mathbf{IMC}$ , it is well-known that the paths space  $\mathbb{X}^{\mathbb{N}}$  can be endowed with a unique probability measure, defined from the sequence of Markov operator  $(P_n)_{n \geq 0}$  and the probability measure  $\theta$ , called the canonical probability measure on the paths space and denoted by  $\mathbf{P}^\theta$ .

Let  $(\Omega, T, \lambda)$  be a dynamical system where  $T$  is a measurable map from  $\Omega$  into itself and  $\lambda$  is a probability measure which, in general, is not assumed to be preserved by the dynamics. Nonetheless, we will suppose that there exists a  $\sigma$ -finite invariant measure  $\rho$  dominating  $\lambda$ .

Let  $\mathbf{G}$  be a locally compact group and denote by Haar the unique, up to a multiplicative constant, Haar measure on  $\mathbf{G}$ . Denote by  $\mathcal{M}^1(\mathbf{G})$  the space of probability measures on  $\mathbf{G}$  endowed with the Borel  $\sigma$ -algebra relatively to the weak topology. A system of probability measures over  $\Omega$  is a measurable map  $\mu : \omega \in \Omega \rightarrow \mu^\omega \in \mathcal{M}^1(\mathbf{G})$ .

If  $\mu$  is a system of probability measures, we define the Markov operator  $\bar{P}$  for all  $f \in \mathbf{L}_{\rho \otimes \text{Haar}}^\infty(\Omega \times \mathbf{G})$  as follows

$$\bar{P}f(\omega, g) = \int_{\mathbf{G}} f(T\omega, gh) d\mu^\omega(h).$$

The system of probability measures  $\mu$  also defines a sequence of Markov operators  $P_n^\omega$  for all  $f \in \mathbf{L}_{\text{Haar}}^\infty(\mathbf{G})$  by

$$P_n^\omega f(x) = \int_{\mathbf{G}} f(xy) \mu^{T^n \omega}(dy), \quad x \in \mathbf{G}, \quad \omega \in \Omega$$

that is  $P_n^\omega$  is the right convolution operator<sup>2</sup>. If  $\nu$  is another system of probability measures, we denote, for all  $\omega$ , the canonical probability measure on the paths space of the inhomogeneous Markov chain  $\text{IMC}(\mathbf{G}, (P_n^\omega)_{n \geq 0}, \nu^\omega)$  by  $\mathbf{P}^{\nu, \omega}$  or for short  $\mathbf{P}^\omega$  if there is no ambiguity on  $\nu$  — for instance if  $\nu^\omega = \delta_{\text{id}}$ , *i.e.* for every  $\omega \in \Omega$  the **IMC** start from the identity of  $\mathbf{G}$ .

**Definition 1.1.** *The triplet  $((\Omega, T, \lambda), \mu, \nu)$  is called a Random Walk with Random Transition Probabilities — on  $\mathbf{G}$  — and will be denoted in sequel by **RWRTP** $((\Omega, T, \lambda), \mu, \nu)$  or if there is no ambiguity **RWRTP** $(\lambda, \mu)$ .*

## 1.2 Inhomogeneous Markov chains as RWRTPs and factor RWRTPs

The elementary Proposition 1.2 shows that any discrete time inhomogeneous Markov chain taking values in a group  $\mathbf{G}$  is a **RWRTP**. Afterward, factor **RWRTP** are defined. This latter notion involves measurable partitions for which the suitable framework is that of Lebesgue spaces (we refer to [4] for the definition of Lebesgue spaces and measurable equivalence relations). Since the set  $\Omega$  shall be the infinite countable product set, we assume that, endowed with a suitable  $\sigma$ -algebra and probability, it is a Lebesgue space.

**Proposition 1.2.** *Let  $\text{IMC}(\mathbf{G}, (P_n)_{n \geq 0}, \theta)$  be an inhomogeneous Markov chain. Then there exists a **RWRTP** $((\Omega, T, \lambda), \mu, \nu)$  such that*

$$\int_{\Omega} \lambda(d\omega) \mathbf{P}^{\nu, \omega} = \mathbf{P}^\theta.$$

*Proof.* Let us set  $\Omega = \mathbf{G}^{\mathbb{N}}$ ,  $\lambda = \mathbf{P}^\theta$ , and  $T$  is the time-shift on the paths space of the **IMC**. Finally, we define  $\mu$  and  $\nu$  for all  $\omega \in \Omega$  as follows

$$\mu^\omega = \delta_{\omega_0^{-1} \omega_1} \quad \text{and} \quad \nu^\omega = \delta_{\omega_0}.$$

Therefore, it defines a **RWRTP** $((\Omega, T, \lambda), \mu, \nu)$  for which the equality of the proposition is trivially satisfied.  $\square$

This proposition essentially relies on the theorem of Ionescu Tulcea that defines the Markovian dynamical system  $(\Omega, T, \mathbf{P}^\theta)$  together with the fact that the considered stochastic process takes its values in a group  $\mathbf{G}$ . Obviously, this proposition is purely theoretical and no more information is given by the resulting **RWRTP**. Below, we introduce the notion of factor **RWRTP** that allows to take into account the internal symmetries of the process and thus simplify in general the analysis.

For a **RWRTP** $((\Omega, T, \lambda), \mu, \nu)$ , consider a measurable equivalence relation denoted by  $\sim$  and denote by  $\text{pr}$  and  $\lambda_{\text{pr}}$  the related projection and conditional measure on  $\Omega/\sim$ . The conditional system of probability measures  $\mu_{\text{pr}}$  over  $\Omega/\sim$  is then defined as follows

$$\mu_{\text{pr}} = \int_{\Omega} \mu^\omega d\lambda_{\text{pr}}(\omega), \tag{2}$$

---

<sup>2</sup>We may define the left convolution operator similarly by the formula  $Pf(x) = \int_{\mathbf{G}} f(yx) \mu^{T^n \omega}(dy)$ .

If in addition, the maps  $T$  can be factorized (in facts, it is  $\sim$ -measurable), that is  $\text{pr } T\omega = \text{pr } T\tilde{\omega}$  as soon as  $\text{pr } \omega = \text{pr } \tilde{\omega}$ , then we shall define the factor  $\mathbf{RWRTP}((\Omega/\sim, \text{pr} \circ T, \lambda_{\text{pr}}, \mu_{\text{pr}}, \nu_{\text{pr}})$ .

**Proposition 1.3.** *Let  $\mathbf{RWRTP}((\Omega, T, \lambda), \mu, \nu)$  be any  $\mathbf{RWRTP}$ , suppose  $T$  can be factorized, and denote by  $\mathbf{RWRTP}((\Omega/\sim, \text{pr} \circ T, \lambda_{\text{pr}}, \mu_{\text{pr}}, \nu_{\text{pr}})$  its factorization under  $\sim$ . Then, the following equality holds*

$$\int_{\Omega} \lambda(d\omega) \mathbf{P}_{\nu, \omega} = \int_{\Omega/\sim} \lambda_{\text{pr}}(d\mathbf{w}) \mathbf{P}_{\nu_{\text{pr}}, \mathbf{w}}. \quad (3)$$

*Proof.* We only need to check the equality of finite dimensional distributions between both probabilities of (3). Thus, let  $\alpha$  and  $\beta$  be any system of probability measures, then the integrated product measures verify for any measurable subset  $A \times B \subset \mathbf{G} \times \mathbf{G}$

$$\begin{aligned} \int_{\Omega/\sim} \lambda_{\text{pr}}(d\mathbf{w}) \alpha_{\text{pr}}^{\mathbf{w}} \otimes \beta_{\text{pr}}^{\mathbf{w}}(A \times B) &= \int_{\Omega/\sim} \lambda_{\text{pr}}(d\mathbf{w}) \alpha_{\text{pr}}^{\mathbf{w}}(A) \beta_{\text{pr}}^{\mathbf{w}}(B) \\ &= \int_{\Omega} \lambda(d\omega) \alpha_{\text{pr}}^{\text{pr}\omega}(A) \beta_{\text{pr}}^{\text{pr}\omega}(B) \\ &= \int_{\Omega} \lambda(d\omega) \alpha^{\omega}(A) \beta^{\omega}(B) \\ &= \int_{\Omega} \lambda(d\omega) \alpha^{\omega} \otimes \beta^{\omega}(A \times B). \end{aligned}$$

Thus the image measures by the maps  $\mathbf{G} \times \mathbf{G} \ni (x, y) \rightarrow xy \in \mathbf{G}$  coincide and it follows that

$$\int_{\Omega} \lambda(d\omega) \alpha^{\omega} * \beta^{\omega} = \int_{\Omega/\sim} \lambda_{\text{pr}}(d\mathbf{w}) \alpha_{\text{pr}}^{\mathbf{w}} * \beta_{\text{pr}}^{\mathbf{w}}.$$

Proceeding by induction, the conclusion of the proposition is proved.  $\square$

**Remark 1.1.** *It is worth noting that the ergodicity of the probability measure  $\lambda$ , that is  $\lambda(A) = 0$  or 1 as soon as  $A$  is a  $T$ -invariant measurable subset of  $\Omega$ , implies its projection  $\lambda_{\text{pr}}$  is ergodic.*

**Remark 1.2.** *We only need here the  $\sim$ -equivalence classes to be invariant under the map  $T$ . However, the conditional probability  $\lambda_{\text{pr}}$ , and hence, the conditional system of probability measures  $\mu_{\text{pr}}$  and  $\nu_{\text{pr}}$  are not in general easily characterized. Actually, it heavily depends on the simplicity of the model considered and thus, the strategy consists of determining the finest equivalence relation (in the complete lattice of measurable partitions up to null sets) whereas  $\lambda_{\text{pr}}$  remains easily computable. In the next section, such equivalence relations shall be given for random walks on directed graphs.*

### 1.3 Random walks on constrained directed graphs

In this section, after introducing standard notions related to graphs, we consider random walks on graphs embedded in a group  $\mathbf{G}$ . As any discrete time stochastic process, it is isomorphic to a  $\mathbf{RWRTP}$ . In the sequel, we give the corresponding  $\mathbf{RWRTP}$  that is minimal in the sense of Remark 1.2.

A directed graph (or di-graph for short)  $\mathbb{G} = (\mathbb{G}^0, \mathbb{G}^1, r, s)$  is a quadruple consisting of a denumerable set  $\mathbb{G}^0$  of vertices, a denumerable set  $\mathbb{G}^1$  of directed edges and a pair of range and source maps, denoted respectively by  $r$  and  $s$ , *i.e.* mappings  $r, s : \mathbb{G}^1 \rightarrow \mathbb{G}^0$ . In the sequel, we only consider graphs without loops (*i.e.* not containing edges  $\alpha \in \mathbb{G}^1$  such that  $r(\alpha) = s(\alpha)$ ) and without multiple edges (*i.e.* if  $\alpha, \beta \in \mathbb{G}^1$  satisfy  $s(\alpha) = s(\beta)$  and  $r(\alpha) = r(\beta)$  then  $\alpha = \beta$ , or in other words, the compound map  $(r, s) : \mathbb{G}^1 \rightarrow \mathbb{G}^0 \times \mathbb{G}^0$  is injective). With these restrictions

in force,  $\mathbb{G}^1$  can be identified with a particular subset of  $\mathbb{G}^0 \times \mathbb{G}^0$  and the maps  $r$  and  $s$  become superfluous:  $s((\mathbf{u}, \mathbf{v})) = \mathbf{u}$  and  $r((\mathbf{u}, \mathbf{v})) = \mathbf{v}$  for all  $(\mathbf{u}, \mathbf{v}) \in \mathbb{G}^1$ . The corresponding directed graph is then termed simple.

We also define respectively, for each vertex  $\mathbf{v} \in \mathbb{G}^0$ , its inwards and outward degree by

$$\deg^+(\mathbf{v}) = \text{card} \{ \alpha \in \mathbb{G}^1 : r(\alpha) = \mathbf{v} \} \text{ and } \deg^-(\mathbf{v}) = \text{card} \{ \alpha \in \mathbb{G}^1 : s(\alpha) = \mathbf{v} \}.$$

A simple graph is termed undirected if the set of edges is symmetric in  $\mathbb{G}^0 \times \mathbb{G}^0$ , that is  $(\mathbf{u}, \mathbf{v}) \in \mathbb{G}^1$  if and only  $(\mathbf{v}, \mathbf{u}) \in \mathbb{G}^1$ . In this case, the inwards and outwards degree coincide, the common value is simply denoted by  $\deg$  and called the degree.

A graph  $\mathbb{G}$  is said to be connected if for any vertices  $\mathbf{u}, \mathbf{v} \in \mathbb{G}^0$  there exists a finite sequence  $\alpha = (\alpha_1, \dots, \alpha_k)$  of edges  $\alpha_i \in \mathbb{G}^1$ , for  $i = 1, \dots, k$ ,  $k \in \mathbb{N}$ , with  $s(\alpha_1) = \mathbf{u}$  and  $r(\alpha_k) = \mathbf{v}$ , such that  $r(\alpha_i) = s(\alpha_{i+1}) \in \mathbb{G}^0$ , for all  $i = 1, \dots, k-1$ . The above sequence  $\alpha$  is called a path of length  $k = |\alpha|$  from  $\mathbf{u}$  to  $\mathbf{v}$ . A directed graph is said to be locally finite if for all  $\mathbf{u} \in \mathbb{G}^0$ , the inwards and outwards degree are finite.

A random walk on a connected locally finite graph  $\mathbb{G}$  is a Markov chain whose state space is given by  $\mathbb{G}^0$  and the transition probabilities satisfy for each  $\mathbf{u}, \mathbf{v} \in \mathbb{G}^0$

$$P(\mathbf{u}, \mathbf{v}) \in (0, 1] \iff (\mathbf{u}, \mathbf{v}) \in \mathbb{G}^1,$$

with the additional property  $\sum_{\mathbf{v} \in \mathbb{G}^0} P(\mathbf{u}, \mathbf{v}) = 1$ . A random walk on  $\mathbb{G}$  is said to be simple if the transition probabilities are given for all  $\mathbf{u}, \mathbf{v} \in \mathbb{G}^0$  by

$$P(\mathbf{u}, \mathbf{v}) = \begin{cases} \frac{1}{\deg^-(\mathbf{u})} & \text{if } (\mathbf{u}, \mathbf{v}) \in \mathbb{G}^1, \\ 0 & \text{otherwise.} \end{cases}$$

Let us denote by  $\mathfrak{S}$  the group of permutations on the set of vertices of  $\mathbb{G}$ . The set of automorphisms of a graph  $\mathbb{G}$  is the subgroup of  $\mathfrak{S}$  denoted by  $\text{Aut}(\mathbb{G})$  defined as follows

$$\text{Aut}(\mathbb{G}) = \{ f \in \mathfrak{S} : (\mathbf{u}, \mathbf{v}) \in \mathbb{G}^1 \iff (f(\mathbf{u}), f(\mathbf{v})) \in \mathbb{G}^1 \}.$$

The group of automorphisms of a random walk on a graph  $\mathbb{G}$  is defined as follows

$$\text{Aut}(\mathbb{G}, P) = \{ f \in \mathfrak{S} : P(\mathbf{u}, \mathbf{v}) = P(f(\mathbf{u}), f(\mathbf{v})) \text{ for all } \mathbf{u}, \mathbf{v} \in \mathbb{G}^0 \}.$$

The latter group is a subgroup of the former and they are isomorphic if the random walk is supposed to be simple. The two groups of automorphisms naturally acts on the graph  $\mathbb{G}$ , however, there is no reason for the action of  $\text{Aut}(\mathbb{G}, P)$  to be transitive. In facts, in the case of a transitive action, the so-called homogeneous case, the random walk on the graph can be transported on a random walk in the classical sense on  $\text{Aut}(\mathbb{G}, P)$  — see [22]. Note that, as soon as the graph is genuinely directed, the group of automorphisms  $\text{Aut}(\mathbb{G})$  cannot act transitively.

In order to see random walks on graphs as **RWRTP**, it is necessary to embed the graph in a group  $\mathbf{G}$ . A natural family of such graphs are constrained graphs, *i.e.* graphs constructed from the Cayley graph of a finitely generated group by deleting some edges with respect to a predicate. More precisely, let  $\mathbf{G}$  be a group generated by a finite set  $\mathcal{S}$  supposed symmetric. A predicate is a  $\{0, 1\}$ -valued map  $\mathbf{p}$  on  $\mathbf{G} \times \mathcal{S}$ . Thus, we may construct a constrained graph  $\mathbb{G} = (\mathbb{G}^0, \mathbb{G}^1)$  where

$$\mathbb{G}^0 \subset \mathbf{G} \text{ countable and } \mathbb{G}^1 = \{ (\mathbf{u}, \mathbf{vs}) \in \mathbb{G}^0 \times \mathbb{G}^0, s \in \mathcal{S}, \mathbf{p}(\mathbf{u}, s) = 1 \}.$$

The resulting graph inherits the simplicity and the local finiteness property of Cayley graphs of finitely generated groups. Though it can be disconnected unless the predicate  $\mathbf{p}$  is suitably chosen which we shall assume in the sequel.

Let  $T$  be the unilateral time shift on  $\Omega = (\mathbb{G}^0)^\mathbb{N}$  and for every fixed  $\mathbf{u}_0 \in \mathbb{G}^0$ , the probability measure  $\lambda = \mathbf{P}^{\mathbf{u}_0}$ . Also, the system of probability measures  $\mu$  and  $\nu$  are defined respectively by

$$\mu^\omega = \delta_{\omega_0^{-1}\omega_1} \quad \text{and} \quad \nu^\omega = \delta_{\omega_0}, \quad \omega \in (\mathbb{G}^0)^\mathbb{N}.$$

Then, by Proposition 1.2,

$$\mathbf{P}^{\mathbf{u}_0} = \int_{(\mathbb{G}^0)^\mathbb{N}} \mathbf{P}_{\nu, \omega} \lambda(d\omega).$$

We shall define an equivalence relation on  $\Omega$ , denoted in the sequel by  $\sim$  as follows

$$\omega \sim \tilde{\omega} \in (\mathbb{G}^0)^\mathbb{N} \iff \forall n \geq 0 : \omega_n = \tilde{\omega}_n \text{ mod } \text{Aut}(\mathbb{G}, P).$$

As a matter of fact, the time shift  $T$  commutes with this equivalence relation. Thus, by Proposition 1.3, the **RWRTP** above can be factorized. More precisely, the map  $\text{pr} \circ T$  is the time shift on the factor space  $\Omega/\sim = \left(\mathbb{G}^0/\text{Aut}(\mathbb{G}, P)\right)^\mathbb{N}$ . Also, we denote by  $\lambda_{\text{pr}}$  the factor probability measure, with respect to  $\sim$ , of the probability measure  $\lambda = \mathbf{P}^{\mathbf{u}_0}$ . It turns out that  $\lambda_{\text{pr}}$  is nothing but the canonical probability measure of the Markov chain on the state space  $\mathbb{G}^0/\text{Aut}(\mathbb{G}, P)$  whose initial distribution is given by  $\delta_{\text{pr}(\mathbf{u}_0)}$  and Markov operator, denoted by  $\dot{P}$ , satisfies  $\dot{P}(\text{pr } \mathbf{u}, \text{pr } \mathbf{v}) = P(\mathbf{u}, \mathbf{v})$  for all  $\mathbf{u}, \mathbf{v} \in \mathbb{G}^0$ . The system of probability measures  $\left\{ \mu_{\text{pr}}^{\mathbf{w}_0, \mathbf{w}_1} : \mathbf{w}_0, \mathbf{w}_1 \in \mathbb{G}^0/\text{Aut}(\mathbb{G}, P) \right\}$  defined in (2) rewrites in our context as

$$\mu_{\text{pr}}^{\mathbf{w}_0, \mathbf{w}_1} = \frac{\sum_{\omega_1 \in \mathbb{G}^0} P(\omega_0, \omega_1) \mathbf{1}_{\{(\mathbf{w}_0, \mathbf{w}_1)\}}(\text{pr } \omega_0, \text{pr } \omega_1) \mu^\omega}{\sum_{\omega_1 \in \mathbb{G}^0} P(\omega_0, \omega_1) \mathbf{1}_{\{(\mathbf{w}_0, \mathbf{w}_1)\}}(\text{pr } \omega_0, \text{pr } \omega_1)}, \quad \text{for any } \omega_0 \in \mathbb{G}^0 \text{ s.t. } \text{pr } \omega_0 = \mathbf{w}_0.$$

Also, we define  $\nu_{\text{pr}}^{\mathbf{w}_0}$  as the Dirac distribution at the point  $\mathbf{w}_0 \in \mathbb{G}^0/\text{Aut}(\mathbb{G}, P)$ .

**Remark 1.3.** *Note that any discrete process can be seen as a Markov chain with respect the filtration containing the whole history. This Markov chain with long memory can be modeled by a random walk on a directed graph whose corresponding factorization reveals the structure of the memory.*

## 2 The type problem

In the previous section, it is shown that any discrete time process (taking values in a group  $\mathbf{G}$ ) is in some sense a **RWRTP**. In this section, we shall consider the restricted class of Markov additive processes, for short **MAP**, that are, roughly speaking, **RWRTP** for which the group  $\mathbf{G}$  is supposed Abelian and the underlying dynamics is Markovian. We shall be even more restrictive by setting  $\mathbf{G} = \mathbb{Z}^d$ . For a non discrete Abelian group, say  $\mathbf{G} = \mathbb{R}^d$  or  $\mathbf{G}$  compact, the Fourier analysis of continuous measure is sensibly different but remains irrelevant for the scope of this paper.

### 2.1 Markov additive processes

In the sequel, we refer to [29] for the notion of Markov chain on general spaces. We shall consider a Markov kernel  $\dot{P}$  relatively to a measurable space  $(\mathbb{X}, \mathcal{X})$  for which the  $\sigma$ -algebra is separable. The space  $(\mathbb{X}, \mathcal{X})$  can be endowed with a  $\sigma$ -finite measure  $m$  dominating  $m\dot{P}$ . Without loss of generality, we may assume that the Markov chain induced by  $\dot{P}$  is  $m$ -irreducible and aperiodic. A stronger assumption is to suppose the Markov chain  $m$ -recurrent. In this case, the time shift on  $\mathbb{X}^\mathbb{N}$  preserves the possibly infinite, but  $\sigma$ -finite, Markov measure  $\mathbf{P}^m$ . The probability

measures  $\mu^\omega$ ,  $\omega \in \mathbb{X}^{\mathbb{N}}$  are supposed to depend only on the two first coordinates<sup>3</sup> of  $\omega \in \mathbb{X}^{\mathbb{N}}$  and shall be alternatively denoted by  $\mu^{x,y}$ ,  $x, y \in \mathbb{X}$ .

A Markov additive process with internal Markov chain  $\mathring{P}$  together with the system of probability measures  $\mu$  is the Markov chain on the space  $\mathbb{X} \times \mathbb{Z}^d$  whose Markov operator is given, for  $f \in \mathbf{L}_m^\infty(\mathbb{X} \times \mathbb{Z}^d)$ , by

$$\bar{P}f(x, \mathbf{u}) = \int_{\mathbb{X} \times \mathbb{Z}^d} \mathring{P}(x, dy) (\delta_y \otimes \mu^{x,y}) (dzd\mathbf{v}) f(z, \mathbf{u} + \mathbf{v}), \quad \text{for } (x, \mathbf{u}) \in \mathbb{X} \times \mathbb{Z}^d.$$

The Markov operator  $\bar{P}$  is invariant by translations of the kind  $\mathbb{X} \times \mathbb{Z}^d \ni (x, \mathbf{u}) \rightarrow (x, \mathbf{u} + \mathbf{v}) \in \mathbb{X} \times \mathbb{Z}^d$ .

In this context, we may introduce the so called Fourier transform operator, denoted by  $\mathcal{F}_t$ , for  $t \in \mathbb{R}^d$ , acting as a contraction on  $\mathbf{L}_m^\infty(\mathbb{X})$  and defined for  $f \in \mathbf{L}_m^\infty(\mathbb{X})$  by

$$\mathcal{F}_t f(x) = \int_{\mathbb{X}} \mathring{P}(x, dy) \widehat{\mu^{x,y}}(t) f(y), \quad \text{for } m\text{-a.e. } x \in \mathbb{X}, \quad (4)$$

where  $\widehat{\mu^{x,y}}$  is the standard Fourier transform of the probability measure  $\mu^{x,y}$  defined, for  $t \in \mathbb{R}^d$  by

$$\widehat{\mu^{x,y}}(t) = \sum_{\mathbf{v} \in \mathbb{Z}^d} \mu^{x,y}(\mathbf{v}) e^{i\langle t, \mathbf{v} \rangle},$$

$\langle \cdot, \cdot \rangle$  standing for the standard inner product.

## 2.2 Basic assumptions

Let  $\sigma$  be a bounded map from  $\mathbb{X}$  to  $\mathbb{Z}^d$  and define

$$\bar{P}_\sigma f(x, \mathbf{u}) = \int_{\mathbb{X} \times \mathbb{Z}^d} \mathring{P}(x, dy) (\delta_y \otimes \mu^{x,y}) (dzd\mathbf{v}) f(z, \mathbf{u} + \mathbf{v} - \sigma(y) + \sigma(x)), \quad \text{for } (x, \mathbf{u}) \in \mathbb{X} \times \mathbb{Z}^d.$$

Intuitively, it corresponds to a change of origin of each fiber parametrized by  $\mathbb{X}$ . Such a function  $\sigma$  is called a change of section. The translated probability shall be denoted  $\mu_\sigma^{x,y} = \mu^{x,y} * \delta_{\sigma(y) - \sigma(x)}$ , or simply  $\mu^{x,y}$  if no ambiguity. Because of the invariance of  $\bar{P}$  under translations of  $\mathbb{Z}^d$ , the changes of section have no fundamental importance in the study of the recurrent or transient behavior of the Markov additive process.

To keep notations light and readable, we shall adopt alternatively the ones related to dynamical system (such as  $\mu^\omega$ ,  $\mathbf{P}^m$ -a.e.,...) or those ones proper to Markov additive processes ( $\mu^{x,y}$ ,  $m$ -a.e.,...).

**Definition 2.1** (Adaptation, aperiodicity, irreducibility). *A MAP is respectively adapted, aperiodic and irreducible if for any change of section  $\sigma$*

1. *there is no proper subgroup  $H \subset \mathbb{Z}^d$  such that  $\mathbf{P}^m$ -a.e.  $\mu^\omega(H) = 1$  ;*
2. *there is no proper subgroup  $H \subset \mathbb{Z}^d$  and no  $a \in \mathbb{R}^d$  such that  $\mathbf{P}^m$ -a.e.  $\mu^\omega(a + H) = 1$ .*
3. *there is no half-space  $H \subset \mathbb{R}^d$  such that  $\mathbf{P}^m$ -a.e.  $\mu^\omega(H) = 1$ .*

The properties of adaptation and aperiodicity can be read on the spectrum of the Fourier transform  $\mathcal{F}_t$  as shown in Corollary 2.3. Below  $\mathbf{W}_d$  denotes the set  $[-\pi, \pi)^d$  and  $\mathbf{W}_d^*$  the set  $\mathbf{W}_d \setminus \{0\}$ .

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<sup>3</sup>Actually, we use here the traditional definition, though it is completely equivalent to suppose that the probabilities  $\mu^\omega$  depends only on the first coordinate providing we consider the 2-order Markov chain induced by  $\mathring{P}$ .

**Proposition 2.2.** *A MAP is adapted if and only if  $\widehat{\mu^\omega}(t) \neq 1$ ,  $\mathbf{P}^m$ -a.e. for any change of section  $\sigma : \mathbb{X} \rightarrow \mathbb{Z}^d$  and any  $t \in \mathbf{W}_d^*$ . A MAP is aperiodic if and only if  $\widehat{\mu^\omega}(t) \neq e^{i\theta}$ ,  $\mathbf{P}^m$ -a.e., for any  $\theta \in \mathbb{R}$ , any change of section  $\sigma$  and any  $t \in \mathbf{W}_d^*$ .*

*Proof.* If a MAP is not aperiodic then there exist a change of section, a proper subgroup  $H$  and  $a \in \mathbb{R}^d$  such that  $\mu^\omega(a + H) = 1$ ,  $\mathbf{P}^m$ -a.e.. It is well known that there exist integers  $n_1, \dots, n_r$ ,  $1 \leq r \leq d$ , such that the subgroup  $H$  is generated by a set  $\{n_1 e_{i_1}, \dots, n_r e_{i_r}\}$  for some indices  $1 \leq i_1 \leq \dots \leq i_r \leq d$ , where  $e_i$  is the  $i^{\text{th}}$  vector of the canonical basis of  $\mathbb{Z}^d$ . Suppose  $r < d$ , then for any  $t \in \mathbf{W}_d^* \cap (\text{span } H)^\perp$  it follows that  $\widehat{\mu^\omega}(t) = e^{i\langle t, a \rangle}$ . If  $r = d$ , set  $t = (\pi/n_1, \dots, \pi/n_d)$  then again  $\widehat{\mu^\omega}(t) = e^{i\langle t, a \rangle}$  and  $t$  obviously belongs to  $\mathbf{W}_d^*$  unless  $n_1 = \dots = n_d = 1$ , i.e.  $H = \mathbb{Z}^d$ .

Conversely, suppose there exists a change of section  $\sigma$  and  $t \in \mathbf{W}_d^*$  such that  $\widehat{\mu^\omega}(t) = e^{i\theta}$ ,  $\mathbf{P}^m$ -a.e., for some  $\theta \in \mathbb{R}$ . It means that  $\mu^\omega(t)$  is an extremal convex combination and since  $\mu^\omega$  is supported by a subset of  $\mathbb{Z}^d$ , one can choose  $a \in \mathbb{Z}^d$  such that  $\theta = \langle t, a \rangle$ . Then, for any  $n \geq 1$

$$e^{i\langle t, na \rangle} = \prod_{k=0}^{n-1} \widehat{\mu^{T^k \omega}}(t) = \sum_{\mathbf{w} \in \mathbb{Z}^d} \mu^\omega * \dots * \mu^{T^{n-1} \omega}(\mathbf{w}) e^{i\langle t, \mathbf{w} \rangle}, \quad \mathbf{P}^m - a.e.,$$

or equivalently,

$$1 = \sum_{\mathbf{w} \in \mathbb{Z}^d} \{\mu^\omega * \delta_{-a}\} * \dots * \{\mu^{T^{n-1} \omega} * \delta_{-a}\}(\mathbf{w}) e^{i\langle t, \mathbf{w} \rangle}, \quad \mathbf{P}^m - a.e..$$

Thus, the convex combination of points on the unit circle on the right hand side is extremal so that each  $\mathbf{w} \in \mathcal{S}_n^\omega = \text{supp } \{\mu^\omega * \delta_{-a}\} * \dots * \{\mu^{T^{n-1} \omega} * \delta_{-a}\}$ ,  $n \geq 1$ , satisfies  $\langle t, \mathbf{w} \rangle = 0$  modulo  $2\pi$ . Setting  $N = \{\omega \in \mathbb{X}^\mathbb{N} : \widehat{\mu^\omega}(t) \neq e^{i\langle t, a \rangle}\}$  and defining  $H$  as the smallest subgroups containing the sets  $\mathcal{S}_n^\omega$ , for all  $n \geq 1$  and  $\omega \in N^{\mathbb{G}}$ , it follows that,  $\mathbf{P}^m$ -a.e.,  $(\mu^\omega * \delta_{-a})(H) = 1$ , i.e.  $\mu^\omega(a + H) = 1$ . If the MAP was aperiodic, the group  $H$  should not be a strict subgroup of  $\mathbb{Z}^d$  which is in contradiction with  $\langle t, \mathbf{w} \rangle = 0$  modulo  $2\pi$  by setting  $\mathbf{w} = e_i$  for  $1 \leq i \leq d$ .

The statement involving the adaptation property follows exactly the same lines by setting  $a = 0$  and  $\theta = 0$ .  $\square$

**Corollary 2.3.** *A MAP is adapted if and only if, for any  $t \in \mathbf{W}_d^*$  there exists a closed  $\mathcal{F}_t$ -invariant subspace  $E$  containing the constants such that one is not an eigenvalue of the operator  $\mathcal{F}_t$  acting on  $E$ . It is aperiodic if and only if, for all  $t \in \mathbf{W}_d^*$  there exists a closed  $\mathcal{F}_t$ -invariant subspace  $E$  containing the constants such that the operator  $\mathcal{F}_t$ , acting on  $E$ , has no eigenvalue of modulus one.*

*Proof.* Let  $t \in \mathbf{W}_d^*$  and suppose there exists  $f \in E$ , where  $E$  is any closed  $\mathcal{F}_t$ -invariant subspace containing the constants, such that

$$\mathcal{F}_t f(x) = \int_{\mathbb{X}} \mathring{P}(x, dy) \widehat{\mu^{x,y}}(t) f(y) = e^{i\theta} f(x), \quad \text{for some } \theta \in \mathbb{R}.$$

By Jensen inequality and the fact that  $|\widehat{\mu^{x,y}}(t)| \leq 1$ , it follows that  $|f| \leq \mathring{P}|f|$ . Thus, the function  $\|f\|_\infty - |f|$  is superharmonic and hence constant  $m$ -a.e. since  $\mathring{P}$  is supposed  $m$ -recurrent (see [29, Proposition 3.13, p. 44]). As a consequence,

$$1 = \int_{\mathbb{X}} |\widehat{\mu^{x,y}}(t)| \mathring{P}(x, dy), \quad m - a.e.,$$

and  $\widehat{\mu^{x,y}}(t)$  is of modulus one,  $\mathbf{P}^m$ -a.e.. By Proposition 2.2, the MAP can not be aperiodic. Conversely, if the MAP is periodic, the same proposition implies that the constants are eigenfunctions for the eigenvalue  $e^{i\langle t, a \rangle}$ .

The proof of the statement involving the adaptation property follows exactly the same lines.  $\square$

**Remark 2.1.** Since the probability measures  $\mu^\omega$  are supposed to be supported by  $\mathbb{Z}^d$ ,  $\mathbf{P}^m$ -a.e., it follows that the operator valued map  $t \rightarrow \mathcal{F}_t$  is  $2\pi$ -periodic along the directions given by the vectors of the standard basis of  $\mathbb{R}^d$ . In fact, the aperiodicity means that it can not be periodic with shorter periods.

**Definition 2.4.** A MAP is said to satisfy condition (S) if for any  $t \in \mathbf{W}_d^*$  there exists a closed  $\mathcal{F}_t$ -invariant subspace  $E$ , containing the constants, for which the spectral values of modulus 1 of  $\mathcal{F}_t$ , operating on  $E$ , consists of eigenvalues.

**Remark 2.2.** If the Markov operator  $\mathring{P}$  is quasi-compact then the Fourier transform operator is also quasi-compact for every  $t$  in a neighborhood of the origin. It is known that the set of spectral values of modulus one consists of isolated eigenvalues. Thus, the condition (S) extends this property to large perturbations. Nonetheless, condition (S) can be satisfied without  $\mathring{P}$  being quasi-compact which is the main motivation of this paper.

For any closed subspace  $E$ , and any bounded operator  $Q$  we denote by  $\|Q\|_E$  the subordinated norm restricted to  $E$  defined as  $\|Q\|_E = \sup_{f \in E: \|f\|=1} \|Qf\|$ . Let  $t \in \mathbf{W}_d$ , we denoted by  $E_t$  the intersection of all closed subspaces  $E$  such that  $\mathcal{F}_t E \subset E$  and  $\mathbf{1} \in E$ . Obviously, the subspace  $E_t$  is itself closed, invariant and contains the constant. Also, we may define the pseudo spectral radius  $\tilde{r}(t)$  of  $\mathcal{F}_t$  by

$$\tilde{r}(t) = \lim_{n \rightarrow \infty} \|\mathcal{F}_t^n\|_{E_t}^{1/n}.$$

**Lemma 2.5.** If a MAP is aperiodic and satisfy condition (S), then the pseudo spectral radius  $\tilde{r}(t)$  of  $\mathcal{F}_t$  is strictly smaller than one for  $t \in \mathbf{W}_d^*$ .

*Proof.* As a matter of fact, the pseudo spectral radius can be defined alternatively as follows

$$\tilde{r}(t) = \inf \left\{ \|\mathcal{F}_t^n\|_E^{1/n}, n \geq 1, E \text{ closed subspace satisfying } \mathcal{F}_t E \subset E \text{ and } \mathbf{1} \in E \right\}.$$

Moreover, let  $t_0 \in \mathbf{W}_d$  and  $E$  be a closed subspace such that  $\mathcal{F}_{t_0} E \subset E$  and  $\mathbf{1} \in E$ . Then, by reverse triangle inequality,

$$\left| \|\mathcal{F}_{t_0}^n\|_E - \|\mathcal{F}_t^n\|_E \right| \leq \|\mathcal{F}_t^n - \mathcal{F}_{t_0}^n\|_E \leq \|\mathcal{F}_t^n - \mathcal{F}_{t_0}^n\|$$

which can be made arbitrarily small since  $t \rightarrow \mathcal{F}_t$  is continuous. It follows that  $\tilde{r}$  is the point-wise infimum of continuous functions, thus  $\tilde{r}$  is upper semi-continuous. Furthermore, the pseudo spectral radius reaches its maximum on compact  $K \subset \mathbf{W}_d$ . Aperiodicity together with condition (S) imply that  $\max_{t \in K} \tilde{r}(t) < 1$  excepted when  $0 \in K$ .  $\square$

From now on, we shall assume the following.

**Assumptions 1.** The MAP is adapted, aperiodic and irreducible.

**Assumptions 2.** The condition (S) is fulfilled.

Additionally, we make the following assumption on the system of probability measure  $\mu$  where  $|\cdot|$  stands for the Euclidean norm.

**Assumptions 3.** Assume that the system of probability measures  $\mu$  admits a uniform third order moment, that is

$$\left\| \sum_{\mathbf{u} \in \mathbb{Z}^d} |\mathbf{u}|^3 \mu(\mathbf{u}) \right\|_\infty < \infty.$$

**Remark 2.3.** Recall that a change of section is a bounded function  $\sigma : \mathbb{X} \rightarrow \mathbb{Z}^d$ , thus Assumption 3 needs not to be stated relatively to any change of section.

For  $n \geq 1$  and  $\mathbf{u}, \mathbf{v} \in \mathbb{Z}^d$ , denote respectively the  $n$ -step transition probability and its Fourier transform by

$$P_{0,n}^\omega(\mathbf{u}, \mathbf{v}) := \mu^\omega * \cdots * \mu^{T^{n-1}\omega}(\mathbf{v} - \mathbf{u}) \quad \text{and} \quad \phi_{0,n}^\omega(t) = \prod_{k=0}^{n-1} \widehat{\mu^{T^k\omega}}(t). \quad (5)$$

Introduce the quantities

$$\Sigma_n(\omega) := -\frac{1}{n} \sum_{k=0}^{n-1} \left[ \nabla^* \nabla \widehat{\mu}^{T^k\omega}(0) - (\nabla \widehat{\mu}^{T^k\omega}(0))^* (\nabla \widehat{\mu}^{T^k\omega}(0)) \right]$$

$$\text{and} \quad \Delta_n(\omega) := -i \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} \nabla \phi_{0,1}^{T^k\omega}(0). \quad (6)$$

**Assumptions 4.** There exists  $\alpha > 0$  such that for all  $t \in \mathbb{R}^d$ ,  $\liminf_{n \rightarrow \infty} t^* \Sigma_n t \geq \alpha |t|^2$ ,  $\mathbf{P}^m$ -a.e..

### 2.3 The type problem : a series criterion

Let  $A \subset \mathbb{X}$  and  $K \subset \mathbb{Z}^d$  be measurable subsets. We are interested in the mean time spent by the Markov additive process — that is nothing but a Markov chain on  $\mathbb{X} \times \mathbb{Z}^d$  — in the product set  $A \times K$ . In facts, it is well known that this quantity, starting from  $(x, \mathbf{u}) \in \mathbb{X} \times \mathbb{Z}^d$ , is actually given by the Green operator defined by

$$G\mathbf{1}_{A \times K}(x, \mathbf{u}) = \sum_{n \geq 0} \bar{P}^n \mathbf{1}_{A \times K}(x, \mathbf{u}).$$

In the special case of  $A = \mathbb{X}$ , using notation of Equation (5), the equation above rewrites

$$G\mathbf{1}_{\mathbb{X} \times K}(x, \mathbf{u}) = \mathbf{1}_{\mathbb{X} \times K}(x, \mathbf{u}) + \sum_{\mathbf{v} \in \mathbb{Z}^d} \sum_{n \geq 1} \int_{\mathbb{X}^{\mathbb{N}}} P_{0,n}(u, v) \mathbf{1}_K(v) d\mathbf{P}^x.$$

**Definition 2.6** (Recurrence and Transience). A Markov additive process is said to be recurrent (resp. transient) if, for any bounded change of section  $\sigma$ ,  $G\mathbf{1}_{\mathbb{X} \times \{0\}}(x, 0) = \infty$ ,  $m$ -a.e. (resp.  $G\mathbf{1}_{\mathbb{X} \times \{0\}}(x, 0) < \infty$ ,  $m$ -a.e.).

**Remark 2.4.** Since the changes of section are supposed bounded, a MAP is simultaneously recurrent or transient for every change of section.

A simple computation gives rise to the identities

$$\begin{aligned} G\mathbf{1}_{\mathbb{X} \times \{0\}}(x, 0) &= \sum_{n \geq 1} \int_{\mathbb{X}^{\mathbb{N}}} \mathbf{P}^x(d\omega) P_{0,n}^\omega(0, 0) = \lim_{r \uparrow 1} \sum_{n \geq 1} r^{n-1} \int_{\mathbb{X}^{\mathbb{N}}} \mathbf{P}^x(d\omega) P_{0,n}^\omega(0, 0) \\ &= \lim_{r \uparrow 1} \frac{1}{(2\pi)^d} \int_{\mathbb{W}_d} \operatorname{Re} \sum_{n \geq 1} r^{n-1} \mathcal{F}_t^n \mathbf{1}(x) dt \end{aligned}$$

where the latter equality is obtained by inverse Fourier transform and Fubini's theorem.

**Theorem 2.7** (Series criterion). *Let  $d \geq 2$  and suppose that the assumptions 1, 2, 3 and 4 are satisfied. In addition, let  $\tau$  be the stopping time defined by  $\tau = \inf\{n \geq 1 : \text{rk } \Sigma_n = d\}$  and assume  $\mathbf{E}^x(\tau) < \infty$  for  $m$ -a.e.  $x \in \mathbb{X}$ . Then,  $G\mathbf{1}_{\mathbb{X} \times \{0\}}(x, 0)$  is finite (resp. infinite) if and only if*

$$\sum_{n \geq 1} \frac{1}{n^{d/2}} \int_{\mathbb{X}^{\mathbb{N}}} \mathbf{1}_{n \geq \tau} \exp\left(-\frac{1}{2} \Delta_n^* \Sigma_n^{-1} \Delta_n\right) d\mathbf{P}^x < \infty \quad (\text{resp. } = \infty),$$

and the **MAP** is transient or recurrent accordingly.

The proof of this theorem is postponed to Section 4

### 3 Applications : random walks on partially directed lattices

Let  $(\epsilon_w)_{w \in \mathbb{Z}}$  be any  $\{-1, 1\}$ -valued random or deterministic sequence (c.f. [5, 16, 14, 31, 7]). Set  $\mathbb{G}^0 = \mathbb{Z}^2$ , and define the set of edges  $\mathbb{G}^1$  as follows, for  $\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2), \mathbf{v} = (\mathbf{v}_1, \mathbf{v}_2) \in \mathbb{G}^0$ ,

$$(\mathbf{u}, \mathbf{v}) \in \mathbb{G}^1 \iff (|\mathbf{v}_2 - \mathbf{u}_2| = 1 \text{ and } \mathbf{v}_1 = \mathbf{u}_1) \text{ or } (\mathbf{v}_2 = \mathbf{u}_2 \text{ and } \mathbf{v}_1 = \mathbf{u}_1 + \epsilon_{\mathbf{u}_2}). \quad (7)$$

This defines a directed graph, denoted in the sequel by  $\mathbb{G}(\epsilon) = (\mathbb{G}^0, \mathbb{G}^1)$ , embedded in  $\mathbb{Z}^2$ . The simple random walk on such a graph, whose Markov operator is denoted by  $M$ , is obviously irreducible if and only if the range of the  $(\epsilon_w)_{w \in \mathbb{Z}}$  contains  $-1$  and  $1$ .

It is straightforward that the automorphism group  $\text{Aut}(\mathbb{G}(\epsilon), M)$  of  $M$ , is a subgroup of  $\mathbb{Z} \times \{0\}$ . Applying the construction of Section 1.3, it turns out that the simple random walk on  $\mathbb{G}(\epsilon)$ , starting from any vertex  $\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2) \in \mathbb{G}^0$ , is equal in distribution to a **MAP**. The underlying dynamics is a random walk on  $\mathbb{Z}$  starting from  $x = \mathbf{u}_2$  whose Markov operator and system of probability measure are given for  $x, y \in \mathbb{Z}$  by

$$\overset{\circ}{P}(x, \cdot) = \frac{1}{3}(\delta_{x-1} + \delta_x + \delta_{x+1}) \quad \text{and} \quad \mu^{x,y} = \begin{cases} \delta_{(0,1)} & \text{if } y = x + 1, \\ \delta_{-(0,1)} & \text{if } y = x - 1, \\ \delta_{(1,0)} & \text{if } y = x, \epsilon_x = 1, \\ \delta_{-(1,0)} & \text{if } y = x, \epsilon_x = -1. \end{cases}$$

The initial system of probability measures  $\nu$  is naturally defined by  $\nu = \delta_{\mathbf{u}_1, \mathbf{u}_2}$ ,  $\mathbf{P}^{\mathbf{u}_2}$ -a.s.. Obviously, Assumption 3 is satisfied.

The Markov operator induces an irreducible, aperiodic and null-recurrent Markov chain  $(S_n)_{n \geq 0}$  with  $S_0 = \mathbf{u}$ . Setting  $\xi_n = S_n - S_{n-1}$  for  $n \geq 1$ , a simple computation gives

$$\Sigma_n = \frac{1}{n} \begin{pmatrix} \sum_{k=0}^{n-1} \mathbf{1}_{\xi_k=0} & 0 \\ 0 & \sum_{k=0}^{n-1} \mathbf{1}_{\xi_k \neq 0} \end{pmatrix}, \quad \text{and} \quad \Delta_n = \frac{1}{\sqrt{n}} \begin{pmatrix} \sum_{k=0}^{n-1} \epsilon_{S_k} \mathbf{1}_{\xi_k=0} \\ S_n \end{pmatrix}.$$

Then, it is not difficult to see that  $\Sigma_n$  converges,  $\mathbf{P}^x$ -a.s., for all  $x \in \mathbb{Z}$ , by the Strong Law of Large Number, to  $\Sigma = \begin{pmatrix} 1/3 & 0 \\ 0 & 2/3 \end{pmatrix}$  so that Assumption 4 is fulfilled.

It is worth noting that even though the **MAP** is adapted and irreducible, the aperiodicity condition fails (consider the change of section  $\sigma : \mathbb{Z} \rightarrow \mathbb{Z}^2$  equal to  $(0, 0)$  on even point and to  $(1, 0)$  on odd ones and choose the subgroup  $H = \mathbb{Z} \times \{0\}$ ). However, without loss of generality, we can consider the “2-fold” **MAP**, i.e. we consider  $\overline{P}^2$  in place of  $\overline{P}$ .

Thus, at this level, we may prove the condition (S), i.e. Assumption 2, is fulfilled.

**Proposition 3.1.** *The condition (S) is fulfilled.*

*Proof.* Let  $t \in \mathbf{W}_d$  and compute for  $f \in \ell^\infty(\mathbb{Z})$  and  $x \in \mathbb{Z}$

$$\begin{aligned}\mathcal{F}_t^2 f(x) &= \left( \frac{2}{9} + \frac{1}{9} e^{2it_1 \varepsilon(x)} \right) f(x) + \frac{1}{9} e^{2it_2} f(x+2) + \frac{1}{9} e^{-2it_2} f(x-2) \\ &\quad + \frac{1}{9} e^{it_2} f(x+1) \left( e^{it_1 \varepsilon(x)} + e^{it_1 \varepsilon(x+1)} \right) \\ &\quad + \frac{1}{9} e^{-it_2} f(x-1) \left( e^{it_1 \varepsilon(x)} + e^{it_1 \varepsilon(x-1)} \right)\end{aligned}$$

First, suppose that  $t_1 \neq 0$ , then it follows that

$$\|\mathcal{F}_t^2\| \leq \frac{2}{3} + \sup_{x \in \mathbb{Z}} \left| \frac{2}{9} + \frac{1}{9} e^{2it_1 \varepsilon(x)} \right| \leq \frac{2}{3} + \frac{1}{9} \sqrt{5 + 4 \cos(2t_1)} < 1.$$

Secondly, if  $t_1 = 0$ , then the expression of  $\mathcal{F}_t^2$  simplifies as follows

$$\mathcal{F}_t^2 f(x) = \frac{1}{3} f(x) + \frac{2}{9} e^{it_2} f(x+1) + \frac{1}{9} e^{-it_2} f(x-1) + \frac{1}{9} e^{2it_2} f(x+2) + \frac{1}{9} e^{-2it_2} f(x-2).$$

As a matter of facts, the space of  $\mathbb{C}$ -valued constant functions is closed and invariant under  $\mathcal{F}_t^2$ . Moreover,

$$\|\mathcal{F}_t^2\|_E \leq \left( \frac{1}{3} + \frac{4}{9} \cos(t_2) + \frac{2}{9} \cos(2t_2) \right).$$

Summarizing, as soon as  $t \in \mathbf{W}_d^*$ , the pseudo spectral radius  $\tilde{r}(t)$  is strictly smaller than 1 and the condition (S) is satisfied.  $\square$

**Remark 3.1.** *In the context of periodic environment  $\varepsilon$  the proposition above is straightforward since the Fourier transform operator  $\mathcal{F}_t^2$ ,  $t \in \mathbf{W}_d$ , acts on the finite dimensional space  $E$  of periodic functions (the dimension of this space is equal to the period of the sequence  $\varepsilon$ ). In fact, in this specific case, using the standard machinery of the literature, the recurrence of the random walk would follow from [18] for instance. Besides, we may obtain a renewal theorem from [1], a large deviation theorem from [28, 26] and sharp estimates of the asymptotics of the Green function (and in turn a determination of the Martin boundary) from [37, 23]. Obviously, these technics can no longer be applied for a generic environment.*

**Remark 3.2.** *When  $t_1 = 0$ , the closed subspace of 2-periodic functions contains the constants and is invariant by  $\mathcal{F}_t^2$ . Still, the function  $x \rightarrow (-1)^x$  is an eigenfunction associated with the eigenvalue 1. Thus, it is crucial that the condition (S) allows to consider different invariant subspaces for different points  $t \in \mathbf{W}_d$ .*

**Theorem 3.2.** *The simple random walk  $M$  is recurrent (resp. transient) if and only if for some (any)  $x \in \mathbb{Z}$*

$$\sum_{n \geq 1} \frac{1}{n} \int_{\mathbb{Z}^{\mathbb{N}}} \mathbf{1}_{n \geq \tau} \exp \left[ - \frac{1}{2} \frac{\left( \sum_{k=0}^{n-1} \varepsilon_{S_k} \mathbf{1}_{\xi_k=0} \right)^2}{\sum_{k=0}^{n-1} \mathbf{1}_{\xi_k=0}} \right] \exp \left[ - \frac{1}{2} \frac{S_n^2}{\sum_{k=0}^{n-1} \mathbf{1}_{\xi_k \neq 0}} \right] d\mathbf{P}^x \quad (8)$$

*is infinite (resp. finite).*

**Remark 3.3.** *Similarly to the classical context of random walks on  $\mathbb{Z}^2$ , the harmonic factor term in the sum reveals the dimension 2 is critical. The exponential factor may improve the convergence rate of the summand in order to make the summation finite. Each of them corresponds to one of the direction given by the axis. More precisely, the first one describes the contribution of the horizontal component, whereas the second one describes that of the vertical one. Note that the Law of Iterated Logarithm implies the second exponential factor is almost surely bounded from below by  $(\log n)^{-1}$  for large  $n$ .*

In the periodic case, one may apply the Law of Iterated Logarithm shown in [12] to the self-normalized martingale  $\Delta^* \Sigma^{-1} \Delta$  and conclude to recurrence. If the sequence  $(\epsilon_y)_{y \in \mathbb{Z}}$  is supposed to be the sign function  $\text{sgn}$ , we shall derive the proposition below from a well known result proved in [35] and obtain the transience of the simple random walk.

**Proposition 3.3.** *The following convergence in distribution holds*

$$\frac{1}{n} \sum_{k=0}^{n-1} \text{sgn}(S_k) \mathbf{1}_{\xi_k=0} \xrightarrow{d} \Gamma,$$

where  $\Gamma$  is a symmetric arcsinus random variable on  $[-1, 1]$ . Therefore, the series (8) of Theorem 3.2 is convergent,  $\mathbf{P}^x$ -a.s., and the resulting MAP and the simple random walk on  $\mathbb{G}(\epsilon)$  are transient.

## 4 Proof of the main theorem

This section is devoted to the proof of Theorem 2.7. We shall be interested in the asymptotics as  $n$  goes to infinity of

$$\int_{\mathbb{X}^{\mathbb{N}}} P_{0,n}(\mathbf{u}, \mathbf{v}) d\mathbf{P}^x = \frac{1}{(2\pi)^d} \int_{\mathbf{W}_d} \mathcal{F}_t^n \mathbf{1}(x) e^{-i\langle t, \mathbf{v} - \mathbf{u} \rangle} dt$$

and more specifically when  $\mathbf{u} = \mathbf{v}$ .

Similarly to the context of classical random walks, let  $\delta > 0$  and split

$$\int_{\mathbf{W}_d} \mathcal{F}_t^n \mathbf{1}(x) e^{-i\langle t, \mathbf{v} - \mathbf{u} \rangle} dt = \underbrace{\int_{(-\delta, \delta)^d} \mathcal{F}_t^n \mathbf{1}(x) e^{-i\langle t, \mathbf{v} - \mathbf{u} \rangle} dt}_{I_1(n)} + \underbrace{\int_{\mathbf{W}_d \setminus (-\delta, \delta)^d} \mathcal{F}_t^n \mathbf{1}(x) e^{-i\langle t, \mathbf{v} - \mathbf{u} \rangle} dt}_{I_2(n)}. \quad (9)$$

### 4.1 Estimates far from the origin

**Proposition 4.1.** *Under Assumptions 1 and 2, there exist constants  $\kappa \in (0, 1)$  and  $C > 0$  such that for all  $\mathbf{v}, \mathbf{u} \in \mathbb{Z}^d$  and all  $\delta > 0$  sufficiently small*

$$\left\| \int_{\mathbf{W}_d \setminus (-\delta, \delta)^d} \mathcal{F}_t^n \mathbf{1}(\cdot) e^{-i\langle t, \mathbf{v} - \mathbf{u} \rangle} dt \right\|_{\infty} \leq C \kappa^n.$$

*Proof.* By Lemma 2.5, under Assumptions 1 and 2, setting  $K := \mathbf{W}_d^* \setminus (-\delta, \delta)^d$ , it follows that  $M := \max\{\tilde{r}(t) : t \in K\} < 1$ . Set  $\kappa := (M + 1)/2$ . It is a matter of fact that  $\kappa^{-n} \|\mathcal{F}_t^n\|_{E_t}$  vanishes as  $n$  goes to infinity so it is for  $\kappa^{-n} \|\mathcal{F}_t^n \mathbf{1}\|_{\infty}$  and the result follows.  $\square$

**Remark 4.1.** *In the literature, it is usually considered that the whole family  $\{\mathcal{F}_t\}_{t \in K}$  shares the same invariant space  $E$ . Allowing different spaces for different points  $t \in K$  improves the estimate of Proposition 4.1 while the pseudo spectral radius  $\tilde{r}$  still satisfies the nice property of upper semi-continuity for continuous perturbations of operators.*

### 4.2 Estimates in the neighborhood of the origin

At this level, it only remains to estimate the first integral term of Equation (9). Setting  $t = \frac{u}{\sqrt{n}}$ , it is given by

$$I_1(n) = \frac{1}{n^{d/2}} \int_{(-\delta\sqrt{n}, \delta\sqrt{n})^d} \mathcal{F}_{\frac{u}{\sqrt{n}}}^n \mathbf{1}(x) e^{-i\langle u/\sqrt{n}, \mathbf{v} - \mathbf{u} \rangle} du.$$

The expected estimate shall be obtained by integrating an almost-sure estimate exploiting the following expression

$$I_1(n) = \underbrace{\int_{\mathbb{X}^{\mathbb{N}}} n^{-d/2} \int_{|u| \leq \delta \sqrt{n}} \phi_{0,n}^{\omega}(u/\sqrt{n}) e^{-i\langle u/\sqrt{n}, \mathbf{v}-\mathbf{u} \rangle} du}_{I_1^{\omega}(n)} \mathbf{P}^x(d\omega). \quad (10)$$

**Proposition 4.2.** *Under Assumption 3, there exist a deterministic  $\delta > 0$  such that for  $|t| \leq \delta$ , the quantity  $\log \phi_{0,1}^{\omega}(t)$  is well defined. In addition, the following approximation formula holds  $\mathbf{P}^m$ -a.e.*

$$\log \phi_{0,1}(t) = \nabla \phi_{0,1}(0)t + \frac{1}{2} t^* [\nabla^* \nabla \phi_{0,1}(0) - \nabla \phi_{0,1}(0)^* \nabla \phi_{0,1}(0)] t + R(t), \quad (11)$$

where the remaining term  $R$  satisfies for  $|t| \leq \delta$  and for any  $\epsilon \in [0, 1)$  :

$$\|R(t)\|_{\infty} \leq \delta^{1-\epsilon} K |t|^{2+\epsilon} \text{ with } K \geq 0.$$

*Proof.* Under Assumption 3, the function  $\phi_{0,1}^{\omega}$  is  $\mathbf{P}^m$ -a.e. three times continuously differentiable. Therefore, the following majoration holds  $\mathbf{P}^m$ -a.e.

$$|\phi_{0,1}(t) - 1| \leq |t| \|\nabla \phi_{0,1}(0)\|_{\infty}.$$

Thus, there exists a deterministic  $\delta > 0$  such that for all  $|t| \leq \delta$  the function  $t \rightarrow \log \phi_{0,1}^{\omega}(t)$  is well defined. In addition, for  $|t| \leq \delta$ , the Taylor formula yields  $\mathbf{P}^m$ -a.e.

$$\log \phi_{0,1}(t) = \nabla \phi_{0,1}(0)t + \frac{1}{2} t^* [\nabla^* \nabla \phi_{0,1}(0) - \nabla \phi_{0,1}(0)^* \nabla \phi_{0,1}(0)] t + R(t),$$

where the remaining term  $R$  satisfies for  $|t| \leq \delta$  and any  $\epsilon \in [0, 1)$

$$\|R(t)\|_{\infty} \leq \delta^{1-\epsilon} K \|t\|^{2+\epsilon}, \quad \text{with } K = \left\| \sum_{\mathbf{u} \in \mathbb{Z}^d} \|\mathbf{u}\|^3 \mu(\mathbf{u}) \right\|_{\infty}.$$

□

This proposition implies that for  $\delta > 0$  sufficiently small and  $|u| \leq \delta \sqrt{n}$

$$\phi_{0,n}(t) = \exp \left\{ i \langle \Delta_n, t \rangle - \frac{1}{2} t^* \Sigma_n t + R_n(t) \right\},$$

with the notation

$$R_n(t) = R_n^{\omega}(t) = \sum_{k=0}^{n-1} R^{T^k \omega}(t/\sqrt{n}).$$

**Proposition 4.3.** *Under Assumption 3 the following properties hold*

1. for all  $n \geq 1$ , the matrix  $\Sigma_n$  is real positive symmetric  $\mathbf{P}^m$ -a.e.,
2. the sequence  $(\Sigma_n)_{n \geq 0}$  remains bounded in the following sense

$$\sup_{n \geq 0} \|\Sigma_n\|_{\infty} < \infty,$$

3.  $\mathbf{P}^m$ -a.e., the rank  $\text{rk } \Sigma_n$  is non decreasing with  $n \geq 1$ ,

4. in addition, under Assumption 1,  $\lim_{n \rightarrow \infty} \text{rk } \Sigma_n = d$ ,  $\mathbf{P}^m$ -a.e..

*Proof.* 1. Under Assumption 3,  $n\Sigma_n$  is a sum of covariance matrices so is positive semidefinite real symmetric.

2. As a matter of facts, the sequence of matrices  $\Sigma_n$  satisfies

$$\text{ess sup}_{\omega \in \Omega} \|\Sigma_n\| \leq \text{ess sup}_{\omega \in \Omega} \|\Sigma_1\| < \infty.$$

3. As a sum of positive semidefinite real symmetric matrices, the kernel of  $\Sigma_n$  is given by

$$\ker \Sigma_n = \bigcap_{k=0}^{n-1} \ker \left[ \nabla^* \nabla \phi_{0,1}^{T^k \omega}(0) - (\nabla \phi_{0,1}^{T^k \omega}(0))^* (\nabla \phi_{0,1}^{T^k \omega}(0)) \right],$$

and the result follows.

4. Since  $\text{rk } \Sigma_n$  is a non decreasing discrete bounded sequence  $\mathbf{P}^m$ -a.e., it suffices to show that

$$\mathbf{P}^x(\liminf \{\text{rk } \Sigma_n = d\}) = \mathbf{P}^x \left( \bigcup_{n \geq 1} \{\text{rk } \Sigma_n = d\} \right) = 1, \quad \text{for } m - \text{a.e. } x \in \mathbb{X}. \quad (12)$$

Moreover, the operator  $P$  is supposed aperiodic and recurrent so that we only need the asymptotic event in (12) holds with positive probability. Thus suppose on the contrary

$$\mathbf{P}^x(N) = 1 \quad \text{with} \quad N = \bigcap_{n \geq 1} \{\text{rk } \Sigma_n \leq d - 1\}.$$

Then the subgroup  $H$  of  $\mathbb{Z}^d$  generated by the supports  $\text{supp } \mu^\omega * \dots * \mu^{T^{n-1}\omega}$ ,  $n \geq 1$ ,  $\omega \in N$ , is  $\mathbf{P}^x$ -a.s. independent of  $\omega \in \mathbb{X}^{\mathbb{N}}$  and satisfies  $\mu^\omega(H) = 1$ ,  $\mathbf{P}^m$ -a.e.. Assumption 1 yields  $H = \mathbb{Z}^d$  which contradicts the non maximality of the asymptotic rank.  $\square$

Recall  $\tau = \inf\{n \geq 1 : \text{rk } \Sigma_n = d\}$  and remark it is finite  $\mathbf{P}^m$ -a.e. by Proposition 4.3. Rewrite  $I_1^\omega(n)$  as the sum of three terms  $I_{11}, I_{12}$  and  $I_{13}$  defined by

$$I_{11}^\omega(n) = n^{-d/2} \int_{|t| \leq \delta\sqrt{n}} e^{-\frac{1}{2}t^* \Sigma_n t} e^{i\langle t, \Delta_n \rangle - i\langle t, (\mathbf{v} - \mathbf{u})/\sqrt{n} \rangle} \left( e^{R_n(t/\sqrt{n})} - 1 \right) dt,$$

$$\text{for } n \geq \tau, \quad I_{12}^\omega(n) = n^{-d/2} \int_{\mathbb{R}^d} e^{-\frac{1}{2}t^* \Sigma_n t} e^{i\langle t, \Delta_n \rangle - i\langle t, (\mathbf{v} - \mathbf{u})/\sqrt{n} \rangle} dt,$$

and,

$$\text{for } n \geq \tau, \quad I_{13}^\omega(n) = -n^{-d/2} \int_{|t| > \delta\sqrt{n}} e^{-\frac{1}{2}t^* \Sigma_n t} e^{i\langle t, \Delta_n \rangle - i\langle t, (\mathbf{v} - \mathbf{u})/\sqrt{n} \rangle} dt.$$

**Proposition 4.4.** *Under Assumptions 1 and 3 for all  $n \geq \tau$  the following holds  $\mathbf{P}^m$ -a.e. for  $\mathbf{u}, \mathbf{v} \in \mathbb{Z}^d$ :*

$$1. \quad I_{12}(n) = \frac{(2\pi)^{d/2}}{n^{d/2} \det(\Sigma_n)^{1/2}} \exp \left\{ \left( -\frac{1}{2} [\Delta_n - (\mathbf{v} - \mathbf{u})/\sqrt{n}]^* \Sigma_n^{-1} [\Delta_n - (\mathbf{v} - \mathbf{u})/\sqrt{n}] \right) \right\}.$$

If additionally Assumption 4 is fulfilled, uniformly in  $\omega \in \Omega$ ,

2. there exists  $\alpha > 0$  such that  $|I_{13}(n)| \leq \mathcal{O}(n^{-d/2} \exp\{-\frac{1}{2}\alpha\delta\sqrt{n}\})$ ,

3. then there exists a deterministic  $\delta > 0$  such that  $|I_{11}(n)| = \mathcal{O}(n^{-\frac{d+\epsilon}{2}})$  for any  $\epsilon \in [0, 1)$ .

*Proof.* First, set  $\mathbf{v} = \mathbf{u}$ .

1. Under Assumptions 1 and 3, Proposition 4.3 implies  $\tau < \infty$   $\mathbf{P}^m$ -a.e. and for  $n \geq \tau$ , there exist orthogonal matrices  $P_n$  and diagonal matrices  $D_n$  such that

$$\Sigma_n = P_n D_n P_n^{-1} \text{ and } D_n = \text{diag}(\alpha_1^2(n), \dots, \alpha_d^2(n))$$

with  $\alpha_i^2(n) > 0$  for all  $i = 1, \dots, d$  and  $n \geq \tau$ . Setting  $t = P_n u$ , we obtain as the Fourier transform of Gaussian vectors

$$\begin{aligned} n^{d/2} I_{12}(n) &= \int_{\mathbb{R}^d} e^{-\frac{1}{2}u^* D_n u} e^{i\langle u, P_n^* \Delta_n \rangle} du \\ &= \det(\Sigma_n)^{-1/2} (2\pi)^{d/2} e^{-\frac{1}{2} \Delta_n^* \Sigma_n^{-1} \Delta_n}. \end{aligned}$$

2. For the term  $I_{13}(n)$  we can proceed analogously and we get the following (not sharp) upper bound for  $n \geq \tau$  :

$$\begin{aligned} n^{d/2} |I_{13}(n)| &\leq \int_{|u| > \delta\sqrt{n}} e^{-\frac{1}{2}u^* D_n u} du \\ &= \mathcal{O}\left(\exp\left\{-\frac{1}{2}\alpha\delta\sqrt{n}\right\}\right), \end{aligned}$$

for  $\alpha > 0$  of Assumption 4.

3. Because of the point (2) of Proposition 4.3, the eigenvalues of  $\Sigma_n$  remain bounded uniformly for  $n \geq 0$ . Thus, with Assumption 4, we deduce the following bound for  $I_{11}(n)$

$$\begin{aligned} n^{d/2} |I_{11}(n)| &\leq \int_{|t| \leq \delta\sqrt{n}} e^{-\frac{1}{2}t^* \Sigma_n t} \left| \exp\{R_n(tn^{-1/2})\} - 1 \right| dt \\ &\leq \int_{|t| \leq \delta\sqrt{n}} e^{-\frac{1}{2}t^* \Sigma_n t} \left| R_n(tn^{-1/2}) \right| \exp\{|R_n(tn^{-1/2})|\} dt \\ &\leq \frac{K\delta^{1-\epsilon}}{n^{\epsilon/2}} \int_{|t| \leq \delta\sqrt{n}} e^{-\frac{1}{2}t^* \Sigma_n t} |t|^{2+\epsilon} e^{|t|^2 \delta K} dt. \end{aligned}$$

The last estimates comes from Proposition 4.2 and holds for any  $\epsilon \in [0, 1)$ . We conclude by choosing  $\delta > 0$  such that  $\delta K \leq \alpha/4$  (where  $\alpha$  is given by Assumption 4). Consequently the integral is convergent and the whole term goes to zero at rate, up to a constant,  $n^{-\epsilon/2}$ .

Setting  $\tilde{\Delta}_n = \Delta_n - (\mathbf{v} - \mathbf{u})/\sqrt{n}$ , the result follows for the general case.  $\square$

*Proof of Theorem 2.7.* Let  $r \in (0, 1)$ , and compute

$$\begin{aligned} &\int_{\mathbf{W}^d} \text{Re} \sum_{n \geq 1} r^{n-1} \mathcal{F}_t^n \mathbf{1}(x) dt - \sum_{n \geq 1} r^{n-1} \int_{\mathbb{X}^{\mathbb{N}}} \mathbf{1}_{\{n \geq \tau\}} I_{12}(n) d\mathbf{P}^x \\ &= \sum_{n \geq 1} r^{n-1} \int_{\mathbb{X}^{\mathbb{N}}} \mathbf{1}_{\{n \geq \tau\}} [I_{11}(n) + I_{13}(n)] d\mathbf{P}^x \\ &+ \sum_{n \geq 1} r^{n-1} I_2(n) \\ &+ \sum_{n \geq 1} r^{n-1} \int_{\mathbb{X}^{\mathbb{N}}} \mathbf{1}_{\{n < \tau\}} I_1(n) d\mathbf{P}^x \end{aligned}$$

Taking absolute values on both side, under Assumptions 1, 2, 3 and 4, using Proposition 4.1, Proposition 4.4 with a suitable  $\delta > 0$ , it follows that, for some  $K \geq 0$ ,  $C > 0$ ,  $\kappa \in (0, 1)$  and any  $\epsilon \in (0, 1)$ ,

$$\begin{aligned} & \left| \int_{\mathbf{W}_d} \operatorname{Re} \sum_{n \geq 1} r^{n-1} \mathcal{F}_t^n \mathbf{1}(x) dt - (2\pi)^{d/2} \sum_{n \geq 1} r^{n-1} \int_{\mathbb{X}^{\mathbb{N}}} \frac{\mathbf{1}_{\{n \geq \tau\}}}{\det(\Sigma_n) n^{d/2}} \exp\left(-\frac{1}{2} \Delta_n^* \Sigma_n^{-1} \Delta_n\right) d\mathbf{P}^x \right| \\ & \leq \sum_{n \geq 1} r^{n-1} \frac{K}{n^{(d+\epsilon)/2}} + \sum_{n \geq 1} r^{n-1} \frac{K}{n^{d/2}} \exp\left\{-\frac{1}{2} \alpha \sqrt{n}\right\} \\ & \quad + C \sum_{n \geq 1} \kappa^{n-1} + \sum_{n \geq 1} r^{n-1} \mathbf{P}^x(\tau > n). \end{aligned}$$

The latter summation follows from the fact  $|I_1(n)| \leq 1$ ,  $\mathbf{P}^m$ -a.e.. Then, letting  $r \uparrow 1$ , the right handside remains bounded in  $\mathbf{L}_m^\infty(\mathbb{X})$  since  $d \geq 2$  and  $\tau$  is integrable with respect to  $\mathbf{P}^x$ ,  $m$ -a.e.  $x \in \mathbb{X}$ . The result follows by letting  $r \uparrow 1$  on the left hand-side and remarking that

$$\begin{aligned} & \lim_{r \uparrow 1} \sum_{n \geq 1} r^{n-1} \int_{\mathbb{X}^{\mathbb{N}}} \mathbf{1}_{n \geq \tau} \frac{1}{\det(\Sigma_n) n^{d/2}} \exp\left(-\frac{1}{2} \Delta_n^* \Sigma_n^{-1} \Delta_n\right) d\mathbf{P}^x \\ & = \sum_{n \geq 1} \int_{\mathbb{X}^{\mathbb{N}}} \frac{\mathbf{1}_{n \geq \tau}}{\det(\Sigma_n) n^{d/2}} \exp\left(-\frac{1}{2} \Delta_n^* \Sigma_n^{-1} \Delta_n\right) d\mathbf{P}^x \in [0, \infty] \end{aligned}$$

$m$ -a.e. by monotone convergence. □

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