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ABSTRACT. Our aim here is to investigate the holomorphic geometric structures on compact complex manifolds which may not be Kähler. We prove that holomorphic geometric structures of affine type on compact Calabi-Yau manifolds with polystable tangent bundle (with respect to some Gauduchon metric on it) are locally homogeneous. In particular, if the geometric structure is rigid in Gromov's sense, then the fundamental group of the manifold must be infinite. We also prove that compact complex manifolds of algebraic dimension one bearing a holomorphic Riemannian metric must have infinite fundamental group.

1. INTRODUCTION

A motivation of this article is the following open question: *If the holomorphic tangent bundle TX of a compact complex manifold X admits a holomorphic connection (also called holomorphic affine connection), does it also admit a flat holomorphic connection ?*

If X is Kähler the classical Chern-Weil theory shows that Chern classes with rational coefficients $c_i(X, \mathbb{Q})$ must vanish and, consequently, there is no topological obstruction for TX to admit a flat connection. Moreover, from Yau's theorem proving Calabi conjecture [Ya] (see also [Be] and [IKO]) X admits a finite unramified cover which is a complex torus. In particular, the standard complex affine structure of the torus produces a flat torsionfree affine connection on X.

It is also proved in [IKO] that compact complex surfaces admitting holomorphic affine connections are biholomorphic to quotients of open sets in $\mathbb{C}^2$ by a discrete group of complex affine transformations acting properly and without fixed points. In particular, these surfaces also admit a torsionfree flat affine connection induced by the complex affine structure of $\mathbb{C}^2$.

Recall that an interesting class of compact non-Kähler manifolds which generalize complex tori are those manifolds whose holomorphic tangent bundle is holomorphically trivial. These, called parallelizable manifolds, are by a result of Wang [Wa] known to be biholomorphic to a quotient of a complex Lie group G by a cocompact lattice Γ in G. Such a quotient G/Γ is Kähler if and only if G is abelian.

All parallelizable manifolds admit an obvious flat affine connection for which all global holomorphic vector fields coming from right invariant vector fields on G are parallel. It was proved in [Du5] that parallelizable manifolds G/Γ of complex semi-simple Lie groups G do not admit *torsionfree* flat affine connections.

Interesting exotic deformations of parallelizable manifolds $SL(2,\mathbb{C})/\Gamma$ were constructed by Ghys in [Gh1]. Those deformations are constructed choosing a group homomorphism $u :$ $\Gamma \longrightarrow SL(2,\mathbb{C})$ and considering the embedding $\gamma \longmapsto (u(\gamma), \gamma)$ of Γ into $SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$ (acting on $SL(2,\mathbb{C})$ by left and right translations). Algebraically, the action is given by:

$$(\gamma, m) \in \Gamma \times SL(2,\mathbb{C}) \longmapsto u(\gamma^{-1})m\gamma \in SL(2,\mathbb{C}).$$

It is proved in [Gh1] that, for u close enough to the trivial morphism, Γ acts properly and freely on $SL(2,\mathbb{C})$ such that the quotient $M(u, \Gamma)$ is a complex compact manifold (covered by $SL(2,\mathbb{C})$). In general, these examples do not admit parallelizable manifolds as finite covers. Moreover, for generic u the space of holomorphic global vector fields is trivial and the holomorphic tangent bundle is *simple* (see definition in Section 3). Notice that the holomorphic flat affine connection on $SL(2,\mathbb{C})$ for which right invariant vector fields are parallel is also left invariant (since left translations preserve parallel vector fields). So this connection descends to the quotients $M(u, \Gamma)$.

But the quotients $M(u, \Gamma)$ also admit non-flat holomorphic affine connections. In order to see that, consider the (non-degenerate) Killing quadratic form on the Lie algebra of $SL(2,\mathbb{C})$ and the associated right invariant holomorphic Riemannian metric g on $SL(2,\mathbb{C})$ (see its definition in Section 2). Since the Killing quadratic form is invariant under the adjoint representation, the induced holomorphic Riemannian metric on $SL(2,\mathbb{C})$ is bi-invariant, so it descends to the quotients $M(u, \Gamma)$. Notice that g is locally isomorphic to the complexification of the spherical metric on S^3 and it has constant non-zero sectional curvature. Its associated holomorphic Levi-Civita connection is not flat, but *locally homogeneous* (since it is induced by a bi-invariant connection on $SL(2,\mathbb{C})$).

It may be mentioned that a positive answer to the open question at the beginning of the introduction implies that compact simply connected manifolds X do not admit holomorphic affine connections. Indeed, since X is simply connected, the parallel transport of a *flat* holomorphic affine connection on TX induces a holomorphic trivialization of TX. By Wang theorem, X is then biholomorphic to a compact quotient of a complex Lie group by a lattice. In particular, the fundamental group of X is infinite.

In this direction, it was recently proved in [DM] that *compact complex simply connected manifolds of algebraic dimension zero do not admit a holomorphic affine connection.*

One of the main results here is a similar theorem for manifolds *of algebraic dimension one*, assuming that the connection is the Levi-Civita connection of a holomorphic Riemannian metric (see Theorem 4.7).

Also Theorem 4.11 proves that *compact complex simply connected manifolds with simple holomorphic tangent bundle do not admit holomorphic Riemannian metrics.*

Another important result is Theorem 3.2 which extends the main result of [Du3] to non-Kähler Calabi-Yau manifolds (see Section 3 and [To]): *Holomorphic geometric structures of affine type on compact Calabi-Yau manifolds with polystable tangent bundle (with respect to some Gauduchon metric) are locally homogeneous.* Notice that this statement is not valid for holomorphic geometric structures of non-affine type, since Ghys constructed in

[Gh2] holomorphic nonsingular foliations of codimension one on complex tori which are not translation invariant and also not locally homogeneous.

From Theorem 3.2 we deduce Corollary 4.8 : *Compact complex simply connected Calabi-Yau manifolds with polystable tangent bundle (with respect to some Gauduchon metric) do not admit rigid holomorphic geometric structures of affine type.* In particular, they do not admit holomorphic affine connections.

The paper is organized in the following way. After this introduction, Section 2 introduces the framework of rigid geometric structures in Gromov's sense with interesting examples. Section 3 recall the notions of stability and polystability of the holomorphic tangent bundle of a complex manifold endowed with a Gauduchon metric and proves the vanishing Lemma 3.1 for (non necessarily Kähler) Calabi-Yau manifolds. We also deduce here Theorem 3.2 from Lemma 3.1. Section 4 contains Theorem 4.1 proving the existence of global Killing fields for holomorphic rigid geometric structure on compact complex simply connected manifolds of non maximal algebraic dimension. Then we deduce Theorem 4.7, Theorem 4.11 and Corollary 4.8.

2. Geometric structures and Killing fields

Holomorphic affine connections are *rigid geometric structures* in Gromov's sense [DG]. Let us briefly recall this definition in the holomorphic category.

Let X be a complex manifold of complex dimension n. For any integer $k \geq 1$, we associate to it the principal bundle of k-frames $R^k(X) \longrightarrow X$, which is the bundle of k-jets of local holomorphic coordinates on X. The corresponding structure group D^k is the group of k-jets of local biholomorphisms of $\mathbb{C}^n$ fixing the origin. This D^k is a complex algebraic group.

Definition 2.1. A *holomorphic geometric structure* ϕ of order k on X is a holomorphic D^k-equivariant map from $R^k(X)$ to a complex algebraic manifold Z endowed with an algebraic action of D^k. The geometric structure ϕ is called of *affine* type if Z is a complex affine manifold.

Holomorphic tensors are holomorphic geometric structures of affine type of order one, and holomorphic affine connections are holomorphic geometric structures of affine type of order two [DG]. Holomorphic foliations and holomorphic projective connections are holomorphic geometric structure of non-affine type.

Another important geometric structure of order one is given in the following definition.

Definition 2.2. A *holomorphic Riemannian metric* on X is a holomorphic section

$$g \in H^0(X, \mathrm{S}^2((TX)^*)),$$

where S^i stands for the i-th symmetric product, such that for every point $x \in X$ the quadratic form $g(x)$ on $T_x X$ is nondegenerate.

As in the Riemannian or pseudo-Riemannian setting, one associates to a holomorphic Riemannian metric g a unique holomorphic affine connection ∇. This connection ∇, called

the Levi-Civita connection for g, is uniquely determined by the following properties: ∇ is torsionfree and g is parallel with respect to ∇. Using ∇ one can compute the curvature tensor of g which vanishes identically if and only if g is locally isomorphic to the standard flat model $dz_1^2 + \ldots + dz_n^2$. For more details about the geometry of holomorphic Riemannian metrics one can see [Du6, DZ]

A more flexible geometric structure is a holomorphic conformal structure.

Definition 2.3. A *holomorphic conformal structure* on a complex manifold X is a holomorphic section ω of the bundle $S^2(T^*X) \otimes L$, where L is some holomorphic line bundle over X, such that at any point x in X the section $\omega(x)$ is nondegenerate.

Roughly speaking this means that X admits an open cover such that on each open set in the cover, X admits a holomorphic Riemannian metric such that on the overlaps of two open sets the two given holomorphic Riemannian metrics agree up to a nonzero multiplicative constant.

Here the flat example is the quadric $z_0^2 + z_1^2 + \ldots + z_{n+1}^2 = 0$ in $P^{n+1}(\mathbb{C})$ with the conformal structure induced by the quadratic form $dz_0^2 + dz_1^2 + \ldots + dz_{n+1}^2$ on the quadric. The automorphism group of the quadric preserving its canonical conformal structure is $\mathbb{P}O(n+2, \mathbb{C})$. A classical result due to Gauss asserts that all conformal structures on surfaces are locally isomorphic to the two-dimensional quadric. Any complex manifold M of complex dimension $n \geq 3$ bearing a flat holomorphic conformal structure (meaning that the Weyl tensor of curvature vanishes on entire M) is locally modelled on the quadric.

A holomorphic conformal structure is a holomorphic geometric structure of non-affine type.

Definition 2.4. A locally defined holomorphic vector field Y is a (local) *Killing field* of a holomorphic geometric structure $\phi : R^k(X) \longrightarrow Z$ if its canonical lift from X to $R^k(X)$ preserves the fibers of ϕ.

In other words, Y is a Killing field of ϕ if and only if its (local) flow preserves ϕ. The Killing vector fields form a Lie algebra with respect to the Lie bracket.

A classical result in Riemannian (and pseudo-Riemannian) geometry shows that Y is a Killing field of a (holomorphic) Riemannian metric g on X if and only if $\nabla.Y$ is a pointwise g-skew-symmetric section of $\mathrm{End}(TX)$, where ∇ is the Levi-Civita connection associated to g [Kob].

The holomorphic geometric structure ϕ is called *locally homogeneous* if the holomorphic tangent bundle TX is spanned by local Killing vector fields of ϕ in the neighborhood of any point $x \in X$. This implies that for any pair of points $x, x' \in X$ there exists a local biholomorphism sending x on x' and preserving ϕ.

A holomorphic geometric structure ϕ is *rigid* of order l in Gromov's sense if any local biholomorphism preserving ϕ is determined by its l-jet at any given point.

Holomorphic affine connections are rigid of order one in Gromov's sense (see the nice expository survey [DG]). The rigidity comes from the fact that local biholomorphisms fixing a point and preserving a connection linearize in exponential coordinates, so they are completely determined by their differential at the fixed point.

Holomorphic Riemannian metrics, holomorphic projective connections and holomorphic conformal structures in dimension ≥ 3 are rigid holomorphic geometric structures. Holomorphic symplectic structures and holomorphic foliations are non-rigid geometric structures [DG].

3. Calabi–Yau manifolds and vanishing results

Let X be a compact complex manifold. A Hermitian structure on the holomorphic tangent bundle TX is called a Gauduchon metric if the corresponding real $(1,1)$–form ω on X satisfies the equation

$$\partial\overline{\partial}\omega^{n-1} = 0,$$

where $n = \dim_{\mathbb{C}} X$. It is known that any compact complex manifold admits a Gauduchon metric [Ga]. Fix a Gauduchon metric on X. Let ω_0 be the corresponding $(1,1)$–form on X.

Let $\Omega_{cl}^{1,1}(X,\mathbb{R})$ denote the space of all globally defined d–closed real $(1,1)$–forms on X. Define the Bott–Chern cohomology for X

$$H_{BC}^{1,1}(X,\mathbb{R}) := \frac{\Omega_{cl}^{1,1}(X,\mathbb{R})}{\{\sqrt{-1}\partial\overline{\partial}\alpha \mid \alpha \in C^{\infty}(X,\mathbb{R})\}}.$$

The functional

$$\Omega_{cl}^{1,1}(X,\mathbb{R}) \longrightarrow \mathbb{R}, \ \alpha \longmapsto \int_X \alpha \wedge \omega_0^{n-1}$$

evidently descends to a functional on $H_{BC}^{1,1}(X,\mathbb{R})$.

Given a holomorphic line bundle L on X, choose a Hermitian structure h_L on L, and consider the element $c(L) \in H_{BC}^{1,1}(X,\mathbb{R})$ given by the curvature of the Chern connection on L associated to h_L. Clearly, $c(L)$ is independent of the choice of h_L.

For a torsionfree coherent analytic sheaf F on X, define

$$\mathrm{degree}(F) := \int_X c(\det F) \wedge \omega_0^{n-1} \in \mathbb{R},$$

where $\det F$ is the determinant line bundle for F [Kob, p. 166, Proposition 6.10]. The real number $\mathrm{degree}(F)/\mathrm{rank}(F)$ is called the *slope* of F and it is denoted by $\mu(F)$. A torsionfree coherent analytic sheaf V on X is called *stable* if

$$\mu(F) < \mu(V)$$

for all coherent analytic sub-sheaf $F \subset V$ with $0 < \mathrm{rank}(F) < \mathrm{rank}(V)$. If V is a direct sum of stable sheaves of same slope, then it called *polystable.* See [LT] for more details.

3.1. **Tensors on Calabi–Yau manifolds.** We recall that X is called *Calabi–Yau* if $c(K_X) \in H^{1,1}_{BC}(X, \mathbb{R})$ vanishes, where K_X is the canonical line bundle of X [To]. In particular, if K_X is trivial or torsion (i.e. of finite order), X is Calabi-Yau, but the converse is not true for non-Kähler manifolds (see [To, Section 3]).

A Kähler Calabi–Yau manifold admits a Ricci flat Kähler metric [Ya], but there is no such result for a general complex Calabi–Yau manifold. Notice that complex surfaces admitting holomorphic affine connections have vanishing first Chern class (in $H^2(X, \mathbb{R})$) [IKO], but they are not always Calabi–Yau: examples are provided by linear Hopf surfaces. The reader will find more information and nice examples in [To, Section 3].

Lemma 3.1. *Let X be a compact Calabi–Yau manifold such that TX is polystable with respect to some Gauduchon metric on X. If a holomorphic section*

$$\psi \in H^0(X, (TX)^{\otimes m} \otimes ((TX)^*)^{\otimes l})$$

for some $m, l \in \mathbb{N}$ vanishes at some point of X, then $\psi = 0$.

Proof. Since TX is polystable, it admits a Hermitian–Yang–Mills metric H [LY, p. 572] (see also [LT, p. 61, Theorem 3.0.1]). The Hermitian structure on $(TX)^{\otimes m} \otimes ((TX)^*)^{\otimes l}$ induced by H also satisfies the Hermitian–Yang–Mills condition. Note that this implies that $(TX)^{\otimes m} \otimes ((TX)^*)^{\otimes l}$ is polystable.

Since X is Calabi–Yau, the Einstein factor e_H for the Hermitian–Yang–Mills metric H is zero. The Einstein factor for the Hermitian–Yang–Mills metric on $(TX)^{\otimes m} \otimes ((TX)^*)^{\otimes l}$ induced by H is $(m - l)e_H$, and hence this Einstein factor is also zero. Now we conclude that the section ψ in the lemma is flat with respect to the Chern connection on $(TX)^{\otimes m} \otimes ((TX)^*)^{\otimes l}$ corresponding to the Hermitian–Yang–Mills structure [LT, p. 50, Theorem 2.2.1]. Therefore, if the section ψ vanishes at some point of X then it vanishes identically.   □

Theorem 3.2. *Let X be a compact Calabi-Yau manifold such that TX is polystable with respect to some Gauduchon metric on X. Then any holomorphic geometric structure of affine type on X is locally homogeneous.*

Proof. We make use of Lemma 3.2 in [Du3, p. 565] which asserts that holomorphic geometric structure are always locally homogeneous on manifolds having the property proved in Lemma 3.1: *holomorphic tensor fields vanishing somewhere are trivial.*   □

A similar result was proved in [Du3] (Theorem 1) for Kähler Calabi-Yau manifolds.

In [Du4] it was exhibited holomorphic torsionfree affine connections on (non-Kähler) elliptic bundles S over Riemann surfaces of genus $g \geq 2$ with odd first Betti number which are not locally homogeneous. More precisely, the Killing algebra of a non-flat connection on S was proved to be of dimension one, generated by the fundamental vector field of the elliptic fibration. Moreover the first Chern class of S vanishes since S admits flat holomorphic affine connections, but S is not Calabi-Yau.

3.2. **Holomorphic Riemannian metrics and simplicity.** A holomorphic vector bundle V on a complex manifold X is called *simple* if $H^0(X, V \times V^*) = \mathbb{C} \cdot \mathrm{Id}_V$.

Lemma 3.3. *Let X be a complex manifold such that TX is simple. If X admits a holomorphic Riemannian metric, then*

$$H^0(X, \Lambda^2(TX)^*) = 0.$$

Proof. Fix a holomorphic Riemannian metric g on X. Let θ be a globally defined holomorphic 2–form on X. Let

$$\theta' = H^0(X, TX \times (TX)^*) = H^0(X, , \mathrm{End}(TX))$$

be the endomorphism defined by

$$g(\theta'(x)(v), w) = \theta(x)(v, w), \ \forall \ x \in X, \ v, w \in T_x X.$$

Since g is fiberwise nondegenerate, this condition uniquely defines θ'.

As TX is simple we have $\theta' = \lambda \cdot \mathrm{Id}_{TX}$ for some $\lambda \in \mathbb{C}$. Since $\theta(x)(v, w) = -\theta(x)(w, v)$, and g is symmetric, it follows that

$$\lambda \cdot g(v, w) = g(\theta'(x)(v), w) = -g(\theta'(x)(w), v) = -\lambda \cdot g(v, w).$$

So, $\lambda = 0$, implying that $\theta = 0$. $\square$

Notice that compact parallelizable manifolds $\mathrm{SL}(2, \mathbb{C})/\Gamma$ admit non-closed holomorphic one-forms induced by the right invariant one-forms on $\mathrm{SL}(2, \mathbb{C})$. Hence Lemma 3.3 is not valid without the hypothesis of simplicity of TX. Nevertheless, those one-forms do not descend on a generic exotic quotient of $\mathrm{SL}(2, \mathbb{C})$ in the sense of Ghys [Gh1]. A generic quotient has simple holomorphic tangent bundle and hence Lemma 3.3 applies.

4. Algebraic dimension and affine connections

Recall that the *algebraic dimension* of a complex manifold X is the transcendence degree of the field of meromorphic functions of X over the field of complex numbers. The algebraic dimension of a projective manifold coincides with its complex dimension. In general, the algebraic dimension of a complex manifold of dimension n may take any value between 0 and n. Complex manifolds of maximal algebraic dimension are called Moishezon manifolds.

Theorem 4.1. *Let X be a compact, connected and simply connected complex manifold of complex dimension n and of algebraic dimension p. Suppose that X admits a holomorphic rigid geometric structure ϕ. Then $H^0(X, TX)$ admits an abelian subalgebra A with generic orbits of dimension $\geq n - p$ and preserving ϕ.*

Proof. By the main theorem in [Du1] (see also [Du2]) the Lie algebra of local holomorphic vector fields on X preserving ϕ has generic orbits of dimension at least $n - p$. Since X is simply connected, by a result due to Nomizu [No] and generalized by Amores [Am] and then by Gromov [DG, p. 73, 5.15] these local vector fields preserving ϕ extend to all of X. We

thus get a finite dimensional complex Lie algebra, formed by holomorphic vector fields v_i preserving ϕ, which acts on X with orbits of generic dimension at least $n - p$.

Now put together ϕ and the v_i to form another rigid holomorphic geometric structure $\phi' = (\phi, v_i)$ (see [DG] for details on the fact that the juxtaposition of a rigid geometric structure with another geometric structure is still a rigid geometric structure in Gromov's sense). Considering ϕ' instead of ϕ and repeating the same proof as before, the complex Lie algebra A of those holomorphic vector fields preserving ϕ' acts on X with generic orbits of dimension at least $n - p$. But preserving ϕ' means preserving both ϕ and the v_i. Hence A lies in the center of the Lie algebra generated by the v_i. In particular A is a complex abelian Lie algebra acting on X with generic orbits of dimension at least $n - p$ and preserving ϕ. $\quad\square$

4.1. Maximal algebraic dimension.
Assume that X is a compact Moishezon manifold. This means that the algebraic dimension of X is $n = \dim_{\mathbb{C}} X$.

The following result gives a simple proof of a particular case of Corollary 2 in [BM].

Proposition 4.2. *If TX admits a holomorphic connection, then X admits a finite unramified covering by a compact complex torus.*

The first step of the proof is:

Lemma 4.3. *Let X be a complex manifold endowed with an affine holomorphic connection. Then X do not admit nontrivial holomorphic maps from $\mathbb{CP}^1$ to X.*

Proof. Let D be a holomorphic connection on X. Let

$$f \,:\, \mathbb{CP}^1 \,\longrightarrow\, X$$

be a holomorphic map. Consider the pulled back connection f^*D on f^*TX. Note that $\bigwedge^2 (T\mathbb{CP}^1)^* = 0$ because $\dim_{\mathbb{C}} \mathbb{CP}^1 = 1$. So, the connection f^*D is flat. Since $\mathbb{CP}^1$ is simply connected, this implies that the holomorphic vector bundle f^*TX is trivial.

Now consider the differential of f

$$df \,:\, T\mathbb{CP}^1 \,\longrightarrow\, f^*TX \,.$$

There is no nonzero holomorphic homomorphism from $T\mathbb{CP}^1$ to the trivial holomorphic line bundle, because $\mathrm{degree}(T\mathbb{CP}^1) > 0$. As f^*TX is trivial, this implies that $df = 0$. Therefore, f is a constant map. $\quad\square$

Proof of Proposition 4.2. Since the Moishezon manifold X does not admit any nonconstant holomorphic map from $\mathbb{CP}^1$, it is a complex projective manifold [Ca, p. 307, Theorem 3.1].

As TX admits a holomorphic connection, $c_i(X, \mathbb{Q}) = 0$ for all $i > 0$ [At, p. 192–193, Theorem 4], where $c_i(X, \mathbb{Q})$ denotes the i–th Chern class of TX with rational coefficients. Therefore, X being complex projective, from Yau's theorem proving Calabi's conjecture, [Ya], it follows that X admits a finite unramified covering by a compact complex torus (see also [Be, p. 759, Theoreme 1] and [IKO]). $\quad\square$

4.2. **Algebraic dimension zero.** We deduce here results about holomorphic geometric structures on manifolds with algebraic dimension zero (compare with Theorem 4.2 in [Du1]).

Proposition 4.4. *Let X be a compact complex manifold of algebraic dimension zero such that the canonical bundle K_X is of finite order. If X admits a holomorphic rigid geometric structure, then the fundamental group of X is infinite.*

Proof. Assume by contradiction that the fundamental group of X is finite. Then up to a finite unramified cover we can assume X simply connected. Since K_X is of finite order and X is simply connected, it follows that K_X is trivial.

By Theorem 4.1, we get a complex abelian algebra A acting on X with a dense open orbit. Let $Y_1, \cdots, Y_n$ be holomorphic vector fields in A which span TX at a generic point. Then $k = Y_1 \wedge \ldots \wedge Y_n$ is a holomorphic section of the anti-canonical bundle $-K_X$. Since $-K_X$ is holomorphically trivial, k does not vanish. Hence, the holomorphic vector fields $Y_1, \cdots, Y_n$ span TX at every point in X and trivialize the holomorphic tangent bundle. The action of the connected complex abelian group of biholomorphisms associated to the Lie algebra generated by $Y_1, \cdots, Y_n$ is transitive on X and hence X is biholomorphic to a compact quotient of a connected complex abelian group by a lattice. Consequently, X is a complex torus. The fundamental group of X is thus infinite, but this is a contradiction. $\square$

Corollary 4.5. *Let X be a compact complex manifold of algebraic dimension zero admitting a holomorphic Riemannian metric. Then the fundamental group of X is infinite.*

Proof. The holomorphic Riemannian metric induces a restriction of the structural group of the frame bundle $R^1(X)$ of X from $\mathrm{GL}(n, \mathbb{C})$ to $O(n, \mathbb{C})$. Up to a double cover we get a restriction to the structural group of $R^1(X)$ to $\mathrm{SO}(n, \mathbb{C})$. Hence, up to double cover of X, the canonical bundle K_X is holomorphically trivial and Proposition 4.4 completes the proof. $\square$

A more general result was proved in [DM]:

Theorem 4.6. *Let X be a compact complex manifold with algebraic dimension zero admitting a holomorphic affine connection. Then the fundamental group of X is infinite.*

4.3. **Algebraic dimension one.**

Theorem 4.7. *Let X be a compact complex manifold of algebraic dimension one admitting a holomorphic Riemannian metric g. Then the fundamental group of X is infinite.*

Proof. Assume by contradiction that X has finite fundamental group. Up to a finite unramified cover, we can assume that X is simply connected.

By Theorem 4.1 there exists a finite dimensional abelian Lie subalgebra $A \subset H^0(X, TX)$ preserving g and acting with generic orbits of dimension at least $n-1$, where n is the complex dimension of X. The case where A acts with an open orbit was settled in Section 4.2.

Let us consider now the case where the generic orbits of A are of dimension $n-1$. Since A is abelian, the isotropy of a point $x \in X$ lying in a generic orbit O acts trivially on $T_x O$

and hence on $T_x X$ (an orthogonal linear map which is identity on a hyperplane is trivial). Consequently, the isotropy of A at x is trivial and A must have dimension $n-1$. Let

$$Y_1, Y_2, \cdots, Y_{n-1} \in H^0(X, TX)$$

be a basis of A.

Denote also by g the symmetric bilinear form associated to the quadratic form g. The functions $g(Y_i, Y_j)$ are holomorphic and hence are constant on X. So

$$V \cdot g(Y_i, Y_j) = 0$$

for any local holomorphic vector field V, so

$$g(\nabla_V Y_i, Y_j) + g(Y_i, \nabla_V Y_j) = 0. \tag{4.1}$$

Since Y_i are Killing vector fields with respect to g, it follows that

$$\varphi_i : TX \longrightarrow TX, \ w \longmapsto \nabla_w Y_i$$

are pointwise g-skew-symmetric. Therefore, we have

$$g(\nabla_V Y_i, W) + g(\nabla_W Y_i, V) = 0 \tag{4.2}$$

for all locally defined holomorphic vector fields V, W. Combining (4.1) and (4.2),

$$0 = g(\nabla_V Y_i, Y_j) + g(Y_i, \nabla_V Y_j) = -g(\nabla_{Y_j} Y_i, V) - g(V, \nabla_{Y_i} Y_j) = -g(\nabla_{Y_j} Y_i + \nabla_{Y_i} Y_j, V),$$

so $\nabla_{Y_j} Y_i + \nabla_{Y_i} Y_j = 0$. On the other hand, the Levi-Civita connection ∇ is torsionfree and the Y_i commute, so $\nabla_{Y_j} Y_i = \nabla_{Y_i} Y_j$. Therefore, we have

$$\nabla_{Y_j} Y_i = 0 \tag{4.3}$$

for all $i, j \in \{1, \cdots, n-1\}$. Now combining (4.1), (4.2) and (4.3),

$$0 = g(\nabla_V Y_i, Y_j) + g(\nabla_V Y_j, Y_i) = g(\nabla_V Y_i, Y_j) - g(\nabla_{Y_i} Y_j, V) = g(\nabla_V Y_i, Y_j)$$

for any local holomorphic vector field V and $i, j \in \{1, \cdots, n-1\}$. So, the image $\varphi_i(TX)$ lies in the orthogonal part $\{Y_1, \cdots, Y_{n-1}\}^\perp$ for all $i \in \{1, \cdots, n-1\}$.

Fix a point $x \in X$ lying on a generic orbit of A. Then $\dim \varphi_i(T_x X) \leq 1$ is at most one as it lies in $\{Y_1(x), \cdots, Y_{n-1}(x)\}^\perp$. On the other hand, the rank of a skew-symmetric g-endomorphism is always even. It now follows that φ_i vanishes identically for all i.

Since $\varphi_i = 0$, the one-form ω_i dual to Y_i, defined by $v \longmapsto g(Y_i, v)$, is closed. Indeed, if V_1 and V_2 are local holomorphic vector fields then

$$d\omega_i(V_1, V_2) = V_1 \cdot \omega_i(V_2) - V_2 \cdot \omega_i(V_1) - \omega_i([V_1, V_2]) = g(\nabla_{V_1} Y_i, V_2) + g(Y_i, \nabla_{V_1} V_2)$$

$$-g(Y_i, \nabla_{V_2} V_1) - g(\nabla_{V_2} Y_i, V_1) - g(Y_i, [V_1, V_2]) = g(Y_i, \nabla_{V_1} V_2 - \nabla_{V_2} V_1 - [V_1, V_2])$$

$$= g(Y_i, T(V_1, V_2)) = g(Y_i, 0) = 0$$

where T is the torsion of the Levi-Civita ∇ connection (which is zero). The above computation shows that the dual form $v \longmapsto g(Y, v)$ is closed if and only if $v \longmapsto \nabla_v Y$ is pointwise g-symmetric. In particular, if Y is a Killing field with respect to g, the associated dual form $v \longmapsto g(Y, v)$ is closed if and only if Y is parallel.

Since X is simply connected, the closed form ω_i must be exact. This implies ω_i vanishes identically because there is no nonconstant holomorphic function on X. Hence Y_i vanishes identically, which is a contradiction. $\qquad\square$

4.4. Other applications. Theorem 3.2 has the following corollaries.

Corollary 4.8. *Let X be a compact Calabi-Yau manifold such that TX is polystable with respect to some Gauduchon metric on X. If X admits a rigid holomorphic geometric structure of affine type ϕ, then the fundamental group of X is infinite.*

Proof. From Theorem 3.2 we know that ϕ is locally homogeneous. Consider local holomorphic Killing vector fields $Y_1, \cdots, Y_n$ which span TX on an open subset in X.

Assume by contradiction that the fundamental group is finite. Then, up to a finite unramified cover, X is simply connected and Nomizu's extension result [No] proves that the Killing vector fields Y_i extend to global holomorphic vector fields $\widetilde{Y}_i \in H^0(X, TX)$. Moreover, Lemma 3.1 shows that the holomorphic tensor $\widetilde{Y}_1 \wedge \cdots \wedge \widetilde{Y}_n$ do not vanish on X. Hence, the $\{\widetilde{Y}_i\}$ trivialize the holomorphic tangent bundle of X. By a result of Wang [Wa], X is biholomorphic to a quotient of a complex Lie group G by a lattice. In particular, the fundamental group of X is infinite, which is a contradiction. $\qquad\square$

Corollary 4.9. *Let X be a compact complex manifold such that TX is polystable with respect to some Gauduchon metric on X. If X admits a holomorphic Riemannian metric g, then g is locally homogeneous and the fundamental group of X is infinite.*

Proof. As in the proof of Corollary 4.5, the manifold X admits a double cover with trivial canonical bundle. Hence, X is Calabi-Yau and Theorem 3.2 and Corollary 4.8 apply. $\qquad\square$

Corollary 4.10. *Let X be a compact complex manifold such that the canonical bundle K_X is of finite order and TX is polystable with respect to some Gauduchon metric on X. Then any holomorphic conformal structure and any holomorphic projective connection on X are locally homogeneous. In particular, if X admits a holomorphic conformal structure or a holomorphic projective connection then the fundamental group of X is infinite.*

Proof. Up to a finite cover, K_X is trivial. Then any holomorphic conformal structure admits a global representative which is a holomorphic Riemannian metric and Corollary 4.9 applies. Also, since K_X is trivial, any holomorphic projective connection admits a global representative which is a holomorphic affine connection (see [Gu, p. 96] and [KO, p. 78–79]) and Corollary 4.8 applies. $\qquad\square$

Theorem 4.11. *Let X be a compact complex manifold such that TX is simple. If X admits a holomorphic Riemannian metric, then the fundamental group of X is infinite.*

Proof. Since TX admits a holomorphic affine connection, if X is Moishezon, then Proposition 4.2 shows that X has a finite cover which is a torus. In particular the fundamental group of X is infinite.

Consider now the case where the algebraic dimension of X is not maximal.

Assume, by contradiction, that the fundamental group of X is finite. Up to a finite cover, we suppose that X is simply connected. Then Theorem 4.1 implies that there exists a nontrivial globally defined Killing field Y on X. Its dual ω, $v \longmapsto g(Y,v)$, is a holomorphic one-form on X. Lemma 3.3 implies that $d\omega = 0$. Since X is simply connected, ω must be exact and so $\omega = 0$ because there is no nonconstant holomorphic function on X: a contradiction (since Y is nontrivial). $\qquad\qquad\square$

4.5. **Nef tangent bundle.** Let X be a compact complex manifold such that the holomorphic tangent bundle TX is nef [DPS, p. 305, Definition 1.9] (see also [DPS, p. 299, Definition 1.2]). Assume that X admits a holomorphic Riemannian metric g. Then g identifies TX with $(TX)^*$, so TX is numerically flat [DPS, p. 311, Definition 1.17]. Hence $(TX)^{\otimes m} \otimes ((TX)^*)^l$ is numerically flat for all nonnegative integers m, l [DPS, p. 307, Proposition 1.14]. Hence any holomorphic section of $(TX)^{\otimes m} \otimes ((TX)^*)^l$ that vanishes at some point of X vanishes identically [DPS, p. 310, Proposition 1.16]. Therefore, any holomorphic geometric structure of affine type on X is locally homogeneous; see the proof of Theorem 3.2. Now we conclude that g is locally homogeneous and the fundamental group of X is infinite (see the proof of Corollary 4.9).

Acknowledgements

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