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The type problem for a class of inhomogeneous random walks : a series criterion by a probabilistic argument

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Abstract In the classical framework, a random walk on a group is a Markov chain with independent and identically distributed increments. In some sense, random walks are time and space homogeneous. In this paper, we are interested in a class of weakly inhomogeneous random walks termed Random Walk with Random Transition Probabilities — we refer to [18] for the terminology. As an application, we give a criterion for the recurrence or transience of these processes in the discrete Abelian case. This criterion is deduced using Fourier analysis of Markov additive processes and a perturbation argument of a Markov operator. The latter extends the results of the literature since it does not involve a quasi-compacity condition on the operator. Finally, we apply this criterion on some well known examples of random walks on directed graphs embedded in $\mathbb{Z}^2$. Despite the type problem has been already solved for these examples, our analysis brought a new insight to this problematic.

Key words Weakly inhomogeneous random walk . Recurrence . Transience . Random Walk with Random Transition Probabilities . Markov additive processes . Fourier analysis . Local Limit Theorem . Operator perturbations

Mathematics Subject Classification (2000) 60F15 . 60J05 . 60J10 . 60J15 . 60J45 . 60J55 . 60K15

Contents

1 The model of Random Walk Random Transition Probabilities **4**
1.1 Random Walk Random Transition Probabilities : definition 4
1.2 Inhomogeneous Markov chains as RWRTPs and factor RWRTPs 4
1.3 Random walk on constrained directed graphs 6

2 The type problem **8**
2.1 Markov additive processes 8
2.2 Basic assumptions and main estimates 9
2.3 The type problem : a series criterion 16

3 Examples **17**

In the classical framework, a random walk on a group $\mathbf{G}$ is a discrete time stochastic process $(Z_n)_{n\geq 0}$ defined as the product of independent and identically distributed (*i.i.d.*) random variables $(\xi_n)_{n\geq 1}$. More precisely, for any $g \in \mathbf{G}$, we set $Z_0 = g$ and

$$Z_n = g\xi_1 \cdots \xi_n.$$

Random walks on groups are obviously Markov chains that are adapted to the group structure in the sense that the underlying Markov operator is invariant under the group action of $\mathbf{G}$. Thus, this homogeneity naturally gives rise to deep connections between stochastic properties of the random walk and algebraic properties of the group. Starting with the seminal paper of Pòlya, [32], a large part of the literature is devoted to the study of these connections in this homogeneous case (for instance [24, 26, 16, 19, 39, 36, 21, 20, 2] and references therein).

In this paper, we aim at investigating weakly inhomogeneous random walks. It turns out that there are at least two ways to introduce inhomogeneity. First, we can consider "spatial" inhomogeneity by weakening the group structure, replacing it, for instance, by a directed graph structure (see [5, 14, 15, 30, 12]). Secondly, we can study "temporally" inhomogeneous random walks by introducing a notion of memory as in the model of reinforced [29, 38], excited [33, 3], self-interacting [10, 31] or also persistent random walks [8, 7, 9]. All these models belong to the larger class of stochastic processes with long range dependency.

The strategy in the sequel consists of making use of the connection between such stochastic processes and Random Walk With Random Transition Probabilities — or for short, **RWRTP** — the terminology comes from [18]. Roughly speaking, a **RWRTP** consists of a dynamical system (Ω, T), endowed with a quasi-invariant (preimages by T of null measure sets have null measure) probability measure λ, together with a family $\{\mu^\omega\}_{\omega\in\Omega}$ on a group $\mathbf{G}$. The dynamics dictate the way the measures of $\{\mu^\omega\}_{\omega\in\Omega}$ can be convoluted[1]. More precisely, choosing at random (with respect to the probability measure λ) a trajectory $\omega \in \Omega$, we are interesting in the asymptotic properties of the sequence of convoluted measures $\mu^\omega * \mu^{T\omega} * \mu^{T^2\omega} * \cdots * \mu^{T^n\omega}$. As a matter of facts, along a trajectory of the dynamical, the resulting stochastic process have independent but, in general, not stationary increments. If in addition, the dynamical system admits a stationary probability measure, in mean, the increments are no longer independent, however, they are in some sense stationary. We refer to Section 1 for the rigorous definition of **RWRTP**. In Proposition 1.2 of Section 1, it is shown that any inhomogeneous Markov chain taking valued in a group $\mathbf{G}$ is in particular a **RWRTP** while Proposition 1.3 is a nice tool to simplify, in some case, the internal dynamics. At the end of this section, an application of this two propositions for directed graphs embedded, in some sense, in a finitely generated group, is given.

As it is noticed in [18], this model is actually a generalization of other well known models such as Random Walk With Internal Degree of Freedom — see [23] — or also in the modern formulation Markov additive processes, for short **MAP** — see [17, 27, 25, 1, 37, 22]. In this context, many results have been proved, however, the basic assumptions are generally too much restrictive to encompass the class of very inhomogeneous random walks (in particular, the model of random walk on directed graphs of Figure 1.

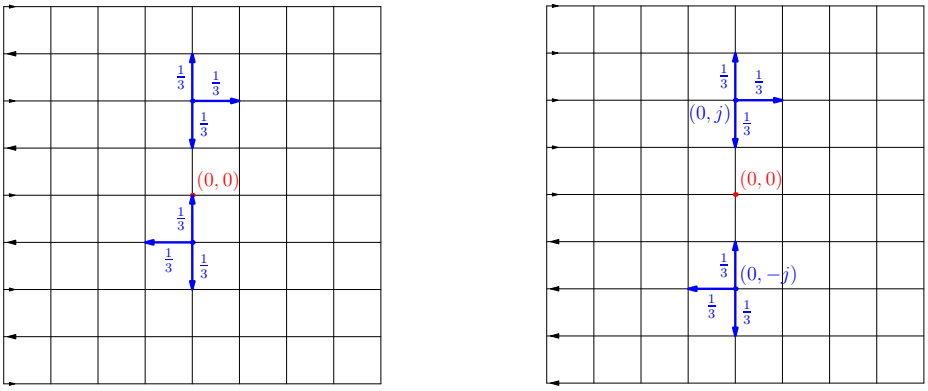
Generally speaking, in the context of Markov additive processes, we may introduce the Fourier Transform operator, denoted by $\mathcal{F}_t$, $t \in \mathbb{R}^d$, which is a continuous perturbation of the Markov operator $\mathring{P}$ defining the underlying dynamics. The rate of the return probability is then estimated, under mild conditions, by inverse Fourier Transform. In the literature, the Markov operator $\mathring{P}$ is usually supposed to be recurrent and quasi-compact. Therefore, $\mathring{P}$ admits a unique invariant probability measure and acts on the more tractable $\mathbf{L}^1$-space. In addition,

[1] In this sense, it is comparable to Random Dynamical Systems: instead of randomly composing transformations, we randomly compose random (continuous) group homomorphisms.

by a perturbation argument the Fourier transform operator remains quasi-compact for all t in a neighborhood of the origin. It allows, under suitable moment conditions on the system of probability measures, to derive a Taylor expansion at the second order of the perturbated dominating eigenvalue $\gamma(t)$ (whose the coefficients are given essentially by the mean and the variance operators). Finally, under an assumption on the spectrum of the Fourier transform operator outside a neighborhood of the origin, it can be concluded that all the needed stochastic information is actually contained in the nature of the singularity at zero of $(1-\gamma(t))^{-1}$.

In this paper, the quasi-compacity condition of $\mathring{P}$ is dropped. It is only assumed that $\mathring{P}$ is irreducible, recurrent and aperiodic. The condition on the spectrum of $\mathcal{F}_t$ for large perturbations remains similar but the nature of the singularity at the origin is analyzed *via* probabilistic methods. These estimates give rise to a criterion for the type problem in terms of the summability of a series (see Theorem 2.11 of section 2 for the rigorous statement). The main term of the series involves the expectation of a functional of two random variables, one can interpreted as the local mean variance of the process and the other as the local drift deviation. The recurrence or transience behavior is interpreted as the result of the competition of this two random quantities. More precisely, the process is recurrent if locally it is sufficiently diffusive in order to balance the local drift deviation.

In Section 3, as a proof of concept, we apply this criterion to the model of random walks on directed graphs introduced in [5] and represented on the figure 1 below.



(a) Periodically directed graph.

(b) Graph with two directed half-plane.

Figure 1: Directed graphs $\mathbb{L}$ et $\mathbb{H}$ de [5].

For the graph (a) of Figure 1, denoted by $\mathbb{L}$, a random walker choose at random one of the nearest neighbour among North, South or West if the ordinate of its current position is odd, and the nearest neighbour among North, South and East if it is even. It is shown in [5] that this random walk is recurrent. For the graph (b) of Figure 1, denoted by $\mathbb{H}$, the possible movements for a random walker are toward the North, South and West on the upper half-plane and North, South and East otherwise and the resulting random walk is shown to be transient.

Using the theorem of factorization of Section 1, we shall show that these random walks are Markov additive processes. The internal Markov chains $\mathring{P}$ are random walks on the group $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}$ respectively. In the periodic case, the internal Markov chain admits a unique stationary probability measure, whereas for the graph with two directed half-plane it is null recurrent. The fundamental difference between the two random walks justifies the necessity to drop the strong quasi-compacity condition. Besides, as a refinement, we prove in the periodic case that the Heat kernel admits Gaussian bounds.

1 The model of Random Walk Random Transition Probabilities

1.1 Random Walk Random Transition Probabilities : definition

Let $(\mathbb{X}, \mathcal{X}, m)$ be a measured space. We denote by $\mathbf{L}^\infty(m)$, endowed with the sup norm $\|\cdot\|_\infty$, the space of real essentially bounded function on $\mathbb{X}$. A linear operator $P : \mathbf{L}^\infty(m) \to \mathbf{L}^\infty(m)$ is called Markov if $P\mathbf{1} = \mathbf{1}$ and $Pf \geq 0$ whenever $f \geq 0$. Let us denote by $\mathbf{IMC}(\mathbb{X}, (P_n)_{n\geq 0}, \theta)$ the Inhomonogeneous Markov Chain on $\mathbb{X}$ where $(P_n)_{n\geq 0}$ is a sequence of Markov operator on $\mathbf{L}^\infty(m)$, θ is absolutely continuous with respect to m — in symbols $\theta \prec m$ — and stands for the initial probability measure. If the sequence of Markov operators $(P_n)_{n\geq 0}$ is constant and equal to P, we denote by $\mathbf{MC}(\mathbb{X}, P, \theta)$ the resulting time homogeneous Markov chain.

For a **IMC**, it is well-known that the paths space $\mathbb{X}^\mathbb{N}$ can be endowed with a unique probability measure, defined from the sequence of Markov operator $(P_n)_{n\geq 0}$ and the probability measure θ, called the canonical probability measure on the paths space and denoted by $\mathbf{P}^\theta$.

Let (Ω, T, λ) be a dynamical system where T is a measurable map from Ω into itself and λ is a probability measure which, in general, is not assumed to be preserved by the dynamics. Nonetheless, we will suppose that there exists a σ-finite invariant measure ρ dominating λ.

Let $\mathbf{G}$ be a locally compact group and denote by Haar the unique, up to a multiplicative constant, Haar measure on $\mathbf{G}$. Denote by $\mathcal{M}^1(\mathbf{G})$ the space of probability measures on $\mathbf{G}$ endowed with the Borel σ-algebra relatively to the weak topology. A system of probability measures over Ω is a measurable map $\mu : \omega \in \Omega \longrightarrow \mu^\omega \in \mathcal{M}^1(\mathbf{G})$.

If μ is a system of probability measures, we define the Markov operator $\overline{P}$ for all $f \in \mathbf{L}^\infty_{\rho\otimes\mathsf{Haar}}(\Omega \times \mathbf{G})$ as follows

$$\overline{P}f(\omega, g) = \int_{\mathbf{G}} f(T\omega, gh)d\mu^\omega(h).$$

The system of probability measures μ also defines a sequence of Markov operators P_n^ω for all $f \in \mathbf{L}^\infty_{\mathsf{Haar}}(\mathbf{G})$ by

$$P_n^\omega f(x) = \int_{\mathbf{G}} f(xy)\mu^{T^n\omega}(dy),\ x \in \mathbf{G},\ \omega \in \Omega$$

that is P_n^ω is a right convolution operator[2]. If ν is another system of probability measures, we denote, for all ω, the canonical probability measure on the paths space of the inhomogeneous Markov chain $\mathbf{IMC}(\mathbf{G}, (P_n^\omega)_{n\geq 0}, \nu^\omega)$ by $\mathbf{P}^{\nu,\omega}$ or for short $\mathbf{P}^\omega$ if there is no ambiguity on ν — for instance if $\nu^\omega = \delta_{\mathsf{id}}$, *i.e.* for every $\omega \in \Omega$ the **IMC** start from the identity id of $\mathbf{G}$.

Definition 1.1. *The triplet $((\Omega, T, \lambda), \mu, \nu)$ is called a Random Walk with Random Transition Probabilities — on $\mathbf{G}$ — and will be denoted in sequel by $\mathbf{RWRTP}((\Omega, T, \lambda), \mu, \nu)$ or if no ambiguity $\mathbf{RWRTP}(\lambda, \mu)$.*

1.2 Inhomogeneous Markov chains as RWRTPs and factor RWRTPs

The elementary Proposition 1.2 shows that any discrete time inhomogeneous Markov chain taking values in a group $\mathbf{G}$ is a **RWRTP**. Afterward, factor **RWRTP** are defined. This latter notion involves measurable partitions for which the suitable framework is that of Lebesgue spaces (we refer to [4] for the definition of Lebesgue spaces and measurable equivalence relations). Since the set Ω shall be the infinite countable product set $\mathbf{G}^\mathbb{N}$ of the locally compact group $\mathbf{G}$, we assume that, endowed with a suitable σ-algebra and a probability, it is a Lebesgue space.

[2] We may define the left convolution operator similarly by the formula $Pf(x) = \int_{\mathbf{G}} f(yx)\mu^{T^n\omega}(dy)$.

Proposition 1.2. *Let* $\mathbf{IMC}(\mathbf{G}, (P_n)_{n\geq 0}, \theta)$ *be an inhomogeneous Markov chain. Then there exists a* $\mathbf{RWRTP}((\Omega, T, \lambda), \mu, \nu)$ *such that*

$$\int_\Omega \lambda(d\omega)\mathbf{P}^{\nu,\omega} = \mathbf{P}^\theta.$$

Proof. Let us set $\Omega = \mathbf{G}^{\mathbb{N}}$, $\lambda = \mathbf{P}^\theta$, and T is the time-shift on the paths space of the **IMC**. Finally, we define μ and ν for all $\omega \in \Omega$ as follows

$$\mu^\omega = \delta_{\omega_0^{-1}\omega_1} \text{ and } \nu^\omega = \delta_{\omega_0}.$$

Therefore, it defines a $\mathbf{RWRTP}((\Omega, T, \lambda), \mu, \nu))$ for which the equality of the proposition is trivially satisfied. □

This proposition essentially relies on the theorem of Ionescu Tulcea that defines the Markovian dynamical system $(\Omega, T, \mathbf{P}^\theta)$ together with the fact that the considered stochastic process takes its values in a group $\mathbf{G}$. Obviously, this proposition is purely theoretical and no more information is given by the resulting **RWRTP**. Below, we introduce the notion of factor **RWRTP** that allows to take into account the internal symmetries of the process and thus simplify in general the analysis.

For a $\mathbf{RWRTP}((\Omega, T, \lambda), \mu, \nu)$, consider a measurable equivalence relation denoted by $\sim$ and its projection denoted by pr. We denote by μ_{pr} the conditional system of probability measures over $\Omega/_\sim$ defined as follows

$$\mu_{\mathsf{pr}} = \int_\Omega \mu^\omega d\lambda_{\mathsf{pr}}(\omega),$$

where λ_{pr} is the probability λ conditioned with respect to the map pr.

In the simple case of a discrete backward probability measure $\lambda_{\mathsf{pr}} = \mathsf{pr}_*\lambda$, we get an explicit expression for the conditional system of probability measures for $\mathsf{w} \in \Omega/_\sim$

$$\mu_{\mathsf{pr}}^{\mathsf{w}} = \int_\Omega \mu^\omega d\lambda(\omega|\mathsf{pr} = \mathsf{w}). \tag{1}$$

If in addition, the maps T can be factorized, that is $\mathsf{pr}\ T\omega = \mathsf{pr}\ T\tilde{\omega}$ as soon as $\mathsf{pr}\ \omega = \mathsf{pr}\ \tilde{\omega}$, then we shall define the factor $\mathbf{RWRTP}((\Omega/_\sim, \mathsf{pr} \circ T, \lambda_{\mathsf{pr}}), \mu_{\mathsf{pr}}, \nu_{\mathsf{pr}})$.

Proposition 1.3. *Let* $\mathbf{RWRTP}((\Omega, T, \lambda), \mu, \nu)$ *be any* **RWRTP**, *suppose* T *can be factorized, and denote by* $\mathbf{RWRTP}((\Omega/_\sim, \mathsf{pr} \circ T, \mathsf{pr}_*\lambda, \mu_{\mathsf{pr}}, \nu_{\mathsf{pr}})$ *its factorization under* $\sim$. *Then, the following equality holds*

$$\int_\Omega \lambda(d\omega)\mathbf{P}_{\nu,\omega} = \int_{\Omega/_\sim} \lambda_{\mathsf{pr}}(d\omega)\mathbf{P}_{\nu_{\mathsf{pr}},\mathsf{w}}. \tag{2}$$

Proof. We only need to check the equality of finite dimensional distributions between both probability of (2). Thus, let α and β be any system of probability measure, then the integrated product measures verify for any measurable subset $A \times B \subset \mathbf{G} \times \mathbf{G}$

$$\begin{aligned}
\int_{\Omega/_\sim} \mathsf{pr}_*\lambda(d\mathsf{w})\alpha_{\mathsf{pr}}^{\mathsf{w}} \otimes \beta_{\mathsf{pr}}^{\mathsf{w}}(A \times B) &= \int_{\Omega/_\sim} \mathsf{pr}_*\lambda(d\mathsf{w})\alpha_{\mathsf{pr}}^{\mathsf{w}}(A)\beta_{\mathsf{pr}}^{\mathsf{w}}(B) \\
&= \int_\Omega \lambda(d\omega)\alpha_{\mathsf{pr}}^{\mathsf{pr}\omega}(A)\beta_{\mathsf{pr}}^{\mathsf{pr}\omega}(B) \\
&= \int_\Omega \lambda(d\omega)\alpha^\omega(A)\beta^\omega(B) \\
&= \int_\Omega \lambda(d\omega)\alpha^\omega \otimes \beta^\omega(A \times B).
\end{aligned}$$

Thus the image measures by the maps $\mathbf{G} \times \mathbf{G} \ni (x, y) \to xy \in \mathbf{G}$ coincide and it follows that

$$\int_{\Omega} \lambda(d\omega)\alpha^{\omega} * \beta^{\omega} = \int_{\Omega/\sim} \lambda_{\mathsf{pr}}(d\mathsf{w})\alpha^{\mathsf{w}}_{\mathsf{pr}} * \beta^{\mathsf{w}}_{\mathsf{pr}}.$$

Proceeding by induction, the conclusion of the proposition is proved. □

Remark 1.1. *It is worth noting that the ergodicity of the probability measure λ, that is $\lambda(A) = 0$ or 1 as soon as A is a T-invariant measurable subset of Ω, implies its projection $\mathsf{pr}_*\lambda$ is ergodic.*

Remark 1.2. *We only need here the $\sim$-equivalence classes to be invariant under the map T. However, the conditional probability λ_{pr}, and hence, the conditional system of probability measures μ_{pr} and ν_{pr} are not in general easily characterized. Actually, it heavily depends on the simplicity of the model considered and thus, the strategy consists of determining the finest equivalence relation (in the complete lattice of measurable partitions up to null sets) wheras λ_{pr} remains easily computable. In the next section, such equivalence relations shall be given for random walks on directed graphs.*

1.3 Random walk on constrained directed graphs

In this section, after introducing standard notions related to graphs, we consider random walks on graphs embedded in a group $\mathbf{G}$. As any discrete time stochastic process, it is isomorphic to a **RWRTP**. In the sequel, we give the corresponding **RWRTP** that is minimal in the sense of Remark 1.2.

A directed graph (or di-graph for short) $\mathbb{G} = (\mathbb{G}^0, \mathbb{G}^1, r, s)$ is a quadruple consisting of a denumerable set $\mathbb{G}^0$ of vertices, a denumerable set $\mathbb{G}^1$ of directed edges and a pair of range and source maps, denoted respectively by r and s, *i.e.* mappings $r, s : \mathbb{G}^1 \to \mathbb{G}^0$. In the sequel, we only consider graphs without loops (*i.e.* not containing edges $\alpha \in \mathbb{G}^1$ such that $r(\alpha) = s(\alpha)$) and without multiple edges (*i.e.* if $\alpha, \beta \in \mathbb{G}^1$ satisfy $s(\alpha) = s(\beta)$ and $r(\alpha) = r(\beta)$ then $\alpha = \beta$, or in other words, the compound map $(r, s) : \mathbb{G}^1 \to \mathbb{G}^0 \times \mathbb{G}^0$ is injective). With these restrictions in force, $\mathbb{G}^1$ can be identified with a particular subset of $\mathbb{G}^0 \times \mathbb{G}^0$ and the maps r and s become superfluous: $s((\mathbf{u}, \mathbf{v})) = \mathbf{u}$ and $r((\mathbf{u}, \mathbf{v})) = \mathbf{v}$ for all $(\mathbf{u}, \mathbf{v}) \in \mathbb{G}^1$. The corresponding directed graph is then termed simple.

We also define respectively, for each vertex $\mathbf{v} \in \mathbb{G}^0$, its inwards and outward degree by

$$\mathsf{deg}^+(\mathbf{v}) = \mathsf{card}\ \{\alpha \in \mathbb{G}^1 : r(\alpha) = \mathbf{v}\} \text{ and } \mathsf{deg}^-(\mathbf{v}) = \mathsf{card}\ \{\alpha \in \mathbb{G}^1 : s(\alpha) = \mathbf{v}\}.$$

A simple graph is termed undirected if the set of edges is symmetric in $\mathbb{G}^0 \times \mathbb{G}^0$, that is $(\mathbf{u}, \mathbf{v}) \in \mathbb{G}^1$ if and only $(\mathbf{v}, \mathbf{u}) \in \mathbb{G}^1$. In this case, the inwards and outwards degree coincide, the common value is simply denoted by deg and called the degree.

A graph $\mathbb{G}$ is said to be connected if for any vertices $\mathbf{u}, \mathbf{v} \in \mathbb{G}^0$ there exists a finite sequence $\alpha = (\alpha_1, \ldots, \alpha_k)$ of edges $\alpha_i \in \mathbb{G}^1$, for $i = 1, \ldots, k$, $k \in \mathbb{N}$, with $s(\alpha_1) = \mathbf{u}$ and $r(\alpha_k) = \mathbf{v}$, such that $r(\alpha_i) = s(\alpha_{i+1}) \in \mathbb{G}^0$, for all $i = 1, \ldots, k-1$. The above sequence α is called a path of length $k = |\alpha|$ from $\mathbf{u}$ to $\mathbf{v}$.

A directed graph is said to be locally finite if for all $\mathbf{v} \in \mathbb{G}^0$, the inwards and outwards degree are finite. It is said to have bounded geometry if it is uniformly locally finite, namely

$$\sup_{\mathbf{v} \in \mathbb{G}^0} \mathsf{deg}^+(\mathbf{v}) < \infty \text{ and } \sup_{\mathbf{v} \in \mathbb{G}^0} \mathsf{deg}^-(\mathbf{v}) < \infty.$$

A random walk on a connected locally finite graph $\mathbb{G}$ is a Markov chain whose state space is given by $\mathbb{G}^0$ and the transition probabilities satisfy for each $\mathbf{u}, \mathbf{v} \in \mathbb{G}^0$

$$P(\mathbf{u}, \mathbf{v}) \in (0, 1] \Longleftrightarrow (\mathbf{u}, \mathbf{v}) \in \mathbb{G}^1,$$

with the additional property $\sum_{\mathbf{v}\in\mathbb{G}^0} P(\mathbf{u},\mathbf{v}) = 1$. A random walk on $\mathbb{G}$ is said to be simple if the transition probabilities are given for all $\mathbf{u},\mathbf{v}\in\mathbb{G}^0$ by

$$P(\mathbf{u},\mathbf{v}) = \begin{cases} \frac{1}{\mathsf{deg}^-(\mathbf{u})} & \text{if } (\mathbf{u},\mathbf{v})\in\mathbb{G}^1, \\ 0 & \text{otherwise.} \end{cases}$$

Let us denote by $\mathfrak{S}$ the group of permutations on the set of vertices of $\mathbb{G}$. The set of automorphisms of a graph $\mathbb{G}$ is the subgroup of $\mathfrak{S}$ denoted by $\mathsf{Aut}(\mathbb{G})$ defined as follows

$$\mathsf{Aut}(\mathbb{G}) = \{f\in\mathfrak{S} : (\mathbf{u},\mathbf{v})\in\mathbb{G}^1 \Leftrightarrow (f(\mathbf{u}),f(\mathbf{v}))\in\mathbb{G}^1\}.$$

The group of automorphisms of a random walk on a graph $\mathbb{G}$ is defined as follows

$$\mathsf{Aut}(\mathbb{G},P) = \{f\in\mathfrak{S} : P(\mathbf{u},\mathbf{v}) = P(f(\mathbf{u}),f(\mathbf{v})) \text{ for all } \mathbf{u},\mathbf{v}\in\mathbb{G}^0\}.$$

The latter group is a subgroup of the former and they are isomorphic if the random walk is supposed to be simple. The two groups of automorphisms naturally acts on the graph $\mathbb{G}$, however, in the sequel, we shall not suppose the action of $\mathsf{Aut}(\mathbb{G},P)$ to be transitive. In facts, in the case of a transitive action, the so-called homogeneous case, the random walk on the graph can be transported on a random walk in the classical sense on $\mathsf{Aut}(\mathbb{G},P)$ — see [21]. Note that, as soon as the graph is genuinely directed, the group of automorphisms $\mathsf{Aut}(\mathbb{G})$ cannot act transitively.

In order to see random walks on graphs as **RWRTP**, it is necessary to embed the graph in a group $\mathbf{G}$. A natural family of such graphs are constrained graphs, *i.e.* graphs constructed from the Cayley graph of a finitely generated group by deleting some edges with respect to a predicate. More precisely, let $\mathbf{G}$ be a group generated by a finite set $\mathcal{S}$ supposed symmetric. A predicate is a $\{0,1\}$-valued map $\mathtt{p}$ on $\mathbf{G}\times\mathcal{S}$. Thus, we may construct a constrained graph $\mathbb{G} = (\mathbb{G}^0,\mathbb{G}^1)$ where

$$\mathbb{G}^0\subset\mathbf{G} \quad \text{countable and} \quad \mathbb{G}^1 = \{(\mathbf{u},\mathbf{v}s)\in\mathbb{G}^0\times\mathbb{G}^0, s\in\mathcal{S}, \mathtt{p}(\mathbf{u},s)=1\}.$$

The resulting graph inherits from the simplicity and the bounded geometry property of Cayley graphs of finitely generated groups, though it can be disconnected unless the predicate $\mathtt{p}$ is suitably chosen which we shall assume in the sequel.

Let T be the unilateral time shift on $\Omega = (\mathbb{G}^0)^{\mathbb{N}}$ and for every fixed $\mathbf{u}_0\in\mathbb{G}^0$, the probability measure $\lambda = \mathbf{P}^{\mathbf{u}_0}$. Also, the system of probability measures μ and ν are defined respectively by

$$\mu^\omega = \delta_{\omega_0^{-1}\omega_1} \quad \text{and} \quad \nu^\omega = \delta_{\omega_0}, \quad \omega\in\left(\mathbb{G}^0\right)^N.$$

Then, by Proposition 1.2,

$$\mathbf{P}^{\mathbf{u}_0} = \int_{\mathbb{X}^{\mathbb{N}}} \mathbf{P}_{\nu,\omega}\lambda(d\omega).$$

We shall define an equivalence relation on Ω, denoted in the sequel by $\sim$, by

$$\omega\sim\tilde{\omega}\in(\mathbb{G}^0)^{\mathbb{N}} \Longleftrightarrow \forall n\geq 0 : \omega_n = \tilde{\omega}_n \ \mathsf{mod}\ \mathsf{Aut}(\mathbb{G},P).$$

As a matter of fact, the time shift T commutes with this equivalence relation. Thus, by Proposition 1.3, the **RWRTP** above can be factorized. More precisely, the map $\mathsf{pr}\circ T$ is the time shift on the factor space $\Omega/_\sim = \left(\mathbb{G}^0/_{\mathsf{Aut}(\mathbb{G},P)}\right)^{\mathbb{N}}$. Also, we denote by λ_{pr} the factor probability measure, with respect to $\sim$, of the probability measure $\lambda = \mathbf{P}^{\mathbf{u}_0}$. It turns out that λ_{pr} is nothing but the canonical probability measure of the Markov chain on the state space

$\mathbb{G}^0/_{\mathsf{Aut}(\mathbb{G},P)}$ whose initial distribution is given by $\delta_{\mathsf{pr}(\mathbf{u}_0)}$ and Markov operator, denoted by $\mathring{P}$, satisfies $\mathring{P}(\mathsf{pr}\ \mathbf{u}, \mathsf{pr}\ \mathbf{v}) = P(\mathbf{u}, \mathbf{v})$ for all $\mathbf{u}, \mathbf{v} \in \mathbb{G}^0$. The system of probability measures $\left\{\mu_{\mathsf{pr}}^{\mathsf{w}_0,\mathsf{w}_1} : \mathsf{w}_0, \mathsf{w}_1 \in \mathbb{G}^0/_{\mathsf{Aut}(\mathbb{G}^0,P)}\right\}$ defined in (1) rewrites in our context as

$$\mu_{\mathsf{pr}}^{\mathsf{w}_0,\mathsf{w}_1} = \frac{\sum_{\omega_1\in\mathbb{G}^0} P(\omega_0,\omega_1)\mathbf{1}_{\{(\mathsf{w}_0,\mathsf{w}_1)\}}(\mathsf{pr}\ \omega_0, \mathsf{pr}\ \omega_1)\mu^{\omega}}{\sum_{\omega_1\in\mathbb{G}^0} P(\omega_0,\omega_1)\mathbf{1}_{\{(\mathsf{w}_0,\mathsf{w}_1)\}}(\mathsf{pr}\ \omega_0, \mathsf{pr}\ \omega_1)}, \quad \text{for any} \quad \omega_0 \in \mathbb{G}^0 \text{ s.t. } \mathsf{pr}\ \omega_0 = \mathsf{w}_0.$$

Also, we define $\nu_{\mathsf{pr}}^{\mathsf{w}}$ as the Dirac distribution at the point $\mathsf{w}_0 \in \mathbb{G}^0/_{\mathsf{Aut}(\mathbb{G},P)}$.

Remark 1.3. *Note that any discrete process can be seen as a Markov chain with respect the filtration containing the whole history. This Markov chain with long memory can be modeled by a random walk on a directed graph whose the corresponding factorization reveals the structure of the memory.*

2 The type problem

In the previous section, it is shown that any discrete time processes (taking values in a group $\mathbf{G}$) is in some sense a **RWRTP**. In this section, we shall consider the restricted class of Markov additive processes, for short **MAP**, that are, roughly speaking, **RWRTP** for which the group $\mathbf{G}$ is supposed Abelian and the underlying dynamics is Markovian. We shall be even more restrictive by setting $\mathbf{G} = \mathbb{Z}^d$. For a non discrete Abelian group, say $\mathbf{G} = \mathbb{R}^d$ or $\mathbf{G}$ compact, the Fourier analysis of continuous measure is sensibly different and is quite irrelevant for the scope of this paper.

2.1 Markov additive processes

In the sequel, we refer to [28] for the notion of Markov chain on general spaces. We shall consider a Markov kernel $\mathring{P}$ relatively to a measurable space $(\mathbb{X}, \mathcal{X})$ for which the σ-algebra is separable. The space $(\mathbb{X}, \mathcal{X})$ can be endowed with a σ-finite measure m dominating $m\mathring{P}$. Without loss of generality, we may assume that the Markov chain induced by $\mathring{P}$ is m-irreducible and aperiodic. A stronger assumption is to supposed the Markov chain m-recurrent. In this case, the time shift on $\mathbb{X}^{\mathbb{N}}$ preserves the possibly infinite, but σ-finite, Markov measure $\mathbf{P}^m$. The probability measures μ^{ω}, $\omega \in \mathbb{X}^{\mathbb{N}}$ are supposed to depend only on the two first coordinates[3] of $\omega \in \mathbb{X}^{\mathbb{N}}$ and shall be alternatively denoted by $\mu^{x,y}$, $x, y \in \mathbb{X}$.

A Markov additive processes with internal Markov chain $\mathring{P}$ together with the system of probability measures μ is the Markov chain on the space $\mathbb{X} \times \mathbb{Z}^d$ whose Markov operator is given, for $f \in \mathbf{L}_m^{\infty}(\mathbb{X} \times \mathbb{Z}^d)$, by

$$\overline{P}f(x, \mathbf{u}) = \int_{\mathbb{X}\times\mathbb{Z}^d} \mathring{P}(x, dy)\delta_y \otimes \mu^{x,y}(dzd\mathbf{v})f(z, \mathbf{u} + \mathbf{v}), \quad \text{for} \quad (x, \mathbf{u}) \in \mathbb{X} \times \mathbb{Z}^d.$$

The Markov operator $\overline{P}$ is invariant by translations of the kind $\mathbb{X} \times \mathbb{Z}^d \ni (x, \mathbf{u}) \to (x, \mathbf{u} + \mathbf{v}) \in \mathbb{X} \times \mathbb{Z}^d$.

In this context, we may introduce the so called Fourier transform operator, denoted by $\mathcal{F}_t$, for $t \in \mathbb{R}^d$, acting as a contraction on $\mathbf{L}_m^{\infty}(\mathbb{X})$ and defined for $f \in \mathbf{L}_m^{\infty}(\mathbb{X})$ by

$$\mathcal{F}_t f(x) = \int_{\mathbb{X}} \mathring{P}(x, dy)\widehat{\mu^{x,y}}(t)f(y), \quad \text{for} \quad m - \text{a.e.} \quad x \in \mathbb{X}, \tag{3}$$

[3]Actually, we use here the tradionnal definition, though it is completely equivalent to suppose that the probabilities μ^{ω} depends only on the first coordinate providing we consider the two order Markov chain induced by $\mathring{P}$.

where $\widehat{\mu^{x,y}}$ is the standard Fourier transform of the probability measure $\mu^{x,y}$ defined, for $t \in \mathbb{R}^d$ by

$$\widehat{\mu^{x,y}}(t) = \sum_{\mathbf{v}\in\mathbb{Z}^d} \mu^{x,y}(\mathbf{v}) e^{i\langle t,\mathbf{v}\rangle},$$

and $\langle \cdot, \cdot \rangle$ stands for the standard inner product.

In the sequel, we introduce usual assumptions, as the irreducibility and aperiodicity, adapted to our context. Will follow a series criterion for the recurrence or transience of Markov additive processes.

2.2 Basic assumptions and main estimates

2.2.1 Basic assumptions

Let σ be a bounded map from $\mathbb{X}$ to $\mathbb{Z}^d$ and define

$$\overline{P}_\sigma f(x,\mathbf{u}) = \int_{\mathbb{X}\times\mathbb{Z}^d} \mathring{P}(x,dy)\delta_y \otimes \mu^{x,y}(dzd\mathbf{v}) f(z, \mathbf{u}+\mathbf{v}-\sigma(y)+\sigma(x)), \quad \text{for} \quad (x,\mathbf{u}) \in \mathbb{X}\times\mathbb{Z}^d.$$

Intuitively, it corresponds to a change of the origin of each fiber parametrized by $\mathbb{X}$. Such a function σ is called a change of section. The translated probability shall be denoted $\mu_\sigma^{x,y} = \mu^{x,y} * \delta_{\sigma(y)-\sigma(x)}$, or simply $\mu^{x,y}$ if no ambiguity. Because of the invariance of $\overline{P}$ under translations of $\mathbb{Z}^d$, the changes of section have no fundamental importance in the study of the recurrent or transient behavior of the Markov additive process.

To keep notations light and readable, we shall adopt alternatively the notations related to dynamical system (such as μ^ω, $\mathbf{P}^m$-a.e.,...) or those ones proper to Markov additive processes ($\mu^{x,y}$, m-a.e.,...).

Definition 2.1 (Adaptation, aperiodicity, irreducibility)**.** *A* **MAP** *is respectively adapted, aperiodic and irreducible if for any change of section σ*

1. *there is no proper subgroup $H \subset \mathbb{Z}^d$ such that $\mathbf{P}^m$-a.e. $\mu^\omega(H) = 1$;*
2. *there is no proper subgroup $H \subset \mathbb{Z}^d$ and no $a \in \mathbb{R}^d$ such that $\mathbf{P}^m$-a.e. $\mu^\omega(a+H) = 1$.*
3. *there is no half-space $H \subset \mathbb{R}^d$ such that $\mathbf{P}^m$-a.e. $\mu^\omega(H) = 1$.*

The properties of adaptation and aperiodicity can be read on the spectrum of the Fourier transform $\mathcal{F}_t$ as shown in the corollary below. In the sequel we denote by $\mathbf{W}_d$ the set $[-\pi,\pi)$ and by $\mathbf{W}_d^*$ the set $\mathbf{W}_d \setminus \{0\}$.

Proposition 2.2. *A* **MAP** *is adapted if and only if for any change of section $\sigma : \mathbb{X} \to \mathbb{Z}^d$ and any $t \in \mathbf{W}_d^*$ it holds that $\widehat{\mu^\omega}(t) \neq 1$, $\mathbf{P}^m$-a.e.. A* **MAP** *is aperiodic if and only if for any change of section σ and any $t \in \mathbf{W}_d^*$ it holds, $\mathbf{P}^m$-a.e., $\widehat{\mu^\omega}(t) \neq e^{i\theta}$ for some $\theta \in \mathbb{R}$.*

Proof. If a **MAP** is not aperiodic then there exist a change of section, a proper subgroup H and $a \in \mathbb{R}^d$ such that $\mu^\omega(a+H) = 1$, $\mathbf{P}^m$-a.e.. It is well known that there exist integers $n_1, \ldots, n_r$, $1 \le r \le d$, such that the subgroup H is generated by a set $\{n_1 e_{i_1}, \ldots, n_r e_{i_r}\}$ for some indices $1 \le i_1 \le \ldots \le i_r \le d$ where e_i is the ith vector of the canonical basis of $\mathbb{Z}^d$. Suppose $r < d$, then for any $t \in \mathbf{W}_d^* \cap (\mathsf{span}\ H)^\perp$ it follows that $\widehat{\mu^\omega}(t) = e^{i\langle t,a\rangle}$. If $r = d$, set $t = (\pi/n_1, \ldots, \pi/n_d)$ then again $\widehat{\mu^\omega}(t) = e^{i\langle t,a\rangle}$ and t obviously belongs to $\mathbf{W}_d^*$ unless $n_1 = \cdots = n_d = 1$, *i.e.* $H = \mathbb{Z}^d$.

Conversely, suppose there exists a change of section σ and $t \in \mathbf{W}_d^*$ such that $\widehat{\mu^\omega}(t) = e^{i\theta}$, $\mathbf{P}^m$-a.e., for some $\theta \in \mathbb{R}$. Since μ^ω is supported by a subset of $\mathbb{Z}^d$, one can choose $a \in \mathbb{Z}^d$ such that $\theta = \langle t, a\rangle$. Then, it holds for any $n \geq 1$

$$e^{i\langle t, na\rangle} = \prod_{k=0}^{n-1} \widehat{\mu^{T^k\omega}}(t) = \sum_{\mathbf{w}\in\mathbb{Z}^d} \mu^\omega * \cdots * \mu^{T^{n-1}\omega}(\mathbf{w}) e^{i\langle t, \mathbf{w}\rangle}, \quad \mathbf{P}^m - a.e.,$$

or equivalently,

$$1 = \sum_{\mathbf{w}\in\mathbb{Z}^d} \{\mu^\omega * \delta_{-a}\} * \cdots * \{\mu^{T^{n-1}\omega} * \delta_{-a}\}(\mathbf{w}) e^{i\langle t, \mathbf{w}\rangle}, \quad \mathbf{P}^m - a.e..$$

Thus, the convex combination of points on the unit circle on the right hand side is extremal so that each $\mathbf{w} \in S_n^\omega = \mathsf{supp}\ \{\mu^\omega * \delta_{-a}\} * \cdots * \{\mu^{T^{n-1}\omega} * \delta_{-a}\}$, $n \geq 1$, satisfies $\langle t, \mathbf{w}\rangle = 0$ modulo 2π. Setting $N = \{\omega \in \mathbb{X}^\mathbb{N} : \widehat{\mu^\omega}(t) \neq e^{i\langle t, a\rangle}\}$ and defining H as the smallest subgroups containing the sets S_n^ω, for all $n \geq 1$ and $\omega \in N^\complement$, it follows that, $\mathbf{P}^m$-a.e., $(\mu^\omega * \delta_{-a})(H) = 1$, *i.e.* $\mu^\omega(a + H) = 1$. If the **MAP** was aperiodic, the group H should not be a strict subgroup of $\mathbb{Z}^d$ which is in contradiction with $\langle t, \mathbf{w}\rangle = 0$ modulo 2π by setting $\mathbf{w} = e_i$ for $1 \leq i \leq d$.

The statement involving the adaptation property follows exactly the same lines by setting $a = 0$ and $\theta = 0$. □

Corollary 2.3. *A* **MAP** *is adapted if and only if, for any $t \in \mathbf{W}_d^*$ there exists a closed $\mathcal{F}_t$-invariant subspace E containing the constants such that one is not an eigenvalue of the operator $\mathcal{F}_t$ acting on E . It is aperiodic if and only if, for all $t \in \mathbf{W}_d^*$ there exists a closed $\mathcal{F}_t$-invariant subspace E containing the constants such that the operator $\mathcal{F}_t$, acting on E, has no eigenvalue of modulus one.*

Proof. Let $t \in \mathbf{W}_d^*$ and suppose there exists $f \in E$, where E is any closed $\mathcal{F}_t$-invariant subspace containing the constants, such that

$$\mathcal{F}_t f(x) = \int_\mathbb{X} \mathring{P}(x, dy) \widehat{\mu^{x,y}}(t) f(y) = e^{i\theta} f(x), \quad \text{for some} \quad \theta \in \mathbb{R}.$$

By Jensen inequality and the fact that $|\widehat{\mu^{x,y}}(t)| \leq 1$, it follows that $|f| \leq \mathring{P}|f|$. Thus, the function $\|f\|_\infty - |f|$ is superharmonic and hence constant m-a.e. since $\mathring{P}$ is supposed m-recurrent (see [28, Proposition 3.13, p. 44]). As a consequence,

$$1 = \int_\mathbb{X} |\widehat{\mu^{x,y}}(t)| \mathring{P}(x, dy), \quad m - \text{a.e.},$$

and $\widehat{\mu^{x,y}}(t)$ is of modulus one, $\mathbf{P}^m$-a.e.. By Proposition 2.2, the **MAP** can not be aperiodic. Conversely, if the **MAP** is periodic, the same proposition implies that the constants are eigenfunctions for the eigenvalue $e^{i\langle t, a\rangle}$.

The proof of the statement involving the adaptation property follows exactly the same lines. □

Remark 2.1. *Since the probability measures μ^ω are supposed to be supported by $\mathbb{Z}^d$, $\mathbf{P}^m$-a.e., it follows that the operator valued map $t \longrightarrow \mathcal{F}_t$ is 2π-periodic along the directions given by the vectors of the standard basis of $\mathbb{R}^d$. In fact, the aperiodicity means that it can not be periodic with shorter periods.*

Definition 2.4. *A* **MAP** *is said to satisfy condition (S) if for any $t \in \mathbf{W}_d^*$ there exists a closed $\mathcal{F}_t$-invariant subspace E, containing the constants, for which the spectral values of modulus 1 of $\mathcal{F}_t$, operating on E, consists of eigenvalues.*

Remark 2.2. *If the Markov operator $\mathring{P}$ is quasi-compact then the Fourier transform operator is also quasi-compact for every t in a neighborhood of the origin. It is known that the set of spectral values of modulus one consists of isolated eigenvalues. Thus, the condition (S) extends this property to large perturbations. Nonetheless, condition (S) can be satisfied without $\mathring{P}$ being quasi-compact which is the main motivation of this paper.*

For any closed subspace E, and any bounded operator Q we denote by $\|Q\|_E$ the subordinated norm restricted to E defined as $\|Q\|_E = \sup_{f\in E:\|f\|=1} \|Qf\|$. Let $t \in \mathbf{W}_d$, we denoted by E_t the intersection of all closed subspaces E such that $\mathcal{F}_t E \subset E$ and $\mathbf{1} \in E$. Obviously, the subspace E_t is itself closed, invariant and contains the constant. Also, we may define the pseudo spectral radius $\tilde{r}(t)$ of $\mathcal{F}_t$ by

$$\tilde{r}(t) = \lim_{n\to\infty} \|\mathcal{F}_t^n\|_{E_t}^{1/n}.$$

Lemma 2.5. *If a* **MAP** *is aperiodic and satisfy condition (S), then the pseudo spectral radius $\tilde{r}(t)$ of $\mathcal{F}_t$ is strictly smaller than one for $t \in \mathbf{W}_d^*$.*

Proof. As a matter of fact, the pseudo spectral radius can be defined alternatively as follows

$$\tilde{r}(t) = \inf\left\{\|\mathcal{F}_t^n\|_E^{1/n}, n \geq 1, E \text{ closed subspace satisfying } \mathcal{F}_t E \subset E \text{ and } \mathbf{1} \in E\right\}.$$

Moreover, let $t_0 \in \mathbf{W}_d$ and E be a closed subspace such that $\mathcal{F}_{t_0} E \subset E$ and $\mathbf{1} \in E$. Then, by reverse triangle inequality,

$$\left|\|\mathcal{F}_{t_0}^n\|_E - \|\mathcal{F}_t\|_E\right| \leq \|\mathcal{F}_t^n - \mathcal{F}_{t_0}^n\|_E \leq \|\mathcal{F}_t - \mathcal{F}_{t_0}\|$$

which can be made arbitrarily small since $t \to \mathcal{F}_t$ is continuous. It follows that $\tilde{r}$ is the pointwise infimum of continuous functions, thus $\tilde{r}$ is upper semi-continuous. Furthermore, the pseudo spectral radius reaches its maximum on compact $K \subset \mathbf{W}_d$. Aperiodicity together with condition (S) imply that $\max_{t\in K} \tilde{r}(t) < 1$ excepted when $0 \in K$. □

From now on, we shall assume the following.

Assumptions 1. *The* **MAP** *is adapted, aperiodic and irreducible.*

Assumptions 2. *The condition (S) is fulfilled.*

Finally, we make the following assumption on the system of probability measure μ where $|\cdot|$ stands for the Euclidean norm.

Assumptions 3. *Assume that the system of probability measures μ admits a uniform third order moment, that is*

$$\left\|\sum_{\mathbf{u}\in\mathbb{Z}^d} |\mathbf{u}|^3 \mu(\mathbf{u})\right\|_\infty < \infty.$$

Recall that a change of section is a bounded function $\sigma : \mathbb{X} \longrightarrow \mathbb{Z}^d$, thus Assumption 3 needs not to be stated relatively to any change of section.

2.2.2 Basic estimates

Set for $n \geq 1$ and $\mathbf{u}, \mathbf{v} \in \mathbb{Z}^d$

$$P^{\omega}_{0,n}(\mathbf{u}, \mathbf{v}) = \prod_{k=0}^{n-1} P^{\omega}_{k,k+1} = \mu^{\omega} * \mu^{T\omega} * \cdots * \mu^{T^{n-1}\omega}(\mathbf{v} - \mathbf{u}). \tag{4}$$

In this section, we shall be interested in the asymptotics as n goes to infinity of

$$\langle \mathbf{P}^x, P_{0,n}(\mathbf{u}, \mathbf{v}) \rangle = \int_{\mathbb{X}^N} \mathbf{P}^x(d\omega) P^{\omega}_{0,n}(\mathbf{u}, \mathbf{v}).$$

By inverse Fourier transform, it follows, using Fubini's theorem

$$\langle \mathbf{P}^x, P_{0,n}(\mathbf{u}, \mathbf{v}) \rangle = \frac{1}{(2\pi)^d} \int_{\mathbf{W}_d} \mathcal{F}^n_t \mathbf{1}(x) e^{-i\langle t, \mathbf{v}-\mathbf{u} \rangle} dt.$$

Similarly to the context of classical random walks, we can split the integrals into two parts, namely, let $\delta > 0$ and set

$$(2\pi)^d \langle \mathbf{P}^x, P_{0,n}(\mathbf{u}, \mathbf{v}) \rangle = \underbrace{\int_{(-\delta,\delta)^d} \mathcal{F}^n_t \mathbf{1}(x) e^{-i\langle t, \mathbf{v}-\mathbf{u} \rangle} dt}_{I_1(n)} + \underbrace{\int_{\mathbf{W}_d \setminus (-\delta,\delta)^d} \mathcal{F}^n_t \mathbf{1}(x) e^{-i\langle t, \mathbf{v}-\mathbf{u} \rangle} dt}_{I_2(n)}.$$

Proposition 2.6. *Under Assumptions 1 and 2, there exist constants $\kappa \in (0,1)$ and $C > 0$ such that for all $\mathbf{v}, \mathbf{u} \in \mathbb{Z}^d$*

$$\left\| \int_{\mathbf{W}_d \setminus (-\delta,\delta)^d} \mathcal{F}^n_t \mathbf{1}(\cdot) e^{-i\langle t, \mathbf{v}-\mathbf{u} \rangle} dt \right\|_{\infty} \leq C\kappa^n.$$

Proof. By Lemma 2.5, under Assumptions 1 and 2, setting $K := \mathbf{W}^*_d \setminus (-\delta,\delta)^d$, it follows that $M := \max\{\tilde{r}(t) : t \in K\} < 1$. Set $\kappa := (M+1)/2$. It is a matter of fact that $\kappa^{-n} \|\mathcal{F}^n_t\|_{E_t}$ vanishes as n goes to infinity so it is for $\kappa^{-n} \|\mathcal{F}^n_t \mathbf{1}\|_{\infty}$ and the result follows. □

Remark 2.3. *In the literature, it is usually considered that the whole family $\{\mathcal{F}_t\}_{t \in K}$ shares the same invariant space E. Allowing different spaces for different points $t \in K$ improves the estimate of Proposition 2.6 while the pseudo spectral radius $\tilde{r}$ still statisfies the nice property of upper semi-continuity for continuous perturbations of operators.*

At this level, it only remains to estimate the first integral term. To this end, it suffices to compute an approximation formula, in the neighborhood of zero, of $\langle \mathbf{P}^x, \mathcal{F}^n_t \mathbf{1} \rangle$. Then, setting $t = \frac{u}{\sqrt{n}}$, the first integral term is given by

$$I_1(n) = \frac{1}{n^{d/2}} \int_{(-\delta\sqrt{n}, \delta\sqrt{n})^d} \mathcal{F}^n_{\frac{u}{\sqrt{n}}} \mathbf{1}(x) e^{-i\langle u/\sqrt{n}, \mathbf{v}-\mathbf{u} \rangle} du.$$

In order to keep notations light, we set:

$$\phi^{\omega}_{0,n}(t) = \prod_{k=0}^{n-1} \widehat{\mu^{T^k\omega}}(t). \tag{5}$$

Proposition 2.7. *Under Assumption 3, there exist a deterministic $\delta > 0$ such that for $|t| \leq \delta$, the quantity $\log \phi^{\omega}_{0,1}(t)$ is well defined. In addition, the following approximation formula holds $\mathbf{P}^m$-a.e.*

$$\log \phi_{0,1}(t) = \nabla \phi_{0,1}(0)t + \frac{1}{2} t^* \nabla^* \nabla \phi_{0,1}(0)t + R(t), \tag{6}$$

where the remaining term R satisfies for $|t| \leq \delta$ and for any $\epsilon \in (0,1)$:

$$\|R(t)\|_\infty \leq \delta^{1-\epsilon} K |t|^{2+\epsilon} \text{ with } K \geq 0.$$

Proof. Under Assumption 3, the function $\phi^{\omega}_{0,1}$ is $\mathbf{P}^m$-a.e. three times continuously differentiable. Therefore, the following majoration holds $\mathbf{P}^m$-a.e.

$$|\phi_{0,1}(t) - 1| \leq |t| \|\nabla \phi_{0,1}(0)\|.$$

Thus, there exists a deterministic $\delta > 0$ such that for all $|t| \leq \delta$ the function $t \to \log \phi^{\omega}_{0,1}(t)$ is well defined. In addition, for $|t| \leq \delta$, the Taylor formula yields $\mathbf{P}^m$-a.e.

$$\log \phi_{0,1}(t) = \nabla \phi_{0,1}(0)t + \frac{1}{2} t^* \nabla^* \nabla \phi_{0,1}(0)t + R(t),$$

where the remaining term R satisfies for $|t| \leq \delta$ and any $\epsilon \in (0,1)$

$$\|R(t)\|_\infty \leq \delta^{1-\epsilon} K \|t\|^{2+\epsilon}, \text{ with } K = \left\| \sum_{\mathbf{u} \in \mathbb{Z}^d} \|\mathbf{u}\|^3 \mu(\mathbf{u}) \right\|_\infty.$$

□

This proposition implies that the product of Equation (5) can be rewritten for $\delta > 0$ sufficiently small and $|t| \leq \delta$ as follows

$$\phi_{0,n}(t) = \exp\left\{ \sum_{k=0}^{n-1} \langle \nabla \phi^{T^k\omega}_{0,1}(0), t \rangle + \frac{1}{2} t^* \sum_{k=0}^{n-1} \nabla^* \nabla \phi^{T^k\omega}_{0,1}(0)t + \sum_{k=0}^{n-1} R^{T^k\omega}(t) \right\} dt.$$

We may introduce the random variables Σ_n, Δ_n and $R_n(t)$, termed respectively the local mean variance, the local drift deviation and the residual function, defined respectively by:

$$\Sigma_n = \Sigma_n(\omega) = -\frac{1}{n} \sum_{k=0}^{n-1} \nabla^* \nabla \phi^{T^k\omega}_{0,1}(0), \tag{7}$$

$$\Delta_n = \Delta_n(\omega) = i \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} \nabla \phi^{T^k\omega}_{0,1}(0), \tag{8}$$

and

$$R_n(t) = R^{\omega}_n(t) = \sum_{k=0}^{n-1} R^{T^k\omega}(t/\sqrt{n}).$$

Proposition 2.8. *Under Assumption 3 the following properties hold*

1. *for all $n \geq 1$, the matrix Σ_n is real positive symmetric $\mathbf{P}^m$-a.e.,*
2. *the sequence $(\Sigma_n)_{n \geq 0}$ remains bounded in the following sense*

$$\sup_{n \geq 0} \| \, \|\Sigma_n\| \, \|_\infty < \infty,$$

3. *the rank* $\mathsf{rk}\ \Sigma_n$ *is non decreasing with* $n \geq 1$ $\mathbf{P}^m$*-a.e.,*

4. *in addition, under Assumption 1,* $\lim_{n\to\infty} \mathsf{rk}\ \Sigma_n = d$, $\mathbf{P}^m$*-a.e..*

Proof. 1. Under Assumption 3, the quantity $-\nabla^*\nabla\phi^\omega_{0,1}(0)$ is real symmetric. In addition, it is non negative since $t \to \mathsf{Re}\ \widehat{\mu^\omega}(t)$ admits a local maximum at $t = 0$.

2. As a matter of facts, the sequence of matrices Σ_n satisfies

$$\operatorname*{ess\ sup}_{\omega\in\Omega} \|\Sigma_n\| \leq \operatorname*{ess\ sup}_{\omega\in\Omega} \|\nabla^*\nabla\phi_{0,1}(0)\| < \infty.$$

3. Because $-\nabla^*\nabla\phi^\omega_{0,1}(0)$ is non negative, the kernel of Σ_n is given by

$$\mathsf{ker}\ \Sigma_n = \bigcap_{k=0}^{n-1} \mathsf{ker}\ \nabla^*\nabla\phi^{T^k\omega}_{0,1}(0),$$

and the result follows.

4. Since $\mathsf{rk}\ \Sigma_n$ is a non decreasing discrete bounded sequence $\mathbf{P}^m$-a.e., it suffices to show that

$$\mathbf{P}^x(\liminf\{\mathsf{rk}\ \Sigma_n = d\}) = \mathbf{P}^x(\bigcup_{n\geq 1}\{\mathsf{rk}\ \Sigma_n = d\}) = 1, \quad \text{for} \quad m-\text{a.e.} \quad x \in \mathbb{X}. \tag{9}$$

Moreover, the operator P is supposed aperiodic and recurrent so that we only need the asymptotic event in (9) holds with positive probability. Thus suppose on the contrary

$$\mathbf{P}^x(N) = 1 \quad \text{with} \quad N = \bigcap_{n\geq 1}\{\mathsf{rk}\ \Sigma_n \leq d-1\}.$$

Then the subgroup H of $\mathbb{Z}^d$ generated by the supports $\mathsf{supp}\ \mu^\omega * \cdots * \mu^{T^{n-1}\omega}$, $n \geq 1$, $\omega \in N$, is $\mathbf{P}^x$-a.s. independent of $\omega \in \mathbb{X}^{\mathbb{N}}$ and satisfies $\mu^\omega(H) = 1$, $\mathbf{P}^m$-a.e.. Assumption 1 yields $H = \mathbb{Z}^d$ which contradicts the non maximality of the asymptotic rank.

□

Let us define the positive random variable $\tau := \inf\{n \geq 1 : \mathsf{rk}\ \Sigma_n = d\}$ which is finite $\mathbf{P}^m$-a.e. by Proposition 2.8. Denote by $I^\omega_1(n)$ the quantity

$$I^\omega_1(n) = n^{-d/2}\int_{|u|\leq\delta\sqrt{n}} \phi^\omega_{0,n}(u/\sqrt{n})e^{-i\langle u/\sqrt{n},\mathbf{v}-\mathbf{u}\rangle}du,$$

and rewrite it as the sum of three terms I_{11}, I_{12} and I_{13} defined by

$$I^\omega_{11}(n) = n^{-d/2}\int_{|t|\leq\delta\sqrt{n}} e^{-\frac{1}{2}t^*\Sigma_n t}e^{i\langle t,\Delta_n\rangle - i\langle t,(\mathbf{v}-\mathbf{u})/\sqrt{n}\rangle}\left(e^{R_n(t/\sqrt{n})} - 1\right)dt,$$

$$\text{for} \quad n \geq \tau: \quad I^\omega_{12}(n) = n^{-d/2}\int_{\mathbb{R}^d} e^{-\frac{1}{2}t^*\Sigma_n t}e^{i\langle t,\Delta_n\rangle - i\langle t,(\mathbf{v}-\mathbf{u})/\sqrt{n}\rangle}dt,$$

and,

$$\text{for} \quad n \geq \tau: \quad I^\omega_{13}(n) = -n^{-d/2}\int_{|t|>\delta\sqrt{n}} e^{-\frac{1}{2}t^*\Sigma_n t}e^{i\langle t,\Delta_n\rangle - i\langle t,(\mathbf{v}-\mathbf{u})/\sqrt{n}\rangle}dt.$$

The following uniform ellipticity condition is necessary for the third estimate of Proposition 2.9 below.

Assumptions 4. *There exists $\alpha > 0$ such that for all $t \in \mathbb{R}^d$, $\liminf_{n\to\infty} t^*\Sigma_n t \geq \alpha|t|^2$, $\mathbf{P}^m$-a.e..*

Proposition 2.9. *Under Assumptions 1 and 3, for all $n \geq \tau$ the following holds $\mathbf{P}^m$-a.e. for $\mathbf{u}, \mathbf{v} \in \mathbb{Z}^d$:*

1. $I_{12}(n) = n^{-d/2}\mathsf{det}\ (\Sigma_n)^{-1/2}(2\pi)^{d/2}\exp\left\{\left(-\frac{1}{2}[\Delta_n - (\mathbf{v}-\mathbf{u})/\sqrt{n}]^*\Sigma_n^{-1}[\Delta_n - (\mathbf{v}-\mathbf{u})/\sqrt{n}]\right)\right\}$,
2. $|I_{13}(n)| \leq n^{-d/2}(2\pi)^{d/2}\exp\left(-\frac{1}{2}\delta\sqrt{n}\ \mathsf{tr}\ (\Sigma_n)\right)$,
3. *if additionally, Assumption 4 is fulfilled, then there exists a deterministic $\delta > 0$ such that $|I_{11}(n)| = \mathscr{O}\left(n^{-\frac{d+\epsilon}{2}}\right)$ uniformly in $\omega \in \Omega$.*

Proof. Setting $\tilde{\Delta}_n = \Delta_n - (\mathbf{v}-\mathbf{u})/\sqrt{n}$, it is only needed to prove the results for $\mathbf{v}-\mathbf{u} = 0$.

1. Under Assumptions 1 and 3, Proposition 2.8 implies $\tau < \infty$ $\mathbf{P}^m$-a.e. and for $n \geq \tau$, there exist orthogonal matrices P_n and diagonal matrices D_n such that

$$\Sigma_n = P_n D_n P_n^{-1} \text{ and } D_n = \mathsf{diag}(\alpha_1^2(n), \ldots, \alpha_d^2(n))$$

with $\alpha_i^2(n) > 0$ for all $i = 1, \ldots, d$ and $n \geq N$. Setting $t = P_n u$, we obtain

$$\begin{aligned} n^{d/2}I_{12}(n) &= \int_{\mathbb{R}^d} e^{-\frac{1}{2}u^*D_n u}e^{i\langle u, P_n^*\Delta_n\rangle}du \\ &= \mathsf{det}\ (\Sigma_n)^{-1/2}(2\pi)^{d/2}e^{-\frac{1}{2}\Delta_n^*\Sigma_n^{-1}\Delta_n}. \end{aligned}$$

2. For the term $I_{13}(n)$ we can proceed analogously, and we get the following upper bound for $n \geq \tau$:

$$\begin{aligned} n^{d/2}|I_{13}(n)| &\leq \int_{|u|>\delta\sqrt{n}} e^{-\frac{1}{2}u^*D_n u}du \\ &\leq (2\pi)^{d/2}\mathsf{det}\ (\Sigma_n)^{-1}\exp\{-\frac{1}{2}\delta\sqrt{n}\ \mathsf{tr}\ (\Sigma_n)\}. \end{aligned}$$

3. Because of the point (2) of Proposition 2.8, the eigenvalues of Σ_n remain bounded uniformly for $n \geq 0$. Thus, with Assumption 4, we deduce the following bound for $I_{11}(n)$

$$\begin{aligned} n^{d/2}|I_{11}(n)| &\leq \int_{|t|\leq\delta\sqrt{n}} e^{-\frac{1}{2}t^*\Sigma_n t}\left|\exp\{R_n(tn^{-1/2})\} - 1\right|dt \\ &\leq \int_{|t|\leq\delta\sqrt{n}} e^{-\frac{1}{2}t^*\Sigma_n t}\left|R_n(tn^{-1/2})\right|\exp\{|R_n(tn^{-1/2})|\}dt \\ &\leq \frac{K\delta^{1-\epsilon}}{n^{\epsilon/2}}\int_{|t|\leq\delta\sqrt{n}} e^{-\frac{1}{2}t^*\Sigma_n t}|t|^{2+\epsilon}e^{|t|^2\delta K}dt. \end{aligned}$$

The last estimates comes from Proposition 2.7 and holds for any $\epsilon \in (0,1)$. We conclude by choosing $\delta > 0$ such that $\delta K \leq \alpha/4$ (where α is given by Assumption 4). Consequently the integral is convergent and the whole term goes to zero at rate, up to a constant, $n^{-\epsilon/2}$.

□

2.3 The type problem : a series criterion

Let $A \subset \mathbb{X}$ and $K \subset \mathbb{Z}^d$ be measurable subsets. We might be interested in the mean time spent by the Markov additive process — that is nothing but a Markov chain on $\mathbb{X} \times \mathbb{Z}^d$ — in the product set $A \times K$. In facts, it is well known that this quantity, starting from $(x, \mathbf{u}) \in \mathbb{X} \times \mathbb{Z}^d$, is actually given by the Green operator defined by

$$G\mathbf{1}_{A\times K}(x, \mathbf{u}) = \sum_{n\geq 0} \overline{P}^n \mathbf{1}_{A\times K}(x, \mathbf{u}).$$

In the special case of $A = \mathbb{X}$, using notation of Equation (4), the equation above rewrites

$$G\mathbf{1}_{\mathbb{X}\times K}(x, \mathbf{u}) = \mathbf{1}_{\mathbb{X}\times K}(x, \mathbf{u}) + \sum_{\mathbf{v}\in\mathbb{Z}^d} \sum_{n\geq 1} \int_{\mathbb{X}^{\mathbb{N}}} P_{0,n}(u, v)\mathbf{1}_K(v) d\mathbf{P}^x.$$

Definition 2.10 (Recurrence and Transience)**.** *A Markov additive process is said to be recurrent (*resp. *transient) if, for any bounded change of section σ, $G^x(0, 0) = \infty$, m-a.e. (*resp. *$G^x(0, 0) < \infty$, m-a.e.).*

Remark 2.4. *Since the changes of section are supposed bounded, a* **MAP** *is simultaneously recurrent or transient for every change of section.*

A simple computation gives rise to the identities[4]

$$\begin{aligned} G^x(0, 0) = \sum_{n\geq 1} \int_{\mathbb{X}^{\mathbb{N}}} \mathbf{P}^x(d\omega) P^{\omega}_{0,n}(0, 0) &= \lim_{r\uparrow 1} \sum_{n\geq 1} r^{n-1} \int_{\mathbb{X}^{\mathbb{N}}} \mathbf{P}^x(d\omega) P^{\omega}_{0,n}(0, 0) \\ &= \lim_{r\uparrow 1} \int_{\mathbb{W}_d} \mathsf{Re} \sum_{n\geq 1} r^{n-1} \mathcal{F}^n_t \mathbf{1}(x) dt. \end{aligned}$$

Theorem 2.11. *Let $d \geq 2$ and suppose that the assumptions 1, 2, 3 and 4 are satisfied. In addition, assume $\mathbf{E}^x(\tau) < \infty$ for $x \in \mathbb{X}$. Then, $G^x(0, 0)$ is finite (*resp. *infinite) if and only if*

$$\sum_{n\geq 1} \frac{1}{n^{d/2}} \int_{\mathbb{X}^{\mathbb{N}}} \mathbf{1}_{\{n\geq\tau\}} \exp\left(-\frac{1}{2}\Delta^*_n \Sigma^{-1}_n \Delta_n\right) d\mathbf{P}^x < \infty \text{ (resp. } = \infty \text{)},$$

and the **MAP** *is recurrent or transient accordingly.*

Proof. Let $r \in (0, 1)$, and compute with the notations of the previous section

$$\begin{aligned} \frac{1}{(2\pi)^d} \int_{\mathbf{W}_d} \mathsf{Re} \sum_{n\geq 1} r^{n-1} \mathcal{F}^n_t \mathbf{1}(x) dt - \frac{1}{(2\pi)^d} \sum_{n\geq 1} r^{n-1} \int_{\mathbb{X}^{\mathbb{N}}} \mathbf{1}_{\{n\geq\tau\}} I_{12}(n) d\mathbf{P}^x \\ = \frac{1}{(2\pi)^d} \sum_{n\geq 1} r^{n-1} \int_{\mathbb{X}^{\mathbb{N}}} \mathbf{1}_{\{n\geq\tau\}} \big[I_{11}(n) + I_{13}(n) + I_2(n)\big] d\mathbf{P}^x \\ + \frac{1}{(2\pi)^d} \sum_{n\geq 1} r^{n-1} \int_{\mathbb{X}^{\mathbb{N}}} \mathbf{1}_{\{n<\tau\}} I_1(n) d\mathbf{P}^x \end{aligned}$$

.

[4]Summations, integrals can be switched, however the limit needs to stay outside.

Taking absolute values on both side, using Proposition 2.9 with a suitable $\delta > 0$ together with Assumption 2, it follows that, for some $K \geq 0$, $\epsilon > 0$, $C > 0$ and $\kappa \in (0,1)$,

$$\begin{aligned}
\left| \frac{1}{(2\pi)^d} \int_{\mathbf{W}_d} \mathsf{Re} \sum_{n\geq 1} r^{n-1} \mathcal{F}_t^n \mathbf{1}(x) dt - \frac{1}{(2\pi)^d} \sum_{n\geq 1} r^{n-1} \int_{\mathbb{X}^{\mathbb{N}}} \frac{1}{\mathsf{det}\ (\Sigma_n) n^{d/2}} \exp\left(-\frac{1}{2}\Delta_n^* \Sigma_n^{-1} \Delta_n\right) d\mathbf{P}^x \right| \\
\leq \frac{1}{(2\pi)^d} \sum_{n\geq 1} r^{n-1} \frac{K}{n^{(d+\epsilon)/2}} \\
+ \frac{1}{(2\pi)^d} \left| \sum_{n\geq 1} r^{n-1} \int_{\mathbb{X}^{\mathbb{N}}} \frac{\exp(-\frac{1}{2}\delta\sqrt{n}\,\mathsf{tr}\ \Sigma_n)}{n^{d/2}} d\mathbf{P}^x \right| \\
+ \frac{C}{(2\pi)^d} \sum_{n\geq 1} \kappa^{n-1} \\
+ \frac{1}{(2\pi)^d} \sum_{n\geq 1} r^{n-1} \mathbf{P}^x(\tau > n).
\end{aligned}$$

The latter summation follows from the fact $|I_1(n)| \leq 1$, $\mathbf{P}^m$-a.e.. Then, letting $r \uparrow 1$, the right handside remains bounded in $\mathbf{L}_m^\infty(\mathbb{X})$ since

- $d \geq 2$ for the first term,
- Σ_n is bounded in $\mathbf{L}_m^\infty(\mathbb{X})$ so is the trace,
- Proposition 2.6,
- τ is integrable with respect to $\mathbf{P}^x$, m-a.e. $x \in \mathbb{X}$.

The result follows by letting $r \uparrow 1$ on the left hand-side and remarking that

$$\begin{aligned}
\lim_{r\uparrow 1} \sum_{n\geq 1} r^{n-1} \int_{\mathbb{X}^{\mathbb{N}}} \mathbf{1}_{n\geq\tau} \frac{1}{\mathsf{det}\ (\Sigma_n) n^{d/2}} \exp\left(-\frac{1}{2}\Delta_n^* \Sigma_n^{-1} \Delta_n\right) d\mathbf{P}^x \\
= \sum_{n\geq 1} \int_{\mathbb{X}^{\mathbb{N}}} \frac{\mathbf{1}_{n\geq\tau}}{\mathsf{det}\ (\Sigma_n) n^{d/2}} \exp\left(-\frac{1}{2}\Delta_n^* \Sigma_n^{-1} \Delta_n\right) d\mathbf{P}^x \in [0,\infty]
\end{aligned}$$

m-a.e. by monotone convergence. □

3 Examples

In this section, we apply Theorem 2.11 to some well known examples of directed graphs originally introduced in [5] but also considered and generalized in [15, 13, 30, 6]. Obviously, these results, excepted the Gaussian bounds in Proposition 3.5, are already known and this section should be mainly seen as a "proof of concept". Nonetheless, we claim it gives a new insight to the type problem in the sense it highlights the competition between the "local drift deviation" and the "local mean variance" as it is suggested by Proposition 3.2.

Random walk on partially directed graphs

Let $(\epsilon_w)_{w\in\mathbb{Z}}$ be any $\{-1,1\}$-valued random or deterministic sequence. Set $\mathbb{G}^0 = \mathbb{Z}^2$, and define the set of edges $\mathbb{G}^1$ as follows, for $\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2), \mathbf{v} = (\mathbf{v}_1, \mathbf{v}_2) \in \mathbb{G}^0$,

$$(\mathbf{u}, \mathbf{v}) \in \mathbb{G}^1 \iff (|\mathbf{v}_2 - \mathbf{u}_2| = 1 \text{ and } \mathbf{v}_1 = \mathbf{u}_1) \text{ or } (\mathbf{v}_2 = \mathbf{u}_2 \text{ and } \mathbf{v}_1 = \mathbf{u}_1 + \epsilon_{\mathbf{u}_2}). \tag{10}$$

This defines a directed graph, denoted in the sequel by $\mathbb{G}(\epsilon) = (\mathbb{G}^0, \mathbb{G}^1)$, embedded in $\mathbb{Z}^2$.

In [5], the simple random walk on such graphs are studied with a sequence $(\epsilon_w)_{w\in\mathbb{Z}}$ chosen periodic, to be equal to the sign sequence on $\mathbb{Z}$ or to be a sequence of centered independent and identically distributed Rademacher random variables. This sequence can be defined with the help of a dynamical system as in [30, 15]. Finally, in [6], it is chosen as a random perturbation of the periodic case.

Note that, in this context, we mainly study the simple random walk in a "centered" environment $(\epsilon_w)_{w\in\mathbb{Z}}$ to avoid trivialities coming from a non zero drift when studying the type problem. However, for more general questions — we may think about the return probability estimate — we can be interested in the non zero drift case and our method suits to this latter case as well. Moreover, the simple random walk is irreducible if and only if the range of the $(\epsilon_w)_{w\in\mathbb{Z}}$ contains -1 and 1.

In the sequel, we denote by M the Markov operator associated with the simple random walk on the graph $\mathbb{G}(\epsilon)$. It is straightforward that the automorphism group $\mathsf{Aut}(\mathbb{G}(\epsilon), M)$ of M, is a subgroup of $\mathbb{Z} \times \{0\}$. Applying the construction of Section 1.3, it turns out that the simple random walk on $\mathbb{G}(\epsilon)$, starting from any vertex $\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2) \in \mathbb{G}^0$, is equal in distribution to a **MAP**. The underlying dynamics is a random walk on $\mathbb{Z}$ starting from $x = \mathbf{u}_2$ whose Markov operator is given by

$$\mathring{P}(x,y) = \begin{cases} \frac{1}{3} & \text{if } |y-x| \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

The system of probability measures μ is given for $x, y \in \mathbb{Z}$ by

$$\mu^{x,y} = \begin{cases} \delta_{(0,1)} & \text{if } y = x+1, \\ \delta_{-(0,1)} & \text{if } y = x-1, \\ \delta_{(1,0)} & \text{if } y = x, \epsilon_x = 1, \\ \delta_{-(1,0)} & \text{if } y = x, \epsilon_y = -1. \end{cases}$$

The initial system of probability measures ν satisfy $\nu = \delta_{\mathbf{u}_1,\mathbf{u}_2}$, $\mathbf{P}^{\mathbf{u}_2}$-a.s..

In the sequel, the random walk $\mathring{P}$ at time $n \geq 0$ will be denoted by S_n ($S_0 = \mathbf{u}$) and its increments $S_{n+1} - S_n = \xi_n$. This random walk is irreducible, aperiodic and null recurrent. Moreover, Assumption 3 is clearly satisfied and by a simple computation, we obtain that

$$\Sigma_n = \frac{1}{n} \begin{pmatrix} \sum_{k=0}^{n-1} \mathbf{1}_{\xi_k=0} & 0 \\ 0 & \sum_{k=0}^{n-1} \mathbf{1}_{\xi_k\neq 0} \end{pmatrix},$$

and

$$\Delta_n = \frac{1}{\sqrt{n}} \begin{pmatrix} \sum_{k=0}^{n-1} \epsilon_{S_k} \mathbf{1}_{\xi_k=0} \\ \sum_{k=0}^{n-1} \xi_k \mathbf{1}_{\xi_k\neq 0} \end{pmatrix} = \begin{pmatrix} \sum_{k=0}^{n-1} \epsilon_{S_k} \mathbf{1}_{\xi_k=0} \\ S_n \end{pmatrix}.$$

Note that the sequence of matrices Σ_n converges $\mathbf{P}^x$-a.s., for all $x \in \mathbb{Z}$, by the Strong Law of Large Number, and the limit is given by

$$\Sigma = \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{2}{3} \end{pmatrix}.$$

Consequently, Assumption 4 is satisfied. Also, it is straightforward that $\mathbf{E}^{\mathbf{u}_2}(\tau) < \infty$.

It is worth noting that even though the **MAP** is adapted and irreducible, the aperiodicity condition fails. In fact, it suffices to consider the change of section $\sigma : \mathbb{Z} \longrightarrow \mathbb{Z}^2$ given by

$\sigma(x) = 0$ if $x \in \mathbb{Z}$ is even and $\sigma(x) = (1,0)$ if x is odd so that

$$\mu_\sigma^{x,y} = \begin{cases} \mu^{x,y} * \delta_{-(0,1)} = \delta_0 & \text{if } y = x+1, \\ \mu^{x,y} * \delta_{(0,1)} = \delta_0 & \text{if } y = x-1, \\ \mu^{x,x} & \text{if } y = x, \end{cases}$$

and choose the subgroup $\mathbb{Z} \times \{0\}$. That is why, in the sequel, we consider the "2-fold" **MAP**, *i.e.* we consider $\overline{P}^2$ in place of $\overline{P}$ which, in terms of recurrence or transient, is irrelevant.

Thus, at this level, we may prove the condition (S), *i.e.* Assumption 2, is fulfilled. In the context of periodic environment ϵ it is straightforward since the Fourier transform operator $\mathcal{F}_t^2$, $t \in \mathbf{W}_d$ acts on the finite dimensional space E of periodic functions (the dimension of this space is equal to the period of the sequence ϵ). In facts, in this specific case, using the standard machinery of the litterature, the recurrence of the random walk would follow from [17]. Besides, we may obtain a renewal theorem from [1], a large deviation theorem from [27, 25] and sharp estimates of the asymptotics of the Green function (and in turn a determination of the Martin boundary) from [37, 22]. Obviously, these technics can no longer be applied for a generic environment.

Proposition 3.1. *The condition (S) is fulfilled.*

Proof. Let $t \in \mathbf{W}_d$ and compute for $f \in \ell^\infty(\mathbb{Z})$ and $x \in \mathbb{Z}$

$$\begin{aligned} \mathcal{F}_t^2 f(x) = &\left(\frac{2}{9} + \frac{1}{9}e^{2it_1\varepsilon(x)}\right) f(x) + \frac{1}{9}e^{2it_2} f(x+2) + \frac{1}{9}e^{-2it_2} f(x-2) \\ &+ \frac{1}{9}e^{it_2} f(x+1)\left(e^{it_1\varepsilon(x)} + e^{it_1\varepsilon(x+1)}\right) \\ &+ \frac{1}{9}e^{-it_2} f(x-1)\left(e^{it_1\varepsilon(x)} + e^{it_1\varepsilon(x-1)}\right) \end{aligned}$$

First, suppose that $t_1 \neq 0$, then it follows that

$$\|\mathcal{F}_t^2\| \leq \frac{2}{3} + \sup_{x\in\mathbb{Z}} \left| \frac{2}{9} + \frac{1}{9}e^{2it_1\varepsilon(x)} \right| \leq \frac{2}{3} + \frac{1}{9}\sqrt{5 + 4\cos(2t_1)} < 1.$$

Secondly, if $t_1 = 0$, then the expression of $\mathcal{F}_t^2$ simplifies as follows

$$\mathcal{F}_t^2 f(x) = \frac{1}{3} f(x) + \frac{2}{9}e^{it_2} f(x+1) + \frac{1}{9}e^{-it_2} f(x-1) + \frac{1}{9}e^{2it_2} f(x+2) + \frac{1}{9}e^{-2it_2} f(x-2).$$

As a matter of facts, the space of $\mathbb{C}$-valued constant functions is closed and invariant under $\mathcal{F}_t^2$. Moreover, it is straightforward that

$$\|\mathcal{F}_t^2\|_E \leq \left(\frac{1}{3} + \frac{4}{9}\cos(t_2) + \frac{2}{9}\cos(2t_2)\right).$$

Summarizing, as soon as $t \in \mathbf{W}_d^*$, the pseudo spectral radius $\tilde{r}(t)$ is strictly smaller than 1 and the condition (S) is satisfied. □

Remark 3.1. *When $t_1 = 0$, the closed subspace of 2-periodic functions contains the constants and is invariant by $\mathcal{F}_t^2$. Still, the function $x \to (-1)^x$ is an eigenfunction associated with the eigenvalue 1. Thus, it is crucial that the condition (S) allows to consider different invariant subspaces for different points $t \in \mathbf{W}_d$.*

Theorem 3.2. *The simple random walk M is recurrent (*resp. *transient) if and only if for some $x \in \mathbb{Z}$*

$$\sum_{n\geq 1} \frac{1}{n} \int_{\mathbb{Z}^{\mathbb{N}}} \mathbf{1}_{n\geq \tau} \exp\left[-\frac{1}{2} \frac{\left(\sum_{k=0}^{n-1} \epsilon_{S_k} \mathbf{1}_{\xi_k=0}\right)^2}{\sum_{k=0}^{n-1} \mathbf{1}_{\xi_k=0}} \right] \exp\left[-\frac{1}{2} \frac{S_n^2}{\sum_{k=0}^{n-1} \mathbf{1}_{\xi_k\neq 0}} \right] d\mathbf{P}^x = \infty \quad (\text{resp. } < \infty). \tag{11}$$

Remark 3.2. *Similarly to the classic context of random walks on $\mathbb{Z}^2$, the harmonic factor term in the sum reveals that the dimension 2 is critical. The exponential factor may improve the convergence rate of the sumand in order to make the sum finite. Each of them corresponds to one of the direction given by the axis. More precisely, the first one describes the contribution of the horizontal component, whereas the second one describes that of the vertical one. The transient behavior can not be explained only by the contribution of the second exponential factor as the following proposition highlights.*

Proposition 3.3. *The second exponential factor in (11) satisfies*

$$\liminf_{n\geq 0} \log(n) \exp\left[-\frac{1}{2} \frac{S_n^2}{\sum_{k=0}^{n-1} \mathbf{1}_{\xi_k\neq 0}} \right] \geq 1, \mathbf{P}^x - a.s.$$

Proof. By the Law of Iterated Logarithm, it follows that

$$\limsup_{n\to\infty} \sqrt{\frac{3}{2n}} \frac{S_n}{\sqrt{2\log\log n}} = 1, \mathbf{P}^x - \text{a.s.}$$

We deduce that for $n \geq 1$ sufficiently large, with probability one,

$$\log(n) \exp\left[-\frac{1}{2} \frac{S_n^2}{\sum_{k=0}^{n-1} \mathbf{1}_{\xi_k\neq 0}} \right] \geq \log(n) \exp\left[-\frac{2n\log\log n}{3\sum_{k=0}^{n-1} \mathbf{1}_{\xi_k\neq 0}} \right],$$

but the right hand side converges $\mathbf{P}^x$-a.s. to 1 since by the Strong Law of Large Number

$$\lim_{n\to\infty} \frac{n}{\sum_{k=0}^{n-1} \mathbf{1}_{\xi_k\neq 0}} = \frac{3}{2}, \mathbf{P}^x - \text{a.s.},$$

and the proposition follows. □

The periodic case

Let us consider the case of a periodic sequence $(\epsilon_y)_{y\in\mathbb{Z}}$. Without loss of generality, we set $\epsilon_y = (-1)^y$ for all $y \in \mathbb{Z}$. The group of automorphisms of the corresponding directed graph is $\mathbb{Z} \times 2\mathbb{Z}$ — for a general periodic sequence, it would be $\mathbb{Z} \times p\mathbb{Z}$ for some integer $p \geq 2$. The **MAP** introduced in the previous section can therefore be factorized. In facts, consider $(\xi_i)_{i\in\mathbb{N}}$ be a sequence of i.i.d. $\mathbb{Z}/2\mathbb{Z}$-valued random variable satisfying

$$\mathbf{P}(\xi_0 = 1) = 2/3 \text{ and } \mathbf{P}(\xi_0 = 0) = 1/3.$$

For $n \geq 0$, let us define $S_n = \left(S_0 + \sum_{k=0}^{n-1} \xi_k\right) \bmod 2$. The dynamical system considered here is the time shift on $\Omega = (\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}}$ endowed with the probability measure $\mathbf{P}^\pi$, with π standing for the unique invariant probability, on the paths space. The system of probability measures μ is then given by

$$\mu^\omega = \begin{cases} \frac{1}{2}(\delta_{e_2} + \delta_{-e_2}) & \text{if } \omega_1 \neq \omega_0, \\ \delta_{e_1} & \text{if } \omega_1 = \omega_0 = 0 \\ \delta_{-e_1} & \text{if } \omega_1 = \omega_0 = 1 \end{cases}.$$

Moreover, one computes

$$\Sigma_n = -\frac{1}{n}\begin{pmatrix} \sum_{k=0}^{n-1} \mathbf{1}_{\omega_k=\omega_{k+1}} & 0 \\ 0 & \sum_{k=0}^{n-1} \mathbf{1}_{\omega_k\neq\omega_{k+1}} \end{pmatrix}$$

and

$$\Delta_n = \frac{1}{\sqrt{n}}\begin{pmatrix} \sum_{k=0}^{n-1} \mathbf{1}_{\omega_k=\omega_{k+1}}(\mathbf{1}_{\omega_k=0} - \mathbf{1}_{\omega_k=1}) \\ 0 \end{pmatrix}.$$

Finally, $\lim \Sigma_n = 1/2\mathsf{Id}$, $\mathbf{P}^\pi$-a.s.

Proposition 3.4. *The following estimates holds $\mathbf{P}^\pi$-a.s.*

$$\liminf_{n\geq 0} \log(n) \exp\left[-\frac{1}{2}\frac{\left(\sum_{k=0}^{n-1} \mathbf{1}_{\xi_k=0}(\mathbf{1}_{S_k=0} - \mathbf{1}_{S_k=1})\right)^2}{\sum_{k=0}^{n-1} \mathbf{1}_{\xi_k=0}}\right] \geq 1.$$

Proof. Applying [35] with $\mathbf{P}^\pi$, we obtain the Law of Iterated Logarithm for martingales, and it follows that

$$\limsup_{n\to\infty} \frac{3\sum_{k=0}^{n-1} \mathbf{1}_{\xi_k=0}(\mathbf{1}_{S_k=0} - \mathbf{1}_{S_k=1})}{\sqrt{2n\log\log n}} = 1 \ , \mathbf{P}^\pi - \text{a.s.}$$

Since $\frac{\sum_{k=0}^{n-1} \mathbf{1}_{\xi_k=0}}{n}$ converges almost surely to $1/3$, we obtain the proposition. □

Remark 3.3. *We consider above the optimal factorization of the graph $\mathbb{G}(\epsilon)$ by taking advantages of the periodicity of the environment. However, the same result can be shown, if we consider the* **MAP** *in the more general setting of the previous section, by applying [11] in place of [35].*

Proposition 3.5. *The following asymptotic equivalent holds as $n \to \infty$, for $\mathbf{u} \in \mathbb{G}^0$, uniformly in $\mathbf{v} \in \mathbb{G}^0$*

$$M^n(\mathbf{u},\mathbf{v}) \sim_{n\to\infty} \frac{1}{4\pi n}\int \mathbf{1}_{\{n\geq\tau\}} \exp\left\{-\frac{1}{2}(\Delta_n - (\mathbf{v}-\mathbf{u})/\sqrt{n})^*\Sigma_n^{-1}(\Delta_n - (\mathbf{v}-\mathbf{u})/\sqrt{n})\right\} d\mathbf{P}^\flat,$$

where $\flat = \mathbf{u}_2 \mathsf{mod}\ 2 \in \mathbb{Z}/_{2\mathbb{Z}}$. In addition, there exist positive constants $C_0, C_1 > 0$ such that, for n large enough

$$\frac{C_0 e^{-\|\mathbf{v}-\mathbf{u}\|^2/n}}{n} \leq M^{2n}(\mathbf{u},\mathbf{v}) \leq \frac{C_1 n e^{-\|\mathbf{v}-\mathbf{u}\|^2/n}}{n}.$$

Proof. In this context the LIL implies that the exponential factor remains bounded away from below so that the quantity 2. and 3. of Proposition 2.9 are negligible with respect the quantity 1. Moreover, the determinant of Σ_n is asymptotically equal to $1/4$.

For the second point, develop the quadratic form, and it follows the expected estimates. □

The proposition above gives, for the simple random walk in the periodic case, Gaussian bounds for the heat kernel that are well known in the standard case of random walks on undirected graphs statisfying a d-dimensional isoperimetric inequality.

The sign function

For this example, the sequence $(\epsilon_y)_{y\in\mathbb{Z}}$ is supposed to be the sign function sgn, that is $\epsilon_y = 1$ for $y \geq 0$ and -1 otherwise. We start with a well known proposition proved in [34].

Proposition 3.6. *The following convergence in distribution holds*

$$\frac{1}{n}\sum_{k=0}^{n-1} \mathsf{sgn}(S_k)\mathbf{1}_{\xi_k=0} \xrightarrow{d} \Gamma,$$

where Γ is a symmetric arcsinus random variable on $[-1,1]$. Therefore, the series (11) of Theorem 3.2 is convergent, $\mathbf{P}^x$-a.s. and the resulting **MAP** *and the simple random walk on $\mathbb{G}(\epsilon)$ are transient.*

In this case, we can not provide a sharp estimate on the heat kernel R^n since the term of point 3. in Proposition 2.9 is no longer negligible. The better bound for the decreasing is of order $n^{-3/2}$ which is undoubtedly far from optimality.

Conclusion

In this paper, we focus on the type problem for Markov additive processes. The main reason is that, generally speaking, this problem appears as an intermediate problem between the Central Limit theorem and the stronger Local Limit theorem. For the former, the spectral condition (S) is not necessary since the weak convergence depends only on the singularity of the Fourrier transform at the origin. In turn, we think our estimating method should apply in this context. A Local Limit theorem is a key ingredient in the estimation of the asymptotics of the Green function which, in turn, leads to the determination of the Martin boundary (actually, a uniform Local limit theorem is needed).

Besides, the spectral condition (S) is trivially satisfied if the Markov chain $\mathring{P}$ is transient. Under mild condition, it would of great interest to study Markov additive processes in this context.

The restriction to $\mathbb{Z}^d$ is essentially irrelevant even though the generalization to $\mathbb{R}^d$ have its own specificities (see for instance [1]). Possibly a more interesting generalization would be to substitute the Markovian dynamics by an arbitrary dynamical system. For general dynamical systems, the Fourier transform operator can be defined as a continuous perturbation of the Koopmann operator. More precisely, we define $\mathcal{F}_t$, acting as a contraction of $\mathbf{L}^\infty_\rho(\Omega)$, for $f \in \mathbf{L}^\infty_\rho(\Omega)$, by

$$\mathcal{F}_t f(\omega) = \widehat{\mu^\omega}(t) f \circ T, \quad \text{for} \quad \rho - \text{a.e.}\omega \in \Omega.$$

The Fourier transform operator $\mathcal{F}_t$ gives rise to a weakly continuous $\mathbb{C}$-bilinear form on $\mathbf{L}^1_\rho(\Omega) \times \mathbf{L}^\infty_\rho(\Omega)$ using duality

$$\langle \lambda, \mathcal{F}_t f\rangle = \int_\Omega \lambda(\omega)\overline{\mathcal{F}_t f(\omega)} d\rho(\omega), \tag{12}$$

where λ is identified with its density with respect to ρ. This generalization is natural in the sense that, in the context of Markov additive processes, we can identify the space $\mathbf{L}^\infty_m(\mathbb{X})$ with the closed subspace of functions in $\mathbf{L}^\infty_{\mathbf{P}^m}(\mathbb{X}^\mathbb{N})$ depending only on the first coordinate so that setting $\lambda = \mathbf{P}^x$ for any $x \in \mathbb{X}$, the Fourier transform rewrites

$$\langle \mathbf{P}^x, \mathcal{F}_t f\rangle = \int_\mathbb{X} \mathring{P}(x, dy)\widehat{\mu^{x,y}}(t) f(y).$$

Our methods could be applied to stochastic processes with long memory such as reinforced random walks. In particular, it is worth noting that, for persistent random walks studied in [9], the dependency with the past can be translated in terms of moment condition on the persistent time. It turns out the new phenomena appear precisely when the internal Markov chain is null recurrent. Unfortunately, the integrability condition on τ fails in this context.

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