



HAL
open science

The type problem for a class of inhomogeneous random walks: a series criterion by a probabilistic argument

Basile de Loynes

► **To cite this version:**

Basile de Loynes. The type problem for a class of inhomogeneous random walks: a series criterion by a probabilistic argument. 2016. hal-01274873v2

HAL Id: hal-01274873

<https://hal.science/hal-01274873v2>

Preprint submitted on 12 May 2016 (v2), last revised 22 Jun 2022 (v6)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



Distributed under a Creative Commons Attribution - NonCommercial - NoDerivatives 4.0 International License

The type problem for a class of inhomogeneous random walks : a series criterion by a probabilistic argument

Basile de Loynes¹

¹Institut de Recherche Mathématique Avancée (IRMA) - UMR CNRS 7501,
Université de Strasbourg, 67084 Strasbourg, France,
deloynes@unistra.fr

Abstract In the classical framework, a random walk on a group is a Markov chain with independent and identically distributed increments. In some sense, random walks are time and space homogeneous. In this paper, a class of weakly inhomogeneous random walks termed Random Walk with Random Transition Probabilities is investigated— *c.f.* [19] for the terminology. As an application, a criterion for the recurrence or transience of these processes in the discrete Abelian case is given. This criterion is deduced using Fourier analysis of Markov additive processes and a perturbation argument of a Markov operator. The latter extends the results of the literature since it does not involve a quasi-compactness condition on the operator. Finally, this criterion is applied to some well known examples of random walks on directed graphs embedded in \mathbb{Z}^2 . Despite the type problem has been already solved for these examples, one claim this analysis brought a new insight to this problematic.

Key words Weakly inhomogeneous random walks . Recurrence . Transience . Random Walks with Random Transition Probabilities . Markov additive processes . Fourier analysis . Local Limit Theorem . Operator perturbations

Mathematics Subject Classification (2000) 60F15 . 60J05 . 60J10 . 60J15 . 60J45 . 60J55 . 60K15

Contents

1	The model of Random Walk with Random Transition Probabilities	4
1.1	Definitions and notations	4
1.2	Inhomogeneous Markov chains as RW RTPs and factor RW RTPs	5
1.3	Random walks on constrained directed graphs	6
2	The type problem	8
2.1	Markov additive processes	8
2.2	Basic assumptions	9
2.3	The type problem : a series criterion	12
3	Applications : random walks on partially directed lattices	13

4 Proof of the main theorem	15
4.1 Estimates far from the origin	15
4.2 Estimates in the neighborhood of the origin	15

Introduction

In the classical framework, a random walk on a group \mathbf{G} is a discrete time stochastic process $(Z_n)_{n \geq 0}$ defined as the product of independent and identically distributed (*i.i.d.*) random variables $(\xi_n)_{n \geq 1}$. More precisely, for any $g \in \mathbf{G}$, we set $Z_0 = g$ and

$$Z_n = g\xi_1 \cdots \xi_n.$$

Random walks on groups are obviously Markov chains that are adapted to the group structure in the sense that the underlying Markov operator is invariant under the group action of \mathbf{G} on itself. Thus, this homogeneity naturally gives rise to deep connections between stochastic properties of the random walk and algebraic properties of the group. Starting with the seminal paper of Pòlya, [33], a large part of the literature is devoted to the study of these connections in this homogeneous case (for instance [25, 27, 17, 20, 39, 36, 22, 21, 2] and references therein).

In this paper, we aim at investigating “weakly” inhomogeneous random walks. It turns out that there are at least two ways to introduce inhomogeneity. First, we can consider “spatial” inhomogeneity by weakening the group structure, replacing it, for instance, by a directed graph structure (see [5, 15, 16, 31, 13]). Secondly, we can study “temporal” inhomogeneous random walks by introducing a notion of memory as in the model of reinforced [30, 38], excited [34, 3], self-interacting [11, 32] or also persistent random walks [9, 8, 10]. All these models belong to the larger class of stochastic processes with long range dependency.

The strategy in the sequel consists of making use of the connection between such stochastic processes and Random Walk with Random Transition Probabilities — or for short, **RWRTP** — the terminology comes from [19]. Roughly speaking, a **RWRTP** consists of a dynamical system (Ω, T) , endowed with a quasi-invariant (preimages by T of null measure sets have null measure) probability measure λ , together with a family $\{\mu^\omega\}_{\omega \in \Omega}$ on a group \mathbf{G} . The dynamics dictate the way the measures of $\{\mu^\omega\}_{\omega \in \Omega}$ can be convoluted¹. More precisely, choosing at random (with respect to the probability measure λ) a trajectory $\omega \in \Omega$, we are interested in the asymptotic properties of the sequence of convoluted measures $\mu^\omega * \mu^{T\omega} * \mu^{T^2\omega} * \dots * \mu^{T^n\omega}$. As a matter of facts, along a trajectory of the dynamical system, the resulting stochastic process have independent but, in general, not stationary increments. If in addition, the dynamical system admits a stationary probability measure, in mean, the increments are no longer independent, however, they are in some sense stationary. We refer to Section 1 for the rigorous definition of **RWRTP**. In Proposition 1.2 of Section 1, it is shown that any inhomogeneous Markov chain taking values in a group \mathbf{G} is in particular a **RWRTP** while Proposition 1.3 is a useful tool to simplify, in some case, the internal dynamics. At the end of this section, an application of this two propositions for directed graphs embedded (in some sense) in a finitely generated group, is given.

As it is noticed in [19], this model is actually a generalization of other well known models such as Random Walk With Internal Degree of Freedom — see [24] — or also in the modern formulation Markov additive processes, for short **MAP** — see [18, 28, 26, 1, 37, 23]. In this context, many results have been proved. Though, the basic assumptions are generally too much

¹In this sense, it is comparable to Random Dynamical Systems: instead of randomly composing transformations, we randomly compose random (continuous) group homomorphisms.

restrictive to encompass the class of very inhomogeneous random walks (in particular, the model of random walk on directed graphs of Figure 1).

Generally speaking, in the context of Markov additive processes, we may introduce the Fourier transform operator, denoted by \mathcal{F}_t , $t \in \mathbb{R}^d$, which is a continuous perturbation of the Markov operator \mathring{P} defining the underlying dynamics. The rate of the return probability is then estimated, under mild conditions, by inverse Fourier transform. In the literature, the Markov operator \mathring{P} is usually supposed to admit an ergodic invariant probability measure and to be quasi-compact. In addition, by a perturbation argument the Fourier transform operator remains quasi-compact for all t in a neighborhood of the origin. It allows, under suitable moment conditions on the system of probability measures, to derive a Taylor expansion at the second order of the perturbed dominating eigenvalue $\gamma(t)$ (whose the coefficients are given essentially by the mean and the variance operators). Finally, under an assumption on the spectrum of the Fourier transform operator outside a neighborhood of the origin, it can be concluded that all the needed stochastic information is actually contained in the nature of the singularity at zero of $(1 - \gamma(t))^{-1}$ (note $\gamma(0) = 1$).

In this paper, the quasi-compactness condition of \mathring{P} is dropped. It is only assumed that \mathring{P} is irreducible, recurrent and aperiodic. The condition on the spectrum of \mathcal{F}_t for large perturbations remains similar but the nature of the singularity at the origin is analyzed *via* probabilistic methods. These estimates give rise to a criterion for the type problem in terms of the summability of a series. More precisely, introduce the following quantities for $n \geq 1$ and $\omega \in \Omega$

$$\Sigma_n := \Sigma_n(\omega) = -\frac{1}{n} \sum_{k=0}^{n-1} \nabla^* \nabla \phi_{0,1}^{T^k \omega}(0), \quad \Delta_n := \Delta_n(\omega) = i \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} \nabla \phi_{0,1}^{T^k \omega}(0), \quad (1)$$

and the (almost surely finite) random variable τ which is the first instant such that the rank Σ_n is maximal. The main theorem of this paper states that, under the assumptions 1, 2, 3 and 4, the Markov additive process is transient or recurrent accordingly as

$$\sum_{n \geq 1} \frac{1}{n^{d/2}} \int_{\Omega} \mathbf{1}_{n \geq \tau} \exp\left(-\frac{1}{2} \Delta_n^* \Sigma_n^{-1} \Delta_n\right) d\lambda$$

is finite or infinite provided τ has a finite expectation. The rigorous assumptions and the statement of the theorem are given in Section 2.

Thus, the main term of the series involves the expectation of a functional of two random variables. The recurrence or transience behavior is interpreted as the result of the competition of this two random quantities. More precisely, the process is recurrent if locally it is sufficiently diffusive in order to balance the instantaneous deviation. It turns out that $\Delta^* \Sigma^{-1} \Delta$ is in some circumstances a self-normalized martingale (see [12] for instance).

In Section 3, applying this criterion to the model of random walks on directed graphs introduced in [5] and represented on the figure 1 below, we aim at illustrating that this theorem is effectively more general than those found in the literature and can be useful in practice.

For the graph (a) of Figure 1, denoted by \mathbb{L} , a random walker choose at random one of the nearest neighbour among North, South or West if the ordinate of its current position is odd, and the nearest neighbour among North, South and East if it is even. It is shown in [5] that this random walk is recurrent. For the graph (b) of Figure 1, denoted by \mathbb{H} , the possible movements for a random walker are toward the North, South and West on the upper half-plane and North, South and East otherwise. The resulting random walk is shown to be transient.

Using the theorem of factorization of Section 1, it follows easily that these random walks are Markov additive processes. The (minimal) internal Markov chains \mathring{P} are random walks on the group $\mathbb{Z}/2\mathbb{Z}$ and \mathbb{Z} respectively. In the periodic case, the internal Markov chain admits a

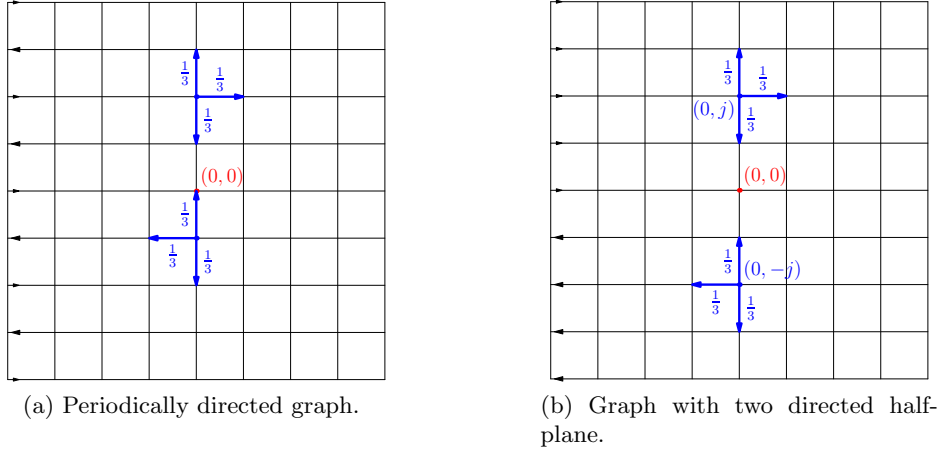


Figure 1: Directed graphs \mathbb{L} et \mathbb{H} de [5].

unique stationary probability measure, whereas for the graph with two directed half-plane it is null recurrent. The fundamental difference between the two random walks imposes to drop the strong quasi-compactness condition. Besides, beyond their own interests in the classical probability theory, it is worth noting the study of these models was initially motivated by natural questions arising in quantum theory as described [6].

1 The model of Random Walk with Random Transition Probabilities

1.1 Definitions and notations

Let $(\mathbb{X}, \mathcal{X}, m)$ be a measured space. We denote by $\mathbf{L}^\infty(m)$ the space of real essentially bounded function on \mathbb{X} . A linear operator $P : \mathbf{L}^\infty(m) \rightarrow \mathbf{L}^\infty(m)$ is called Markov if $P\mathbf{1} = \mathbf{1}$ and $Pf \geq 0$ whenever $f \geq 0$. Let us denote by $\mathbf{IMC}(\mathbb{X}, (P_n)_{n \geq 0}, \theta)$ the Inhomogeneous Markov Chain on \mathbb{X} where $(P_n)_{n \geq 0}$ is a sequence of Markov operator on $\mathbf{L}^\infty(m)$, θ is absolutely continuous with respect to m — in symbols $\theta \prec m$ — and stands for the initial probability measure. If the sequence of Markov operators $(P_n)_{n \geq 0}$ is constant and equal to P , we denote by $\mathbf{MC}(\mathbb{X}, P, \theta)$ the resulting time homogeneous Markov chain.

For a \mathbf{IMC} , it is well-known that the paths space $\mathbb{X}^{\mathbb{N}}$ can be endowed with a unique probability measure, defined from the sequence of Markov operator $(P_n)_{n \geq 0}$ and the probability measure θ , called the canonical probability measure on the paths space and denoted by \mathbf{P}^θ .

Let (Ω, T, λ) be a dynamical system where T is a measurable map from Ω into itself and λ is a probability measure which, in general, is not assumed to be preserved by the dynamics. Nonetheless, we will suppose that there exists a σ -finite invariant measure ρ dominating λ .

Let \mathbf{G} be a locally compact group and denote by \mathbf{Haar} the unique, up to a multiplicative constant, Haar measure on \mathbf{G} . Denote by $\mathcal{M}^1(\mathbf{G})$ the space of probability measures on \mathbf{G} endowed with the Borel σ -algebra relatively to the weak topology. A system of probability measures over Ω is a measurable map $\mu : \omega \in \Omega \rightarrow \mu^\omega \in \mathcal{M}^1(\mathbf{G})$.

If μ is a system of probability measures, we define the Markov operator \bar{P} for all $f \in \mathbf{L}_{\rho \otimes \mathbf{Haar}}^\infty(\Omega \times \mathbf{G})$ as follows

$$\bar{P}f(\omega, g) = \int_{\mathbf{G}} f(T\omega, gh) d\mu^\omega(h).$$

The system of probability measures μ also defines a sequence of Markov operators P_n^ω for all $f \in \mathbf{L}_{\text{Haar}}^\infty(\mathbf{G})$ by

$$P_n^\omega f(x) = \int_{\mathbf{G}} f(xy) \mu^{T^n \omega}(dy), \quad x \in \mathbf{G}, \quad \omega \in \Omega$$

that is P_n^ω is the right convolution operator². If ν is another system of probability measures, we denote, for all ω , the canonical probability measure on the paths space of the inhomogeneous Markov chain $\mathbf{IMC}(\mathbf{G}, (P_n^\omega)_{n \geq 0}, \nu^\omega)$ by $\mathbf{P}^{\nu, \omega}$ or for short \mathbf{P}^ω if there is no ambiguity on ν — for instance if $\nu^\omega = \delta_{\text{id}}$, *i.e.* for every $\omega \in \Omega$ the \mathbf{IMC} start from the identity of \mathbf{G} .

Definition 1.1. *The triplet $((\Omega, T, \lambda), \mu, \nu)$ is called a Random Walk with Random Transition Probabilities — on \mathbf{G} — and will be denoted in sequel by $\mathbf{RW RTP}((\Omega, T, \lambda), \mu, \nu)$ or if there is no ambiguity $\mathbf{RW RTP}(\lambda, \mu)$.*

1.2 Inhomogeneous Markov chains as $\mathbf{RW RTP}$ s and factor $\mathbf{RW RTP}$ s

The elementary Proposition 1.2 shows that any discrete time inhomogeneous Markov chain taking values in a group \mathbf{G} is a $\mathbf{RW RTP}$. Afterward, factor $\mathbf{RW RTP}$ are defined. This latter notion involves measurable partitions for which the suitable framework is that of Lebesgue spaces (we refer to [4] for the definition of Lebesgue spaces and measurable equivalence relations). Since the set Ω shall be the infinite countable product set, we assume that, endowed with a suitable σ -algebra and probability, it is a Lebesgue space.

Proposition 1.2. *Let $\mathbf{IMC}(\mathbf{G}, (P_n)_{n \geq 0}, \theta)$ be an inhomogeneous Markov chain. Then there exists a $\mathbf{RW RTP}((\Omega, T, \lambda), \mu, \nu)$ such that*

$$\int_{\Omega} \lambda(d\omega) \mathbf{P}^{\nu, \omega} = \mathbf{P}^\theta.$$

Proof. Let us set $\Omega = \mathbf{G}^{\mathbb{N}}$, $\lambda = \mathbf{P}^\theta$, and T is the time-shift on the paths space of the \mathbf{IMC} . Finally, we define μ and ν for all $\omega \in \Omega$ as follows

$$\mu^\omega = \delta_{\omega_0^{-1} \omega_1} \quad \text{and} \quad \nu^\omega = \delta_{\omega_0}.$$

Therefore, it defines a $\mathbf{RW RTP}((\Omega, T, \lambda), \mu, \nu)$ for which the equality of the proposition is trivially satisfied. \square

This proposition essentially relies on the theorem of Ionescu Tulcea that defines the Markovian dynamical system $(\Omega, T, \mathbf{P}^\theta)$ together with the fact that the considered stochastic process takes its values in a group \mathbf{G} . Obviously, this proposition is purely theoretical and no more information is given by the resulting $\mathbf{RW RTP}$. Below, we introduce the notion of factor $\mathbf{RW RTP}$ that allows to take into account the internal symmetries of the process and thus simplify in general the analysis.

For a $\mathbf{RW RTP}((\Omega, T, \lambda), \mu, \nu)$, consider a measurable equivalence relation denoted by \sim and denote by pr and λ_{pr} the related projection and conditional measure on Ω/\sim . The conditional system of probability measures μ_{pr} over Ω/\sim is then defined as follows

$$\mu_{\text{pr}} = \int_{\Omega} \mu^\omega d\lambda_{\text{pr}}(\omega), \quad (2)$$

If in addition, the maps T can be factorized (in facts, it is \sim -measurable), that is $\text{pr } T\omega = \text{pr } T\tilde{\omega}$ as soon as $\text{pr } \omega = \text{pr } \tilde{\omega}$, then we shall define the factor $\mathbf{RW RTP}((\Omega/\sim, \text{pr} \circ T, \lambda_{\text{pr}}), \mu_{\text{pr}}, \nu_{\text{pr}})$.

²We may define the left convolution operator similarly by the formula $Pf(x) = \int_{\mathbf{G}} f(yx) \mu^{T^n \omega}(dy)$.

Proposition 1.3. *Let $\mathbf{RW RTP}((\Omega, T, \lambda), \mu, \nu)$ be any $\mathbf{RW RTP}$, suppose T can be factorized, and denote by $\mathbf{RW RTP}((\Omega/\sim, \text{pr} \circ T, \lambda_{\text{pr}}, \mu_{\text{pr}}, \nu_{\text{pr}})$ its factorization under \sim . Then, the following equality holds*

$$\int_{\Omega} \lambda(d\omega) \mathbf{P}_{\nu, \omega} = \int_{\Omega/\sim} \lambda_{\text{pr}}(d\omega) \mathbf{P}_{\nu_{\text{pr}}, \omega}. \quad (3)$$

Proof. We only need to check the equality of finite dimensional distributions between both probability of (3). Thus, let α and β be any system of probability measures, then the integrated product measures verify for any measurable subset $A \times B \subset \mathbf{G} \times \mathbf{G}$

$$\begin{aligned} \int_{\Omega/\sim} \lambda_{\text{pr}}(d\omega) \alpha_{\text{pr}}^{\omega} \otimes \beta_{\text{pr}}^{\omega}(A \times B) &= \int_{\Omega/\sim} \lambda_{\text{pr}}(d\omega) \alpha_{\text{pr}}^{\omega}(A) \beta_{\text{pr}}^{\omega}(B) \\ &= \int_{\Omega} \lambda(d\omega) \alpha_{\text{pr}}^{\text{pr}\omega}(A) \beta_{\text{pr}}^{\text{pr}\omega}(B) \\ &= \int_{\Omega} \lambda(d\omega) \alpha^{\omega}(A) \beta^{\omega}(B) \\ &= \int_{\Omega} \lambda(d\omega) \alpha^{\omega} \otimes \beta^{\omega}(A \times B). \end{aligned}$$

Thus the image measures by the maps $\mathbf{G} \times \mathbf{G} \ni (x, y) \rightarrow xy \in \mathbf{G}$ coincide and it follows that

$$\int_{\Omega} \lambda(d\omega) \alpha^{\omega} * \beta^{\omega} = \int_{\Omega/\sim} \lambda_{\text{pr}}(d\omega) \alpha_{\text{pr}}^{\omega} * \beta_{\text{pr}}^{\omega}.$$

Proceeding by induction, the conclusion of the proposition is proved. \square

Remark 1.1. *It is worth noting that the ergodicity of the probability measure λ , that is $\lambda(A) = 0$ or 1 as soon as A is a T -invariant measurable subset of Ω , implies its projection $\text{pr}_* \lambda$ is ergodic.*

Remark 1.2. *We only need here the \sim -equivalence classes to be invariant under the map T . However, the conditional probability λ_{pr} , and hence, the conditional system of probability measures μ_{pr} and ν_{pr} are not in general easily characterized. Actually, it heavily depends on the simplicity of the model considered and thus, the strategy consists of determining the finest equivalence relation (in the complete lattice of measurable partitions up to null sets) whereas λ_{pr} remains easily computable. In the next section, such equivalence relations shall be given for random walks on directed graphs.*

1.3 Random walks on constrained directed graphs

In this section, after introducing standard notions related to graphs, we consider random walks on graphs embedded in a group \mathbf{G} . As any discrete time stochastic process, it is isomorphic to a $\mathbf{RW RTP}$. In the sequel, we give the corresponding $\mathbf{RW RTP}$ that is minimal in the sense of Remark 1.2.

A directed graph (or di-graph for short) $\mathbb{G} = (\mathbb{G}^0, \mathbb{G}^1, r, s)$ is a quadruple consisting of a denumerable set \mathbb{G}^0 of vertices, a denumerable set \mathbb{G}^1 of directed edges and a pair of range and source maps, denoted respectively by r and s , *i.e.* mappings $r, s : \mathbb{G}^1 \rightarrow \mathbb{G}^0$. In the sequel, we only consider graphs without loops (*i.e.* not containing edges $\alpha \in \mathbb{G}^1$ such that $r(\alpha) = s(\alpha)$) and without multiple edges (*i.e.* if $\alpha, \beta \in \mathbb{G}^1$ satisfy $s(\alpha) = s(\beta)$ and $r(\alpha) = r(\beta)$ then $\alpha = \beta$, or in other words, the compound map $(r, s) : \mathbb{G}^1 \rightarrow \mathbb{G}^0 \times \mathbb{G}^0$ is injective). With these restrictions in force, \mathbb{G}^1 can be identified with a particular subset of $\mathbb{G}^0 \times \mathbb{G}^0$ and the maps r and s become superfluous: $s((\mathbf{u}, \mathbf{v})) = \mathbf{u}$ and $r((\mathbf{u}, \mathbf{v})) = \mathbf{v}$ for all $(\mathbf{u}, \mathbf{v}) \in \mathbb{G}^1$. The corresponding directed graph is then termed simple.

We also define respectively, for each vertex $\mathbf{v} \in \mathbb{G}^0$, its inwards and outward degree by

$$\deg^+(\mathbf{v}) = \text{card} \{ \alpha \in \mathbb{G}^1 : r(\alpha) = \mathbf{v} \} \text{ and } \deg^-(\mathbf{v}) = \text{card} \{ \alpha \in \mathbb{G}^1 : s(\alpha) = \mathbf{v} \}.$$

A simple graph is termed undirected if the set of edges is symmetric in $\mathbb{G}^0 \times \mathbb{G}^0$, that is $(\mathbf{u}, \mathbf{v}) \in \mathbb{G}^1$ if and only if $(\mathbf{v}, \mathbf{u}) \in \mathbb{G}^1$. In this case, the inwards and outwards degree coincide, the common value is simply denoted by \deg and called the degree.

A graph \mathbb{G} is said to be connected if for any vertices $\mathbf{u}, \mathbf{v} \in \mathbb{G}^0$ there exists a finite sequence $\alpha = (\alpha_1, \dots, \alpha_k)$ of edges $\alpha_i \in \mathbb{G}^1$, for $i = 1, \dots, k$, $k \in \mathbb{N}$, with $s(\alpha_1) = \mathbf{u}$ and $r(\alpha_k) = \mathbf{v}$, such that $r(\alpha_i) = s(\alpha_{i+1}) \in \mathbb{G}^0$, for all $i = 1, \dots, k - 1$. The above sequence α is called a path of length $k = |\alpha|$ from \mathbf{u} to \mathbf{v} . A directed graph is said to be locally finite if for all $\mathbf{u} \in \mathbb{G}^0$, the inwards and outwards degree are finite.

A random walk on a connected locally finite graph \mathbb{G} is a Markov chain whose state space is given by \mathbb{G}^0 and the transition probabilities satisfy for each $\mathbf{u}, \mathbf{v} \in \mathbb{G}^0$

$$P(\mathbf{u}, \mathbf{v}) \in (0, 1] \iff (\mathbf{u}, \mathbf{v}) \in \mathbb{G}^1,$$

with the additional property $\sum_{\mathbf{v} \in \mathbb{G}^0} P(\mathbf{u}, \mathbf{v}) = 1$. A random walk on \mathbb{G} is said to be simple if the transition probabilities are given for all $\mathbf{u}, \mathbf{v} \in \mathbb{G}^0$ by

$$P(\mathbf{u}, \mathbf{v}) = \begin{cases} \frac{1}{\deg^-(\mathbf{u})} & \text{if } (\mathbf{u}, \mathbf{v}) \in \mathbb{G}^1, \\ 0 & \text{otherwise.} \end{cases}$$

Let us denote by \mathfrak{S} the group of permutations on the set of vertices of \mathbb{G} . The set of automorphisms of a graph \mathbb{G} is the subgroup of \mathfrak{S} denoted by $\text{Aut}(\mathbb{G})$ defined as follows

$$\text{Aut}(\mathbb{G}) = \{ f \in \mathfrak{S} : (\mathbf{u}, \mathbf{v}) \in \mathbb{G}^1 \iff (f(\mathbf{u}), f(\mathbf{v})) \in \mathbb{G}^1 \}.$$

The group of automorphisms of a random walk on a graph \mathbb{G} is defined as follows

$$\text{Aut}(\mathbb{G}, P) = \{ f \in \mathfrak{S} : P(\mathbf{u}, \mathbf{v}) = P(f(\mathbf{u}), f(\mathbf{v})) \text{ for all } \mathbf{u}, \mathbf{v} \in \mathbb{G}^0 \}.$$

The latter group is a subgroup of the former and they are isomorphic if the random walk is supposed to be simple. The two groups of automorphisms naturally acts on the graph \mathbb{G} , however, there is no reason for the action of $\text{Aut}(\mathbb{G}, P)$ to be transitive. In facts, in the case of a transitive action, the so-called homogeneous case, the random walk on the graph can be transported on a random walk in the classical sense on $\text{Aut}(\mathbb{G}, P)$ — see [22]. Note that, as soon as the graph is genuinely directed, the group of automorphisms $\text{Aut}(\mathbb{G})$ cannot act transitively.

In order to see random walks on graphs as **RWRTP**, it is necessary to embed the graph in a group \mathbf{G} . A natural family of such graphs are constrained graphs, *i.e.* graphs constructed from the Cayley graph of a finitely generated group by deleting some edges with respect to a predicate. More precisely, let \mathbf{G} be a group generated by a finite set \mathcal{S} supposed symmetric. A predicate is a $\{0, 1\}$ -valued map \mathbf{p} on $\mathbf{G} \times \mathcal{S}$. Thus, we may construct a constrained graph $\mathbb{G} = (\mathbb{G}^0, \mathbb{G}^1)$ where

$$\mathbb{G}^0 \subset \mathbf{G} \text{ countable and } \mathbb{G}^1 = \{ (\mathbf{u}, \mathbf{v}s) \in \mathbb{G}^0 \times \mathbb{G}^0, s \in \mathcal{S}, \mathbf{p}(\mathbf{u}, s) = 1 \}.$$

The resulting graph inherits the simplicity and the local finiteness property of Cayley graphs of finitely generated groups. Though it can be disconnected unless the predicate \mathbf{p} is suitably chosen which we shall assume in the sequel.

Let T be the unilateral time shift on $\Omega = (\mathbb{G}^0)^\mathbb{N}$ and for every fixed $\mathbf{u}_0 \in \mathbb{G}^0$, the probability measure $\lambda = \mathbf{P}^{\mathbf{u}_0}$. Also, the system of probability measures μ and ν are defined respectively by

$$\mu^\omega = \delta_{\omega_0^{-1}\omega_1} \text{ and } \nu^\omega = \delta_{\omega_0}, \quad \omega \in (\mathbb{G}^0)^\mathbb{N}.$$

Then, by Proposition 1.2,

$$\mathbf{P}^{\mathbf{u}_0} = \int_{(\mathbb{G}^0)^{\mathbb{N}}} \mathbf{P}_{\nu, \omega} \lambda(d\omega).$$

We shall define an equivalence relation on Ω , denoted in the sequel by \sim as follows

$$\omega \sim \tilde{\omega} \in (\mathbb{G}^0)^{\mathbb{N}} \iff \forall n \geq 0 : \omega_n = \tilde{\omega}_n \bmod \text{Aut}(\mathbb{G}, P).$$

As a matter of fact, the time shift T commutes with this equivalence relation. Thus, by Proposition 1.3, the **RWRTP** above can be factorized. More precisely, the map $\text{pr} \circ T$ is the time shift on the factor space $\Omega/\sim = \left(\mathbb{G}^0/\text{Aut}(\mathbb{G}, P)\right)^{\mathbb{N}}$. Also, we denote by λ_{pr} the factor probability measure, with respect to \sim , of the probability measure $\lambda = \mathbf{P}^{\mathbf{u}_0}$. It turns out that λ_{pr} is nothing but the canonical probability measure of the Markov chain on the state space $\mathbb{G}^0/\text{Aut}(\mathbb{G}, P)$ whose initial distribution is given by $\delta_{\text{pr}(\mathbf{u}_0)}$ and Markov operator, denoted by \dot{P} , satisfies $\dot{P}(\text{pr } \mathbf{u}, \text{pr } \mathbf{v}) = P(\mathbf{u}, \mathbf{v})$ for all $\mathbf{u}, \mathbf{v} \in \mathbb{G}^0$. The system of probability measures $\left\{ \mu_{\text{pr}}^{\mathbf{w}_0, \mathbf{w}_1} : \mathbf{w}_0, \mathbf{w}_1 \in \mathbb{G}^0/\text{Aut}(\mathbb{G}^0, P) \right\}$ defined in (2) rewrites in our context as

$$\mu_{\text{pr}}^{\mathbf{w}_0, \mathbf{w}_1} = \frac{\sum_{\omega_1 \in \mathbb{G}^0} P(\omega_0, \omega_1) \mathbf{1}_{\{(\mathbf{w}_0, \mathbf{w}_1)\}}(\text{pr } \omega_0, \text{pr } \omega_1) \mu^\omega}{\sum_{\omega_1 \in \mathbb{G}^0} P(\omega_0, \omega_1) \mathbf{1}_{\{(\mathbf{w}_0, \mathbf{w}_1)\}}(\text{pr } \omega_0, \text{pr } \omega_1)}, \quad \text{for any } \omega_0 \in \mathbb{G}^0 \text{ s.t. } \text{pr } \omega_0 = \mathbf{w}_0.$$

Also, we define $\nu_{\text{pr}}^{\mathbf{w}}$ as the Dirac distribution at the point $\mathbf{w}_0 \in \mathbb{G}^0/\text{Aut}(\mathbb{G}, P)$.

Remark 1.3. *Note that any discrete process can be seen as a Markov chain with respect the filtration containing the whole history. This Markov chain with long memory can be modeled by a random walk on a directed graph whose corresponding factorization reveals the structure of the memory.*

2 The type problem

In the previous section, it is shown that any discrete time process (taking values in a group \mathbf{G}) is in some sense a **RWRTP**. In this section, we shall consider the restricted class of Markov additive processes, for short **MAP**, that are, roughly speaking, **RWRTP** for which the group \mathbf{G} is supposed Abelian and the underlying dynamics is Markovian. We shall be even more restrictive by setting $\mathbf{G} = \mathbb{Z}^d$. For a non discrete Abelian group, say $\mathbf{G} = \mathbb{R}^d$ or \mathbf{G} compact, the Fourier analysis of continuous measure is sensibly different but remains irrelevant for the scope of this paper.

2.1 Markov additive processes

In the sequel, we refer to [29] for the notion of Markov chain on general spaces. We shall consider a Markov kernel \dot{P} relatively to a measurable space $(\mathbb{X}, \mathcal{X})$ for which the σ -algebra is separable. The space $(\mathbb{X}, \mathcal{X})$ can be endowed with a σ -finite measure m dominating $m\dot{P}$. Without loss of generality, we may assume that the Markov chain induced by \dot{P} is m -irreducible and aperiodic. A stronger assumption is to suppose the Markov chain m -recurrent. In this case, the time shift on $\mathbb{X}^{\mathbb{N}}$ preserves the possibly infinite, but σ -finite, Markov measure \mathbf{P}^m . The probability measures μ^ω , $\omega \in \mathbb{X}^{\mathbb{N}}$ are supposed to depend only on the two first coordinates³ of $\omega \in \mathbb{X}^{\mathbb{N}}$ and shall be alternatively denoted by $\mu^{x,y}$, $x, y \in \mathbb{X}$.

³Actually, we use here the traditional definition, though it is completely equivalent to suppose that the probabilities μ^ω depends only on the first coordinate providing we consider the 2-order Markov chain induced by \dot{P} .

A Markov additive process with internal Markov chain \mathring{P} together with the system of probability measures μ is the Markov chain on the space $\mathbb{X} \times \mathbb{Z}^d$ whose Markov operator is given, for $f \in \mathbf{L}_m^\infty(\mathbb{X} \times \mathbb{Z}^d)$, by

$$\bar{P}f(x, \mathbf{u}) = \int_{\mathbb{X} \times \mathbb{Z}^d} \mathring{P}(x, dy) (\delta_y \otimes \mu^{x,y}) (dzd\mathbf{v}) f(z, \mathbf{u} + \mathbf{v}), \quad \text{for } (x, \mathbf{u}) \in \mathbb{X} \times \mathbb{Z}^d.$$

The Markov operator \bar{P} is invariant by translations of the kind $\mathbb{X} \times \mathbb{Z}^d \ni (x, \mathbf{u}) \rightarrow (x, \mathbf{u} + \mathbf{v}) \in \mathbb{X} \times \mathbb{Z}^d$.

In this context, we may introduce the so called Fourier transform operator, denoted by \mathcal{F}_t , for $t \in \mathbb{R}^d$, acting as a contraction on $\mathbf{L}_m^\infty(\mathbb{X})$ and defined for $f \in \mathbf{L}_m^\infty(\mathbb{X})$ by

$$\mathcal{F}_t f(x) = \int_{\mathbb{X}} \mathring{P}(x, dy) \widehat{\mu^{x,y}}(t) f(y), \quad \text{for } m\text{-a.e. } x \in \mathbb{X}, \quad (4)$$

where $\widehat{\mu^{x,y}}$ is the standard Fourier transform of the probability measure $\mu^{x,y}$ defined, for $t \in \mathbb{R}^d$ by

$$\widehat{\mu^{x,y}}(t) = \sum_{\mathbf{v} \in \mathbb{Z}^d} \mu^{x,y}(\mathbf{v}) e^{i\langle t, \mathbf{v} \rangle},$$

$\langle \cdot, \cdot \rangle$ standing for the standard inner product.

2.2 Basic assumptions

Let σ be a bounded map from \mathbb{X} to \mathbb{Z}^d and define

$$\bar{P}_\sigma f(x, \mathbf{u}) = \int_{\mathbb{X} \times \mathbb{Z}^d} \mathring{P}(x, dy) (\delta_y \otimes \mu^{x,y}) (dzd\mathbf{v}) f(z, \mathbf{u} + \mathbf{v} - \sigma(y) + \sigma(x)), \quad \text{for } (x, \mathbf{u}) \in \mathbb{X} \times \mathbb{Z}^d.$$

Intuitively, it corresponds to a change of origin of each fiber parametrized by \mathbb{X} . Such a function σ is called a change of section. The translated probability shall be denoted $\mu_\sigma^{x,y} = \mu^{x,y} * \delta_{\sigma(y) - \sigma(x)}$, or simply $\mu^{x,y}$ if no ambiguity. Because of the invariance of \bar{P} under translations of \mathbb{Z}^d , the changes of section have no fundamental importance in the study of the recurrent or transient behavior of the Markov additive process.

To keep notations light and readable, we shall adopt alternatively the ones related to dynamical system (such as μ^ω , \mathbf{P}^m -a.e.,...) or those ones proper to Markov additive processes ($\mu^{x,y}$, m -a.e.,...).

Definition 2.1 (Adaptation, aperiodicity, irreducibility). *A MAP is respectively adapted, aperiodic and irreducible if for any change of section σ*

1. *there is no proper subgroup $H \subset \mathbb{Z}^d$ such that \mathbf{P}^m -a.e. $\mu^\omega(H) = 1$;*
2. *there is no proper subgroup $H \subset \mathbb{Z}^d$ and no $a \in \mathbb{R}^d$ such that \mathbf{P}^m -a.e. $\mu^\omega(a + H) = 1$.*
3. *there is no half-space $H \subset \mathbb{R}^d$ such that \mathbf{P}^m -a.e. $\mu^\omega(H) = 1$.*

The properties of adaptation and aperiodicity can be read on the spectrum of the Fourier transform \mathcal{F}_t as shown in the corollary below \mathbf{W}_d denotes the set $[-\pi, \pi)$ and \mathbf{W}_d^* the set $\mathbf{W}_d \setminus \{0\}$.

Proposition 2.2. *A MAP is adapted if and only if for any change of section $\sigma : \mathbb{X} \rightarrow \mathbb{Z}^d$ and any $t \in \mathbf{W}_d^*$ it holds that $\widehat{\mu^\omega}(t) \neq 1$, \mathbf{P}^m -a.e.. A MAP is aperiodic if and only if for any change of section σ and any $t \in \mathbf{W}_d^*$ it holds, \mathbf{P}^m -a.e., $\widehat{\mu^\omega}(t) \neq e^{i\theta}$ for all $\theta \in \mathbb{R}$.*

Proof. If a **MAP** is not aperiodic then there exist a change of section, a proper subgroup H and $a \in \mathbb{R}^d$ such that $\mu^\omega(a + H) = 1$, \mathbf{P}^m -a.e.. It is well known that there exist integers n_1, \dots, n_r , $1 \leq r \leq d$, such that the subgroup H is generated by a set $\{n_1 e_{i_1}, \dots, n_r e_{i_r}\}$ for some indices $1 \leq i_1 \leq \dots \leq i_r \leq d$, where e_i is the i^{th} vector of the canonical basis of \mathbb{Z}^d . Suppose $r < d$, then for any $t \in \mathbf{W}_d^* \cap (\text{span } H)^\perp$ it follows that $\widehat{\mu^\omega}(t) = e^{i\langle t, a \rangle}$. If $r = d$, set $t = (\pi/n_1, \dots, \pi/n_d)$ then again $\widehat{\mu^\omega}(t) = e^{i\langle t, a \rangle}$ and t obviously belongs to \mathbf{W}_d^* unless $n_1 = \dots = n_d = 1$, i.e. $H = \mathbb{Z}^d$.

Conversely, suppose there exists a change of section σ and $t \in \mathbf{W}_d^*$ such that $\widehat{\mu^\omega}(t) = e^{i\theta}$, \mathbf{P}^m -a.e., for some $\theta \in \mathbb{R}$. Since μ^ω is supported by a subset of \mathbb{Z}^d , one can choose $a \in \mathbb{Z}^d$ such that $\theta = \langle t, a \rangle$. Then, it holds for any $n \geq 1$

$$e^{i\langle t, na \rangle} = \prod_{k=0}^{n-1} \widehat{\mu^{T^k \omega}}(t) = \sum_{\mathbf{w} \in \mathbb{Z}^d} \mu^\omega * \dots * \mu^{T^{n-1} \omega}(\mathbf{w}) e^{i\langle t, \mathbf{w} \rangle}, \quad \mathbf{P}^m - a.e.,$$

or equivalently,

$$1 = \sum_{\mathbf{w} \in \mathbb{Z}^d} \{\mu^\omega * \delta_{-a}\} * \dots * \{\mu^{T^{n-1} \omega} * \delta_{-a}\}(\mathbf{w}) e^{i\langle t, \mathbf{w} \rangle}, \quad \mathbf{P}^m - a.e..$$

Thus, the convex combination of points on the unit circle on the right hand side is extremal so that each $\mathbf{w} \in \mathcal{S}_n^\omega = \text{supp } \{\mu^\omega * \delta_{-a}\} * \dots * \{\mu^{T^{n-1} \omega} * \delta_{-a}\}$, $n \geq 1$, satisfies $\langle t, \mathbf{w} \rangle = 0$ modulo 2π . Setting $N = \{\omega \in \mathbb{X}^{\mathbb{N}} : \widehat{\mu^\omega}(t) \neq e^{i\langle t, a \rangle}\}$ and defining H as the smallest subgroups containing the sets \mathcal{S}_n^ω , for all $n \geq 1$ and $\omega \in N^c$, it follows that, \mathbf{P}^m -a.e., $(\mu^\omega * \delta_{-a})(H) = 1$, i.e. $\mu^\omega(a + H) = 1$. If the **MAP** was aperiodic, the group H should not be a strict subgroup of \mathbb{Z}^d which is in contradiction with $\langle t, \mathbf{w} \rangle = 0$ modulo 2π by setting $\mathbf{w} = e_i$ for $1 \leq i \leq d$.

The statement involving the adaptation property follows exactly the same lines by setting $a = 0$ and $\theta = 0$. \square

Corollary 2.3. *A **MAP** is adapted if and only if, for any $t \in \mathbf{W}_d^*$ there exists a closed \mathcal{F}_t -invariant subspace E containing the constants such that one is not an eigenvalue of the operator \mathcal{F}_t acting on E . It is aperiodic if and only if, for all $t \in \mathbf{W}_d^*$ there exists a closed \mathcal{F}_t -invariant subspace E containing the constants such that the operator \mathcal{F}_t , acting on E , has no eigenvalue of modulus one.*

Proof. Let $t \in \mathbf{W}_d^*$ and suppose there exists $f \in E$, where E is any closed \mathcal{F}_t -invariant subspace containing the constants, such that

$$\mathcal{F}_t f(x) = \int_{\mathbb{X}} \mathring{P}(x, dy) \widehat{\mu^{x,y}}(t) f(y) = e^{i\theta} f(x), \quad \text{for some } \theta \in \mathbb{R}.$$

By Jensen inequality and the fact that $|\widehat{\mu^{x,y}}(t)| \leq 1$, it follows that $|f| \leq \mathring{P}|f|$. Thus, the function $\|f\|_\infty - |f|$ is superharmonic and hence constant m -a.e. since \mathring{P} is supposed m -recurrent (see [29, Proposition 3.13, p. 44]). As a consequence,

$$1 = \int_{\mathbb{X}} |\widehat{\mu^{x,y}}(t)| \mathring{P}(x, dy), \quad m - a.e.,$$

and $\widehat{\mu^{x,y}}(t)$ is of modulus one, \mathbf{P}^m -a.e.. By Proposition 2.2, the **MAP** can not be aperiodic. Conversely, if the **MAP** is periodic, the same proposition implies that the constants are eigenfunctions for the eigenvalue $e^{i\langle t, a \rangle}$.

The proof of the statement involving the adaptation property follows exactly the same lines. \square

Remark 2.1. Since the probability measures μ^ω are supposed to be supported by \mathbb{Z}^d , \mathbf{P}^m -a.e., it follows that the operator valued map $t \rightarrow \mathcal{F}_t$ is 2π -periodic along the directions given by the vectors of the standard basis of \mathbb{R}^d . In fact, the aperiodicity means that it can not be periodic with shorter periods.

Definition 2.4. A MAP is said to satisfy condition (S) if for any $t \in \mathbf{W}_d^*$ there exists a closed \mathcal{F}_t -invariant subspace E , containing the constants, for which the spectral values of modulus 1 of \mathcal{F}_t , operating on E , consists of eigenvalues.

Remark 2.2. If the Markov operator \mathring{P} is quasi-compact then the Fourier transform operator is also quasi-compact for every t in a neighborhood of the origin. It is known that the set of spectral values of modulus one consists of isolated eigenvalues. Thus, the condition (S) extends this property to large perturbations. Nonetheless, condition (S) can be satisfied without \mathring{P} being quasi-compact which is the main motivation of this paper.

For any closed subspace E , and any bounded operator Q we denote by $\|Q\|_E$ the subordinated norm restricted to E defined as $\|Q\|_E = \sup_{f \in E: \|f\|=1} \|Qf\|$. Let $t \in \mathbf{W}_d$, we denoted by E_t the intersection of all closed subspaces E such that $\mathcal{F}_t E \subset E$ and $\mathbf{1} \in E$. Obviously, the subspace E_t is itself closed, invariant and contains the constant. Also, we may define the pseudo spectral radius $\tilde{r}(t)$ of \mathcal{F}_t by

$$\tilde{r}(t) = \lim_{n \rightarrow \infty} \|\mathcal{F}_t^n\|_{E_t}^{1/n}.$$

Lemma 2.5. If a MAP is aperiodic and satisfy condition (S), then the pseudo spectral radius $\tilde{r}(t)$ of \mathcal{F}_t is strictly smaller than one for $t \in \mathbf{W}_d^*$.

Proof. As a matter of fact, the pseudo spectral radius can be defined alternatively as follows

$$\tilde{r}(t) = \inf \left\{ \|\mathcal{F}_t^n\|_E^{1/n}, n \geq 1, E \text{ closed subspace satisfying } \mathcal{F}_t E \subset E \text{ and } \mathbf{1} \in E \right\}.$$

Moreover, let $t_0 \in \mathbf{W}_d$ and E be a closed subspace such that $\mathcal{F}_{t_0} E \subset E$ and $\mathbf{1} \in E$. Then, by reverse triangle inequality,

$$\left| \|\mathcal{F}_{t_0}^n\|_E - \|\mathcal{F}_t^n\|_E \right| \leq \|\mathcal{F}_t^n - \mathcal{F}_{t_0}^n\|_E \leq \|\mathcal{F}_t^n - \mathcal{F}_{t_0}^n\|$$

which can be made arbitrarily small since $t \rightarrow \mathcal{F}_t$ is continuous. It follows that \tilde{r} is the point-wise infimum of continuous functions, thus \tilde{r} is upper semi-continuous. Furthermore, the pseudo spectral radius reaches its maximum on compact $K \subset \mathbf{W}_d$. Aperiodicity together with condition (S) imply that $\max_{t \in K} \tilde{r}(t) < 1$ excepted when $0 \in K$. \square

From now on, we shall assume the following.

Assumptions 1. The MAP is adapted, aperiodic and irreducible.

Assumptions 2. The condition (S) is fulfilled.

Finally, we make the following assumption on the system of probability measure μ where $|\cdot|$ stands for the Euclidean norm.

Assumptions 3. Assume that the system of probability measures μ admits a uniform third order moment, that is

$$\left\| \sum_{\mathbf{u} \in \mathbb{Z}^d} |\mathbf{u}|^3 \mu(\mathbf{u}) \right\|_\infty < \infty.$$

Remark 2.3. Recall that a change of section is a bounded function $\sigma : \mathbb{X} \rightarrow \mathbb{Z}^d$, thus Assumption 3 needs not to be stated relatively to any change of section.

For $n \geq 1$ and $\mathbf{u}, \mathbf{v} \in \mathbb{Z}^d$, denote respectively the n -step transition probability and its Fourier transform by

$$P_{0,n}^\omega(\mathbf{u}, \mathbf{v}) = \prod_{k=0}^{n-1} P_{k,k+1}^\omega = \mu^\omega * \dots * \mu^{T^{n-1}\omega}(\mathbf{v} - \mathbf{u}) \quad \text{and} \quad \phi_{0,n}^\omega(t) = \prod_{k=0}^{n-1} \widehat{\mu^{T^k\omega}}(t). \quad (5)$$

Introduce the quantities

$$\Sigma_n = \Sigma_n(\omega) = -\frac{1}{n} \sum_{k=0}^{n-1} \nabla^* \nabla \phi_{0,1}^{T^k\omega}(0) \quad \text{and} \quad \Delta_n = \Delta_n(\omega) = i \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} \nabla \phi_{0,1}^{T^k\omega}(0). \quad (6)$$

Assumptions 4. There exists $\alpha > 0$ such that for all $t \in \mathbb{R}^d$, $\liminf_{n \rightarrow \infty} t^* \Sigma_n t \geq \alpha |t|^2$, \mathbf{P}^m -a.e..

2.3 The type problem : a series criterion

Let $A \subset \mathbb{X}$ and $K \subset \mathbb{Z}^d$ be measurable subsets. We are interested in the mean time spent by the Markov additive process — that is nothing but a Markov chain on $\mathbb{X} \times \mathbb{Z}^d$ — in the product set $A \times K$. In facts, it is well known that this quantity, starting from $(x, \mathbf{u}) \in \mathbb{X} \times \mathbb{Z}^d$, is actually given by the Green operator defined by

$$G\mathbf{1}_{A \times K}(x, \mathbf{u}) = \sum_{n \geq 0} \overline{P}^n \mathbf{1}_{A \times K}(x, \mathbf{u}).$$

In the special case of $A = \mathbb{X}$, using notation of Equation (5), the equation above rewrites

$$G\mathbf{1}_{\mathbb{X} \times K}(x, \mathbf{u}) = \mathbf{1}_{\mathbb{X} \times K}(x, \mathbf{u}) + \sum_{\mathbf{v} \in \mathbb{Z}^d} \sum_{n \geq 1} \int_{\mathbb{X}^{\mathbb{N}}} P_{0,n}(u, v) \mathbf{1}_K(v) d\mathbf{P}^x.$$

Definition 2.6 (Recurrence and Transience). A Markov additive process is said to be recurrent (resp. transient) if, for any bounded change of section σ , $G\mathbf{1}_{\mathbb{X} \times \{0\}}(x, 0) = \infty$, m -a.e. (resp. $G\mathbf{1}_{\mathbb{X} \times \{0\}}(x, 0) < \infty$, m -a.e.).

Remark 2.4. Since the changes of section are supposed bounded, a MAP is simultaneously recurrent or transient for every change of section.

A simple computation gives rise to the identities

$$\begin{aligned} G\mathbf{1}_{\mathbb{X} \times \{0\}}(x, 0) &= \sum_{n \geq 1} \int_{\mathbb{X}^{\mathbb{N}}} \mathbf{P}^x(d\omega) P_{0,n}^\omega(0, 0) = \lim_{r \uparrow 1} \sum_{n \geq 1} r^{n-1} \int_{\mathbb{X}^{\mathbb{N}}} \mathbf{P}^x(d\omega) P_{0,n}^\omega(0, 0) \\ &= \lim_{r \uparrow 1} \int_{\mathbb{W}_d} \operatorname{Re} \sum_{n \geq 1} r^{n-1} \mathcal{F}_t^n \mathbf{1}(x) dt. \end{aligned}$$

Theorem 2.7 (Series criterion). Let $d \geq 2$ and suppose that the assumptions 1, 2, 3 and 4 are satisfied. In addition, let τ be the stopping time defined by $\tau = \inf\{n \geq 1 : \operatorname{rk} \Sigma_n = d\}$ and assume $\mathbf{E}^x(\tau) < \infty$ for m -a.e. $x \in \mathbb{X}$. Then, $G\mathbf{1}_{\mathbb{X} \times \{0\}}(x, 0)$ is finite (resp. infinite) if and only if

$$\sum_{n \geq 1} \frac{1}{n^{d/2}} \int_{\mathbb{X}^{\mathbb{N}}} \mathbf{1}_{n \geq \tau} \exp\left(-\frac{1}{2} \Delta_n^* \Sigma_n^{-1} \Delta_n\right) d\mathbf{P}^x < \infty \quad (\text{resp.} \quad = \infty),$$

and the MAP is transient or recurrent accordingly.

3 Applications : random walks on partially directed lattices

Let $(\epsilon_w)_{w \in \mathbb{Z}}$ be any $\{-1, 1\}$ -valued random or deterministic sequence (*c.f.* [5, 16, 14, 31, 7]). Set $\mathbb{G}^0 = \mathbb{Z}^2$, and define the set of edges \mathbb{G}^1 as follows, for $\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2), \mathbf{v} = (\mathbf{v}_1, \mathbf{v}_2) \in \mathbb{G}^0$,

$$(\mathbf{u}, \mathbf{v}) \in \mathbb{G}^1 \iff (|\mathbf{v}_2 - \mathbf{u}_2| = 1 \text{ and } \mathbf{v}_1 = \mathbf{u}_1) \text{ or } (\mathbf{v}_2 = \mathbf{u}_2 \text{ and } \mathbf{v}_1 = \mathbf{u}_1 + \epsilon_{\mathbf{u}_2}). \quad (7)$$

This defines a directed graph, denoted in the sequel by $\mathbb{G}(\epsilon) = (\mathbb{G}^0, \mathbb{G}^1)$, embedded in \mathbb{Z}^2 . The simple random walk on such a graph, whose Markov operator is denoted by M , is obviously irreducible if and only if the range of the $(\epsilon_w)_{w \in \mathbb{Z}}$ contains -1 and 1 .

It is straightforward that the automorphism group $\text{Aut}(\mathbb{G}(\epsilon), M)$ of M , is a subgroup of $\mathbb{Z} \times \{0\}$. Applying the construction of Section 1.3, it turns out that the simple random walk on $\mathbb{G}(\epsilon)$, starting from any vertex $\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2) \in \mathbb{G}^0$, is equal in distribution to a **MAP**. The underlying dynamics is a random walk on \mathbb{Z} starting from $x = \mathbf{u}_2$ whose Markov operator and system of probability measure are given for $x, y \in \mathbb{Z}$ by

$$\mathring{P}(x, \cdot) = \frac{1}{3}(\delta_{x-1} + \delta_x + \delta_{x+1}) \quad \text{and} \quad \mu^{x,y} = \begin{cases} \delta_{(0,1)} & \text{if } y = x + 1, \\ \delta_{-(0,1)} & \text{if } y = x - 1, \\ \delta_{(1,0)} & \text{if } y = x, \epsilon_x = 1, \\ \delta_{-(1,0)} & \text{if } y = x, \epsilon_x = -1. \end{cases}$$

The initial system of probability measures ν is naturally defined by $\nu = \delta_{\mathbf{u}_1, \mathbf{u}_2}$, $\mathbf{P}^{\mathbf{u}_2}$ -a.s.. Obviously, Assumption 3 is satisfied.

The Markov operator induces an irreducible, aperiodic and null-recurrent Markov chain $(S_n)_{n \geq 0}$ with $S_0 = \mathbf{u}$. Setting $\xi_n = S_n - S_{n-1}$ for $n \geq 1$, a simple computation gives

$$\Sigma_n = \frac{1}{n} \begin{pmatrix} \sum_{k=0}^{n-1} \mathbf{1}_{\xi_k=0} & 0 \\ 0 & \sum_{k=0}^{n-1} \mathbf{1}_{\xi_k \neq 0} \end{pmatrix}, \quad \text{and} \quad \Delta_n = \frac{1}{\sqrt{n}} \begin{pmatrix} \sum_{k=0}^{n-1} \epsilon_{S_k} \mathbf{1}_{\xi_k=0} \\ S_n \end{pmatrix}.$$

Then, it is not difficult to see that Σ_n converges, \mathbf{P}^x -a.s., for all $x \in \mathbb{Z}$, by the Strong Law of Large Number, to $\Sigma = \begin{pmatrix} 1/3 & 0 \\ 0 & 2/3 \end{pmatrix}$ so that Assumption 4 is fulfilled.

It is worth noting that even though the **MAP** is adapted and irreducible, the aperiodicity condition fails (consider the change of section $\sigma : \mathbb{Z} \rightarrow \mathbb{Z}^2$ equal to $(0, 0)$ on even point and to $(1, 0)$ on odd ones and choose the subgroup $H = \mathbb{Z} \times \{0\}$). However, without loss of generality, we can consider the “2-fold” **MAP**, *i.e.* we consider \overline{P}^2 in place of \overline{P} .

Thus, at this level, we may prove the condition (S), *i.e.* Assumption 2, is fulfilled.

Proposition 3.1. *The condition (S) is fulfilled.*

Proof. Let $t \in \mathbf{W}_d$ and compute for $f \in \ell^\infty(\mathbb{Z})$ and $x \in \mathbb{Z}$

$$\begin{aligned} \mathcal{F}_t^2 f(x) &= \left(\frac{2}{9} + \frac{1}{9} e^{2it_1 \epsilon(x)} \right) f(x) + \frac{1}{9} e^{2it_2} f(x+2) + \frac{1}{9} e^{-2it_2} f(x-2) \\ &\quad + \frac{1}{9} e^{it_2} f(x+1) \left(e^{it_1 \epsilon(x)} + e^{it_1 \epsilon(x+1)} \right) \\ &\quad + \frac{1}{9} e^{-it_2} f(x-1) \left(e^{it_1 \epsilon(x)} + e^{it_1 \epsilon(x-1)} \right) \end{aligned}$$

First, suppose that $t_1 \neq 0$, then it follows that

$$\|\mathcal{F}_t^2\| \leq \frac{2}{3} + \sup_{x \in \mathbb{Z}} \left| \frac{2}{9} + \frac{1}{9} e^{2it_1 \epsilon(x)} \right| \leq \frac{2}{3} + \frac{1}{9} \sqrt{5 + 4 \cos(2t_1)} < 1.$$

Secondly, if $t_1 = 0$, then the expression of \mathcal{F}_t^2 simplifies as follows

$$\mathcal{F}_t^2 f(x) = \frac{1}{3}f(x) + \frac{2}{9}e^{it_2}f(x+1) + \frac{1}{9}e^{-it_2}f(x-1) + \frac{1}{9}e^{2it_2}f(x+2) + \frac{1}{9}e^{-2it_2}f(x-2).$$

As a matter of facts, the space of \mathbb{C} -valued constant functions is closed and invariant under \mathcal{F}_t^2 . Moreover,

$$\|\mathcal{F}_t^2\|_E \leq \left(\frac{1}{3} + \frac{4}{9} \cos(t_2) + \frac{2}{9} \cos(2t_2) \right).$$

Summarizing, as soon as $t \in \mathbf{W}_d^*$, the pseudo spectral radius $\tilde{r}(t)$ is strictly smaller than 1 and the condition (S) is satisfied. \square

Remark 3.1. *In the context of periodic environment ϵ the proposition above is straightforward since the Fourier transform operator \mathcal{F}_t^2 , $t \in \mathbf{W}_d$, acts on the finite dimensional space E of periodic functions (the dimension of this space is equal to the period of the sequence ϵ). In fact, in this specific case, using the standard machinery of the literature, the recurrence of the random walk would follow from [18] for instance. Besides, we may obtain a renewal theorem from [1], a large deviation theorem from [28, 26] and sharp estimates of the asymptotics of the Green function (and in turn a determination of the Martin boundary) from [37, 23]. Obviously, these technics can no longer be applied for a generic environment.*

Remark 3.2. *When $t_1 = 0$, the closed subspace of 2-periodic functions contains the constants and is invariant by \mathcal{F}_t^2 . Still, the function $x \rightarrow (-1)^x$ is an eigenfunction associated with the eigenvalue 1. Thus, it is crucial that the condition (S) allows to consider different invariant subspaces for different points $t \in \mathbf{W}_d$.*

Theorem 3.2. *The simple random walk M is recurrent (resp. transient) if and only if for some (any) $x \in \mathbb{Z}$*

$$\sum_{n \geq 1} \frac{1}{n} \int_{\mathbb{Z}^{\mathbb{N}}} \mathbf{1}_{n \geq \tau} \exp \left[-\frac{1}{2} \frac{\left(\sum_{k=0}^{n-1} \epsilon_{S_k} \mathbf{1}_{\xi_k=0} \right)^2}{\sum_{k=0}^{n-1} \mathbf{1}_{\xi_k=0}} \right] \exp \left[-\frac{1}{2} \frac{S_n^2}{\sum_{k=0}^{n-1} \mathbf{1}_{\xi_k \neq 0}} \right] d\mathbf{P}^x \quad (8)$$

is infinite (resp. finite).

Remark 3.3. *Similarly to the classical context of random walks on \mathbb{Z}^2 , the harmonic factor term in the sum reveals the dimension 2 is critical. The exponential factor may improve the convergence rate of the summand in order to make the summation finite. Each of them corresponds to one of the direction given by the axis. More precisely, the first one describes the contribution of the horizontal component, whereas the second one describes that of the vertical one. Note that the Law of Iterated Logarithm implies the second exponential factor is almost surely bounded from below by $(\log n)^{-1}$.*

In order to prove the recurrence of the simple random walk in the periodic case, one may apply the Law of Iterated Logarithm shown in [12] to the self-normalized martingale $\Delta^* \Sigma^{-1} \Delta$. Alternatively, one can remark that the group of automorphisms of the corresponding directed graph is $\mathbb{Z} \times p\mathbb{Z}$, where p is period of ϵ . We obtain, by factorisation, a **MAP** with a finite state set Markov chain so that the usual machinery of Markov additive processes under quasi-compactness condition holds (recurrence criterion, renewal theorem ...).

If the sequence $(\epsilon_y)_{y \in \mathbb{Z}}$ is supposed to be the sign function sgn , we shall derive the following result from a well known proposition proved in [35].

Proposition 3.3. *The following convergence in distribution holds*

$$\frac{1}{n} \sum_{k=0}^{n-1} \operatorname{sgn}(S_k) \mathbf{1}_{\xi_k=0} \xrightarrow{d} \Gamma,$$

where Γ is a symmetric arcsinus random variable on $[-1, 1]$. Therefore, the series (8) of Theorem 3.2 is convergent, \mathbf{P}^x -a.s., and the resulting MAP and the simple random walk on $\mathbb{G}(\epsilon)$ are transient.

4 Proof of the main theorem

This section is devoted to the proof of Theorem 2.7. We shall be interested in the asymptotics as n goes to infinity of

$$\langle \mathbf{P}^x, P_{0,n}(\mathbf{u}, \mathbf{v}) \rangle = \int_{\mathbb{X}^N} \mathbf{P}^x(d\omega) P_{0,n}^\omega(\mathbf{u}, \mathbf{v}).$$

By inverse Fourier transform, it follows, using Fubini's theorem

$$\langle \mathbf{P}^x, P_{0,n}(\mathbf{u}, \mathbf{v}) \rangle = \frac{1}{(2\pi)^d} \int_{\mathbf{W}_d} \mathcal{F}_t^n \mathbf{1}(x) e^{-i\langle t, \mathbf{v} - \mathbf{u} \rangle} dt.$$

Similarly to the context of classical random walks, we can split the integrals into two parts, namely, let $\delta > 0$ and set

$$(2\pi)^d \langle \mathbf{P}^x, P_{0,n}(\mathbf{u}, \mathbf{v}) \rangle = \underbrace{\int_{(-\delta, \delta)^d} \mathcal{F}_t^n \mathbf{1}(x) e^{-i\langle t, \mathbf{v} - \mathbf{u} \rangle} dt}_{I_1(n)} + \underbrace{\int_{\mathbf{W}_d \setminus (-\delta, \delta)^d} \mathcal{F}_t^n \mathbf{1}(x) e^{-i\langle t, \mathbf{v} - \mathbf{u} \rangle} dt}_{I_2(n)}. \quad (9)$$

4.1 Estimates far from the origin

Proposition 4.1. *Under Assumptions 1 and 2, there exist constants $\kappa \in (0, 1)$ and $C > 0$ such that for all $\mathbf{v}, \mathbf{u} \in \mathbb{Z}^d$ and all $\delta > 0$ sufficiently small*

$$\left\| \int_{\mathbf{W}_d \setminus (-\delta, \delta)^d} \mathcal{F}_t^n \mathbf{1}(\cdot) e^{-i\langle t, \mathbf{v} - \mathbf{u} \rangle} dt \right\|_\infty \leq C \kappa^n.$$

Proof. By Lemma 2.5, under Assumptions 1 and 2, setting $K := \mathbf{W}_d^* \setminus (-\delta, \delta)^d$, it follows that $M := \max\{\tilde{r}(t) : t \in K\} < 1$. Set $\kappa := (M + 1)/2$. It is a matter of fact that $\kappa^{-n} \|\mathcal{F}_t^n\|_{E_t}$ vanishes as n goes to infinity so it is for $\kappa^{-n} \|\mathcal{F}_t^n \mathbf{1}\|_\infty$ and the result follows. \square

Remark 4.1. *In the literature, it is usually considered that the whole family $\{\mathcal{F}_t\}_{t \in K}$ shares the same invariant space E . Allowing different spaces for different points $t \in K$ improves the estimate of Proposition 4.1 while the pseudo spectral radius \tilde{r} still satisfies the nice property of upper semi-continuity for continuous perturbations of operators.*

4.2 Estimates in the neighborhood of the origin

At this level, it only remains to estimate the first integral term of Equation 9. Setting $t = \frac{u}{\sqrt{n}}$, it is given by

$$I_1(n) = \frac{1}{n^{d/2}} \int_{(-\delta\sqrt{n}, \delta\sqrt{n})^d} \mathcal{F}_{\frac{u}{\sqrt{n}}}^n \mathbf{1}(x) e^{-i\langle u/\sqrt{n}, \mathbf{v} - \mathbf{u} \rangle} du.$$

The expected estimate shall be obtained by integrating an almost-sure estimate exploiting the following expression

$$I_1(n) = \underbrace{\int_{\mathbb{X}^N} n^{-d/2} \int_{|u| \leq \delta\sqrt{n}} \phi_{0,n}^\omega(u/\sqrt{n}) e^{-i\langle u/\sqrt{n}, \mathbf{v}-\mathbf{u} \rangle} du \mathbf{P}^x(d\omega)}_{I_1^\omega(n)}. \quad (10)$$

Proposition 4.2. *Under Assumption 3, there exist a deterministic $\delta > 0$ such that for $|t| \leq \delta$, the quantity $\log \phi_{0,1}^\omega(t)$ is well defined. In addition, the following approximation formula holds \mathbf{P}^m -a.e.*

$$\log \phi_{0,1}(t) = \nabla \phi_{0,1}(0)t + \frac{1}{2} t^* \nabla^* \nabla \phi_{0,1}(0)t + R(t), \quad (11)$$

where the remaining term R satisfies for $|t| \leq \delta$ and for any $\epsilon \in (0, 1)$:

$$\|R(t)\|_\infty \leq \delta^{1-\epsilon} K |t|^{2+\epsilon} \text{ with } K \geq 0.$$

Proof. Under Assumption 3, the function $\phi_{0,1}^\omega$ is \mathbf{P}^m -a.e. three times continuously differentiable. Therefore, the following majoration holds \mathbf{P}^m -a.e.

$$|\phi_{0,1}(t) - 1| \leq |t| \|\nabla \phi_{0,1}(0)\|_\infty.$$

Thus, there exists a deterministic $\delta > 0$ such that for all $|t| \leq \delta$ the function $t \rightarrow \log \phi_{0,1}^\omega(t)$ is well defined. In addition, for $|t| \leq \delta$, the Taylor formula yields \mathbf{P}^m -a.e.

$$\log \phi_{0,1}(t) = \nabla \phi_{0,1}(0)t + \frac{1}{2} t^* \nabla^* \nabla \phi_{0,1}(0)t + R(t),$$

where the remaining term R satisfies for $|t| \leq \delta$ and any $\epsilon \in (0, 1)$

$$\|R(t)\|_\infty \leq \delta^{1-\epsilon} K \|t\|^{2+\epsilon}, \quad \text{with } K = \left\| \sum_{\mathbf{u} \in \mathbb{Z}^d} \|\mathbf{u}\|^3 \mu(\mathbf{u}) \right\|_\infty.$$

□

This proposition implies that for $\delta > 0$ sufficiently small and $|u| \leq \delta\sqrt{n}$

$$\phi_{0,n}(t) = \exp \left\{ -i \langle \Delta_n, u \rangle - \frac{1}{2} u^* \Sigma_n u + R_n(u) \right\},$$

with the notation

$$R_n(u) = R_n^\omega(u) = \sum_{k=0}^{n-1} R^{T^k \omega}(u/\sqrt{n}).$$

Proposition 4.3. *Under Assumption 3 the following properties hold*

1. for all $n \geq 1$, the matrix Σ_n is real positive symmetric \mathbf{P}^m -a.e.,
2. the sequence $(\Sigma_n)_{n \geq 0}$ remains bounded in the following sense

$$\sup_{n \geq 0} \|\Sigma_n\|_\infty < \infty,$$

3. \mathbf{P}^m -a.e., the rank $\text{rk } \Sigma_n$ is non decreasing with $n \geq 1$,
4. in addition, under Assumption 1, $\lim_{n \rightarrow \infty} \text{rk } \Sigma_n = d$, \mathbf{P}^m -a.e..

Proof. 1. Under Assumption 3, the quantity $-\nabla^* \nabla \phi_{0,1}^\omega(0)$ is real symmetric. In addition, it is non negative since $t \rightarrow \operatorname{Re} \widehat{\mu}^\omega(t)$ admits a local maximum at $t = 0$.

2. As a matter of facts, the sequence of matrices Σ_n satisfies

$$\operatorname{ess\,sup}_{\omega \in \Omega} \|\Sigma_n\| \leq \operatorname{ess\,sup}_{\omega \in \Omega} \|\nabla^* \nabla \phi_{0,1}^\omega(0)\| < \infty.$$

3. Because $-\nabla^* \nabla \phi_{0,1}^\omega(0)$ is non negative, the kernel of Σ_n is given by

$$\ker \Sigma_n = \bigcap_{k=0}^{n-1} \ker \nabla^* \nabla \phi_{0,1}^{T^k \omega}(0),$$

and the result follows.

4. Since $\operatorname{rk} \Sigma_n$ is a non decreasing discrete bounded sequence \mathbf{P}^m -a.e., it suffices to show that

$$\mathbf{P}^x(\liminf\{\operatorname{rk} \Sigma_n = d\}) = \mathbf{P}^x\left(\bigcup_{n \geq 1} \{\operatorname{rk} \Sigma_n = d\}\right) = 1, \quad \text{for } m\text{-a.e. } x \in \mathbb{X}. \quad (12)$$

Moreover, the operator P is supposed aperiodic and recurrent so that we only need the asymptotic event in (12) holds with positive probability. Thus suppose on the contrary

$$\mathbf{P}^x(N) = 1 \quad \text{with} \quad N = \bigcap_{n \geq 1} \{\operatorname{rk} \Sigma_n \leq d-1\}.$$

Then the subgroup H of \mathbb{Z}^d generated by the supports $\operatorname{supp} \mu^\omega * \dots * \mu^{T^{n-1}\omega}$, $n \geq 1$, $\omega \in N$, is \mathbf{P}^x -a.s. independent of $\omega \in \mathbb{X}^{\mathbb{N}}$ and satisfies $\mu^\omega(H) = 1$, \mathbf{P}^m -a.e.. Assumption 1 yields $H = \mathbb{Z}^d$ which contradicts the non maximality of the asymptotic rank. \square

Recall $\tau = \inf\{n \geq 1 : \operatorname{rk} \Sigma_n = d\}$ and remark it is finite \mathbf{P}^m -a.e. by Proposition 4.3. Rewrite $I_1^\omega(n)$ as the sum of three terms I_{11}, I_{12} and I_{13} defined by

$$I_{11}^\omega(n) = n^{-d/2} \int_{|t| \leq \delta \sqrt{n}} e^{-\frac{1}{2} t^* \Sigma_n t} e^{i\langle t, \Delta_n \rangle - i\langle t, (\mathbf{v} - \mathbf{u}) / \sqrt{n} \rangle} \left(e^{R_n(t/\sqrt{n})} - 1 \right) dt,$$

$$\text{for } n \geq \tau, \quad I_{12}^\omega(n) = n^{-d/2} \int_{\mathbb{R}^d} e^{-\frac{1}{2} t^* \Sigma_n t} e^{i\langle t, \Delta_n \rangle - i\langle t, (\mathbf{v} - \mathbf{u}) / \sqrt{n} \rangle} dt,$$

and,

$$\text{for } n \geq \tau, \quad I_{13}^\omega(n) = -n^{-d/2} \int_{|t| > \delta \sqrt{n}} e^{-\frac{1}{2} t^* \Sigma_n t} e^{i\langle t, \Delta_n \rangle - i\langle t, (\mathbf{v} - \mathbf{u}) / \sqrt{n} \rangle} dt.$$

Proposition 4.4. *Under Assumptions 1 and 3, for all $n \geq \tau$ the following holds \mathbf{P}^m -a.e. for $\mathbf{u}, \mathbf{v} \in \mathbb{Z}^d$:*

1. $I_{12}(n) = n^{-d/2} \det(\Sigma_n)^{-1/2} (2\pi)^{d/2} \exp \left\{ \left(-\frac{1}{2} [\Delta_n - (\mathbf{v} - \mathbf{u}) / \sqrt{n}]^* \Sigma_n^{-1} [\Delta_n - (\mathbf{v} - \mathbf{u}) / \sqrt{n}] \right) \right\}$.
2. $|I_{13}(n)| \leq n^{-d/2} (2\pi)^{d/2} \exp \left(-\frac{1}{2} \delta \sqrt{n} \operatorname{tr}(\Sigma_n) \right)$,
3. if additionally Assumption 4 is fulfilled, then there exists a deterministic $\delta > 0$ such that $|I_{11}(n)| = \mathcal{O} \left(n^{-\frac{d+\epsilon}{2}} \right)$ uniformly in $\omega \in \Omega$.

Proof. Setting $\tilde{\Delta}_n = \Delta_n - (\mathbf{v} - \mathbf{u})/\sqrt{n}$, it is only needed to prove the results for $\mathbf{v} - \mathbf{u} = 0$.

1. Under Assumptions 1 and 3, Proposition 4.3 implies $\tau < \infty$ \mathbf{P}^m -a.e. and for $n \geq \tau$, there exist orthogonal matrices P_n and diagonal matrices D_n such that

$$\Sigma_n = P_n D_n P_n^{-1} \text{ and } D_n = \text{diag}(\alpha_1^2(n), \dots, \alpha_d^2(n))$$

with $\alpha_i^2(n) > 0$ for all $i = 1, \dots, d$ and $n \geq \tau$. Setting $t = P_n u$, we obtain

$$\begin{aligned} n^{d/2} I_{12}(n) &= \int_{\mathbb{R}^d} e^{-\frac{1}{2} u^* D_n u} e^{i \langle u, P_n^* \Delta_n \rangle} du \\ &= \det(\Sigma_n)^{-1/2} (2\pi)^{d/2} e^{-\frac{1}{2} \Delta_n^* \Sigma_n^{-1} \Delta_n}. \end{aligned}$$

2. For the term $I_{13}(n)$ we can proceed analogously, and we get the following upper bound for $n \geq \tau$:

$$\begin{aligned} n^{d/2} |I_{13}(n)| &\leq \int_{|u| > \delta \sqrt{n}} e^{-\frac{1}{2} u^* D_n u} du \\ &\leq (2\pi)^{d/2} \det(\Sigma_n)^{-1} \exp\left\{-\frac{1}{2} \delta \sqrt{n} \text{tr}(\Sigma_n)\right\}. \end{aligned}$$

3. Because of the point (2) of Proposition 4.3, the eigenvalues of Σ_n remain bounded uniformly for $n \geq 0$. Thus, with Assumption 4, we deduce the following bound for $I_{11}(n)$

$$\begin{aligned} n^{d/2} |I_{11}(n)| &\leq \int_{|t| \leq \delta \sqrt{n}} e^{-\frac{1}{2} t^* \Sigma_n t} \left| \exp\{R_n(tn^{-1/2})\} - 1 \right| dt \\ &\leq \int_{|t| \leq \delta \sqrt{n}} e^{-\frac{1}{2} t^* \Sigma_n t} \left| R_n(tn^{-1/2}) \right| \exp\{|R_n(tn^{-1/2})|\} dt \\ &\leq \frac{K \delta^{1-\epsilon}}{n^{\epsilon/2}} \int_{|t| \leq \delta \sqrt{n}} e^{-\frac{1}{2} t^* \Sigma_n t} |t|^{2+\epsilon} e^{|t|^2 \delta K} dt. \end{aligned}$$

The last estimates comes from Proposition 4.2 and holds for any $\epsilon \in (0, 1)$. We conclude by choosing $\delta > 0$ such that $\delta K \leq \alpha/4$ (where α is given by Assumption 4). Consequently the integral is convergent and the whole term goes to zero at rate, up to a constant, $n^{-\epsilon/2}$.

□

Proof of Theorem 2.7. Let $r \in (0, 1)$, and compute with the notations of the previous section

$$\begin{aligned} &\frac{1}{(2\pi)^d} \int_{\mathbf{W}_d} \text{Re} \sum_{n \geq 1} r^{n-1} \mathcal{F}_t^n \mathbf{1}(x) dt - \frac{1}{(2\pi)^d} \sum_{n \geq 1} r^{n-1} \int_{\mathbb{X}^{\mathbb{N}}} \mathbf{1}_{\{n \geq \tau\}} I_{12}(n) d\mathbf{P}^x \\ &= \frac{1}{(2\pi)^d} \sum_{n \geq 1} r^{n-1} \int_{\mathbb{X}^{\mathbb{N}}} \mathbf{1}_{\{n \geq \tau\}} [I_{11}(n) + I_{13}(n)] d\mathbf{P}^x + I_2(n) \\ &+ \frac{1}{(2\pi)^d} \sum_{n \geq 1} r^{n-1} \int_{\mathbb{X}^{\mathbb{N}}} \mathbf{1}_{\{n < \tau\}} I_1(n) d\mathbf{P}^x \end{aligned}$$

Taking absolute values on both side, using Proposition 4.1, Proposition 4.4 with a suitable $\delta > 0$ together with Assumption 2, it follows that, for some $K \geq 0$, $\epsilon > 0$, $C > 0$ and $\kappa \in (0, 1)$,

$$\begin{aligned} & \left| \frac{1}{(2\pi)^d} \int_{\mathbf{W}_d} \operatorname{Re} \sum_{n \geq 1} r^{n-1} \mathcal{F}_t^n \mathbf{1}(x) dt - \frac{1}{(2\pi)^d} \sum_{n \geq 1} r^{n-1} \int_{\mathbb{X}^{\mathbb{N}}} \frac{1}{\det(\Sigma_n) n^{d/2}} \exp\left(-\frac{1}{2} \Delta_n^* \Sigma_n^{-1} \Delta_n\right) d\mathbf{P}^x \right| \\ & \leq \frac{1}{(2\pi)^d} \sum_{n \geq 1} r^{n-1} \frac{K}{n^{(d+\epsilon)/2}} \\ & \quad + \frac{1}{(2\pi)^d} \left| \sum_{n \geq 1} r^{n-1} \int_{\mathbb{X}^{\mathbb{N}}} \frac{\exp(-\frac{1}{2} \delta \sqrt{n} \operatorname{tr} \Sigma_n)}{n^{d/2}} d\mathbf{P}^x \right| \\ & \quad + \frac{C}{(2\pi)^d} \sum_{n \geq 1} \kappa^{n-1} \\ & \quad + \frac{1}{(2\pi)^d} \sum_{n \geq 1} r^{n-1} \mathbf{P}^x(\tau > n). \end{aligned}$$

The latter summation follows from the fact $|I_1(n)| \leq 1$, \mathbf{P}^m -a.e.. Then, letting $r \uparrow 1$, the right handside remains bounded in $\mathbf{L}_m^\infty(\mathbb{X})$ since

- $d \geq 2$ for the first term,
- under Assumption 4, the trace is bounded from below by a strictly positive constant,
- $\kappa \in (0, 1)$,
- τ is integrable with respect to \mathbf{P}^x , m -a.e. $x \in \mathbb{X}$.

The result follows by letting $r \uparrow 1$ on the left hand-side and remarking that

$$\begin{aligned} & \lim_{r \uparrow 1} \sum_{n \geq 1} r^{n-1} \int_{\mathbb{X}^{\mathbb{N}}} \mathbf{1}_{n \geq \tau} \frac{1}{\det(\Sigma_n) n^{d/2}} \exp\left(-\frac{1}{2} \Delta_n^* \Sigma_n^{-1} \Delta_n\right) d\mathbf{P}^x \\ & = \sum_{n \geq 1} \int_{\mathbb{X}^{\mathbb{N}}} \frac{\mathbf{1}_{n \geq \tau}}{\det(\Sigma_n) n^{d/2}} \exp\left(-\frac{1}{2} \Delta_n^* \Sigma_n^{-1} \Delta_n\right) d\mathbf{P}^x \in [0, \infty] \end{aligned}$$

m -a.e. by monotone convergence. □

References

- [1] M. Babillot. Théorie du renouvellement pour des chaînes semi-markoviennes transientes. *Ann. Inst. H. Poincaré Probab. Statist.*, 24(4):507–569, 1988. URL: http://www.numdam.org/item?id=AIHPB_1988__24_4_507_0. 2, 14
- [2] M. Babillot. An introduction to Poisson boundaries of Lie groups. In *Probability measures on groups: recent directions and trends*, pages 1–90. Tata Inst. Fund. Res., Mumbai, 2006. 2
- [3] Itai Benjamini, Gady Kozma, and Bruno Schapira. A balanced excited random walk. *C. R. Math. Acad. Sci. Paris*, 349(7-8):459–462, 2011. URL: <http://dx.doi.org/10.1016/j.crma.2011.02.018>, doi:10.1016/j.crma.2011.02.018. 2
- [4] V. I. Bogachev. *Measure theory. Vol. I, II*. Springer-Verlag, Berlin, 2007. 5

- [5] M. Campanino and D. Petritis. Random walks on randomly oriented lattices. *Markov Process. Related Fields*, 9(3):391–412, 2003. [2](#), [3](#), [4](#), [13](#)
- [6] Massimo Campanino and Dimitri Petritis. On the physical relevance of random walks: an example of random walks on a randomly oriented lattice. In *Random walks and geometry*, pages 393–411. Walter de Gruyter GmbH & Co. KG, Berlin, 2004. [4](#)
- [7] Massimo Campanino and Dimitri Petritis. Type transition of simple random walks on randomly directed regular lattices. *J. Appl. Probab.*, 51(4):1065–1080, 2014. URL: <http://dx.doi.org/10.1239/jap/1421763328>, doi:10.1239/jap/1421763328. [13](#)
- [8] P. Cénac, B. Chauvin, S. Herrmann, and P. Vallois. Persistent random walks, variable length Markov chains and piecewise deterministic Markov processes. *Markov Process. Related Fields*, 19(1):1–50, 2013. [2](#)
- [9] Peggy Cénac, Brigitte Chauvin, Frédéric Paccaut, and Nicolas Pouyanne. Context trees, variable length Markov chains and dynamical sources. In *Séminaire de Probabilités XLIV*, volume 2046 of *Lecture Notes in Math.*, pages 1–39. Springer, Heidelberg, 2012. URL: http://dx.doi.org/10.1007/978-3-642-27461-9_1, doi:10.1007/978-3-642-27461-9_1. [2](#)
- [10] Peggy Cénac, Basile De Loynes, Arnaud Le Ny, and Yoann Offret. Persistent random walks I: recurrence *versus* transience. March 2015. URL: <https://hal.archives-ouvertes.fr/hal-01135794>. [2](#)
- [11] Francis Comets, Mikhail V. Menshikov, Stanislav Volkov, and Andrew R. Wade. Random walk with barycentric self-interaction. *J. Stat. Phys.*, 143(5):855–888, 2011. URL: <http://dx.doi.org/10.1007/s10955-011-0218-7>, doi:10.1007/s10955-011-0218-7. [2](#)
- [12] Victor H. de la Peña, Michael J. Klass, and Tze Leung Lai. Theory and applications of multivariate self-normalized processes. *Stochastic Process. Appl.*, 119(12):4210–4227, 2009. URL: <http://dx.doi.org/10.1016/j.spa.2009.10.003>, doi:10.1016/j.spa.2009.10.003. [3](#), [14](#)
- [13] Basile de Loynes. Marche aléatoire sur un di-graphe et frontière de Martin. *C. R. Math. Acad. Sci. Paris*, 350(1-2):87–90, 2012. URL: <http://dx.doi.org/10.1016/j.crma.2011.12.005>, doi:10.1016/j.crma.2011.12.005. [2](#)
- [14] Alexis Devulder and Françoise Pène. Random walk in random environment in a two-dimensional stratified medium with orientations. *Electron. J. Probab.*, 18:no. 18, 23, 2013. URL: <http://dx.doi.org/10.1214/EJP.v18-2459>, doi:10.1214/EJP.v18-2459. [13](#)
- [15] N. Guillotin-Plantard and A. Le Ny. Transient random walks on 2D-oriented lattices. *Teor. Veroyatn. Primen.*, 52(4):815–826, 2007. URL: <http://dx.doi.org/10.1137/S0040585X97983353>, doi:10.1137/S0040585X97983353. [2](#)
- [16] Nadine Guillotin-Plantard and Arnaud Le Ny. A functional limit theorem for a 2D-random walk with dependent marginals. *Electron. Commun. Probab.*, 13:337–351, 2008. [2](#), [13](#)
- [17] Yves Guivarc’h. Groupes de Lie à croissance polynomiale. *C. R. Acad. Sci. Paris Sér. A-B*, 271:A237–A239, 1970. [2](#)
- [18] Yves Guivarc’h. Application d’un théorème limite local à la transience et à la récurrence de marches de Markov. In *Théorie du potentiel (Orsay, 1983)*, volume 1096 of *Lecture Notes in Math.*, pages 301–332. Springer, Berlin, 1984. URL: <http://dx.doi.org/10.1007/BFb0100117>, doi:10.1007/BFb0100117. [2](#), [14](#)

- [19] V. A. Kaimanovich, Y. Kifer, and B.-Z. Rubshtein. Boundaries and harmonic functions for random walks with random transition probabilities. *J. Theoret. Probab.*, 17(3):605–646, 2004. 1, 2
- [20] V. A. Kaimanovich and A. M. Vershik. Random walks on discrete groups: boundary and entropy. *Ann. Probab.*, 11(3):457–490, 1983. 2
- [21] Vadim A. Kaimanovich. Amenability and the Liouville property. *Israel J. Math.*, 149:45–85, 2005. Probability in mathematics. 2
- [22] Vadim A. Kaimanovich and Wolfgang Woess. Boundary and entropy of space homogeneous Markov chains. *Ann. Probab.*, 30(1):323–363, 2002. 2, 7
- [23] Takahiro Kazami and Kohei Uchiyama. Random walks on periodic graphs. *Trans. Amer. Math. Soc.*, 360(11):6065–6087, 2008. URL: <http://dx.doi.org/10.1090/S0002-9947-08-04451-6>, doi:10.1090/S0002-9947-08-04451-6. 2, 14
- [24] András Krámli and Domokos Szász. Random walks with internal degrees of freedom. I. Local limit theorems. *Z. Wahrsch. Verw. Gebiete*, 63(1):85–95, 1983. URL: <http://dx.doi.org/10.1007/BF00534179>, doi:10.1007/BF00534179. 2
- [25] G. A. Margulis. Positive harmonic functions on nilpotent groups. *Soviet Math. Dokl.*, 7:241–244, 1966. 2
- [26] P. Ney and E. Nummelin. Markov additive processes II. Large deviations. *Ann. Probab.*, 15(2):593–609, 1987. URL: [http://links.jstor.org/sici?sici=0091-1798\(198704\)15:2<593:MAPILD>2.0.CO;2-0&origin=MSN](http://links.jstor.org/sici?sici=0091-1798(198704)15:2<593:MAPILD>2.0.CO;2-0&origin=MSN). 2, 14
- [27] P. Ney and F. Spitzer. The Martin boundary for random walk. *Trans. Amer. Math. Soc.*, 121:116–132, 1966. 2
- [28] Peter Ney and Esa Nummelin. Markov additive processes: large deviations for the continuous time case. In *Probability theory and mathematical statistics, Vol. II (Vilnius, 1985)*, pages 377–389. VNU Sci. Press, Utrecht, 1987. 2, 14
- [29] Esa Nummelin. *General irreducible Markov chains and nonnegative operators*, volume 83 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1984. URL: <http://dx.doi.org/10.1017/CB09780511526237>, doi:10.1017/CB09780511526237. 8, 10
- [30] Robin Pemantle. Vertex-reinforced random walk. *Probab. Theory Related Fields*, 92(1):117–136, 1992. URL: <http://dx.doi.org/10.1007/BF01205239>, doi:10.1007/BF01205239. 2
- [31] F. Pène. Transient random walk in with stationary orientations. *ESAIM: Probability and Statistics*, 13:417–436, 2009. 2, 13
- [32] Yuval Peres, Serguei Popov, and Perla Sousi. On recurrence and transience of self-interacting random walks. *Bull. Braz. Math. Soc. (N.S.)*, 44(4):841–867, 2013. URL: <http://dx.doi.org/10.1007/s00574-013-0036-4>, doi:10.1007/s00574-013-0036-4. 2
- [33] Georg Pólya. Über eine Aufgabe der Wahrscheinlichkeitsrechnung betreffend die Irrfahrt im Straßennetz. *Math. Ann.*, 84(1-2):149–160, 1921. 2

- [34] Olivier Raimond and Bruno Schapira. Random walks with occasionally modified transition probabilities. *Illinois J. Math.*, 54(4):1213–1238 (2012), 2010. [2](#)
- [35] Frank Spitzer. *Principles of random walks*. Springer-Verlag, New York, second edition, 1976. Graduate Texts in Mathematics, Vol. 34. [14](#)
- [36] Kôhei Uchiyama. Green’s functions for random walks on \mathbf{Z}^N . *Proc. London Math. Soc.* (3), 77(1):215–240, 1998. [2](#)
- [37] Kohei Uchiyama. Asymptotic estimates of the Green functions and transition probabilities for Markov additive processes. *Electron. J. Probab.*, 12:no. 6, 138–180, 2007. URL: <http://dx.doi.org/10.1214/EJP.v12-396>, [doi:10.1214/EJP.v12-396](https://doi.org/10.1214/EJP.v12-396). [2](#), [14](#)
- [38] Stanislav Volkov. Vertex-reinforced random walk on arbitrary graphs. *Ann. Probab.*, 29(1):66–91, 2001. URL: <http://dx.doi.org/10.1214/aop/1008956322>, [doi:10.1214/aop/1008956322](https://doi.org/10.1214/aop/1008956322). [2](#)
- [39] Wolfgang Woess. The Martin boundary for harmonic functions on groups of automorphisms of a homogeneous tree. *Monatsh. Math.*, 120(1):55–72, 1995. [2](#)