An error estimate for a finite-volume scheme for the Cahn-Hilliard equation with dynamic boundary conditions

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AN ERROR ESTIMATE FOR A FINITE-VOLUME SCHEME FOR THE CAHN-HILLIARD EQUATION WITH DYNAMIC BOUNDARY CONDITIONS

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Abstract. In the paper we consider a finite-volume approximation for the Cahn-Hilliard equation with dynamic boundary conditions. We prove an error estimate for the fully-discrete scheme. We also give numerical simulations which validate the theoretical result.

Key words. Cahn-Hilliard equation, dynamic boundary condition, finite-volume method, error estimate.

AMS subject classifications. 35K55, 65M08, 65M15, 76M12

1. Introduction.

1.1. The Cahn-Hilliard model. We consider the following Cahn-Hilliard equation with a dynamic boundary condition which describe the phase separation process of a binary mixture. Find the concentration of one of the two components $c : (0,T) \times \Omega \to \mathbb{R}$ and the chemical potential $\mu : (0,T) \times \Omega \to \mathbb{R}$ such that

$$
\begin{cases}
\partial_t c = \Gamma_b \Delta \mu; & \text{in } (0,T)\times\Omega; \quad (1.1a)\\
\mu = -\dfrac{3}{2}\varepsilon\sigma_b \Delta c + \dfrac{12}{\varepsilon}\sigma_b f_b'(c); & \text{in } (0,T)\times\Omega; \quad (1.1b)\\
\dfrac{\varepsilon^3}{64\Gamma_b\Gamma_s}\partial_t c_{|\Gamma} = \dfrac{3}{8}\varepsilon^2\sigma_b\sigma_s\Delta_\Gamma c_{|\Gamma} - 6\sigma_b f_s'(c_{|\Gamma}) - \dfrac{3}{2}\varepsilon\sigma_b\partial_n c; & \text{on } (0,T)\times\Gamma; \quad (1.1c)\\
\partial_n \mu = 0; & \text{on } (0,T)\times\Gamma; \quad (1.1d)\\
c(0,.) = c^0; & \text{in } \Omega; \quad (1.1e)
\end{cases}
$$

where Ω is a connected and bounded domain of $\mathbb{R}^2$, ∂_n is the normal derivative operator, $c_{|\Gamma}$ is the trace of c on the boundary $\Gamma = \partial\Omega$ and Δ_Γ is the Laplace-Beltrami operator.

The Cahn-Hilliard model is a diffuse interface model that means that the interfaces have a small but non-zero thickness $\varepsilon > 0$ (see Fig. 1.1a). Several physical parameters which describe the physical properties of the mixture components and the wall appear in the model: a diffusion coefficient called the mobility (supposed to be constant here) $\Gamma_b > 0$, the binary surface tension coefficient $\sigma_b > 0$ between the two components (which is the density interfacial energy), a capillarity coefficient $\sigma_s > 0$ and a relaxation coefficient $\Gamma_s > 0$. The bulk and surface potentials f_b (typically $f_b(c) = c^2(1-c)^2$, see Fig. 1.1b) and f_s respectively satisfy the following dissipativity assumption (useful to prove the discrete dissipation of the energy given in Proposition 4.3),

$$\liminf_{|c|\to\infty} f_b''(c) > 0 \quad \text{and} \quad \liminf_{|c|\to\infty} f_s''(c) > 0,$$

and the polynomial growth condition for f_b,

$$|f_b'(c)| \le C(1+|c|^p), \ \ \forall c \in \mathbb{R}, \tag{1.2}$$

for some $C > 0$ and $p \in [1, +\infty[$.

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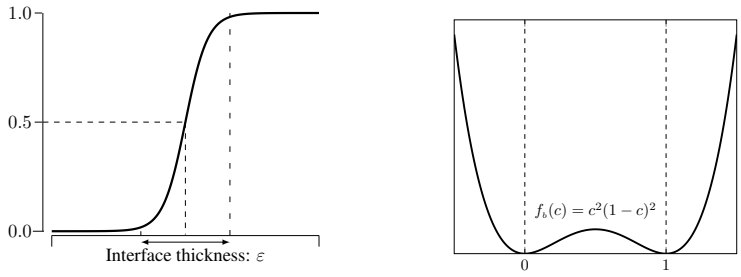


(a) Interface thickness (b) Bulk potential

Fig. 1.1: Definition of the interface thickness and double-well structure of f_b

The total Cahn-Hilliard energy is written as follows:

$$\mathcal{F}(c) = \underbrace{\int_\Omega \left(\frac{3}{4}\varepsilon\sigma_b \left|\nabla c\right|^2 + \frac{12}{\varepsilon}\sigma_b f_b(c)\right)}_{:=\mathcal{F}_b(c)} + \underbrace{\int_\Gamma \left(\frac{3}{16}\varepsilon^2\sigma_b\sigma_s \left|\nabla_\Gamma c_{|\Gamma}\right|^2 + 6\sigma_b f_s(c_{|\Gamma})\right)}_{:=\mathcal{F}_s(c)}, \tag{1.3}$$

and this energy is dissipated over time:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}(c(t,.)) = -\Gamma_b \int_\Omega \left|\nabla\mu(t,.)\right|^2 - \frac{\varepsilon^3}{64\Gamma_b\Gamma_s}\int_\Gamma \left|\partial_t c_{|\Gamma}(t,.)\right|^2, \quad t \in [0,T[. \tag{1.4}$$

We can remark that the bulk energy $\mathcal{F}_b$ is the energy associated with the Cahn-Hilliard equation with Neumann boundary conditions. Definition (1.3) of the total energy induces us to introduce the function spaces $H^1_\Gamma(\Omega) = \{u \in H^1(\Omega) : u_{|\Gamma} \in H^1(\Gamma)\}$ and $H^2_\Gamma(\Omega) = \{u \in H^2(\Omega) : u_{|\Gamma} \in H^2(\Gamma)\}$ and the corresponding norms, for $i \in \{1,2\}$:

$$\|u\|_{H^i_\Gamma(\Omega)} = \left(\|u\|^2_{H^i(\Omega)} + \|u_{|\Gamma}\|^2_{H^i(\Gamma)}\right)^{\frac{1}{2}}, \ \forall u \in H^i_\Gamma(\Omega).$$

In the analysis to follow, for the sake of simplicity, all the coefficients in problem (1.1) will be taken equal to one.

1.2. Former results and outline. In the past 30 years, the Cahn-Hilliard equation associated with the homogeneous Neumann boundary condition on the order parameter c has been extensively studied. Recently physicists [14, 15, 19] have introduced the dynamic boundary condition which allows to take into account the interaction between the components and the wall, especially the contact-line dynamics (see [17]). In the case of Neumann boundary conditions the numerical analysis with finite-difference and finite-element methods is well-understood (see [3, 4, 5, 6, 8, 7, 9, 11, 12, 13, 16, 21] and the references therein). However, to our knowledge, as far as the problem we are focusing there is no error estimate for the fully-discrete scheme on a curved domain. There exists finite-difference methods but without proof of stability or convergence (see [14, 15, 19]). A numerical analysis for the semi-discrete scheme using a spatial finite-element scheme is done in [2] in a slab with periodic boundary conditions in the longitudinal direction. In [20] the author propose a finite-volume scheme and prove the convergence of the numerical scheme towards a weak solution of problem (1.1) for a smooth non-polygonal domain. In order to prove an error estimate for this problem, we use the same finite-volume scheme that the one introduced in [20]. Therefore in Section 2 we present the finite-volume framework namely the finite-volume notation on a curved domain,

the associated discrete inner products and norms and the functional inequalities used in the paper. Then Section 3 is devoted to the presentation of the scheme. In Section 4 (and in the Appendix) we prove the main result of the paper: an error estimate for the fully-discrete scheme (Theorem 4.2). Finally in Section 5 we present a numerical error estimate for this model in accordance with the error estimate theorem proved in the previous section.

2. The finite-volume framework.

2.1. Mesh and notation. We recall here the main finite-volume notations used in the paper (see Fig. 2.1). The usual notation for a polygonal domain can be found for example in [10] and the notation associated to a curved domain in [20].

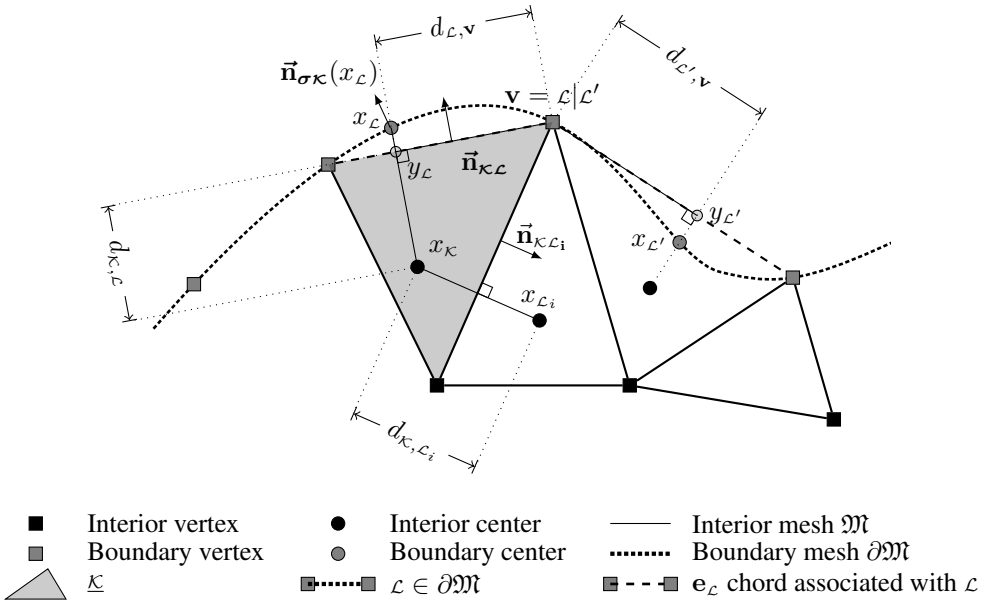


Fig. 2.1: Mesh $\mathcal{T}$ associated with Ω

An admissible mesh $\mathcal{T}$ of Ω is given by an interior mesh $\mathfrak{M}$ and a boundary mesh $\partial\mathfrak{M}$. The interior mesh $\mathfrak{M}$ is a set of disjoint open subsets of Ω, denoted by κ and called interior control volumes, which satisfy:

- $\overline{\Omega}=\cup_{\kappa\in\mathfrak{M}}\kappa$;
- if $\kappa,\mathcal{L}\in\mathfrak{M},\kappa\neq\mathcal{L}$, then $\kappa\cap\mathcal{L}=\emptyset$;
- if $\kappa,\mathcal{L}\in\mathfrak{M},\kappa\neq\mathcal{L}$ such that the dimension of $\bar{\kappa}\cap\bar{\mathcal{L}}$ is equal to 1, then $\bar{\kappa}\cap\bar{\mathcal{L}}$ is the edge of the mesh separating the control volumes κ and $\mathcal{L}$;
- if $\kappa\cap\Gamma$ contains a finite number of points, then κ is polygonal;
- for any $\kappa\in\mathfrak{M}$, we associate a point x_κ (referred to as the center of κ) such that if $\kappa,\mathcal{L}$ are two neighboring interior control volumes, the edge which separates κ and $\mathcal{L}$, which is denoted by $\sigma=\kappa|\mathcal{L}$, is orthogonal to the straight line going through x_κ and $x_\mathcal{L}$.

Let $\mathcal{E}$ be the set of edges of the interior mesh $\mathfrak{M}$. We decompose $\mathcal{E}$ into two disjoints subsets: the set of interior (flat) edges $\mathcal{E}_{int}=\{\sigma\in\mathcal{E}:\sigma\not\subset\Gamma\}$ and the set of exterior (curved) edges $\mathcal{E}_{ext}=\{\sigma\in\mathcal{E}:\sigma\subset\Gamma\}$. Similarly we use the notations $\mathcal{E}_\kappa^{int}$ and $\mathcal{E}_\kappa^{ext}$ for

the edges of a given control volume $\kappa \in \mathfrak{M}$. For any $\sigma \in \mathcal{E}$, we note m_σ its length. For each edge $\sigma \in \mathcal{E}$, we associate a diamond cell $\mathcal{D}$ defined as follows:

- $\mathcal{D} = \mathcal{D}_\sigma$ the quadrangle whose diagonals are the edge σ and the line segment $[x_\kappa, x_\mathcal{L}]$ if $\sigma \in \mathcal{E}_{int}$;
- $\mathcal{D} = \mathcal{D}_\sigma = \{tx + (1-t)x_\kappa, t \in [0,1], x \in \sigma\}$ if $\sigma \in \mathcal{E}_{ext} \cap \mathcal{E}_\kappa$.

We note $m_\mathcal{D}$ the Lebesgue measure of $\mathcal{D}$ and $\mathfrak{D}$ is the set of all diamond cells.

Since the domain Ω is not polygonal, we have to introduce an approximate domain $\underline{\Omega} = \cup_{\kappa \in \mathfrak{M}} \underline{\kappa}$ of Ω where

- $\underline{\kappa} = \kappa$ if $\mathcal{E}_\kappa \cap \mathcal{E}_{ext} = \emptyset$;
- $\underline{\kappa}$ is the polygon obtained by joining all the vertex of κ if $\mathcal{E}_\kappa \cap \mathcal{E}_{ext} \neq \emptyset$. We can notice that in this case κ may be not convex and that $\underline{\kappa}$ may be not included in Ω.

We denote by m_κ (resp. $m_{\underline{\kappa}}$) the Lebesgue measure of κ (resp. $\underline{\kappa}$) and by $m_\mathfrak{m} \in \mathbb{R}^\mathfrak{m}$ (resp. $m_{\underline{\mathfrak{m}}} \in \mathbb{R}^\mathfrak{m}$) the vector $(m_\kappa)_{\kappa \in \mathfrak{m}}$ (resp. $(m_{\underline{\kappa}})_{\kappa \in \mathfrak{m}}$). The Lebesgue measures of κ and $\underline{\kappa}$ are then related by the following relation.

PROPOSITION 2.1. *For any interior control volume $\kappa \in \mathfrak{M}$, we have*

$$m_{\underline{\kappa}} - m_\kappa = \mathcal{O}\left(\mathrm{diam}(\kappa)^3\right).$$

Therefore, there exists $C_1, C_2 > 0$ depending only on Γ such that

$$C_1 m_{\underline{\kappa}} \leq m_\kappa \leq C_2 m_{\underline{\kappa}}.$$

The boundary mesh $\partial\mathfrak{M}$ is equal to the set of exterior edges $\mathcal{E}_{ext}$. Thus, the exterior edges are also boundary control volumes. When we consider them as edges, we denote them by $\sigma \in \mathcal{E}_{ext}$, and, when we consider them as control volumes of the boundary mesh, we denote them by $\mathcal{L} \in \partial\mathfrak{M}$ (and its length by $m_\mathcal{L}$). The chord associated with $\mathcal{L}$ is then denoted by $\mathbf{e}_\mathcal{L}$ (and its length by $m_{\mathbf{e}_\mathcal{L}}$), and, the quantities $m_\mathcal{L}$ and $m_{\mathbf{e}_\mathcal{L}}$ are related by the following relation.

PROPOSITION 2.2. *For any boundary control volume $\mathcal{L} \in \partial\mathfrak{M}$, we have*

$$m_{\mathbf{e}_\mathcal{L}} - m_\mathcal{L} = \mathcal{O}\left(m_\mathcal{L}^3\right).$$

In particular, there exists $C_3 > 0$ independent of $\mathrm{size}(\mathcal{T})$ *such that*

$$m_{\mathbf{e}_\mathcal{L}} \leq m_\mathcal{L} \leq C_3 m_{\mathbf{e}_\mathcal{L}}.$$

Let $m_{\partial\mathfrak{m}}$ (resp. $m_{\underline{\partial\mathfrak{m}}}$) be the vector $(m_\mathcal{L})_{\mathcal{L} \in \partial\mathfrak{m}}$ (resp. $(m_{\mathbf{e}_\mathcal{L}})_{\mathcal{L} \in \partial\mathfrak{m}}$). We set $m_\mathcal{T} = (m_\mathfrak{m}, m_{\partial\mathfrak{m}}) \in \mathbb{R}^\mathcal{T}$ (resp. $m_{\underline{\mathcal{T}}} = (m_{\underline{\mathfrak{m}}}, m_{\underline{\partial\mathfrak{m}}}) \in \mathbb{R}^\mathcal{T}$).

For any control volume $\mathcal{L} \in \partial\mathfrak{M}$, we associate a point $x_\mathcal{L} \in \mathcal{L}$, called the center of the control volume. For any boundary control volume $\mathcal{L} \in \partial\mathfrak{M}$, let $\kappa \in \mathfrak{M}$ the interior control volume such that $\mathcal{L} = \sigma$ is an edge of κ. Then, we impose that the straight line going through x_κ and $x_\mathcal{L}$ is orthogonal to the chord $\mathbf{e}_\mathcal{L}$ associated with $\mathcal{L}$. Moreover, we define $y_\mathcal{L}$ as the orthogonal projection of $x_\mathcal{L}$ on the chord $\mathbf{e}_\mathcal{L}$.

Let $\mathcal{V}$ be the set of vertices of the mesh $\mathfrak{M}$ which belongs to Γ. We denote by $\mathbf{v} = \mathcal{L}|\mathcal{L}'$ the vertex which separates the boundary control volumes $\mathcal{L}$ and $\mathcal{L}'$. For any $\mathbf{v} = \mathcal{L}|\mathcal{L}'$, let $d_{\mathcal{L},\mathbf{v}} = d(x_\mathcal{L}, \mathbf{v})$ be the approximation of the length $m_{\gamma_{\mathcal{L}\mathbf{v}}}$ of the arc $\gamma_{\mathcal{L}\mathbf{v}}$ included in the boundary control volume $\mathcal{L}$ whose ends are $x_\mathcal{L}$ and $\mathbf{v}$. The measure of the arc $\gamma_{\mathcal{L}\mathcal{L}'}$ (which is the arc whose ends are $x_\mathcal{L}$ and $x_{\mathcal{L}'}$ and passing through the vertex $\mathbf{v}$) is then approximate by the distance $d_{\mathcal{L},\mathcal{L}'} = d_{\mathcal{L},\mathbf{v}} + d_{\mathcal{L}',\mathbf{v}}$.

If $\sigma = \kappa|\mathcal{L} \in \mathcal{E}_{int}$ is an interior edge, we note $d_{\kappa,\mathcal{L}}$ the distance between the centers x_κ and $x_\mathcal{L}$; and $\vec{\mathbf{n}}_{\kappa\mathcal{L}}$ the normal vector to σ going from κ to $\mathcal{L}$. If $\sigma = \mathcal{L} \in \mathcal{E}_{ext} \cap \mathcal{E}_\kappa$ is an

exterior edge, we note $d_{\kappa,\mathcal{L}}$ the distance between the center x_κ and the point $y_\mathcal{L}$; and $\vec{\mathbf{n}}_{\kappa\mathcal{L}}$ the normal vector to the chord $\mathbf{e}_\mathcal{L}$ outward to κ. In this case, the distance $d(x_\kappa, x_\mathcal{L})$ and its approximation $d_{\kappa,\mathcal{L}}$ satisfy the following relation.

PROPOSITION 2.3. *Let us consider a boundary control volume $\mathcal{L} \in \partial\mathfrak{M}$ such that $\mathcal{L} = \sigma \in \mathcal{E}_\kappa$ where $\kappa \in \mathfrak{M}$ is an interior control volume, then we have:*

$$d(x_\mathcal{L}, y_\mathcal{L}) = |d(x_\kappa, x_\mathcal{L}) - d_{\kappa,\mathcal{L}}| = \mathcal{O}(m_\mathcal{L} m_{\gamma_{\mathcal{L}\mathbf{v}}}).$$

Let $\mathrm{size}(\mathcal{T})$ be the maximum of the diameters of the interior control volumes κ. We introduce a positive number $\mathrm{reg}(\mathcal{T})$ that measures the regularity of a given mesh and is useful to perform the numerical analysis of finite-volume schemes:

$$\mathrm{reg}(\mathcal{T}) := \max\left(\max_{\kappa\in\mathfrak{M}} \frac{\mathrm{diam}(\kappa)}{\sqrt{m_\kappa}}, \max_{\substack{\kappa\in\mathfrak{M}\\ \sigma\in\mathcal{E}_\kappa}} \frac{\mathrm{diam}(\kappa)}{d(x_\kappa,\sigma)}, \max_{\substack{\kappa\in\mathfrak{M}\\ \sigma\in\mathcal{E}_\kappa}} \frac{d_{\kappa,\mathcal{L}}}{d(x_\kappa,\sigma)}\right). \tag{2.1}$$

The number $\mathrm{reg}(\mathcal{T})$ should be uniformly bounded when $\mathrm{size}(\mathcal{T}) \to 0$.

For the needs of the proof of error estimate theorem, we also define a family of quasi-uniform meshes.

DEFINITION 2.4 (Quasi-uniform mesh family of Ω).

Let $\mathcal{T}$ be a family of meshes of Ω, we define the number $\mathrm{reg}_{\mathrm{unif}}(\mathcal{T})$ as follows,

$$\mathrm{reg}_{\mathrm{unif}}(\mathcal{T}) \overset{\mathrm{def}}{=} \sup\left(\mathrm{reg}(\mathcal{T}), \sup_{\kappa\in\mathfrak{M}} \frac{\mathrm{size}(\mathcal{T})^2}{m_\kappa}\right).$$

We say that a mesh family $\left(\mathcal{T}^{(i)}\right)_{i\in\mathbb{N}}$ is quasi-uniform if $\mathrm{reg}_{\mathrm{unif}}(\mathcal{T}^{(i)})$ is bounded.

2.2. Inner-products and norms. Since the domain Ω is not polygonal, we introduce a L^2-inner product on the domain Ω (and on its boundary $\Gamma = \partial\Omega$) but also on the approximate polygonal domain $\underline{\Omega}$ (and on its boundary $\partial\underline{\Omega}$).

For the space discretization, the finite-volume method associates an unknown value $u_\kappa \in \mathbb{R}$ (resp. $u_\mathcal{L} \in \mathbb{R}$) to each interior (resp. boundary) control volume $\kappa \in \mathfrak{M}$ (resp. $\mathcal{L} \in \partial\mathfrak{M}$). Thus we note

$$u_\mathcal{T} = (u_\mathfrak{M}, u_{\partial\mathfrak{M}}) = ((u_\kappa)_{\kappa\in\mathfrak{M}}, (u_\mathcal{L})_{\mathcal{L}\in\partial\mathfrak{M}}) \in \mathbb{R}^\mathcal{T} = \mathbb{R}^\mathfrak{M} \times \mathbb{R}^{\partial\mathfrak{M}}.$$

DEFINITION 2.5 (Discrete L^2-inner products).

- *We define the inner product $(.,.)_{L^2(\Omega)}$ on $L^2(\Omega)$ and the inner product $(.,.)_\mathfrak{M}$ on $L^2(\underline{\Omega})$ as follows. For any $u_\mathfrak{M}, v_\mathfrak{M} \in \mathbb{R}^\mathfrak{M}$, we have*

$$(u_\mathfrak{M}, v_\mathfrak{M})_{L^2(\Omega)} = \sum_{\kappa\in\mathfrak{M}} m_\kappa u_\kappa v_\kappa \quad \text{and} \quad (u_\mathfrak{M}, v_\mathfrak{M})_\mathfrak{M} = \sum_{\kappa\in\mathfrak{M}} m_{\underline{\kappa}} u_\kappa v_\kappa.$$

  We denote by $\|.\|_{L^2(\Omega)}$ and $\|.\|_{0,\mathfrak{M}}$ the associated norms.
- *We define the inner product $(.,.)_{L^2(\Gamma)}$ on $L^2(\Gamma)$ and the inner product $(.,.)_{\partial\mathfrak{M}}$ on $L^2(\partial\underline{\Omega})$ as follows. For any $u_{\partial\mathfrak{M}}, v_{\partial\mathfrak{M}} \in \mathbb{R}^{\partial\mathfrak{M}}$, we have*

$$(u_{\partial\mathfrak{M}}, v_{\partial\mathfrak{M}})_{L^2(\Gamma)} = \sum_{\mathcal{L}\in\partial\mathfrak{M}} m_\mathcal{L} u_\mathcal{L} v_\mathcal{L} \quad \text{and} \quad (u_{\partial\mathfrak{M}}, v_{\partial\mathfrak{M}})_{\partial\mathfrak{M}} = \sum_{\mathcal{L}\in\partial\mathfrak{M}} m_{\mathbf{e}_\mathcal{L}} u_\mathcal{L} v_\mathcal{L}.$$

  We denote by $\|.\|_{L^2(\Gamma)}$ and $\|.\|_{0,\partial\mathfrak{M}}$ the associated norms.

Owing to Proposition 2.1 (resp. Proposition 2.2), the norms $\|.\|_{L^2(\Omega)}$ and $\|.\|_{0,\mathfrak{M}}$ (resp. $\|.\|_{L^2(\Gamma)}$ and $\|.\|_{0,\partial\mathfrak{M}}$) are equivalent (with constants independent on the mesh size).

We also define semi-inner products in $\mathbb{R}^{\mathcal{T}}$ and $\mathbb{R}^{\partial\mathfrak{M}}$.

DEFINITION 2.6 (Discrete H^1-semidefinite inner products). *We define the H^1-semidefinite inner product in $\mathbb{R}^{\mathcal{T}}$ as follows. For any $u_{\mathcal{T}}, v_{\mathcal{T}} \in \mathbb{R}^{\mathcal{T}}$,*

$$[\![u_{\mathcal{T}}, v_{\mathcal{T}}]\!]_{1,\mathcal{T}} = \sum_{\sigma=\mathcal{K}|\mathcal{L}\in\mathcal{E}_{int}} \frac{m_\sigma}{d_{\mathcal{K},\mathcal{L}}}(u_{\mathcal{K}} - u_{\mathcal{L}})(v_{\mathcal{K}} - v_{\mathcal{L}}) + \sum_{\sigma=\mathcal{L}\in\mathcal{E}_{ext}} \frac{m_{\mathbf{e}_{\mathcal{L}}}}{d_{\mathcal{K},\mathcal{L}}}(u_{\mathcal{K}} - u_{\mathcal{L}})(v_{\mathcal{K}} - v_{\mathcal{L}}).$$

We also define the H^1-semidefinite inner product in $\mathbb{R}^{\partial\mathfrak{M}}$ as follows.

$$[\![u_{\partial\mathfrak{M}}, v_{\partial\mathfrak{M}}]\!]_{1,\partial\mathfrak{M}} = \sum_{\mathbf{v}=\mathcal{L}|\mathcal{L}'\in\mathcal{V}} \frac{1}{d_{\mathcal{L},\mathcal{L}'}}(u_{\mathcal{L}} - u_{\mathcal{L}'})(v_{\mathcal{L}} - v_{\mathcal{L}'}),\ \forall u_{\partial\mathfrak{M}}, v_{\partial\mathfrak{M}} \in \mathbb{R}^{\partial\mathfrak{M}}.$$

We denote by $|.|_{1,\mathcal{T}}$ and $|.|_{1,\partial\mathfrak{M}}$ the associated seminorms. Moreover, we define the H^1-norms in $\mathbb{R}^{\mathcal{T}}$ and $\mathbb{R}^{\partial\mathfrak{M}}$ as follows:

$$\|u_{\mathcal{T}}\|_{1,\mathcal{T}} = \left(\|u_{\mathcal{T}}\|_{0,\mathfrak{M}}^2 + |u_{\mathcal{T}}|_{1,\mathcal{T}}^2\right)^{\frac{1}{2}},\ \forall u_{\mathcal{T}} \in \mathbb{R}^{\mathcal{T}},$$
$$\|u_{\partial\mathfrak{M}}\|_{1,\partial\mathfrak{M}} = \left(\|u_{\partial\mathfrak{M}}\|_{0,\partial\mathfrak{M}}^2 + |u_{\partial\mathfrak{M}}|_{1,\partial\mathfrak{M}}^2\right)^{\frac{1}{2}},\ \forall u_{\partial\mathfrak{M}} \in \mathbb{R}^{\partial\mathfrak{M}}.$$

2.3. Functional inequalities. We give here some functional inequalities that we will use all along the paper. The proof are available in the literature and so we do not give them here. We consider an admissible mesh $\mathcal{T}$ of Ω.

LEMMA 2.7 (Discrete mean Poincaré inequality, [10, Lemma 3.7]). *There exists $C_4 > 0$ depending only on Ω such that for any $u_{\mathcal{T}} \in \mathbb{R}^{\mathcal{T}}$,*

$$\|u_{\mathfrak{M}} - m_\Omega(u_{\mathfrak{M}})\|_{L^2(\Omega)} \le C_4 |u_{\mathcal{T}}|_{1,\mathcal{T}} \text{ with } m_\Omega(u_{\mathfrak{M}}) = \frac{1}{|\Omega|}\sum_{\mathcal{K}\in\mathfrak{M}} m_{\mathcal{K}} u_{\mathcal{K}},$$
$$\left\|u_{\mathfrak{M}} - m_{\underline{\Omega}}(u_{\mathfrak{M}})\right\|_{0,\mathfrak{M}} \le C_4 |u_{\mathcal{T}}|_{1,\mathcal{T}} \text{ with } m_{\underline{\Omega}}(u_{\mathfrak{M}}) = \frac{1}{|\underline{\Omega}|}\sum_{\mathcal{K}\in\mathfrak{M}} m_{\underline{\mathcal{K}}} u_{\mathcal{K}}.$$

Thus, we also have:

$$\|u_{\mathfrak{M}}\|_{L^2(\Omega)} \le C_4 |u_{\mathcal{T}}|_{1,\mathcal{T}} + |\Omega|^{\frac{1}{2}} |m_\Omega(u_{\mathfrak{M}})|,$$
$$\|u_{\mathfrak{M}}\|_{0,\mathfrak{M}} \le C_4 |u_{\mathcal{T}}|_{1,\mathcal{T}} + |\underline{\Omega}|^{\frac{1}{2}} |m_{\underline{\Omega}}(u_{\mathfrak{M}})|. \tag{2.2}$$

LEMMA 2.8 (Poincaré-Sobolev inequality, [1, Theorem 3.2]). *Let $1 \le q < +\infty$, there exists $C_5 > 0$ depending only on q, Ω and $\mathrm{reg}(\mathcal{T})$ such that*

$$\|u_{\mathfrak{M}}\|_{L^q(\Omega)} := \left(\sum_{\mathcal{K}\in\mathfrak{M}} m_{\mathcal{K}} |u_{\mathcal{K}}|^q\right)^{\frac{1}{q}} \le C_5 \|u_{\mathcal{T}}\|_{1,\mathcal{T}}\ , \quad \forall u_{\mathcal{T}} \in \mathbb{R}^{\mathcal{T}}. \tag{2.3}$$

We can also easily prove the following Sobolev inequality on the one dimensional manifold Γ.

LEMMA 2.9. *There exists $C_6 > 0$ depending only on Γ such that for any $u_{\partial\mathfrak{M}} \in \mathbb{R}^{\partial\mathfrak{M}}$,*

$$\sup_{\mathcal{L}\in\partial\mathfrak{M}} |u_{\mathcal{L}}| \le C_6 \|u_{\partial\mathfrak{M}}\|_{1,\partial\mathfrak{M}}.$$

The proof of the following lemma for a polygonal domain is done in [10, Lemma 3.10] but it can be adapt in our case.

LEMMA 2.10 (Trace inequality). *There exists $C_7 > 0$ depending only on Ω such that for any $u_{\mathcal{T}} \in \mathbb{R}^{\mathcal{T}}$ we have,*

$$\|u_{\partial\mathfrak{M}}\|_{L^2(\Gamma)} \leq C_7 \left(|u_{\mathcal{T}}|_{1,\mathcal{T}} + \|u_{\mathfrak{M}}\|_{L^2(\Omega)} \right).$$

We can also remark that for a quasi-uniform mesh family $\left(\mathcal{T}^{(i)}\right)_{i\in\mathbb{N}}$ (see Definition 2.4) and for any $q \geq 1$, there exists a uniform constant $C_8 > 0$ (depending on q and $\mathrm{reg}_{\mathrm{unif}}(\mathcal{T})$) such that,

$$\sup_{\mathcal{K}\in\mathfrak{M}} |u_{\mathcal{K}}| \leq \frac{C_8}{\mathrm{size}(\mathcal{T}^{(i)})^{2/q}} \|u_{\mathfrak{M}}\|_{L^q(\Omega)}, \quad \forall u_{\mathfrak{M}} \in \mathbb{R}^{\mathfrak{M}}. \tag{2.4}$$

3. The finite-volume scheme . The section is devoted to the presentation of the numerical scheme. We refer the reader to [20] for the proofs of the energy stability, the existence of a discrete solution and the convergence analysis.

For the time discretization, let $N \in \mathbb{N}$ and $\Delta t = \frac{T}{N}$ be the time step. For any $n \in \{0, \cdots, N\}$ we define $t^n = n\Delta t$. Then, at time t^n, the unknowns are denoted by:

$$c_{\mathcal{T}}^n = \begin{pmatrix} (c_{\mathcal{K}}^n)_{\mathcal{K}\in\mathfrak{M}} \\ (c_{\mathcal{L}}^n)_{\mathcal{L}\in\partial\mathfrak{M}} \end{pmatrix} \text{ and } \mu_{\mathcal{T}}^n = \begin{pmatrix} (\mu_{\mathcal{K}}^n)_{\mathcal{K}\in\mathfrak{M}} \\ (\mu_{\mathcal{L}}^n)_{\mathcal{L}\in\partial\mathfrak{M}} \end{pmatrix}.$$

Since μ is associated with the homogeneous Neumann boundary condition, for any $\mathcal{L} \in \partial\mathfrak{M}$ we have $\mu_{\mathcal{L}}^n = \mu_{\mathcal{K}}^n$ where $\mathcal{K} \in \mathfrak{M}$ is the interior control volume such that $\mathcal{L} \subset \partial\mathcal{K}$.

To obtain the finite-volume approximation of problem (1.1) we integrate the continuous equations (1.1a) and (1.1b) on all interior control volumes $\mathcal{K} \in \mathfrak{M}$ and we use a consistent two-point flux approximation for the Laplace operators (associated with the Neumann boundary condition for μ). Then we integrate dynamic boundary condition (1.1c) on all boundary control volumes $\mathcal{L} \in \partial\mathfrak{M}$ and we use a consistent two-point flux approximation for the Laplace-Beltrami operator.

As regards the discretization of nonlinear terms f_b' and f_s' (denoted by d^{f_b} and d^{f_s} respectively) we use two different discretizations (see Definition 3.1): the classical implicit discretization and a semi-implicit discretization which enables us to obtain an energy estimate unconditionally stable.

The problem is then written as follows.

PROBLEM 3.1. *For a given $c_{\mathcal{T}}^n \in \mathbb{R}^{\mathcal{T}}$, find $(c_{\mathcal{T}}^{n+1}, \mu_{\mathcal{T}}^{n+1}) \in \mathbb{R}^{\mathcal{T}} \times \mathbb{R}^{\mathcal{T}}$ such that for any $\mathcal{K} \in \mathfrak{M}$, $\mathcal{L} \in \partial\mathfrak{M}$,*

$$\begin{cases} m_{\underline{\mathcal{K}}} \dfrac{c_{\mathcal{K}}^{n+1} - c_{\mathcal{K}}^n}{\Delta t} = -\displaystyle\sum_{\sigma\in\mathcal{E}_{\mathcal{K}}^{int}} \frac{m_\sigma}{d_{\mathcal{K},\mathcal{L}}} \left(\mu_{\mathcal{K}}^{n+1} - \mu_{\mathcal{L}}^{n+1}\right); & \text{(3.1a)} \\ m_{\underline{\mathcal{K}}} \mu_{\mathcal{K}}^{n+1} = \displaystyle\sum_{\sigma\in\mathcal{E}_{\mathcal{K}}^{int}} \frac{m_\sigma}{d_{\mathcal{K},\mathcal{L}}} \left(c_{\mathcal{K}}^{n+1} - c_{\mathcal{L}}^{n+1}\right) + \sum_{\sigma\in\mathcal{E}_{\mathcal{K}}^{ext}} \frac{m_{\mathbf{e}_\mathcal{L}}}{d_{\mathcal{K},\mathcal{L}}} \left(c_{\mathcal{K}}^{n+1} - c_{\mathcal{L}}^{n+1}\right) & \text{(3.1b)} \\ \qquad + m_{\underline{\mathcal{K}}} d^{f_b}(c_{\mathcal{K}}^n, c_{\mathcal{K}}^{n+1}); \\ m_{\mathbf{e}_\mathcal{L}} \dfrac{c_{\mathcal{L}}^{n+1} - c_{\mathcal{L}}^n}{\Delta t} = -\displaystyle\sum_{\mathbf{v}\in\mathcal{V}_\mathcal{L}} \frac{\left(c_{\mathcal{L}}^{n+1} - c_{\mathcal{L}'}^{n+1}\right)}{d_{\mathcal{L},\mathcal{L}'}} - m_{\mathbf{e}_\mathcal{L}} d^{f_s}(c_{\mathcal{L}}^n, c_{\mathcal{L}}^{n+1}) & \text{(3.1c)} \\ \qquad - \dfrac{m_{\mathbf{e}_\mathcal{L}}}{d_{\mathcal{K},\mathcal{L}}} \left(c_{\mathcal{L}}^{n+1} - c_{\mathcal{K}}^{n+1}\right). \end{cases}$$

DEFINITION 3.1 (Discretization of nonlinear terms). *The implicit discretization is define as follows: for any $\mathcal{K} \in \mathfrak{M}$, $\mathcal{L} \in \partial\mathfrak{M}$,*

$$d^{f_b}(c_{\mathcal{K}}^n, c_{\mathcal{K}}^{n+1}) = f_b'(c_{\mathcal{K}}^{n+1}) \text{ and } d^{f_s}(c_{\mathcal{L}}^n, c_{\mathcal{L}}^{n+1}) = f_s'(c_{\mathcal{L}}^{n+1}).$$

As regards the semi-implicit discretization, for $* \in \{b, s\}$ we note

$$d^{f_*}(x,y) = \frac{f_*(x) - f_*(y)}{x-y}, \forall x, y, x \neq y \text{ and } d^{f_*}(x,x) = f_*'(x), \forall x.$$

We remark that in practice we use a polynomial function for the potential f_* and that $d^{f_*}(x,y)$ is a polynomial function in the variables x, y. Thus, from a computational point of view, we do not have numerical instability when x is too close to y.

PROPOSITION 3.2. *For both discretizations the discrete energy is dissipated as follows: if $c_{\mathcal{T}}^n$ is given and $(c_{\mathcal{T}}^{n+1}, \mu_{\mathcal{T}}^{n+1})$ is solution to Problem 3.1, then there exists $C_9 > 0$ independent of Δt and $\mathcal{T}$ such that,*

$$\mathcal{F}_{\mathcal{T}}(c_{\mathcal{T}}^{n+1}) - \mathcal{F}_{\mathcal{T}}(c_{\mathcal{T}}^{n}) + C_9\Big(\Delta t \left|\mu_{\mathcal{T}}^{n+1}\right|_{1,\mathcal{T}}^2 + \frac{1}{\Delta t}\left\|c_{\partial\mathfrak{M}}^{n+1} - c_{\partial\mathfrak{M}}^{n}\right\|_{0,\partial\mathfrak{M}}^2 + \frac{1}{2}\left|c_{\mathcal{T}}^{n+1} - c_{\mathcal{T}}^{n}\right|_{1,\mathcal{T}}^2 + \frac{1}{2}\left|c_{\partial\mathfrak{M}}^{n+1} - c_{\partial\mathfrak{M}}^{n}\right|_{1,\partial\mathfrak{M}}^2\Big) \leq 0,$$

with a condition $\Delta t \leq \Delta t_0$ (with Δt_0 depending on the parameters of the equation) for the implicit discretization.

We note that summing equation (3.1a) for all $\kappa \in \mathfrak{M}$ we have the volume conservation at the discrete level,

$$|\underline{\Omega}| m_{\underline{\Omega}}\left(c_{\mathfrak{M}}^n\right) := \sum_{\kappa\in\mathfrak{M}} m_{\underline{\kappa}} c_{\kappa}^n = \sum_{\kappa\in\mathfrak{M}} m_{\underline{\kappa}} c_{\kappa}^0 := |\underline{\Omega}| m_{\underline{\Omega}}\left(c_{\mathfrak{M}}^0\right), \ \forall n \in \{1, \ldots, N\}. \tag{3.2}$$

4. Error estimate for the fully-discrete scheme. We can now enter in the heart of the matter. The most delicate point in the proof of the error estimate (Theorem 4.2) comes from the nonlinear term $f_b'(c)$ at which we have to pay a special attention. For this, we draw on methods described in [8, 18] for the Neumann boundary condition and a finite-element approximation. However some supplementary difficulties are added in our case. First, the finite-volume framework complicates the study of this term. Indeed, when we use a conform finite-element method we work (for the space discretization at least) on H^1-conform spaces. That is not the case when we use a finite-volume method where we use discrete spaces. Moreover, we use here two different discretizations for the nonlinear term f_b' (see Definition 3.1). The second one, namely the semi-implicit discrization, is more difficult to study that complicates again the proof of Theorem 4.2.

4.1. Main result. In the section we present the error estimate (Theorem 4.2) between the center-value projection (see Definition 4.1) of the exact solution and the solution to Problem 3.1.

DEFINITION 4.1 (Center-value projection).

The center-value projection $\mathbb{P}_{\mathcal{T}}^c : \mathcal{C}^0(\overline{\Omega}) \to \mathbb{R}^{\mathcal{T}}$ is defined as follows. For any $u \in \mathcal{C}^0(\overline{\Omega})$, we set $\mathbb{P}_{\mathcal{T}}^c u = (\mathbb{P}_{\mathfrak{M}}^c u, \mathbb{P}_{\partial\mathfrak{M}}^c u)$ with

$$\mathbb{P}_{\mathfrak{M}}^c u = \left(\mathbb{P}_{\kappa}^c u\right)_{\kappa\in\mathfrak{M}} = \left(u(x_\kappa)\right)_{\kappa\in\mathfrak{M}} \text{ and } \mathbb{P}_{\partial\mathfrak{M}}^c u = \left(\mathbb{P}_{\mathcal{L}}^c u\right)_{\mathcal{L}\in\partial\mathfrak{M}} = \left(u_{|\Gamma}(x_{\mathcal{L}})\right)_{\mathcal{L}\in\partial\mathfrak{M}}.$$

THEOREM 4.2 (Error estimate). *Let $c^0 \in \mathcal{C}^2(\overline{\Omega})$ and $(c, \mu) \in \mathcal{C}^3([0,T] \times \overline{\Omega}) \times \mathcal{C}^2([0,T] \times \overline{\Omega})$ be a solution to the Cahn-Hilliard equation (1.1). Let $M > 0$ be such that $\|c\|_{L^\infty(0,T;L^\infty(\Omega))} \leq M$ and $M' > M$. Then setting $c_{\mathcal{T}}^0 = \mathbb{P}_{\mathcal{T}}^c c^0$, for any solution $(c_{\mathcal{T}}^n, \mu_{\mathcal{T}}^n)$ to Problem 3.1 satisfying*

$$\sup\left(\sup_{\kappa\in\mathfrak{M}} |c_\kappa^n|, \sup_{\mathcal{L}\in\partial\mathfrak{M}} |c_{\mathcal{L}}^n|\right) \leq M', \quad \forall n \in [\![0, N]\!]. \tag{4.1}$$

There exists $C_{10} > 0$ (depending on M and M') such that the following estimate holds (with $\Delta t \leq \Delta t^e$),

$$\sup_{n\in[\![0,N]\!]} \|\mathbb{P}^c_{\mathcal{T}} c(t^n) - c^n_{\mathcal{T}}\|_{1,\mathcal{T}} + \sup_{n\in[\![0,N]\!]} \|\mathbb{P}^c_{\partial\mathfrak{M}} c(t^n) - c^n_{\partial\mathfrak{M}}\|_{1,\partial\mathfrak{M}} \leq C_{10}(\Delta t + \mathrm{size}(\mathcal{T})). \tag{4.2}$$

With a view to the proof we need some regularity on the potentials, indeed they have to be Lipschitz continuous functions. With that in mind we will introduce truncated Lipschitz continuous functions.

The exact solution c of continuous problem (1.1) is supposed to belong to $\mathcal{C}^3([0,T]\times\overline{\Omega})$. Thus, there exists $M > 0$ such that $\|c\|_{L^\infty(0,T;L^\infty(\Omega))} \leq M$. Let $M' > M$, we choose truncated potentials $\widetilde{f_b}$ and $\widetilde{f_s}$ of the initial potentials f_b and f_s which satisfy

$$\widetilde{f_b} = f_b \text{ and } \widetilde{f_s} = f_s \text{ on } [-M', M']$$

and which are constants at the infinity. These truncated functions, and all their derivatives, are Lipschitz continuous. Moreover, the definition of the semi-implicit discretization of nonlinear terms implies,

$$d^{f_b}(x,y) = \int_0^1 f_b'(x + s(y-x))\mathrm{d}s \text{ and } d^{f_s}(x,y) = \int_0^1 f_s'(x + s(y-x))\mathrm{d}s.$$

Thus the semi-implicit discretization of $\widetilde{f_b'}$ (resp. $\widetilde{f_s'}$) coincides with d^{f_b} (resp. d^{f_s}) on $[-M', M']^2$ and is Lipschitz continuous.

Therefore, we can achieve the proof of Theorem 4.2 with the truncated potentials instead of the initial potentials (and so with $c^n_{\mathcal{T}}$ a solution to discrete Problem 3.1 but with the truncated potentials). We obtain estimate (4.2) for this problem with constant C_{10} depending on M'. Thus for a mesh belonging to a quasi-uniform mesh family, gathering estimate (2.4), Lemma 2.8 and Theorem 4.2 we deduce,

$$\sup_{\mathcal{K}\in\mathfrak{M}} |\mathbb{P}^c_{\mathcal{K}} c(t^n) - c^n_{\mathcal{K}}| \leq C_5 C_{10} C_8 \left(\frac{\Delta t}{\mathrm{size}(\mathcal{T})^{2/q}} + \mathrm{size}(\mathcal{T})^{1-2/q} \right).$$

Moreover, thanks to Lemma 2.9 and Theorem 4.2 we also have:

$$\sup_{\mathcal{L}\in\partial\mathfrak{M}} |\mathbb{P}^c_{\mathcal{L}} c(t^n) - c^n_{\mathcal{L}}| \leq C_6 C_{10}(\Delta t + \mathrm{size}(\mathcal{T})).$$

Thus, if Δt and $\mathrm{size}(\mathcal{T})$ tend to 0 with $\Delta t \leq C\mathrm{size}(\mathcal{T})^\alpha$ for a certain $\alpha > 0$, these estimates allow us to affirm that bound (4.1) is satisfied for any approximate solution. Moreover, since the functions f_b (resp. f_s) and $\widetilde{f_b}$ (resp. $\widetilde{f_s}$) coincide on $[-M', M']$, if $c^n_{\mathcal{T}}$ is solution to discrete Problem 3.1 with the truncated potentials, it is also with the initial potentials (and reciprocally).

In conclusion, for a quasi-uniform mesh family, if Δt and $\mathrm{size}(\mathcal{T})$ are related by the relation $\Delta t \leq \mathrm{size}(\mathcal{T})^\alpha$ (for an arbitrary value $\alpha > 0$), then assumption (4.1) is still satisfied for Δt and $\mathrm{size}(\mathcal{T})$ small enough and so there exists a solution $c^n_{\mathcal{T}}$ to discrete Problem 3.1 which satisfies it (even if it can be exist a solution to Problem 3.1 which does not satisfy bound (4.1)). Thus we can carry out the proof of Theorem 4.2 with the truncated functions which satisfy all the necessary regularity assumptions. For the sake of simplicity we will omit the tilde sign in the sequel.

REMARK 4.1. *From a computational point of view, we can check that assumption* (4.1) *holds when we use the potentials f_b and f_s.*

In order to prove the error estimate theorem we have to use *a priori* bounds on the discrete solution (obtained thanks to the discrete energy estimate given in Proposition 3.2).

PROPOSITION 4.3 (Bounds of the discrete solutions, [20, Proposition 4.4]). *For any $c^0 \in \mathcal{C}^2(\overline{\Omega})$, let $c_\mathcal{T}^0 = \mathbb{P}_\mathcal{T}^c c^0$ and $(c_\mathcal{T}^n, \mu_\mathcal{T}^n) \in \mathbb{R}^\mathcal{T} \times \mathbb{R}^\mathcal{T}$ be a solution to Problem 3.1. Then, there exist positive constants M_1, M_2, M_3, M_4 and M_5 independent of Δt and $\mathrm{size}(\mathcal{T})$ such that,*

$$\begin{aligned} &\sup_{n\le N} \|c_\mathcal{T}^n\|_{1,\mathcal{T}} \le M_1, \qquad \sup_{n\le N} \|c_{\partial\mathfrak{M}}^n\|_{1,\partial\mathfrak{M}} \le M_2, \\ &\sum_{n=0}^{N-1} \Delta t \left\|\mu_\mathcal{T}^{n+1}\right\|_{1,\mathcal{T}}^2 \le M_3, \qquad \sum_{n=0}^{N-1} \Delta t \left\|\frac{c_\mathcal{T}^{n+1}-c_\mathcal{T}^n}{\Delta t}\right\|_{1,\mathcal{T}}^2 \le \frac{M_4}{\Delta t} \\ \text{and} \qquad &\sum_{n=0}^{N-1} \Delta t \left\|\frac{c_{\partial\mathfrak{M}}^{n+1}-c_{\partial\mathfrak{M}}^n}{\Delta t}\right\|_{0,\partial\mathfrak{M}}^2 + \Delta t^2 \sum_{n=0}^{N-1} \left|\frac{c_{\partial\mathfrak{M}}^{n+1}-c_{\partial\mathfrak{M}}^n}{\Delta t}\right|_{1,\partial\mathfrak{M}}^2 \le M_5. \end{aligned} \tag{4.3}$$

We can remark that in [20] the proposition is proved by choosing the mean-value projection on all control volumes as initial discrete data for the initial concentration c^0. Indeed, when the initial data is not enough regular, we have to choose this projection to obtain the *a priori* bounds. However, when the initial data belongs to $\mathcal{C}^2(\overline{\Omega})$ the center-value projection is sufficient. Indeed, thanks to the mean-value theorem, we obtain a bound on the H^1-seminorm of $c_\mathcal{T}^0$ which allows us to prove the proposition as in [20].

4.2. Discrete projections. With a view to prove Theorem 4.2 we have to define another projection: the elliptic projection. These projection is in fact the solution of a suitable Laplace problem and thus depends on the boundary condition that we want to impose. Therefore, the definition of the elliptic projection for the chemical potential (see Definition 4.4) and for the order parameter (see Definition 4.5) are different.

DEFINITION 4.4 (Elliptic projection with Neumann boundary conditions).

We define the space $H_N^2(\Omega) = \{u \in H^2(\Omega) : \nabla u \cdot \vec{\mathbf{n}} = 0 \ \text{ on } \Gamma\}$, then the elliptic projection $\mathbb{P}_\mathcal{T}^{ell,N} : H_N^2(\Omega) \to \mathbb{R}^\mathcal{T}$ is defined as follows. For any $u \in H_N^2(\Omega)$, $\mathbb{P}_\mathcal{T}^{ell,N} u$ is the solution to the following discrete Laplace problem.

Find $v_\mathcal{T} \in \mathbb{R}^\mathcal{T}$ such that $\sum_{\kappa\in\mathfrak{M}} m_\kappa v_\kappa = \int_\Omega u$ and

$$\sum_{\sigma\in\mathcal{E}_\kappa^{int}} m_\sigma \frac{v_\kappa - v_\mathcal{L}}{d_{\kappa,\mathcal{L}}} = -\int_\kappa \Delta u(x)\mathrm{d}x, \quad \forall \kappa \in \mathfrak{M}. \tag{4.4}$$

DEFINITION 4.5 (Elliptic projection with Ventcell boundary conditions).

The elliptic projection $\mathbb{P}_\mathcal{T}^{ell,D} : H_\Gamma^2(\Omega) \to \mathbb{R}^\mathcal{T}$ is defined as follows. For any $u \in H_\Gamma^2(\Omega)$, $\mathbb{P}_\mathcal{T}^{ell,D} u$ is the solution to the following discrete Laplace problem.

Find $v_\mathcal{T} \in \mathbb{R}^\mathcal{T}$ such that $\sum_{\kappa\in\mathfrak{M}} m_\kappa v_\kappa = \int_\Omega u$ and for any $\kappa \in \mathfrak{M}$, $\mathcal{L} \in \partial\mathfrak{M}$,

$$\begin{aligned} &\sum_{\sigma\in\mathcal{E}_\kappa^{int}} m_\sigma \frac{v_\kappa - v_\mathcal{L}}{d_{\kappa,\mathcal{L}}} + \sum_{\sigma\in\mathcal{E}_\kappa^{ext}} m_{\mathbf{e}_\mathcal{L}} \frac{v_\kappa - v_\mathcal{L}}{d_{\kappa,\mathcal{L}}} = -\int_\kappa \Delta u(x)\mathrm{d}x; \\ &\sum_{\mathbf{v}\in\mathcal{V}_\mathcal{L}} \frac{v_\mathcal{L} - v_{\mathcal{L}'}}{d_{\mathcal{L},\mathcal{L}'}} + m_{\mathbf{e}_\mathcal{L}} \frac{v_\mathcal{L} - v_\kappa}{d_{\kappa,\mathcal{L}}} = -\int_\mathcal{L} \Delta_\Gamma u_{|\Gamma}(x)\mathrm{d}\sigma(x) + \int_\mathcal{L} \nabla u(x)\cdot\vec{\mathbf{n}}(x)\mathrm{d}\sigma(x). \end{aligned} \tag{4.5}$$

REMARK 4.2. *The elliptic projection is the solution to the finite-volume two-point flux approximation of the continuous problem*

$$\begin{cases} -\Delta v = f & \text{in } \Omega, \\ -\alpha\Delta_\Gamma v + \partial_n v = g & \text{on } \Gamma, \end{cases} \tag{4.6}$$

with $f = -\Delta u$ *and*

- $\alpha = 0$, $g = 0$ *for the elliptic projection* $\mathbb{P}_{\mathcal{T}}^{ell,N}$*;*
- $\alpha = 1$, $g = -\Delta_\Gamma u + \partial_n u$ *for the elliptic projection* $\mathbb{P}_{\mathcal{T}}^{ell,D}$.

When u is a time-dependent function, for a fixed time $t \in \mathbb{R}$, we will denote $\mathbb{P}_{\mathcal{T}}^{ell,N}(u(t))$ and $\mathbb{P}_{\mathcal{T}}^{ell,D}(u(t))$ the elliptic projections of the function $v = u(t,.)$.

In order to prove Theorem 4.2 we have to related the different discrete projections to the solution of the continuous problem (1.1). With this in mind we give below several properties which will be used all along the proof in Section 4.3.

Thanks to the Taylor's formulas and the Jensen inequality we can easily prove the following estimate.

LEMMA 4.6. *Let* $u \in H^2(\Omega)$*, there exists* $C_{11} > 0$ *independent of* $\mathrm{size}(\mathcal{T})$ *such that*

$$\|u - \mathbb{P}_{\mathfrak{M}}^c u\|_{L^2(\Omega)} \leq C_{11}\mathrm{size}(\mathcal{T}) \|\nabla u\|_{H^1(\Omega)}.$$

LEMMA 4.7. *Let* $u_{\mid\Gamma} \in H^1(\Gamma)$*, then there exists* $C_{12} > 0$ *independent of* $\mathrm{size}(\mathcal{T})$ *such that,*

$$\|u - \mathbb{P}_{\partial\mathfrak{M}}^c u\|_{L^2(\Gamma)} \leq C_{12}\mathrm{size}(\mathcal{T}) \|\nabla_\Gamma u_{\mid\Gamma}\|_{L^2(\Gamma)}.$$

PROPOSITION 4.8. *Let* $u \in H_N^2(\Omega)$*, there exists* $C_{13} > 0$ *depending only on* Ω *such that*

$$\left\|\mathbb{P}_{\mathfrak{M}}^{ell,N} u - u\right\|_{L^2(\Omega)} \leq C_{13}\mathrm{size}(\mathcal{T}) \|u\|_{H^2(\Omega)}$$

and.

$$\left|\mathbb{P}_{\mathcal{T}}^{ell,N} u - \mathbb{P}_{\mathcal{T}}^c u\right|_{1,\mathcal{T}} \leq C_{13}\mathrm{size}(\mathcal{T}) \|u\|_{H^2(\Omega)}.$$

Proof . Thanks to Definition 4.4 of the elliptic projection, $\mathbb{P}_{\mathcal{T}}^{ell,N} u$ is solution to discrete problem (4.4). Thus the difference $\mathbb{P}_{\mathcal{T}}^{ell,N} u - \mathbb{P}_{\mathcal{T}}^c u$ is the error associated to problem (4.6) with $\alpha = g = 0$. Thanks to the error estimate for the Laplace problem (4.6) for the two-point flux approximation scheme (see for example [10, Section 3.2.3]) we obtain the expected estimates. □

As regards the analogous proposition in the case of Ventcell boundary condition, the proof does not seem available in the literature and so we detail it in the Appendix.

PROPOSITION 4.9. *Let* $u \in H_\Gamma^2(\Omega)$*, there exists* $C_{14} > 0$ *depending only on* Ω *and* $\mathrm{reg}(\mathcal{T})$ *such that*

$$\left\|\mathbb{P}_{\mathcal{T}}^{ell,D} u - u\right\|_{L^2(\Omega)} + \left\|\mathbb{P}_{\partial\mathfrak{M}}^{ell,D} u - u\right\|_{L^2(\Gamma)} \leq C_{14}\mathrm{size}(\mathcal{T}) \|u\|_{H_\Gamma^2(\Omega)}.$$

and

$$\left|\mathbb{P}_{\mathcal{T}}^{ell,D} u - \mathbb{P}_{\mathcal{T}}^c u\right|_{1,\mathcal{T}} + \left|\mathbb{P}_{\partial\mathfrak{M}}^{ell,D} u - \mathbb{P}_{\partial\mathfrak{M}}^c u\right|_{1,\partial\mathfrak{M}} \leq C_{14}\mathrm{size}(\mathcal{T}) \|u\|_{H_\Gamma^2(\Omega)}.$$

4.3. Proof of Theorem 4.2. This section is devoted to the proof of Theorem 4.2. We decompose the proof in three steps. First, we use the scheme to decompose the different components of the error. Then, we control all the terms and prove that they tend to 0 when the mesh size and the time step tend to 0. Finally we use the discrete Gronwall lemma to conclude the proof.

In the proof different components of the total error will appear. Thus we decompose the error as follows.

DEFINITION 4.10 (Error). *Let $u : (0,T)\times\Omega \to \mathbb{R}$ and $u_\mathcal{T}^n$ be a finite-volume approximation of u at time t^n. We denote by $e_\mathcal{T}^{u,n} \in \mathbb{R}^\mathcal{T}$ the error associated with u at time $t = t^n$ defined as follows,*

$$e_\mathcal{T}^{u,n} = \ddot{e}_\mathcal{T}^{u,n} + \dot{e}_\mathcal{T}^{u,n} \text{ with } \ddot{e}_\mathcal{T}^{u,n} = \mathbb{P}_\mathcal{T}^c u(t^n) - \mathbb{P}_\mathcal{T}^{ell,*} u(t^n) \text{ and } \dot{e}_\mathcal{T}^{u,n} = \mathbb{P}_\mathcal{T}^{ell,*} u(t^n) - u_\mathcal{T}^n,$$

with $* = \{N, D\}$ depending on the boundary condition associated with u.

We also define $\bar{e}_\mathcal{T}^{u,n} = u(t^n,\cdot) - \frac{m_\mathcal{T}}{m_\mathcal{T}} \mathbb{P}_\mathcal{T}^{ell,*} u(t^n)$.

4.3.1. Different contributions of the error. First, subtracting the scheme and the continuous problem we identify the different components of the error (Proposition 4.11). Then, we separate the error into two parts: the error $\dot{e}_\mathcal{T}$ between the elliptic projection of the exact solution and the discrete solution in the left hand side, and all the other contributions in the right hand side (Proposition 4.12).

PROPOSITION 4.11. *Let us consider a couple (c,μ) solution to the continuous Cahn-Hilliard equation* (1.1) *and a couple $(c_\mathcal{T}^{n+1}, \mu_\mathcal{T}^{n+1})$ solution to the finite-volume scheme* (3.1)*. Then, the following equality holds*

$$\begin{aligned}
&\Delta t \left|\dot{e}_\mathcal{T}^{\mu,n+1}\right|_{1,\mathcal{T}}^2 + \frac{1}{\Delta t}\left\|\dot{e}_{\partial\mathfrak{M}}^{c,n+1} - \dot{e}_{\partial\mathfrak{M}}^{c,n}\right\|_{0,\partial\mathfrak{M}}^2 \\
&\quad + \frac{1}{2}\left(\left|\dot{e}_\mathcal{T}^{c,n+1}\right|_{1,\mathcal{T}}^2 - \left|\dot{e}_\mathcal{T}^{c,n}\right|_{1,\mathcal{T}}^2 + \left|\dot{e}_\mathcal{T}^{c,n+1} - \dot{e}_\mathcal{T}^{c,n}\right|_{1,\mathcal{T}}^2\right) \\
&\quad + \frac{1}{2}\left(\left|\dot{e}_{\partial\mathfrak{M}}^{c,n+1}\right|_{1,\partial\mathfrak{M}}^2 - \left|\dot{e}_{\partial\mathfrak{M}}^{c,n}\right|_{1,\partial\mathfrak{M}}^2 + \left|\dot{e}_{\partial\mathfrak{M}}^{c,n+1} - \dot{e}_{\partial\mathfrak{M}}^{c,n}\right|_{1,\partial\mathfrak{M}}^2\right) \\
&= \Delta t \left(R_c^{n+1}, \dot{e}_\mathfrak{M}^{\mu,n+1}\right)_{L^2(\Omega)} - \left(\bar{e}_\mathfrak{M}^{c,n+1} - \bar{e}_\mathfrak{M}^{c,n}, \dot{e}_\mathfrak{M}^{\mu,n+1}\right)_{L^2(\Omega)} \\
&\quad + \left(\bar{e}_\mathfrak{M}^{\mu,n+1}, \dot{e}_\mathfrak{M}^{c,n+1} - \dot{e}_\mathfrak{M}^{c,n}\right)_{L^2(\Omega)} + \left(R_{c_{|\Gamma}}^{n+1}, \dot{e}_{\partial\mathfrak{M}}^{c,n+1} - \dot{e}_{\partial\mathfrak{M}}^{c,n}\right)_{L^2(\Gamma)} \\
&\quad - \left(\frac{\bar{e}_{\partial\mathfrak{M}}^{c,n+1} - \bar{e}_{\partial\mathfrak{M}}^{c,n}}{\Delta t}, \dot{e}_{\partial\mathfrak{M}}^{c,n+1} - \dot{e}_{\partial\mathfrak{M}}^{c,n}\right)_{L^2(\Gamma)} \\
&\quad - \left(f_b'(c(t^{n+1},\cdot)) - \frac{m_\mathfrak{M}}{m_\mathfrak{M}} d^{f_b}(c_\mathfrak{M}^n, c_\mathfrak{M}^{n+1}), \dot{e}_\mathfrak{M}^{c,n+1} - \dot{e}_\mathfrak{M}^{c,n}\right)_{L^2(\Omega)} \\
&\quad - \left(f_s'(c_{|\Gamma}(t^{n+1},\cdot)) - \frac{m_{\partial\mathfrak{M}}}{m_{\partial\mathfrak{M}}} d^{f_s}(c_{\partial\mathfrak{M}}^n, c_{\partial\mathfrak{M}}^{n+1}), \dot{e}_{\partial\mathfrak{M}}^{c,n+1} - \dot{e}_{\partial\mathfrak{M}}^{c,n}\right)_{L^2(\Gamma)};
\end{aligned} \tag{4.7}$$

where the terms R_c^{n+1} and $R_{c_{|\Gamma}}^{n+1}$ are defined as follows:

$$R_{c_{(|\Gamma)}}^{n+1} = \frac{c_{(|\Gamma)}(t^{n+1},x) - c_{(|\Gamma)}(t^n,x)}{\Delta t} - \partial_t c_{(|\Gamma)}(t^{n+1},x). \tag{4.8}$$

Proof. Let (c,μ) be a solution to the Cahn-Hilliard equation (1.1). Applying Definition 4.4 of the elliptic projection with Neumann boundary condition to μ, for any $\kappa \in \mathfrak{M}$ we have

$$\int_\mathcal{K} \partial_t c(t^{n+1},x)\mathrm{d}x + \sum_{\sigma\in\mathcal{E}_\mathcal{K}^{int}} m_\sigma \frac{\mathbb{P}_\mathcal{K}^{ell,N}\mu(t^{n+1}) - \mathbb{P}_\mathcal{L}^{ell,N}\mu(t^{n+1})}{d_{\mathcal{K},\mathcal{L}}} = 0. \tag{4.9}$$

In the same way, applying Definition 4.5 of the elliptic projection with Ventcell boundary condition to c, for any $\kappa \in \mathfrak{M}$ we have

$$\begin{aligned}
&\sum_{\sigma\in\mathcal{E}_{\mathcal{K}}^{int}} m_\sigma \frac{\mathbb{P}_{\mathcal{K}}^{ell,D}c(t^{n+1}) - \mathbb{P}_{\mathcal{L}}^{ell,D}c(t^{n+1})}{d_{\mathcal{K},\mathcal{L}}} \\
&\quad + \sum_{\sigma\in\mathcal{E}_{\mathcal{K}}^{ext}} m_{\mathbf{e}_{\mathcal{L}}} \frac{\mathbb{P}_{\mathcal{K}}^{ell,D}c(t^{n+1}) - \mathbb{P}_{\mathcal{L}}^{ell,D}c(t^{n+1})}{d_{\mathcal{K},\mathcal{L}}} \\
&\quad + \int_{\mathcal{K}} f_b'(c(t^{n+1},x))\mathrm{d}x - \int_{\mathcal{K}} \mu(t^{n+1},x)\mathrm{d}x = 0,
\end{aligned} \tag{4.10}$$

and for any $\mathcal{L} \in \partial\mathfrak{M}$,

$$\begin{aligned}
&\sum_{\mathbf{v}\in\mathcal{V}_{\mathcal{L}}} \frac{\mathbb{P}_{\mathcal{L}}^{ell,D}c(t^{n+1}) - \mathbb{P}_{\mathcal{L}'}^{ell,D}c(t^{n+1})}{d_{\mathcal{L},\mathcal{L}'}} + m_{\mathbf{e}_{\mathcal{L}}} \frac{\mathbb{P}_{\mathcal{L}}^{ell,D}c(t^{n+1}) - \mathbb{P}_{\mathcal{K}}^{ell,D}c(t^{n+1})}{d_{\mathcal{K},\mathcal{L}}} \\
&\quad + \int_{\mathcal{L}} f_s'(c_{|\Gamma}(t^{n+1},x))\mathrm{d}\sigma(x) + \int_{\mathcal{L}} \partial_t c_{|\Gamma}(t^{n+1},x)\mathrm{d}\sigma(x) = 0,
\end{aligned} \tag{4.11}$$

where $\kappa \in \mathfrak{M}$ is the interior control volume such that $\mathcal{L} \in \partial\mathcal{K}$. Subtracting equation (3.1a) of discrete problem and equation (4.9) and using definition (4.8) of R_c^{n+1} imply,

$$\begin{aligned}
\sum_{\sigma\in\mathcal{E}_{\mathcal{K}}^{int}} m_\sigma \frac{\dot{e}_{\mathcal{K}}^{\mu,n+1} - \dot{e}_{\mathcal{L}}^{\mu,n+1}}{d_{\mathcal{K},\mathcal{L}}} + m_{\underline{\mathcal{K}}} \frac{\dot{e}_{\mathcal{K}}^{c,n+1} - \dot{e}_{\mathcal{K}}^{c,n}}{\Delta t} &= \int_{\mathcal{K}} R_c^{n+1}(x)\mathrm{d}x \\
&\quad - \int_{\mathcal{K}} \frac{\bar{e}_{\mathcal{K}}^{c,n+1}(x) - \bar{e}_{\mathcal{K}}^{c,n}(x)}{\Delta t}\mathrm{d}x.
\end{aligned} \tag{4.12}$$

Now, we subtract equation (3.1b) of the discrete problem and equation (4.10), then we obtain

$$\begin{aligned}
&\sum_{\sigma\in\mathcal{E}_{\mathcal{K}}^{int}} m_\sigma \frac{\dot{e}_{\mathcal{K}}^{c,n+1} - \dot{e}_{\mathcal{L}}^{c,n+1}}{d_{\mathcal{K},\mathcal{L}}} + \sum_{\sigma\in\mathcal{E}_{\mathcal{K}}^{ext}} m_{\mathbf{e}_{\mathcal{L}}} \frac{\dot{e}_{\mathcal{K}}^{c,n+1} - \dot{e}_{\mathcal{L}}^{c,n+1}}{d_{\mathcal{K},\mathcal{L}}} - m_{\underline{\mathcal{K}}} \dot{e}_{\mathcal{K}}^{\mu,n+1} \\
&\quad = \int_{\mathcal{K}} \bar{e}_{\mathfrak{M}}^{\mu,n+1}\mathrm{d}x - \int_{\mathcal{K}} \left(f_b'(c(t^{n+1},x)) - \frac{m_{\underline{\mathcal{K}}}}{m_{\mathcal{K}}} d^{f_b}(c_{\mathcal{K}}^n, c_{\mathcal{K}}^{n+1}) \right) \mathrm{d}x.
\end{aligned} \tag{4.13}$$

Subtracting equation (3.1c) of the discrete problem and equation (4.11) we have,

$$\begin{aligned}
&\sum_{\mathbf{v}\in\mathcal{V}_{\mathcal{L}}} \frac{\dot{e}_{\mathcal{L}}^{c,n+1} - \dot{e}_{\mathcal{L}'}^{c,n+1}}{d_{\mathcal{L},\mathcal{L}'}} + m_{\mathbf{e}_{\mathcal{L}}} \frac{\dot{e}_{\mathcal{L}}^{c,n+1} - \dot{e}_{\mathcal{K}}^{c,n+1}}{d_{\mathcal{K},\mathcal{L}}} + m_{\mathbf{e}_{\mathcal{L}}} \frac{\dot{e}_{\mathcal{L}}^{c,n+1} - \dot{e}_{\mathcal{L}}^{c,n}}{\Delta t} \\
&\quad = \int_{\mathcal{L}} R_{c_{|\Gamma}}^{n+1}(x)\mathrm{d}\sigma(x) - \int_{\mathcal{L}} \frac{\bar{e}_{\mathcal{L}}^{c,n+1}(x) - \bar{e}_{\mathcal{L}}^{c,n}(x)}{\Delta t}\mathrm{d}\sigma(x) \\
&\quad\quad - \int_{\mathcal{L}} \left(f_s'(c_{|\Gamma}(t^{n+1},x)) - \frac{m_{\mathbf{e}_{\mathcal{L}}}}{m_{\mathcal{L}}} d^{f_s}(c_{\mathcal{L}}^n, c_{\mathcal{L}}^{n+1}) \right) \mathrm{d}\sigma(x).
\end{aligned} \tag{4.14}$$

We multiply equation (4.12) by $v_{\mathcal{K}}$ and we sum up over all interior control volumes $\kappa \in \mathfrak{M}$, we obtain

$$\begin{aligned}
[\![\dot{e}_{\mathcal{T}}^{\mu,n+1}, v_{\mathcal{T}}]\!]_{1,\mathcal{T}} + \left(\frac{\dot{e}_{\mathfrak{M}}^{c,n+1} - \dot{e}_{\mathfrak{M}}^{c,n}}{\Delta t}, v_{\mathfrak{M}} \right)_{\mathfrak{M}} &= \left(R_c^{n+1}, v_{\mathfrak{M}} \right)_{L^2(\Omega)} \\
&\quad - \left(\frac{\bar{e}_{\mathfrak{M}}^{c,n+1} - \bar{e}_{\mathfrak{M}}^{c,n}}{\Delta t}, v_{\mathfrak{M}} \right)_{L^2(\Omega)}.
\end{aligned} \tag{4.15}$$

Now, we first multiply equation (4.13) by $u_{\mathcal{K}}$ and we sum up over all interior control volumes $\mathcal{K} \in \mathfrak{M}$. Then, we multiply equation (4.14) by $u_{\mathcal{L}}$ and we sum up over all boundary control volumes $\mathcal{L} \in \partial\mathfrak{M}$. Summing the resulting equalities, we have

$$
\begin{aligned}
&[\![\dot{e}_{\mathcal{T}}^{c,n+1}, u_{\mathcal{T}}]\!]_{1,\mathcal{T}} - \left(\dot{e}_{\mathfrak{M}}^{\mu,n+1}, u_{\mathfrak{M}}\right)_{\mathfrak{M}} \\
&\quad + [\![\dot{e}_{\partial\mathfrak{M}}^{c,n+1}, u_{\partial\mathfrak{M}}]\!]_{1,\partial\mathfrak{M}} + \left(\frac{\dot{e}_{\partial\mathfrak{M}}^{c,n+1} - \dot{e}_{\partial\mathfrak{M}}^{c,n}}{\Delta t}, u_{\partial\mathfrak{M}}\right)_{\partial\mathfrak{M}} \\
&= \left(\bar{e}_{\mathfrak{M}}^{\mu,n+1}, u_{\mathfrak{M}}\right)_{L^2(\Omega)} + \left(R_{c_{|\Gamma}}^{n+1}, u_{\partial\mathfrak{M}}\right)_{L^2(\Gamma)} - \left(\frac{\bar{e}_{\partial\mathfrak{M}}^{c,n+1} - \bar{e}_{\partial\mathfrak{M}}^{c,n}}{\Delta t}, u_{\partial\mathfrak{M}}\right)_{L^2(\Omega)} \\
&\quad - \left(f_b'(c(t^{n+1},\cdot)) - \frac{m_{\underline{\mathfrak{M}}}}{m_{\mathfrak{M}}} d^{f_b}(c_{\mathfrak{M}}^n, c_{\mathfrak{M}}^{n+1}), u_{\mathfrak{M}}\right)_{L^2(\Omega)} \\
&\quad - \left(f_s'(c_{|\Gamma}(t^{n+1},\cdot) - \frac{m_{\underline{\partial\mathfrak{M}}}}{m_{\partial\mathfrak{M}}} d^{f_s}(c_{\partial\mathfrak{M}}^n, c_{\partial\mathfrak{M}}^{n+1}), u_{\partial\mathfrak{M}}\right)_{L^2(\Gamma)}.
\end{aligned} \tag{4.16}
$$

Choosing $v_{\mathcal{T}} = \Delta t \dot{e}_{\mathcal{T}}^{\mu,n+1}$ in equation (4.15) and $u_{\mathcal{T}} = \dot{e}_{\mathcal{T}}^{c,n+1} - \dot{e}_{\mathcal{T}}^{c,n}$ in equation (4.16) and adding the two resulting equalities the claim is proved. □

PROPOSITION 4.12. *Let (c,μ) be solution to the Cahn-Hilliard equation* (1.1) *and $(c_{\mathcal{T}}^{n+1}, \mu_{\mathcal{T}}^{n+1})$ be solution to the discrete scheme* (3.1)*. Then, for any $n_0 \in [\![0,N]\!]$ we have:*

$$
\begin{aligned}
&\frac{1}{2}\sum_{n=0}^{n_0} \Delta t \left|\dot{e}_{\mathcal{T}}^{\mu,n+1}\right|_{1,\mathcal{T}}^2 + \frac{1}{2}\sum_{n=0}^{n_0} \Delta t \left\|\frac{\dot{e}_{\partial\mathfrak{M}}^{c,n+1} - \dot{e}_{\partial\mathfrak{M}}^{c,n}}{\Delta t}\right\|_{0,\partial\mathfrak{M}}^2 \\
&\quad + \frac{1}{2}\left(\frac{1}{2}\left|\dot{e}_{\mathcal{T}}^{c,n_0+1}\right|_{1,\mathcal{T}}^2 - \left|\dot{e}_{\mathcal{T}}^{c,0}\right|_{1,\mathcal{T}}^2 + \sum_{n=0}^{n_0}\left|\dot{e}_{\mathcal{T}}^{c,n+1} - \dot{e}_{\mathcal{T}}^{c,n}\right|_{1,\mathcal{T}}^2\right) \\
&\quad + \frac{1}{2}\left(\left|\dot{e}_{\partial\mathfrak{M}}^{c,n_0+1}\right|_{1,\partial\mathfrak{M}}^2 - \left|\dot{e}_{\partial\mathfrak{M}}^{c,0}\right|_{1,\partial\mathfrak{M}}^2 + \sum_{n=0}^{n_0}\left|\dot{e}_{\partial\mathfrak{M}}^{c,n+1} - \dot{e}_{\partial\mathfrak{M}}^{c,n}\right|_{1,\partial\mathfrak{M}}^2\right) \\
&\leq T_{c^0} + \sum_{n=1}^{n_0} \Delta t \left|\dot{e}_{\mathcal{T}}^{c,n}\right|_{1,\mathcal{T}}^2 + T_{R_c^{n+1}} + T_{\bar{e}\mathcal{T}} + T_{g\mathcal{T}} + T_{f_b} + T_{f_s} + T_{\mathcal{T}},
\end{aligned} \tag{4.17}
$$

where the different error terms are defined as follows:

$$
\begin{aligned}
T_{c^0} =& \left\|\dot{e}_{\mathfrak{M}}^{c,0}\right\|_{L^2(\Omega)}^2 + \frac{5|\Omega|}{8C_1C_4^2} m_{\underline{\Omega}}\left(\dot{e}_{\mathfrak{M}}^{c,0}\right)^2 \\
T_{R_c^{n+1}} =& \frac{7}{2}C_4^2 \sum_{n=0}^{n_0} \Delta t \left\|R_c^{n+1}\right\|_{L^2(\Omega)}^2 + 2C_3 \sum_{n=0}^{n_0} \Delta t \left\|R_{c_{|\Gamma}}^{n+1}\right\|_{L^2(\Gamma)}^2 \\
T_{\bar{e}\mathcal{T}} =& \frac{7}{2}C_4^2 \sum_{n=0}^{n_0} \Delta t \left\|\frac{\bar{e}_{\mathfrak{M}}^{c,n+1} - \bar{e}_{\mathfrak{M}}^{c,n}}{\Delta t}\right\|_{L^2(\Omega)}^2 + 2C_3 \sum_{n=0}^{n_0} \Delta t \left\|\frac{\bar{e}_{\partial\mathfrak{M}}^{c,n+1} - \bar{e}_{\partial\mathfrak{M}}^{c,n}}{\Delta t}\right\|_{L^2(\Gamma)}^2 \\
&+ \frac{1}{2}\left\|\bar{e}_{\mathfrak{M}}^{\mu,1}\right\|_{L^2(\Omega)}^2 + 2C_4^2\left\|\bar{e}_{\mathfrak{M}}^{\mu,n_0+1}\right\|_{L^2(\Omega)}^2 + \frac{C_4^2}{2}\sum_{n=1}^{n_0} \Delta t \left\|\frac{\bar{e}_{\mathfrak{M}}^{\mu,n} - \bar{e}_{\mathfrak{M}}^{\mu,n+1}}{\Delta t}\right\|_{L^2(\Omega)}^2 \\
T_{g\mathcal{T}} =& \frac{1}{2}\left\|g_{\mathfrak{M}}(t^1,\cdot)\right\|_{L^2(\Omega)}^2 + 4C_2^2C_4^2\left\|g_{\mathfrak{M}}(t^{n_0+1},\cdot)\right\|_{L^2(\Omega)}^2 \\
&+ C_2^2C_4^2 \sum_{n=1}^{n_0} \Delta t \left\|\frac{g_{\mathfrak{M}}(t^n,\cdot) - g_{\mathfrak{M}}(t^{n+1},\cdot)}{\Delta t}\right\|_{L^2(\Omega)}^2 + 2C_3 \sum_{n=0}^{n_0} \Delta t \left\|g_{\partial\mathfrak{M}}(t^{n+1},\cdot)\right\|_{L^2(\Gamma)}^2 \\
T_{f_b} =& 4\sum_{n=0}^{n_0} \Delta t \left(\left|f_b'(\mathbb{P}_{\mathcal{T}}^c(c(t^{n+1}))) - f_b'(c_{\mathcal{T}}^{n+1})\right|_{1,\mathcal{T}}^2 + \left|d^{f_b}(c_{\mathcal{T}}^{n+1}, c_{\mathcal{T}}^{n+1}) - d^{f_b}(c_{\mathcal{T}}^n, c_{\mathcal{T}}^{n+1})\right|_{1,\mathcal{T}}^2\right)
\end{aligned}
$$

$$
\begin{aligned}
T_{f_s} =& 4C_3 \sum_{n=0}^{n_0} \Delta t \left\| f_s' \left(\mathbb{P}^c_{\partial\mathfrak{M}}(c(t^{n+1})) \right) - d^{f_s}(c^n_{\partial\mathfrak{M}}, c^{n+1}_{\partial\mathfrak{M}}) \right\|^2_{L^2(\Gamma)} \\
T_{\mathcal{T}} =& 4 \sum_{n=0}^{n_0} \Delta t \left| \frac{m_{\mathcal{T}} - m_{\underline{\mathcal{T}}}}{m_{\underline{\mathcal{T}}}} f_b'(\mathbb{P}^c_{\mathcal{T}}(c(t^{n+1}))) \right|^2_{1,\mathcal{T}} \\
&+ 4C_3 \sum_{n=0}^{n_0} \Delta t \left\| \frac{m_{\partial\mathfrak{M}} - m_{\underline{\partial\mathfrak{M}}}}{m_{\partial\mathfrak{M}}} d^{f_s}(c^n_{\partial\mathfrak{M}}, c^{n+1}_{\partial\mathfrak{M}}) \right\|^2_{L^2(\Gamma)}
\end{aligned}
$$

The term $g_{\mathfrak{M}}$ (resp. $g_{\partial\mathfrak{M}}$) is such that for any $\kappa \in \mathfrak{M}$ (resp. $\mathcal{L} \in \partial\mathfrak{M}$),

$$
\begin{aligned}
g_{\mathfrak{M}}(t,x) &= f_b'\left(c(t,x)\right) - f_b'\left(c(t,x_\kappa)\right), \quad \forall x \in \kappa, t \in \mathbb{R}; \\
g_{\partial\mathfrak{M}}(t,x) &= f_s'\left(c(t,x)\right) - f_s'\left(c(t,x_{\mathcal{L}})\right), \quad \forall x \in \mathcal{L}, t \in \mathbb{R}.
\end{aligned} \tag{4.18}
$$

Proof . To begin we sum identity (4.7) for n going from 0 to n_0, then thanks to definition (4.18) of the functions $g_{\mathfrak{M}}$ and $g_{\partial\mathfrak{M}}$ we obtain,

$$
\begin{aligned}
&\sum_{n=0}^{n_0} \Delta t \left| \dot{e}^{\mu,n+1}_{\mathcal{T}} \right|^2_{1,\mathcal{T}} + \sum_{n=0}^{n_0} \Delta t \left\| \frac{\dot{e}^{c,n+1}_{\partial\mathfrak{M}} - \dot{e}^{c,n}_{\partial\mathfrak{M}}}{\Delta t} \right\|^2_{0,\partial\mathfrak{M}} \\
&\quad + \frac{1}{2} \left(\left| \dot{e}^{c,n_0+1}_{\mathcal{T}} \right|^2_{1,\mathcal{T}} - \left| \dot{e}^{c,0}_{\mathcal{T}} \right|^2_{1,\mathcal{T}} + \sum_{n=0}^{n_0} \left| \dot{e}^{c,n+1}_{\mathcal{T}} - \dot{e}^{c,n}_{\mathcal{T}} \right|^2_{1,\mathcal{T}} \right) \\
&\quad + \frac{1}{2} \left(\left| \dot{e}^{c,n_0+1}_{\partial\mathfrak{M}} \right|^2_{1,\partial\mathfrak{M}} - \left| \dot{e}^{c,0}_{\partial\mathfrak{M}} \right|^2_{1,\partial\mathfrak{M}} + \sum_{n=0}^{n_0} \left| \dot{e}^{c,n+1}_{\partial\mathfrak{M}} - \dot{e}^{c,n}_{\partial\mathfrak{M}} \right|^2_{1,\partial\mathfrak{M}} \right) \\
&= \sum_{n=0}^{n_0} \Delta t \left(R_c^{n+1}, \dot{e}^{\mu,n+1}_{\mathfrak{M}} \right)_{L^2(\Omega)} - \sum_{n=0}^{n_0} \left(\bar{e}^{c,n+1}_{\mathfrak{M}} - \bar{e}^{c,n}_{\mathfrak{M}}, \dot{e}^{\mu,n+1}_{\mathfrak{M}} \right)_{L^2(\Omega)} \\
&\quad + \sum_{n=0}^{n_0} \left(\bar{e}^{\mu,n+1}_{\mathfrak{M}}, \dot{e}^{c,n+1}_{\mathfrak{M}} - \dot{e}^{c,n}_{\mathfrak{M}} \right)_{L^2(\Omega)} + \sum_{n=0}^{n_0} \left(R^{n+1}_{c_{|\Gamma}}, \dot{e}^{c,n+1}_{\partial\mathfrak{M}} - \dot{e}^{c,n}_{\partial\mathfrak{M}} \right)_{L^2(\Gamma)} \\
&\quad - \sum_{n=0}^{n_0} \left(\frac{\bar{e}^{c,n+1}_{\partial\mathfrak{M}} - \bar{e}^{c,n}_{\partial\mathfrak{M}}}{\Delta t}, \dot{e}^{c,n+1}_{\partial\mathfrak{M}} - \dot{e}^{c,n}_{\partial\mathfrak{M}} \right)_{L^2(\Gamma)} \\
&\quad - \sum_{n=0}^{n_0} \left(g_{\mathfrak{M}}(t^{n+1}, \cdot), \dot{e}^{c,n+1}_{\mathfrak{M}} - \dot{e}^{c,n}_{\mathfrak{M}} \right)_{L^2(\Omega)} \\
&\quad - \sum_{n=0}^{n_0} \left(g_{\partial\mathfrak{M}}(t^{n+1}, \cdot), \dot{e}^{c,n+1}_{\partial\mathfrak{M}} - \dot{e}^{c,n}_{\partial\mathfrak{M}} \right)_{L^2(\Gamma)} \\
&\quad - \sum_{n=0}^{n_0} \left(f_b' \left(\mathbb{P}^c_{\mathfrak{M}}(c(t^{n+1})) \right) - \frac{m_{\underline{\mathfrak{M}}}}{m_{\mathfrak{M}}} d^{f_b}(c^n_{\mathfrak{M}}, c^{n+1}_{\mathfrak{M}}), \dot{e}^{c,n+1}_{\mathfrak{M}} - \dot{e}^{c,n}_{\mathfrak{M}} \right)_{L^2(\Omega)} \\
&\quad - \sum_{n=0}^{n_0} \left(f_s' \left(\mathbb{P}^c_{\partial\mathfrak{M}}(c(t^{n+1})) \right) - \frac{m_{\underline{\partial\mathfrak{M}}}}{m_{\partial\mathfrak{M}}} d^{f_s}(c^n_{\partial\mathfrak{M}}, c^{n+1}_{\partial\mathfrak{M}}), \dot{e}^{c,n+1}_{\partial\mathfrak{M}} - \dot{e}^{c,n}_{\partial\mathfrak{M}} \right)_{L^2(\Gamma)}.
\end{aligned} \tag{4.19}
$$

To obtain estimate (4.17), we have to apply the Young inequality to all the terms in the right hand side of (4.19).

- Let us begin by the two first terms in the right hand side of (4.19), namely the terms where the L^2-inner product with $\dot{e}^{\mu,n+1}_{\mathfrak{M}}$ appears. Noting that if $v_{\mathfrak{M}} \in \mathbb{R}^{\mathfrak{M}}$ is a null average function, for any $u_{\mathfrak{M}} \in \mathbb{R}^{\mathfrak{M}}$, we have

$$
\left(u_{\mathfrak{M}}, v_{\mathfrak{M}} \right)_{L^2(\Omega)} = \left(u_{\mathfrak{M}} - m_\Omega\left(u_{\mathfrak{M}}\right), v_{\mathfrak{M}} \right)_{L^2(\Omega)} \leq \left\| u_{\mathfrak{M}} - m_\Omega\left(u_{\mathfrak{M}}\right) \right\|_{L^2(\Omega)} \left\| v_{\mathfrak{M}} \right\|_{L^2(\Omega)},
$$

then thanks to the discrete mean Poincaré inequality given in Lemma 2.7, we obtain

$$(u_{\mathfrak{M}}, v_{\mathfrak{M}})_{L^2(\Omega)} \leq C_4 \left|u_{\mathcal{T}}\right|_{1,\mathcal{T}} \left\|v_{\mathfrak{M}}\right\|_{L^2(\Omega)}. \tag{4.20}$$

The couple (c, μ) is solution to the Cahn-Hilliard equation (1.1) and μ satisfies the homogeneous Neumann boundary condition which implies for any $n \in [\![0, n_0]\!]$,

$$\int_\Omega \partial_t c(t^{n+1}, x)\mathrm{d}x = \int_{\partial\Omega} \nabla\mu(t^{n+1}, x) \cdot \vec{\mathbf{n}} = 0, \tag{4.21}$$

and

$$\int_\Omega (c(t^{n+1}, x) - c(t^n, x))\mathrm{d}x = \int_\Omega \int_{t^n}^{t^{n+1}} \partial_t c(s, x)\mathrm{d}s = 0. \tag{4.22}$$

Then we deduce that for any $n \in [\![0, n_0]\!]$,

$$m_\Omega\left(R_c^{n+1}\right) = \int_\Omega \left(\frac{c(t^{n+1}, x) - c(t^n, x)}{\Delta t} - \partial_t c(t^{n+1}, x)\right)\mathrm{d}x = 0, \tag{4.23}$$

and estimate (4.20) allows us to obtain,

$$\left(R_c^{n+1}, \dot{e}_{\mathfrak{M}}^{\mu,n+1}\right)_{L^2(\Omega)} \leq C_4 \left|\dot{e}_{\mathcal{T}}^{\mu,n+1}\right|_{1,\mathcal{T}} \left\|R_c^{n+1}\right\|_{L^2(\Omega)}.$$

Applying the Young inequality we finally get for any $n \in [\![0, n_0]\!]$,

$$\Delta t \left(R_c^{n+1}, \dot{e}_{\mathfrak{M}}^{\mu,n+1}\right)_{L^2(\Omega)} \leq \frac{1}{8}\Delta t \left|\dot{e}_{\mathcal{T}}^{\mu,n+1}\right|_{1,\mathcal{T}}^2 + 2C_4^2 \Delta t \left\|R_c^{n+1}\right\|_{L^2(\Omega)}^2. \tag{4.24}$$

Thanks to Definition 4.5 of the elliptic projection, for any $n \in [\![0, n_0]\!]$ we also have

$$m_\Omega\left(\bar{e}_{\mathfrak{M}}^{c,n}\right) = \int_\Omega c(t^n, \cdot) - \sum_{\kappa \in \mathfrak{M}} m_{\underline{\kappa}} \mathbb{P}_\kappa^{ell,D} c(t^n) = 0.$$

Since $m_\Omega\left(\bar{e}_{\mathfrak{M}}^{c,n+1} - \bar{e}_{\mathfrak{M}}^{c,n}\right) = 0$ estimate (4.20) gives,

$$\left(\bar{e}_{\mathfrak{M}}^{c,n+1} - \bar{e}_{\mathfrak{M}}^{c,n}, \dot{e}_{\mathcal{T}}^{\mu,n+1}\right)_{L^2(\Omega)} \leq C_4 \left|\dot{e}_{\mathcal{T}}^{\mu,n+1}\right|_{1,\mathcal{T}} \left\|\bar{e}_{\mathfrak{M}}^{c,n+1} - \bar{e}_{\mathfrak{M}}^{c,n}\right\|_{L^2(\Omega)}$$

and the Young inequality implies for any $n \in [\![0, n_0]\!]$,

$$\begin{aligned}\left(\bar{e}_{\mathfrak{M}}^{c,n+1} - \bar{e}_{\mathfrak{M}}^{c,n}, \dot{e}_{\mathfrak{M}}^{\mu,n+1}\right)_{L^2(\Omega)} \leq & \frac{1}{8}\Delta t \left|\dot{e}_{\mathcal{T}}^{\mu,n+1}\right|_{1,\mathcal{T}}^2 \\ & + 2C_4^2 \Delta t \left\|\frac{\bar{e}_{\mathfrak{M}}^{c,n+1} - \bar{e}_{\mathfrak{M}}^{c,n}}{\Delta t}\right\|_{L^2(\Omega)}^2.\end{aligned} \tag{4.25}$$

- Now we focus on the terms whose term $\dot{e}_{\mathfrak{M}}^{c,n+1} - \dot{e}_{\mathfrak{M}}^{c,n}$ appears in the inner product on $L^2(\Omega)$. For the two first we perform a discrete time integration by parts.
Let us begin by the term where the function $\bar{e}_{\mathfrak{M}}^{\mu,n+1}$ intervenes.

$$\begin{aligned}\sum_{n=0}^{n_0} \left(\bar{e}_{\mathfrak{M}}^{\mu,n+1}, \dot{e}_{\mathfrak{M}}^{c,n+1} - \dot{e}_{\mathfrak{M}}^{c,n}\right)_{L^2(\Omega)} = & -\left(\bar{e}_{\mathfrak{M}}^{\mu,1}, \dot{e}_{\mathfrak{M}}^{c,0}\right)_{L^2(\Omega)} + \left(\bar{e}_{\mathfrak{M}}^{\mu,n_0+1}, \dot{e}_{\mathfrak{M}}^{c,n_0+1}\right)_{L^2(\Omega)} \\ & + \sum_{n=1}^{n_0} \left(\bar{e}_{\mathfrak{M}}^{\mu,n} - \bar{e}_{\mathfrak{M}}^{\mu,n+1}, \dot{e}_{\mathfrak{M}}^{c,n}\right)_{L^2(\Omega)}.\end{aligned}$$

Noting that by definition of the elliptic projection (see Definition 4.4) $m_\Omega\left(\bar{e}_{\mathfrak{M}}^{\mu,n}\right) = 0$, the Cauchy-Schwartz inequality, inequality (4.20) and the Young inequality give,

$$
\begin{aligned}
\left| \sum_{n=0}^{n_0} \left(\bar{e}_{\mathfrak{M}}^{\mu,n+1}, \dot{e}_{\mathfrak{M}}^{c,n+1} - \dot{e}_{\mathfrak{M}}^{c,n}\right)_{L^2(\Omega)} \right| &\leq \frac{1}{2}\left\|\bar{e}_{\mathfrak{M}}^{\mu,1}\right\|_{L^2(\Omega)}^2 + \frac{1}{2}\left\|\dot{e}_{\mathfrak{M}}^{c,0}\right\|_{L^2(\Omega)}^2 \\
&+ \frac{C_4^2}{2}\sum_{n=1}^{n_0}\Delta t\left\|\frac{\bar{e}_{\mathfrak{M}}^{\mu,n} - \bar{e}_{\mathfrak{M}}^{\mu,n+1}}{\Delta t}\right\|_{L^2(\Omega)}^2 + \frac{1}{2}\sum_{n=1}^{n_0}\Delta t\left|\dot{e}_{\mathcal{T}}^{c,n}\right|_{1,\mathcal{T}}^2 \\
&+ \frac{1}{8}\left|\dot{e}_{\mathcal{T}}^{c,n_0+1}\right|_{1,\mathcal{T}}^2 + 2C_4^2\left\|\bar{e}_{\mathfrak{M}}^{\mu,n_0+1}\right\|_{L^2(\Omega)}^2.
\end{aligned} \tag{4.26}
$$

Considering now the inner product with the term $g_{\mathfrak{M}}(t^{n+1},\cdot)$. Noting that the mean-value of the function $g_{\mathfrak{M}}(t,\cdot)$ is not equal to 0, thanks to the Cauchy-Schwarz inequality we have,

$$
\begin{aligned}
\left| \sum_{n=0}^{n_0} \left(g_{\mathfrak{M}}(t^{n+1},\cdot), \dot{e}_{\mathfrak{M}}^{c,n+1} - \dot{e}_{\mathfrak{M}}^{c,n}\right)_{L^2(\Omega)} \right| &\leq \left\|g_{\mathfrak{M}}(t^1,\cdot)\right\|_{L^2(\Omega)}\left\|\dot{e}_{\mathfrak{M}}^{c,0}\right\|_{L^2(\Omega)} \\
&+ \sum_{n=1}^{n_0}\left\|g_{\mathfrak{M}}(t^n,\cdot) - g_{\mathfrak{M}}(t^{n+1},\cdot)\right\|_{L^2(\Omega)}\left\|\dot{e}_{\mathfrak{M}}^{c,n}\right\|_{L^2(\Omega)} \\
&+ \left\|g_{\mathfrak{M}}(t^{n_0+1},\cdot)\right\|_{L^2(\Omega)}\left\|\dot{e}_{\mathfrak{M}}^{c,n_0+1}\right\|_{L^2(\Omega)}.
\end{aligned}
$$

Definition 4.5 of the elliptic projection and the discrete volume conservation (3.2) imply that for any $n \in [\![0,N]\!]$, $m_{\underline{\Omega}}\left(\dot{e}_{\mathfrak{M}}^{c,n}\right) = m_{\underline{\Omega}}\left(\dot{e}_{\mathfrak{M}}^{c,0}\right)$. Then Proposition 2.1, Poincaré inequality (2.2) and the Young inequality get,

$$
\begin{aligned}
\left| \sum_{n=0}^{n_0} \left(g_{\mathfrak{M}}(t^{n+1},\cdot), \dot{e}_{\mathfrak{M}}^{c,n+1} - \dot{e}_{\mathfrak{M}}^{c,n}\right)_{L^2(\Omega)} \right| &\leq \frac{1}{2}\left\|g_{\mathfrak{M}}(t^1,\cdot)\right\|_{L^2(\Omega)}^2 \\
&+ \frac{1}{2}\left\|\dot{e}_{\mathfrak{M}}^{c,0}\right\|_{L^2(\Omega)}^2 + 4C_2^2C_4^2\left\|g_{\mathfrak{M}}(t^{n_0+1},\cdot)\right\|_{L^2(\Omega)}^2 + \frac{1}{8}\left|\dot{e}_{\mathcal{T}}^{c,n_0+1}\right|_{1,\mathcal{T}}^2 \\
&+ \frac{5|\Omega|}{8C_1C_4^2}m_{\underline{\Omega}}\left(\dot{e}_{\mathfrak{M}}^{c,0}\right)^2 + \frac{1}{2}\sum_{n=1}^{n_0}\Delta t\left|\dot{e}_{\mathcal{T}}^{c,n}\right|_{1,\mathcal{T}}^2 \\
&+ C_2^2C_4^2\sum_{n=1}^{n_0}\Delta t\left\|\frac{g_{\mathfrak{M}}(t^n,\cdot) - g_{\mathfrak{M}}(t^{n+1},\cdot)}{\Delta t}\right\|_{L^2(\Omega)}^2.
\end{aligned} \tag{4.27}
$$

As regards the last term where the inner product with $\dot{e}_{\mathfrak{M}}^{c,n+1} - \dot{e}_{\mathfrak{M}}^{c,n}$ appears we have to use the scheme. Noting that

$$
\begin{aligned}
&\left(f_b'\left(\mathbb{P}_{\mathfrak{M}}^c(c(t^{n+1}))\right) - \frac{m_{\mathfrak{M}}}{m_{\underline{\mathfrak{M}}}}d^{f_b}(c_{\mathfrak{M}}^n, c_{\mathfrak{M}}^{n+1}), \dot{e}_{\mathfrak{M}}^{c,n+1} - \dot{e}_{\mathfrak{M}}^{c,n}\right)_{L^2(\Omega)} \\
&= \left(\frac{m_{\mathfrak{M}}}{m_{\underline{\mathfrak{M}}}}f_b'\left(\mathbb{P}_{\mathfrak{M}}^c(c(t^{n+1}))\right) - d^{f_b}(c_{\mathfrak{M}}^n, c_{\mathfrak{M}}^{n+1}), \dot{e}_{\mathfrak{M}}^{c,n+1} - \dot{e}_{\mathfrak{M}}^{c,n}\right)_{\mathfrak{M}},
\end{aligned}
$$

and choosing $v_{\mathcal{T}} = \Delta t\left(\frac{m_{\mathfrak{M}}}{m_{\underline{\mathfrak{M}}}}f_b'(\mathbb{P}_{\mathcal{T}}^c(c(t^{n+1}))) - d^{f_b}(c_{\mathcal{T}}^n, c_{\mathcal{T}}^{n+1})\right)$ in identity (4.15)

we obtain,

$$\begin{aligned}
&\left(f_b'\left(\mathbb{P}^c_{\mathfrak{M}}(c(t^{n+1}))\right) - \frac{m_{\underline{\mathfrak{M}}}}{m_{\mathfrak{M}}} d^{f_b}(c^n_{\mathfrak{M}}, c^{n+1}_{\mathfrak{M}}), \dot{e}^{c,n+1}_{\mathfrak{M}} - \dot{e}^{c,n}_{\mathfrak{M}}\right)_{L^2(\Omega)} \\
&\qquad = -\Delta t [\![\dot{e}^{\mu,n+1}_{\mathcal{T}}, \frac{m_{\mathcal{T}}}{m_{\underline{\mathcal{T}}}} f_b'(\mathbb{P}^c_{\mathcal{T}}(c(t^{n+1}))) - d^{f_b}(c^n_{\mathcal{T}}, c^{n+1}_{\mathcal{T}})]\!]_{1,\mathcal{T}} \\
&\qquad\quad + \Delta t \left(R^{n+1}_c, \frac{m_{\mathfrak{M}}}{m_{\underline{\mathfrak{M}}}} f_b'(\mathbb{P}^c_{\mathfrak{M}}(c(t^{n+1}))) - d^{f_b}(c^n_{\mathfrak{M}}, c^{n+1}_{\mathfrak{M}})\right)_{L^2(\Omega)} \\
&\qquad\quad - \left(\bar{e}^{c,n+1}_{\mathfrak{M}} - \bar{e}^{c,n}_{\mathfrak{M}}, \frac{m_{\mathfrak{M}}}{m_{\underline{\mathfrak{M}}}} f_b'(\mathbb{P}^c_{\mathfrak{M}}(c(t^{n+1}))) - d^{f_b}(c^n_{\mathfrak{M}}, c^{n+1}_{\mathfrak{M}})\right)_{L^2(\Omega)}.
\end{aligned}$$

Since $m_\Omega\left(R^{n+1}_c\right) = m_\Omega\left(\bar{e}^{c,n+1}_{\mathfrak{M}} - \bar{e}^{c,n}_{\mathfrak{M}}\right) = 0$, the Cauchy-Schwarz inequality and inequality (4.20) imply,

$$\begin{aligned}
&\left(f_b'\left(\mathbb{P}^c_{\mathfrak{M}}(c(t^{n+1}))\right) - \frac{m_{\underline{\mathfrak{M}}}}{m_{\mathfrak{M}}} d^{f_b}(c^n_{\mathfrak{M}}, c^{n+1}_{\mathfrak{M}}), \dot{e}^{c,n+1}_{\mathfrak{M}} - \dot{e}^{c,n}_{\mathfrak{M}}\right)_{L^2(\Omega)} \\
&\qquad \leq \Delta t \left(\left|\dot{e}^{\mu,n+1}_{\mathcal{T}}\right|_{1,\mathcal{T}} + C_4 \left\|R^{n+1}_c\right\|_{L^2(\Omega)} + C_4 \left\|\frac{\bar{e}^{c,n+1}_{\mathfrak{M}} - \bar{e}^{c,n}_{\mathfrak{M}}}{\Delta t}\right\|_{L^2(\Omega)}\right) \\
&\qquad\quad \left|\frac{m_{\mathcal{T}}}{m_{\underline{\mathcal{T}}}} f_b'(\mathbb{P}^c_{\mathcal{T}}(c(t^{n+1}))) - d^{f_b}(c^n_{\mathcal{T}}, c^{n+1}_{\mathcal{T}})\right|_{1,\mathcal{T}}.
\end{aligned}$$

Recalling that for any $x \in \mathbb{R}$, $d^{f_b}(x,x) = f_b'(x)$ (see Definition 3.1) and applying the Young inequality we deduce,

$$\begin{aligned}
&\left(f_b'\left(\mathbb{P}^c_{\mathfrak{M}}(c(t^{n+1}))\right) - \frac{m_{\underline{\mathfrak{M}}}}{m_{\mathfrak{M}}} d^{f_b}(c^n_{\mathfrak{M}}, c^{n+1}_{\mathfrak{M}}), \dot{e}^{c,n+1}_{\mathfrak{M}} - \dot{e}^{c,n}_{\mathfrak{M}}\right)_{L^2(\Omega)} \\
&\leq \frac{1}{4}\Delta t \left|\dot{e}^{\mu,n+1}_{\mathcal{T}}\right|^2_{1,\mathcal{T}} + \frac{3}{2} C_4^2 \Delta t \left(\left\|R^{n+1}_c\right\|^2_{L^2(\Omega)} + \left\|\frac{\bar{e}^{c,n+1}_{\mathfrak{M}} - \bar{e}^{c,n}_{\mathfrak{M}}}{\Delta t}\right\|^2_{L^2(\Omega)}\right) \\
&\quad + 4\Delta t \left|\frac{m_{\mathcal{T}} - m_{\underline{\mathcal{T}}}}{m_{\underline{\mathcal{T}}}} f_b'(\mathbb{P}^c_{\mathcal{T}}(c(t^{n+1})))\right|^2_{1,\mathcal{T}} \\
&\quad + 4\Delta t \left|f_b'(\mathbb{P}^c_{\mathcal{T}}(c(t^{n+1}))) - f_b'(c^{n+1}_{\mathcal{T}})\right|^2_{1,\mathcal{T}} \\
&\quad + 4\Delta t \left|d^{f_b}(c^{n+1}_{\mathcal{T}}, c^{n+1}_{\mathcal{T}}) - d^{f_b}(c^n_{\mathcal{T}}, c^{n+1}_{\mathcal{T}})\right|^2_{1,\mathcal{T}}.
\end{aligned} \tag{4.28}$$

- Finally we focus on the terms due to the dynamic boundary condition, namely the terms where the inner product in $L^2(\Gamma)$ appears.
  According to the Cauchy-Scwarz inegality, the Young inequality and the equivalence of norms $\|.\|_{0,\partial\mathfrak{M}}$ and $\|.\|_{L^2(\Gamma)}$, for any $u_{\partial\mathfrak{M}} \in \mathbb{R}^{\partial\mathfrak{M}}$ we have

$$\begin{aligned}
\left(u_{\partial\mathfrak{M}}, \dot{e}^{c,n+1}_{\partial\mathfrak{M}} - \dot{e}^{c,n}_{\partial\mathfrak{M}}\right)_{L^2(\Gamma)} \leq & \frac{1}{8}\Delta t \left\|\frac{\dot{e}^{c,n+1}_{\partial\mathfrak{M}} - \dot{e}^{c,n}_{\partial\mathfrak{M}}}{\Delta t}\right\|^2_{0,\partial\mathfrak{M}} \\
& + 2C_3 \Delta t \left\|u_{\partial\mathfrak{M}}\right\|^2_{L^2(\Gamma)}.
\end{aligned} \tag{4.29}$$

Choosing $u_{\partial\mathfrak{M}} = R^{n+1}_{c_{|\Gamma}}$, then $u_{\partial\mathfrak{M}} = \frac{\bar{e}^{c,n+1}_{\partial\mathfrak{M}} - \bar{e}^{c,n}_{\partial\mathfrak{M}}}{\Delta t}$ and finally $u_{\partial\mathfrak{M}} = g_{\partial\mathfrak{M}}(t^{n+1}, \cdot)$ in equation (4.29) we can control the first three terms.

For the last term we set $u_{\partial\mathfrak{M}} = f_s'\left(\mathbb{P}^c_{\partial\mathfrak{M}}(c(t^{n+1}))\right) - \frac{m_{\underline{\partial\mathfrak{M}}}}{m_{\partial\mathfrak{M}}} d^{f_s}(c^n_{\partial\mathfrak{M}}, c^{n+1}_{\partial\mathfrak{M}})$ in equation (4.29) and noting that

$$\begin{aligned}\left\| f_s'\left(\mathbb{P}^c_{\partial\mathfrak{M}}(c(t^{n+1}))\right) - \frac{m_{\underline{\partial\mathfrak{M}}}}{m_{\partial\mathfrak{M}}} d^{f_s}(c^n_{\partial\mathfrak{M}}, c^{n+1}_{\partial\mathfrak{M}}) \right\|_{L^2(\Gamma)} &\leq \left\| \frac{m_{\partial\mathfrak{M}} - m_{\underline{\partial\mathfrak{M}}}}{m_{\partial\mathfrak{M}}} d^{f_s}(c^n_{\partial\mathfrak{M}}, c^{n+1}_{\partial\mathfrak{M}}) \right\|_{L^2(\Gamma)} \\ &\quad + \left\| f_s'\left(\mathbb{P}^c_{\partial\mathfrak{M}}(c(t^{n+1}))\right) - d^{f_s}(c^n_{\partial\mathfrak{M}}, c^{n+1}_{\partial\mathfrak{M}}) \right\|_{L^2(\Gamma)},\end{aligned}$$

we obtain

$$\begin{aligned}&- \left(f_s'\left(\mathbb{P}^c_{\partial\mathfrak{M}}(c(t^{n+1}))\right) - \frac{m_{\underline{\partial\mathfrak{M}}}}{m_{\partial\mathfrak{M}}} d^{f_s}(c^n_{\partial\mathfrak{M}}, c^{n+1}_{\partial\mathfrak{M}}), \dot{e}^{c,n+1}_{\partial\mathfrak{M}} - \dot{e}^{c,n}_{\partial\mathfrak{M}} \right)_{L^2(\Gamma)} \\ &\leq \frac{1}{8}\Delta t \left\| \frac{\dot{e}^{c,n+1}_{\partial\mathfrak{M}} - \dot{e}^{c,n}_{\partial\mathfrak{M}}}{\Delta t} \right\|^2_{0,\partial\mathfrak{M}} \\ &\quad + 4C_3\Delta t \left\| \frac{m_{\partial\mathfrak{M}} - m_{\underline{\partial\mathfrak{M}}}}{m_{\partial\mathfrak{M}}} d^{f_s}(c^n_{\partial\mathfrak{M}}, c^{n+1}_{\partial\mathfrak{M}}) \right\|^2_{L^2(\Gamma)} \\ &\quad + 4C_3\Delta t \left\| f_s'\left(\mathbb{P}^c_{\partial\mathfrak{M}}(c(t^{n+1}))\right) - d^{f_s}(c^n_{\partial\mathfrak{M}}, c^{n+1}_{\partial\mathfrak{M}}) \right\|^2_{L^2(\Gamma)}.\end{aligned} \tag{4.30}$$

Gathering inequalities (4.24)–(4.30) in equation (4.19), we obtain estimate (4.17).

□

4.3.2. Estimate of the residual terms. In order to apply the discrete Gronwall lemma, we have to estimate all the terms in the right hand side of inequality (4.17) independently of n.

We begin by a bound on the initial data.

PROPOSITION 4.13. *Let $c^0 \in \mathcal{C}^2(\overline{\Omega})$ and $c^0_{\mathcal{T}} = \mathbb{P}^c_{\mathcal{T}} c^0$. Then for some $C_{15} > 0$ we have*

$$m_\Omega\left(\dot{e}^{c,0}_{\mathfrak{M}}\right) + m_{\underline{\Omega}}\left(\dot{e}^{c,0}_{\mathfrak{M}}\right) + \left\|\dot{e}^{c,0}_{\mathfrak{M}}\right\|_{L^2(\Omega)} + \left|\dot{e}^{c,0}_{\mathcal{T}}\right|_{1,\mathcal{T}} + \left|\dot{e}^{c,0}_{\partial\mathfrak{M}}\right|_{1,\partial\mathfrak{M}} \leq C_{15}\mathrm{size}(\mathcal{T}) \left\|c^0\right\|_{H^2_\Gamma(\Omega)}.$$

Proof. Estimates of the (semi)norms are only a direct consequence of the definition of $c^0_{\mathcal{T}}$ and Proposition 4.9 (associated with Lemma 4.6 for the L^2-norm).

Then owing to Proposition 2.1 and the Cauchy-Schwarz inequality we deduce

$$m_{\underline{\Omega}}\left(\dot{e}^{c,0}_{\mathfrak{M}}\right) \leq \left(\frac{C_2}{C_1|\Omega|}\right)^{\frac{1}{2}} \left\|\dot{e}^{c,0}_{\mathfrak{M}}\right\|_{L^2(\Omega)} \text{ and } m_\Omega\left(\dot{e}^{c,0}_{\mathfrak{M}}\right) \leq \frac{1}{|\Omega|^{\frac{1}{2}}} \left\|\dot{e}^{c,0}_{\mathfrak{M}}\right\|_{L^2(\Omega)}$$

and the L^2-estimate gives the claim. □

Then we study the term $T_{R^{n+1}_c}$ about the error of the time discretization.

PROPOSITION 4.14. *For any $n_0 \in [\![0, N]\!]$, the remainder terms R^{n+1}_c and $R^{n+1}_{c_{|\Gamma}}$ defined by* (4.8) *satisfy*

$$\begin{aligned}\sum_{n=0}^{n_0} \Delta t \left\|R^{n+1}_c\right\|^2_{L^2(\Omega)} &\leq \left\|\partial_{tt} c\right\|^2_{L^2(0,T;L^2(\Omega))} \Delta t^2, \\ \sum_{n=0}^{n_0} \Delta t \left\|R^{n+1}_{c_{|\Gamma}}\right\|^2_{L^2(\Gamma)} &\leq \left\|\partial_{tt} c_{|\Gamma}\right\|^2_{L^2(0,T;L^2(\Gamma))} \Delta t^2.\end{aligned} \tag{4.31}$$

Proof. The reasoning is the same for the two estimates. Thus, we only give here the proof for the first term. Applying the Taylor's theorem to the term R_c^{n+1} defined by (4.8), we deduce

$$R_c^{n+1}(x) = \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} (t^n - s)\partial_{tt}c(s,x)\mathrm{d}s.$$

Then we have

$$\begin{aligned}\left\|R_c^{n+1}\right\|_{L^2(\Omega)}^2 \leq & \frac{1}{\Delta t^2}\left(\int_{t^n}^{t^{n+1}} (t^n - s)^2\mathrm{d}s\right)\left(\int_{t^n}^{t^{n+1}} \left\|\partial_{tt}c(s,\cdot)\right\|_{L^2(\Omega)}^2 \mathrm{d}s\right)\\ \leq & \Delta t\left(\int_{t^n}^{t^{n+1}} \left\|\partial_{tt}c(s,\cdot)\right\|_{L^2(\Omega)}^2 \mathrm{d}s\right),\end{aligned}$$

and summing these inequalities for n going from 0 to n_0 we conclude the proof. □

Now we are interested in the term $T_{\bar{e}_{\mathcal{T}}}$ concerning the error between the exact solution and its elliptic projection.

PROPOSITION 4.15. *There exists $C_{16} > 0$ such that for any $n_0 \in [\![0, N]\!]$ the following estimates hold*

$$\sum_{n=0}^{n_0} \Delta t \left\|\frac{\bar{e}_{\mathfrak{M}}^{c,n+1} - \bar{e}_{\mathfrak{M}}^{c,n}}{\Delta t}\right\|_{L^2(\Omega)}^2 \leq C_{16}^2 \mathrm{size}(\mathcal{T})^2 \left\|\partial_t c\right\|_{L^2(0,T;H^2_\Gamma(\Omega))}^2, \tag{4.32a}$$

$$\sum_{n=0}^{n_0} \Delta t \left\|\frac{\bar{e}_{\partial\mathfrak{M}}^{c,n+1} - \bar{e}_{\partial\mathfrak{M}}^{c,n}}{\Delta t}\right\|_{L^2(\Gamma)}^2 \leq C_{16}^2 \mathrm{size}(\mathcal{T})^2 \left\|\partial_t c\right\|_{L^2(0,T;H^2_\Gamma(\Omega))}^2, \tag{4.32b}$$

$$\sum_{n=0}^{n_0} \Delta t \left\|\frac{\bar{e}_{\mathfrak{M}}^{\mu,n+1} - \bar{e}_{\mathfrak{M}}^{\mu,n}}{\Delta t}\right\|_{L^2(\Omega)}^2 \leq C_{16}^2 \mathrm{size}(\mathcal{T})^2 \left\|\partial_t \mu\right\|_{L^2(0,T;H^2(\Omega))}^2. \tag{4.32c}$$

Moreover, for any $n \in [\![0, N]\!]$ we have

$$\left\|\bar{e}_{\mathfrak{M}}^{\mu,n}\right\|_{L^2(\Omega)} \leq C_{16}\mathrm{size}(\mathcal{T}) \left\|\mu\right\|_{L^\infty(0,T;H^2(\Omega))}. \tag{4.33}$$

Proof.

We begin by proving estimates (4.32). Thanks to Definition 4.10 we have

$$\begin{aligned}&\left\|\bar{e}_{\mathfrak{M}}^{c,n+1} - \bar{e}_{\mathfrak{M}}^{c,n}\right\|_{L^2(\Omega)}^2\\ &\qquad \leq 2\left\|\left(c(t^{n+1},\cdot) - c(t^n,\cdot)\right) - \left(\mathbb{P}_{\mathfrak{M}}^{ell,D}c(t^{n+1},\cdot) - \mathbb{P}_{\mathfrak{M}}^{ell,D}c(t^n,\cdot)\right)\right\|_{L^2(\Omega)}^2\\ &\qquad\quad +2\sum_{\kappa\in\mathfrak{M}}\int_{\mathcal{K}}\left(\frac{m_{\mathcal{K}} - m_{\underline{\mathcal{K}}}}{m_{\mathcal{K}}}\right)^2\left(\mathbb{P}_{\mathcal{K}}^{ell,D}c(t^{n+1},\cdot) - \mathbb{P}_{\mathcal{K}}^{ell,D}c(t^n,\cdot)\right)^2.\end{aligned}$$

Owing to the linearity of the elliptic projection (see Definition 4.5) and Proposition 4.9, the first term in the right-hand side satisfies

$$\begin{aligned}&\left\|\left(c(t^{n+1},\cdot) - c(t^n,\cdot)\right) - \left(\mathbb{P}_{\mathfrak{M}}^{ell,D}c(t^{n+1},\cdot) - \mathbb{P}_{\mathfrak{M}}^{ell,D}c(t^n,\cdot)\right)\right\|_{L^2(\Omega)}^2\\ &\qquad\qquad \leq C_{14}^2\Delta t\mathrm{size}(\mathcal{T})^2\int_{t^n}^{t^{n+1}} \left\|\partial_t c(t,\cdot)\right\|_{H^2_\Gamma(\Omega)}^2\mathrm{d}t.\end{aligned} \tag{4.34}$$

Moreover, thanks to Proposition 2.1 and the mesh regularity (2.1) for any $\kappa \in \mathfrak{M}$ we have $m_\kappa - m_{\underline{\kappa}} \leq C\mathrm{reg}(\mathcal{T})^2 m_\kappa \mathrm{diam}(\kappa)$ thus,

$$\sum_{\kappa\in\mathfrak{M}} \int_\kappa \left(\frac{m_\kappa - m_{\underline{\kappa}}}{m_\kappa}\right)^2 \left(\mathbb{P}_\kappa^{ell,D} c(t^{n+1},\cdot) - \mathbb{P}_\kappa^{ell,D} c(t^n,\cdot)\right)^2 \\ \leq C(\mathrm{reg}(\mathcal{T}))\mathrm{size}(\mathcal{T})^2 \left\| \mathbb{P}_{\mathfrak{M}}^{ell,D} c(t^{n+1},\cdot) - \mathbb{P}_{\mathfrak{M}}^{ell,D} c(t^n,\cdot)\right\|^2_{L^2(\Omega)}$$

and so Proposition 4.9 yields,

$$\sum_{\kappa\in\mathfrak{M}} \int_\kappa \left(\frac{m_\kappa - m_{\underline{\kappa}}}{m_\kappa}\right)^2 \left(\mathbb{P}_\kappa^{ell,D} c(t^{n+1},\cdot) - \mathbb{P}_\kappa^{ell,D} c(t^n,\cdot)\right)^2 \\ \leq 2C(\mathrm{reg}(\mathcal{T}))(C_{14}^2+1)\Delta t \mathrm{size}(\mathcal{T})^2 \int_{t^n}^{t^{n+1}} \|\partial_t c(t,\cdot)\|^2_{H^2_\Gamma(\Omega)}\,\mathrm{d}t. \tag{4.35}$$

Summing inequalities (4.34) and (4.35) for n going from 0 to n_0 provides estimate (4.32a).

Estimate (4.32c) is obtained with exactly the same reasoning using Definition 4.4 of the elliptic projection (instead of Definition 4.5) and Proposition 4.8 instead of Proposition 4.9.

In the same way we deduce estimate (4.32b) with a similar reasoning on Γ and Proposition 2.2 instead of Proposition 2.1.

It remains to prove estimate (4.33). For any $n \in [\![0,N]\!]$,

$$\|\bar{e}_{\mathfrak{M}}^{\mu,n}\|^2_{L^2(\Omega)} \leq 2\left\|\mu(t^n,\cdot) - \mathbb{P}_{\mathfrak{M}}^{ell,N}\mu(t^n,\cdot)\right\|^2_{L^2(\Omega)} \\ + 2\sum_{\kappa\in\mathfrak{M}} \int_\kappa \left(\frac{m_\kappa - m_{\underline{\kappa}}}{m_\kappa}\right)^2 \left(\mathbb{P}_\kappa^{ell,N}\mu(t^n,\cdot)\right)^2.$$

Then, applying Proposition 4.8 to the function $\mu(t^n,\cdot)$ the first term in the right hand side is written as follows

$$\left\|\mu(t^n,\cdot) - \mathbb{P}_{\mathfrak{M}}^{ell,N}\mu(t^n)\right\|_{L^2(\Omega)} \leq C_{13}\mathrm{size}(\mathcal{T})\,\|\mu\|_{L^\infty(0,T;H^2(\Omega))}.$$

The second term is treated as estimate (4.35) that concludes the proof.

□

We can now we concentrate on the error due to the discretization of nonlinear terms. In this way we begin with the term $T_{g_\mathcal{T}}$ related to the terms $g_\mathfrak{M}$ and $g_{\partial\mathfrak{M}}$.

PROPOSITION 4.16. *For any $t \in [0,T]$, the functions $g_\mathfrak{M}$ and $g_{\partial\mathfrak{M}}$ defined by* (4.18) *satisfy the following estimates*

$$\|g_\mathfrak{M}(t,\cdot)\|^2_{L^2(\Omega)} \leq 2L^2_{f_b'} C_{11}^2 \mathrm{size}(\mathcal{T})^2 \|\nabla c(t,\cdot)\|^2_{H^1(\Omega)}, \tag{4.36a}$$

$$\|g_{\partial\mathfrak{M}}(t,\cdot)\|^2_{L^2(\Gamma)} \leq L^2_{f_s'} C_{12}^2 \mathrm{size}(\mathcal{T})^2 \|\nabla_\Gamma c_{|\Gamma}(t,\cdot)\|^2_{L^2(\Gamma)}, \tag{4.36b}$$

where $L_{f_b'} > 0$ (respectively $L_{f_s'}$) is the Lipschitz constant of the function f_b' (respectively f_s').

Moreover, there exists $C_{17} > 0$ depending only on f_b (and on its derivatives) such that for any $n_0 \in [\![0,N]\!]$,

$$\sum_{n=0}^{n_0} \Delta t \left\| \frac{g_\mathfrak{M}(t^{n+1},\cdot) - g_\mathfrak{M}(t^n,\cdot)}{\Delta t}\right\|^2_{L^2(\Omega)} \leq C_{17} M_\infty^c \|\partial_t c\|^2_{L^2(0,T;H^2(\Omega))} \mathrm{size}(\mathcal{T})^2 \tag{4.37}$$

where,

$$M_\infty^c = \max\left(1, \|\nabla c\|^4_{L^\infty(0,T;L^\infty(\Omega))}, \left\|D^2 c\right\|^2_{L^\infty(0,T;L^\infty(\Omega))}\right).$$

Proof. Thanks to definition (4.18) of $g_{\mathfrak{M}}$ and $g_{\partial\mathfrak{M}}$ and since f_b' and f_s' are Lipschitz continuous functions, Lemmas 4.6 and 4.7 yield estimates (4.36).

We concentrate now on estimate (4.37). For any $n \in [\![0, N-1]\!]$ we define the function h^n such that for any $x \in \Omega$,

$$h^n(x) = f_b'(c(t^{n+1}, x)) - f_b'(c(t^n, x)),$$

Therefore, for any $\kappa \in \mathfrak{M}$ and for any $n \in [\![0, N-1]\!]$,

$$g_{\mathfrak{M}}(t^{n+1}, x) - g_{\mathfrak{M}}(t^n, x) = h^n(x) - h^n(x_{\mathcal{K}}), \quad \forall x \in \kappa.$$

Thus, thanks to Lemma 4.6 we have

$$\left\| g_{\mathfrak{M}}(t^{n+1}, \cdot) - g_{\mathfrak{M}}(t^n, \cdot) \right\|_{L^2(\Omega)}^2 \leq C_{11}^2 \mathrm{size}(\mathcal{T})^2 \left\| \nabla h^n \right\|_{H^1(\Omega)}^2.$$

Moreover, there exists M_{f_b} depending only on the Lipschitz constants of the derivatives of f_b (up to 3) such that

$$\begin{aligned}\left\| \nabla h^n \right\|_{H^1(\Omega)}^2 \leq M_{f_b} \Delta t \Bigg(& \left(\left\| \nabla c \right\|_{L^\infty(0,T;L^\infty(\Omega))}^4 + \left\| D^2 c \right\|_{L^\infty(0,T;L^\infty(\Omega))}^2 \right) \int_{t^n}^{t^{n+1}} \left\| \partial_t c \right\|_{L^2(\Omega)}^2 \\ & + \left\| \nabla c \right\|_{L^\infty(0,T;L^\infty(\Omega))}^2 \int_{t^n}^{t^{n+1}} \left\| \partial_t \nabla c \right\|_{L^2(\Omega)}^2 + \int_{t^n}^{t^{n+1}} \left\| \partial_t c \right\|_{H^2(\Omega)}^2 \Bigg).\end{aligned}$$

Summing these inequalities for n going from 0 to n_0 gives estimate (4.37). □

Now we have to deal with the main difficulty of the discretization of nonlinear terms. In each case the reasoning is similar. Thus we begin by proving the following general result.

LEMMA 4.17. *Let us consider a function $\phi \in \mathcal{C}^2(\mathbb{R}^2, \mathbb{R})$ such that all the derivatives up to the second order are bounded. Then, there exists $C_{18} > 0$ depending only on ϕ and on the mesh regularity* $\mathrm{reg}(\mathcal{T})$*, such that for any $a_\mathcal{T}, b_\mathcal{T} \in \mathbb{R}^\mathcal{T}$ the following estimates hold*

$$|\phi(a_\mathcal{T}, b_\mathcal{T})|_{1,\mathcal{T}} \leq C_{18} \left(|a_\mathcal{T}|_{1,\mathcal{T}} + |b_\mathcal{T}|_{1,\mathcal{T}} \right) \tag{4.38}$$

and for any $\widetilde{b}_\mathcal{T} \in \mathbb{R}^\mathcal{T}$,

$$\begin{aligned}|\phi(a_\mathcal{T}, b_\mathcal{T}) - \phi(b_\mathcal{T}, b_\mathcal{T})|_{1,\mathcal{T}}^2 \leq C_{18} \Bigg(& |a_\mathcal{T} - b_\mathcal{T}|_{1,\mathcal{T}}^2 + \left| b_\mathcal{T} - \widetilde{b}_\mathcal{T} \right|_{1,\mathcal{T}}^2 \\ & + \mathrm{Lip}_\mathcal{T}\left(\widetilde{b}_\mathcal{T} \right)^2 \left\| a_{\mathfrak{M}} - b_{\mathfrak{M}} \right\|_{L^2(\Omega)}^2 \Bigg),\end{aligned} \tag{4.39}$$

where

$$\mathrm{Lip}_\mathcal{T}\left(\widetilde{b}_\mathcal{T} \right) = \sup_{\sigma \in \mathcal{E}} \left| \frac{\widetilde{b}_\mathcal{K} - \widetilde{b}_\mathcal{L}}{d_{\mathcal{K},\mathcal{L}}} \right|.$$

Moreover for any $a_{\partial\mathfrak{M}}, b_{\partial\mathfrak{M}} \in \mathbb{R}^{\partial\mathfrak{M}}$ we have

$$\left\| \phi(a_{\partial\mathfrak{M}}, b_{\partial\mathfrak{M}}) - \phi(b_{\partial\mathfrak{M}}, b_{\partial\mathfrak{M}}) \right\|_{L^2(\Gamma)} \leq C_{18} \left\| a_{\partial\mathfrak{M}} - b_{\partial\mathfrak{M}} \right\|_{L^2(\Gamma)}. \tag{4.40}$$

Proof . We first give the proof of estimate (4.39). Thanks to the definition of the discrete H^1-seminorm we have

$$|\phi(a_\mathcal{T},b_\mathcal{T})-\phi(b_\mathcal{T},b_\mathcal{T})|_{1,\mathcal{T}}^2=\sum_{\sigma=\mathcal{K}|\mathcal{L}\in\mathcal{E}_{int}} m_\sigma d_{\mathcal{K},\mathcal{L}}\left(\frac{\phi_\sigma(a_\mathcal{T},b_\mathcal{T})}{d_{\mathcal{K},\mathcal{L}}}\right)^2 + \sum_{\sigma=\mathcal{L}\in\mathcal{E}_{ext}} m_{\mathbf{e}_\mathcal{L}} d_{\mathcal{K},\mathcal{L}}\left(\frac{\phi_\sigma(a_\mathcal{T},b_\mathcal{T})}{d_{\mathcal{K},\mathcal{L}}}\right)^2.$$

where for any $\sigma=\mathcal{K}|\mathcal{L}\in\mathcal{E}_{int}$ or $\sigma=\mathcal{L}\in\mathcal{E}_{ext}\cap\mathcal{E}_\mathcal{K}$ (with $\mathcal{K}\in\mathfrak{M}$) we note:

$$\phi_\sigma(a_\mathcal{T},b_\mathcal{T})=[\phi(a_\mathcal{K},b_\mathcal{K})-\phi(a_\mathcal{L},b_\mathcal{L})]-[\phi(b_\mathcal{K},b_\mathcal{K})-\phi(b_\mathcal{L},b_\mathcal{L})]\,.$$

Noting that

$$\begin{aligned}\phi(a_\mathcal{K},b_\mathcal{K})-\phi(a_\mathcal{L},b_\mathcal{L})=\int_0^1\Big((a_\mathcal{K}-a_\mathcal{L})\partial_1\phi(a_\mathcal{L}+s(a_\mathcal{K}-a_\mathcal{L}),b_\mathcal{L}+s(b_\mathcal{K}-b_\mathcal{L}))\\ +(b_\mathcal{K}-b_\mathcal{L})\partial_2\phi(a_\mathcal{L}+s(a_\mathcal{K}-a_\mathcal{L}),b_\mathcal{L}+s(b_\mathcal{K}-b_\mathcal{L}))\Big)\mathrm{d}s\end{aligned}\tag{4.41}$$

and

$$\begin{aligned}\phi(b_\mathcal{K},b_\mathcal{K})-\phi(b_\mathcal{L},b_\mathcal{L})=\int_0^1\Big((b_\mathcal{K}-b_\mathcal{L})\partial_1\phi(b_\mathcal{L}+s(b_\mathcal{K}-b_\mathcal{L}),b_\mathcal{L}+s(b_\mathcal{K}-b_\mathcal{L}))\\ +(b_\mathcal{K}-b_\mathcal{L})\partial_2\phi(b_\mathcal{L}+s(b_\mathcal{K}-b_\mathcal{L}),b_\mathcal{L}+s(b_\mathcal{K}-b_\mathcal{L}))\Big)\mathrm{d}s.\end{aligned}$$

Then for any $\sigma\in\mathcal{E}$ the term $\phi_\sigma(a_\mathcal{T},b_\mathcal{T})$ can be written as follows

$$\phi_\sigma(a_\mathcal{T},b_\mathcal{T})=\phi_\sigma^1(a_\mathcal{T},b_\mathcal{T})+\phi_\sigma^2(a_\mathcal{T},b_\mathcal{T})\tag{4.42}$$

with

$$\begin{aligned}\phi_\sigma^1(a_\mathcal{T},b_\mathcal{T})=\int_0^1(b_\mathcal{K}-b_\mathcal{L})\Big(&\partial_2\phi(a_\mathcal{L}+s(a_\mathcal{K}-a_\mathcal{L}),b_\mathcal{L}+s(b_\mathcal{K}-b_\mathcal{L}))\\ &-\partial_2\phi(b_\mathcal{L}+s(b_\mathcal{K}-b_\mathcal{L}),b_\mathcal{L}+s(b_\mathcal{K}-b_\mathcal{L}))\\ &+\partial_1\phi(a_\mathcal{L}+s(a_\mathcal{K}-a_\mathcal{L}),b_\mathcal{L}+s(b_\mathcal{K}-b_\mathcal{L}))\\ &-\partial_1\phi(b_\mathcal{L}+s(b_\mathcal{K}-b_\mathcal{L}),b_\mathcal{L}+s(b_\mathcal{K}-b_\mathcal{L}))\Big)\mathrm{d}s\end{aligned}$$

and

$$\phi_\sigma^2(a_\mathcal{T},b_\mathcal{T})=\int_0^1\Big((a_\mathcal{K}-a_\mathcal{L})-(b_\mathcal{K}-b_\mathcal{L})\Big)\partial_1\phi(a_\mathcal{L}+s(a_\mathcal{K}-a_\mathcal{L}),b_\mathcal{L}+s(b_\mathcal{K}-b_\mathcal{L}))\mathrm{d}s.$$

First, we study the term $\phi_\sigma^2(a_\mathcal{T},b_\mathcal{T})$. Since the function $\partial_1\phi$ is bounded we obtain,

$$\left|\phi_\sigma^2(a_\mathcal{T},b_\mathcal{T})\right|_{1,\mathcal{T}}\leq\|\partial_1\phi\|_{L^\infty}\,|a_\mathcal{T}-b_\mathcal{T}|_{1,\mathcal{T}}\,.$$

As regards the term $\phi_\sigma^1(a_\mathcal{T},b_\mathcal{T})$, we write:

$$\phi_\sigma^1(a_\mathcal{T},b_\mathcal{T})=\phi_\sigma^{1,1}(a_\mathcal{T},b_\mathcal{T})+\phi_\sigma^{1,2}(a_\mathcal{T},b_\mathcal{T})$$

with

$$\phi_\sigma^{1,1}(a_\tau,b_\tau)=\int_0^1\left((b_\kappa-\widetilde{b}_\kappa)-(b_\iota-\widetilde{b}_\iota)\right)\Big(\partial_2\phi(a_\iota+s(a_\kappa-a_\iota),b_\iota+s(b_\kappa-b_\iota))\\ -\partial_2\phi(b_\iota+s(b_\kappa-b_\iota),b_\iota+s(b_\kappa-b_\iota))\\ +\partial_1\phi(a_\iota+s(a_\kappa-a_\iota),b_\iota+s(b_\kappa-b_\iota))\\ -\partial_1\phi(b_\iota+s(b_\kappa-b_\iota),b_\iota+s(b_\kappa-b_\iota))\Big)\mathrm{d}s$$

and

$$\phi_\sigma^{1,2}(a_\tau,b_\tau)=\int_0^1(\widetilde{b}_\kappa-\widetilde{b}_\iota)\Big(\partial_2\phi(a_\iota+s(a_\kappa-a_\iota),b_\iota+s(b_\kappa-b_\iota))\\ -\partial_2\phi(b_\iota+s(b_\kappa-b_\iota),b_\iota+s(b_\kappa-b_\iota))\\ +\partial_1\phi(a_\iota+s(a_\kappa-a_\iota),b_\iota+s(b_\kappa-b_\iota))\\ -\partial_1\phi(b_\iota+s(b_\kappa-b_\iota),b_\iota+s(b_\kappa-b_\iota))\Big)\mathrm{d}s.$$

Since the functions $\partial_1\phi$ and $\partial_2\phi$, are bounded we get

$$\left|\phi_\sigma^{1,1}(a_\tau,b_\tau)\right|_{1,\tau}\leq\max\left(\left\|\partial_1\phi\right\|_{L^\infty},\left\|\partial_2\phi\right\|_{L^\infty}\right)\left|b_\tau-\widetilde{b}_\tau\right|_{1,\tau}.$$

Finally we have to deal with the term $\phi_\sigma^{1,2}(a_\tau,b_\tau)$. For any $i=1,2$, since $\partial_{1,i}\phi$ is bounded we have

$$\left|\partial_i\phi(a_\iota+s(a_\kappa-a_\iota),b_\iota+s(b_\kappa-b_\iota))-\partial_i\phi(b_\iota+s(b_\kappa-b_\iota),b_\iota+s(b_\kappa-b_\iota))\right|\\ \leq\left\|\partial_{1,i}\phi\right\|_{L^\infty}\left|(1-s)(a_\iota-b_\iota)+s(a_\kappa-b_\kappa)\right|.$$

and so for any $\sigma\in\mathcal{E}$, we obtain

$$\left(\frac{\phi_\sigma^{1,2}(a_\tau,b_\tau)}{d_{\kappa,\iota}}\right)^2\leq 2\left(\max\left(\left\|\partial_{1,1}\phi\right\|_{L^\infty},\left\|\partial_{1,2}\phi\right\|_{L^\infty}\right)\right)^2\left|\frac{\widetilde{b}_\kappa-\widetilde{b}_\iota}{d_{\kappa,\iota}}\right|^2\\ \left(|a_\kappa-b_\kappa|^2+|a_\iota-b_\iota|^2\right).$$

Since $d_{\kappa,\iota}\leq\mathrm{reg}(\mathcal{T})d(x_\kappa,\sigma)$ for any $\kappa\in\mathfrak{M}$, for any $\sigma\in\mathcal{E}_\kappa$ (see definition (2.1)), there exists $C(\mathrm{reg}(\mathcal{T}))$ such that

$$\left|\phi_\sigma^{1,2}(a_\tau,b_\tau)\right|_{1,\tau}\leq C(\mathrm{reg}(\mathcal{T}))\max\left(\left\|\partial_{1,1}\phi\right\|_{L^\infty},\left\|\partial_{1,2}\phi\right\|_{L^\infty}\right)\mathrm{Lip}_\tau\left(\widetilde{b}_\tau\right)\left\|a_{\mathfrak{m}}-b_{\mathfrak{m}}\right\|_{L^2(\Omega)},$$

and estimate (4.39) yields.

Owing to (4.41) we also obtain estimate (4.38) and a similar reasoning gives estimate (4.40).
□

With this result at hand we can now bounded the terms T_{f_b} (Proposition 4.18) and T_{f_s} (Proposition 4.19).

PROPOSITION 4.18. *For any $n_0 \in [\![0, N]\!]$, there exists $C_{19} > 0$ (depending on f_b but not on Δt and* size$(\mathcal{T})$*) such that,*

$$\begin{aligned}
&\sum_{n=0}^{n_0} \Delta t \left(\left| f_b'(\mathbb{P}_\mathcal{T}^c(c(t^{n+1}))) - f_b'(c_\mathcal{T}^{n+1}) \right|_{1,\mathcal{T}}^2 + \left| d^{f_b}(c_\mathcal{T}^{n+1}, c_\mathcal{T}^{n+1}) - d^{f_b}(c_\mathcal{T}^n, c_\mathcal{T}^{n+1}) \right|_{1,\mathcal{T}}^2 \right) \\
&\quad \leq C_{19} \left(1 + \|\nabla c\|_{L^\infty(0,T;L^\infty(\Omega))}^2 \right) \left(\sum_{n=1}^{n_0+1} \Delta t \left| \dot{e}_\mathcal{T}^{c,n} \right|_{1,\mathcal{T}}^2 + \sum_{n=0}^{n_0} \Delta t \left| \dot{e}_\mathcal{T}^{c,n+1} - \dot{e}_\mathcal{T}^{c,n} \right|_{1,\mathcal{T}}^2 \right) \\
&\qquad + C_{19} \text{size}(\mathcal{T})^2 \|c\|_{L^\infty(0,T;H_\Gamma^2(\Omega))}^2 \\
&\qquad + C_{19} \|\nabla c\|_{L^\infty(0,T;L^\infty(\Omega))}^2 \left(\left\| c^0 \right\|_{H_\Gamma^2(\Omega)}^2 + \|c\|_{L^\infty(0,T;H_\Gamma^2(\Omega))}^2 \right) \text{size}(\mathcal{T})^2 \\
&\qquad + C_{19} \left(1 + \|\nabla c\|_{L^\infty(0,T;L^\infty(\Omega))}^2 \right) \|\partial_t \nabla c\|_{L^2(0,T;L^\infty(\Omega))}^2 \Delta t^2 \\
&\qquad + C_{19} \left(1 + \|\nabla c\|_{L^\infty(0,T;L^\infty(\Omega))}^2 \right) \|\partial_t c\|_{L^2(0,T;H_\Gamma^2(\Omega))}^2 \Delta t^2 \text{size}(\mathcal{T})^2.
\end{aligned}$$

Proof. We will apply Lemma 4.17 twice with a good choice of the functions which intervene. First, we apply the lemma to the function ϕ defined by $\phi(x, y) = f_b'(x)$ for any $x, y \in \mathbb{R}$. Then we choose $a_\mathcal{T} = c_\mathcal{T}^{n+1}$ and $b_\mathcal{T} = \widetilde{b}_\mathcal{T} = \mathbb{P}_\mathcal{T}^c(c(t^{n+1}))$, and so thanks to (4.39) we obtain

$$\begin{aligned}
&\left| f_b'(\mathbb{P}_\mathcal{T}^c(c(t^{n+1}))) - f_b'(c_\mathcal{T}^{n+1}) \right|_{1,\mathcal{T}}^2 \\
&\qquad \leq C_{18} \Big(\left| e_\mathcal{T}^{c,n+1} \right|_{1,\mathcal{T}}^2 + \text{Lip}_\mathcal{T} \left(\mathbb{P}_\mathcal{T}^c(c(t^{n+1})) \right)^2 \left\| e_\mathfrak{M}^{c,n+1} \right\|_{L^2(\Omega)}^2 \Big).
\end{aligned} \tag{4.43}$$

Now we apply Lemma 4.17 to the function $\phi = d^{f_b}$ with $a_\mathcal{T} = c_\mathcal{T}^n$, $b_\mathcal{T} = c_\mathcal{T}^{n+1}$ and $\widetilde{b}_\mathcal{T} = \mathbb{P}_\mathcal{T}^c(c(t^{n+1}))$. Inequality (4.39) implies,

$$\begin{aligned}
\left| d^{f_b}(c_\mathcal{T}^{n+1}, c_\mathcal{T}^{n+1}) - d^{f_b}(c_\mathcal{T}^n, c_\mathcal{T}^{n+1}) \right|_{1,\mathcal{T}}^2 &\leq C_{18} \Big(\left| c_\mathcal{T}^{n+1} - c_\mathcal{T}^n \right|_{1,\mathcal{T}}^2 + \left| e_\mathcal{T}^{c,n+1} \right|_{1,\mathcal{T}}^2 \\
&\quad + \text{Lip}_\mathcal{T} \left(\mathbb{P}_\mathcal{T}^c(c(t^{n+1})) \right)^2 \left\| c_\mathfrak{M}^{n+1} - c_\mathfrak{M}^n \right\|_{L^2(\Omega)}^2 \Big).
\end{aligned} \tag{4.44}$$

Thanks to the mean-value theorem, we have

$$\text{Lip}_\mathcal{T} \left(\mathbb{P}_\mathcal{T}^c(c(t^{n+1})) \right) \leq \|\nabla c\|_{L^\infty(0,T;L^\infty(\Omega))}. \tag{4.45}$$

Moreover for any $n \in [\![0, N]\!]$, we have

$$\|e_\mathfrak{M}^{c,n}\|_{L^2(\Omega)}^2 \leq 2 \|\dot{e}_\mathfrak{M}^{c,n}\|_{L^2(\Omega)}^2 + 2 \|\ddot{e}_\mathfrak{M}^{c,n}\|_{L^2(\Omega)}^2. \tag{4.46}$$

The definition of the elliptic projection and the volume conservation at the discrete level (3.2) imply $m_{\underline{\Omega}}(\dot{e}_\mathfrak{M}^{c,n}) = m_{\underline{\Omega}}\left(\dot{e}_\mathfrak{M}^{c,0}\right)$, thus thanks to the bound on the initial data (Proposition 4.13), Lemma 2.7 and Proposition 2.1 we deduce,

$$\|\dot{e}_\mathfrak{M}^{c,n}\|_{L^2(\Omega)}^2 \leq 2C_2C_4^2 \left| \dot{e}_\mathcal{T}^{c,n} \right|_{1,\mathcal{T}}^2 + \frac{2C_2C_{15}^2}{C_1} |\Omega| \text{size}(\mathcal{T})^2 \left\| c^0 \right\|_{H_\Gamma^2(\Omega)}^2. \tag{4.47}$$

Moreover Lemma 4.6 and Proposition 4.9 give,

$$\begin{aligned}
\|\ddot{e}_\mathfrak{M}^{c,n}\|_{L^2(\Omega)} &\leq (C_{11} + C_{14}) \text{size}(\mathcal{T}) \|c(t^n, \cdot)\|_{H_\Gamma^2(\Omega)} \\
\left| \ddot{e}_\mathfrak{M}^{c,n} \right|_{1,\mathcal{T}} &\leq C_{14} \text{size}(\mathcal{T}) \|c(t^n, \cdot)\|_{H_\Gamma^2(\Omega)}.
\end{aligned} \tag{4.48}$$

Thus, gathering estimates (4.46), (4.47) and (4.48) there exists $C_{20} > 0$ such that for any $n \in [\![0, N]\!]$,

$$\|e_{\mathfrak{M}}^{c,n}\|_{L^2(\Omega)}^2 \leq C_{20} \left|\dot{e}_{\mathcal{T}}^{c,n}\right|_{1,\mathcal{T}}^2 + C_{20}\text{size}(\mathcal{T})^2 \left(\left\|c^0\right\|_{H_\Gamma^2(\Omega)}^2 + \|c\|_{L^\infty(0,T;H_\Gamma^2(\Omega))}^2\right). \tag{4.49}$$

Now, we have to deal with the terms on the right hand side of (4.44). First, we write

$$\left|c_{\mathcal{T}}^{n+1} - c_{\mathcal{T}}^n\right|_{1,\mathcal{T}} \leq \left|\dot{e}_{\mathcal{T}}^{c,n+1} - \dot{e}_{\mathcal{T}}^{c,n}\right|_{1,\mathcal{T}} + \left|\ddot{e}_{\mathcal{T}}^{c,n+1} - \ddot{e}_{\mathcal{T}}^{c,n}\right|_{1,\mathcal{T}} + \left|\mathbb{P}_{\mathcal{T}}^c c(t^{n+1}) - \mathbb{P}_{\mathcal{T}}^c c(t^n)\right|_{1,\mathcal{T}}.$$

Since the projections $\mathbb{P}_{\mathcal{T}}^{ell,D}$ and $\mathbb{P}_{\mathcal{T}}^c$ are linear we can apply Proposition 4.9 that implies

$$\left|\ddot{e}_{\mathcal{T}}^{c,n+1} - \ddot{e}_{\mathcal{T}}^{c,n}\right|_{1,\mathcal{T}}^2 \leq C_{14}^2 \Delta t \text{size}(\mathcal{T})^2 \int_{t^n}^{t^{n+1}} \|\partial_t c(t,\cdot)\|_{H_\Gamma^2(\Omega)}^2 \,\mathrm{d}t.$$

Moreover, thanks to Proposition 2.3 there exists $C > 0$ depending only on Ω and $\text{reg}(\mathcal{T})$ such that,

$$\begin{aligned}
&\left|\mathbb{P}_{\mathcal{T}}^c c(t^{n+1}) - \mathbb{P}_{\mathcal{T}}^c c(t^n)\right|_{1,\mathcal{T}}^2 \\
&\leq \Delta t \sum_{\sigma=\mathcal{K}|\mathcal{L}\in\mathcal{E}_{int}} m_\sigma d_{\mathcal{K},\mathcal{L}} \int_{t^n}^{t^{n+1}} \left|\int_0^1 \partial_t \nabla c(t, x_\mathcal{L} + s(x_\mathcal{K} - x_\mathcal{L})) \cdot \vec{\mathbf{n}}_{\boldsymbol{\mathcal{K}\mathcal{L}}} \mathrm{d}s\right|^2 \mathrm{d}t \\
&+ \Delta t \sum_{\sigma=\mathcal{L}\in\mathcal{E}_{ext}} m_{\mathbf{e}_\mathcal{L}} d_{\mathcal{K},\mathcal{L}} \int_{t^n}^{t^{n+1}} \left|\frac{d(x_\mathcal{K}, x_\mathcal{L})}{d_{\mathcal{K},\mathcal{L}}} \int_0^1 \partial_t \nabla c(t, x_\mathcal{L} + s(x_\mathcal{K} - x_\mathcal{L})) \cdot \vec{\mathbf{n}}_{\boldsymbol{\mathcal{K}\mathcal{L}}} \mathrm{d}s\right|^2 \mathrm{d}t \\
&\leq C\Delta t \int_{t^n}^{t^{n+1}} \|\partial_t \nabla c(t,\cdot)\|_{L^\infty(\Omega)}^2 \,\mathrm{d}t.
\end{aligned}$$

Therefore we have,

$$\begin{aligned}
\left|c_{\mathcal{T}}^{n+1} - c_{\mathcal{T}}^n\right|_{1,\mathcal{T}}^2 \leq& 3\left|\dot{e}_{\mathcal{T}}^{c,n+1} - \dot{e}_{\mathcal{T}}^{c,n}\right|_{1,\mathcal{T}}^2 + 3C_{14}^2 \Delta t \text{size}(\mathcal{T})^2 \int_{t^n}^{t^{n+1}} \|\partial_t c(t,\cdot)\|_{H_\Gamma^2(\Omega)}^2 \\
&+ 3C\Delta t \int_{t^n}^{t^{n+1}} \|\partial_t \nabla c(t,\cdot)\|_{L^\infty(\Omega)}^2 \,\mathrm{d}t.
\end{aligned} \tag{4.50}$$

Finally, the discrete conservation of the volume (3.2) implies $m_{\underline{\Omega}}\left(c_{\mathfrak{M}}^{n+1} - c_{\mathfrak{M}}^n\right) = 0$ and so Proposition 2.1 and Lemma 2.7 give,

$$\left\|c_{\mathfrak{M}}^{n+1} - c_{\mathfrak{M}}^n\right\|_{L^2(\Omega)} \leq \sqrt{C_2} C_4 \left|c_{\mathcal{T}}^{n+1} - c_{\mathcal{T}}^n\right|_{1,\mathcal{T}}. \tag{4.51}$$

Summing (4.43) and (4.44), gathering estimates (4.45), (4.48), (4.49), (4.50) and (4.51) and summing the resulting inequality for n going from 0 to n_0 conclude the proof.

□

PROPOSITION 4.19. *For any $n_0 \in [\![0, N]\!]$, there exists $C_{21} > 0$ independent of Δt and* $\text{size}(\mathcal{T})$ *(but depending on f_s) such that,*

$$\begin{aligned}
&\sum_{n=0}^{n_0} \Delta t \left\|f_s'\left(\mathbb{P}_{\partial\mathfrak{M}}^c(c(t^{n+1}))\right) - d^{f_s}(c_{\partial\mathfrak{M}}^n, c_{\partial\mathfrak{M}}^{n+1})\right\|_{L^2(\Gamma)} \\
&\qquad \leq C_{21} \sum_{n=0}^{n_0} \Delta t \left|\dot{e}_{\mathcal{T}}^{c,n+1}\right|_{1,\mathcal{T}}^2 + C_{21}\Delta t^2 \sum_{n=0}^{n_0} \Delta t \left\|\frac{\dot{e}_{\partial\mathfrak{M}}^{c,n+1} - \dot{e}_{\partial\mathfrak{M}}^{c,n}}{\Delta t}\right\|_{0,\partial\mathfrak{M}}^2 \\
&\qquad + C_{21}\text{size}(\mathcal{T})^2 \left\|c^0\right\|_{H_\Gamma^2(\Omega)}^2 + C_{21}\text{size}(\mathcal{T})^2 \|c\|_{L^\infty(0,T;H_\Gamma^2(\Omega))}^2 \\
&\qquad + C_{21}\text{size}(\mathcal{T})^2 \Delta t^2 \|\partial_t c\|_{L^2(0,T;H_\Gamma^2(\Omega))}^2 + C_{21}\Delta t^2 \left\|\partial_t c_{|\Gamma}\right\|_{L^2(0,T;H^1(\Gamma))}^2.
\end{aligned}$$

Proof.

Applying inequality (4.40) to the function $\phi = f_s'$ at first, and then, to $\phi = d^{f_s}$ give

$$\left\| f_s' \left(\mathbb{P}^c_{\partial\mathfrak{M}}(c(t^{n+1})) \right) - d^{f_s}(c^n_{\partial\mathfrak{M}}, c^{n+1}_{\partial\mathfrak{M}}) \right\|_{L^2(\Gamma)} \leq C_{18} \left(\left\| e^{c,n+1}_{\partial\mathfrak{M}} \right\|_{L^2(\Gamma)} + \left\| c^{n+1}_{\partial\mathfrak{M}} - c^n_{\partial\mathfrak{M}} \right\|_{L^2(\Gamma)} \right).$$

Owing to Lemma 2.10 and estimate (4.47) we have

$$\left\| \dot{e}^{c,n+1}_{\partial\mathfrak{M}} \right\|^2_{L^2(\Gamma)} \leq 2C_7^2 \left((1 + 2C_2C_4^2) \left| \dot{e}^{c,n+1}_{\mathcal{T}} \right|^2_{1,\mathcal{T}} + \frac{2C_2C_{15}^2}{C_1} |\Omega| \mathrm{size}(\mathcal{T})^2 \left\| c^0 \right\|^2_{H^2_\Gamma(\Omega)} \right).$$

Moreover, Lemma 4.7 and Proposition 4.9 we obtain

$$\left\| \ddot{e}^{c,n+1}_{\partial\mathfrak{M}} \right\|_{L^2(\Gamma)} \leq (C_{12} + C_{14}) \mathrm{size}(\mathcal{T}) \left\| c(t^{n+1}, \cdot) \right\|_{H^2_\Gamma(\Omega)}.$$

Finally we write

$$\begin{aligned} \left\| c^{n+1}_{\partial\mathfrak{M}} - c^n_{\partial\mathfrak{M}} \right\|_{L^2(\Gamma)} \leq & \left\| \dot{e}^{c,n+1}_{\partial\mathfrak{M}} - \dot{e}^{c,n}_{\partial\mathfrak{M}} \right\|_{L^2(\Gamma)} + \left\| \ddot{e}^{c,n+1}_{\partial\mathfrak{M}} - \ddot{e}^{c,n}_{\partial\mathfrak{M}} \right\|_{L^2(\Gamma)} \\ & + \left\| \mathbb{P}^c_{\partial\mathfrak{M}} c(t^{n+1}) - \mathbb{P}^c_{\partial\mathfrak{M}} c(t^n) \right\|_{L^2(\Gamma)}. \end{aligned}$$

Then, the linearity of the projection $\mathbb{P}^{ell,D}_{\partial\mathfrak{M}}$ and $\mathbb{P}^c_{\partial\mathfrak{M}}$, Lemma 4.7 and Proposition 4.9 give

$$\left\| \ddot{e}^{c,n+1}_{\partial\mathfrak{M}} - \ddot{e}^{c,n}_{\partial\mathfrak{M}} \right\|^2_{L^2(\Gamma)} \leq (C_{12} + C_{14})^2 \mathrm{size}(\mathcal{T})^2 \Delta t \int_{t^n}^{t^{n+1}} \left\| \partial_t c(t, \cdot) \right\|^2_{H^2_\Gamma(\Omega)} \mathrm{d}t,$$

and

$$\begin{aligned} \left\| \mathbb{P}^c_{\partial\mathfrak{M}} c(t^{n+1}) - \mathbb{P}^c_{\partial\mathfrak{M}} c(t^n) \right\|^2_{L^2(\Gamma)} &\leq \Delta t \int_{t^n}^{t^{n+1}} \left\| \mathbb{P}^c_{\partial\mathfrak{M}} \partial_t c(t, \cdot) \right\|^2_{L^2(\Gamma)} \mathrm{d}t \\ &\leq 2(C_{12}^2 + 1) \Delta t \int_{t^n}^{t^{n+1}} \left\| \partial_t c_{\mid\Gamma}(t, \cdot) \right\|^2_{H^1(\Gamma)} \mathrm{d}t. \end{aligned}$$

Thus gathering these estimates and summing the resulting inequality between 0 and n_0 conclude the proof.

□

It remains to be bounded the error $T_{\mathcal{T}}$ due to the fact that the domain Ω is not polygonal.

PROPOSITION 4.20. *There exists $C_{22} > 0$ depending only on $\mathrm{reg}(\mathcal{T})$, f_b and f_s such that*

$$\begin{aligned} \sum_{n=0}^{n_0} \Delta t \left| \frac{m_{\mathcal{T}} - m_{\underline{\mathcal{T}}}}{m_{\mathcal{T}}} f_b' \left(\mathbb{P}^c_{\mathcal{T}} c(t^{n+1}) \right) \right|^2_{1,\mathcal{T}} + \sum_{n=0}^{n_0} \Delta t \left\| \frac{m_{\partial\mathfrak{M}} - m_{\underline{\partial\mathfrak{M}}}}{m_{\partial\mathfrak{M}}} d^{f_s}(c^n_{\partial\mathfrak{M}}, c^{n+1}_{\partial\mathfrak{M}}) \right\|^2_{L^2(\Gamma)} \\ \leq C_{22} \mathrm{size}(\mathcal{T})^2. \end{aligned}$$

Proof. Thanks to the mesh regularity (2.1), for any $\kappa \in \mathfrak{M}$ we have $\mathrm{diam}(\kappa) \leq \mathrm{reg}(\mathcal{T})^2 m_\kappa$ thus owing to Proposition 2.1 we obtain,

$$\frac{m_\kappa - m_{\underline{\kappa}}}{m_\kappa} \leq C(\mathrm{reg}(\mathcal{T})) \mathrm{size}(\mathcal{T}).$$

Similarly, thanks to Proposition 2.2 for any $\mathcal{L} \in \partial\mathfrak{M}$,

$$\frac{m_\mathcal{L} - m_{\mathbf{e}_\mathcal{L}}}{m_\mathcal{L}} \leq C \mathrm{size}(\mathcal{T})^2.$$

Thus the definition of the H^1-seminorm on Ω and Lemma 4.17 give,

$$\begin{aligned}\left|\frac{m_{\mathcal{T}}-m_{\underline{\mathcal{T}}}}{m_{\mathcal{T}}}f_b'\left(\mathbb{P}^c_{\mathcal{T}}c(t^{n+1})\right)\right|^2_{1,\mathcal{T}} \leq & C_{18}C(\mathrm{reg}(\mathcal{T}))^2\mathrm{size}(\mathcal{T})^2\left|\mathbb{P}^c_{\mathcal{T}}c(t^{n+1})\right|^2_{1,\mathcal{T}}\\ \leq & C_{18}C'(\mathrm{reg}(\mathcal{T}))^2\mathrm{size}(\mathcal{T})^2\left\|\nabla c\right\|^2_{L^\infty(0,T;L^\infty(\Omega))},\end{aligned}$$

and

$$\left\|\frac{m_{\partial\mathfrak{M}}-m_{\underline{\partial\mathfrak{M}}}}{m_{\partial\mathfrak{M}}}d^{f_s}(c^n_{\partial\mathfrak{M}},c^{n+1}_{\partial\mathfrak{M}})\right\|^2_{L^2(\Gamma)}\leq C^2\mathrm{size}(\mathcal{T})^4\left\|d^{f_s}(c^n_{\partial\mathfrak{M}},c^{n+1}_{\partial\mathfrak{M}})\right\|^2_{L^2(\Gamma)}.$$

Thanks to the bounds on the discrete solutions (see Proposition 4.3) for any $\mathcal{L}\in\partial\mathfrak{M}$ we have

$$\left|d^{f_s}(c^n_{\mathcal{L}},c^{n+1}_{\mathcal{L}})\right|\leq\max_{\overline{B}(0,M_2)}|f_s'|,$$

that concludes the proof.

□

4.3.3. End of the proof of Theorem 4.2:. We are now able to apply the discrete Gronwall lemma and to conclude the proof.

Gathering Propositions 4.13, 4.14, 4.15, 4.16, 4.18, 4.19 and 4.20, we can estimate term (4.17) (cf Proposition 4.12). Thus, there exists $C_{23}>0$ (depending on c and μ) independent of $\mathrm{size}(\mathcal{T})$ and Δt such that for any $n_0\in[\![0,N-1]\!]$,

$$\begin{aligned}&\frac{1}{2}\sum_{n=0}^{n_0}\Delta t\left|\dot{e}^{\mu,n+1}_{\mathcal{T}}\right|^2_{1,\mathcal{T}}+\frac{1}{4}\left|\dot{e}^{c,n_0+1}_{\mathcal{T}}\right|^2_{1,\mathcal{T}}+\frac{1}{2}\left|\dot{e}^{c,n_0+1}_{\partial\mathfrak{M}}\right|^2_{1,\partial\mathfrak{M}}+\frac{1}{2}\sum_{n=0}^{n_0}\left|\dot{e}^{c,n+1}_{\mathcal{T}}-\dot{e}^{c,n}_{\mathcal{T}}\right|^2_{1,\mathcal{T}}\\ &\quad+\frac{1}{2}\sum_{n=0}^{n_0}\Delta t\left\|\frac{\dot{e}^{c,n+1}_{\partial\mathfrak{M}}-\dot{e}^{c,n}_{\partial\mathfrak{M}}}{\Delta t}\right\|^2_{0,\partial\mathfrak{M}}+\frac{1}{2}\sum_{n=0}^{n_0}\left|\dot{e}^{c,n+1}_{\partial\mathfrak{M}}-\dot{e}^{c,n}_{\partial\mathfrak{M}}\right|^2_{1,\partial\mathfrak{M}}\\ &\leq C_{23}\left(\mathrm{size}(\mathcal{T})^2+\Delta t^2\right)+C_{23}\Delta t\left|\dot{e}^{c,n_0+1}_{\mathcal{T}}\right|^2_{1,\mathcal{T}}+(1+C_{23})\sum_{n=0}^{n_0}\Delta t\left|\dot{e}^{c,n}_{\mathcal{T}}\right|^2_{1,\mathcal{T}}\\ &\quad+C_{23}\Delta t\sum_{n=0}^{n_0}\left|\dot{e}^{c,n+1}_{\mathcal{T}}-\dot{e}^{c,n}_{\mathcal{T}}\right|^2_{1,\mathcal{T}}+C_{23}\Delta t^2\sum_{n=0}^{n_0}\Delta t\left\|\frac{\dot{e}^{c,n+1}_{\partial\mathfrak{M}}-\dot{e}^{c,n}_{\partial\mathfrak{M}}}{\Delta t}\right\|^2_{0,\partial\mathfrak{M}}.\end{aligned}$$

Thus by choosing $\Delta t\leq\min\left(\frac{1}{8C_{23}},\frac{1}{2\sqrt{C_{23}}}\right)$ we deduce

$$\frac{1}{8}\left|\dot{e}^{c,n_0+1}_{\mathcal{T}}\right|^2_{1,\mathcal{T}}\leq(1+C_{23})\sum_{n=0}^{n_0}\Delta t\left|\dot{e}^{c,n}_{\mathcal{T}}\right|^2_{1,\mathcal{T}}+C_{23}\left(\mathrm{size}(\mathcal{T})^2+\Delta t^2\right).$$

Thanks to the discrete Gronwall lemma, we have:

$$\left|\dot{e}^{c,n_0+1}_{\mathcal{T}}\right|^2_{1,\mathcal{T}}\leq 8(1+C_{23})\left(\mathrm{size}(\mathcal{T})^2+\Delta t^2+\Delta t\left|\dot{e}^{c,0}_{\mathcal{T}}\right|^2_{1,\mathcal{T}}\right)e^{8T(1+C_{23})}.$$

Estimates (4.47) and (4.48), Proposition 4.13 on the initial data and Proposition 4.9 conclude the proof of Theorem 4.2.

5. Numerical error estimate. In the section we present numerical error estimate which illustrate the previous result. Several qualitative numerical results which show the different behaviours of the scheme in function of the properties of the components and the wall are performed in [20].

The domain Ω is the unit circle and we consider a Delaunay triangular mesh for which for any $\kappa \in \mathfrak{M}$ the center x_κ is the circumcenter of the triangle κ (or $\underline{\kappa}$ is $\partial\kappa \cap \mathcal{E}_{ext} \neq \emptyset$) and for any $\mathcal{L} \in \partial\mathfrak{M}$, $y_\mathcal{L}$ is the middle of the chord $\mathbf{e}_\mathcal{L}$. Since we do not know non-trivial solution of problem (1.1) we choose an analytical solution (see Fig. 5.1)

$$c(t,(x,y)) = \frac{1}{2}\cos(2\pi x + 4t)\sin(2\pi y) + \frac{1}{2},\ \forall t \in (0,T), \forall (x,y), \in \Omega$$

and we modify equation (1.1) by adding non-zero source terms in equations (1.1a) and (1.1c) and by considering non-homogeneous Neumann boundary condition for μ.

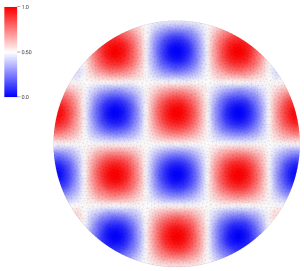


Fig. 5.1: Analytic solution at time $T = 0.5$

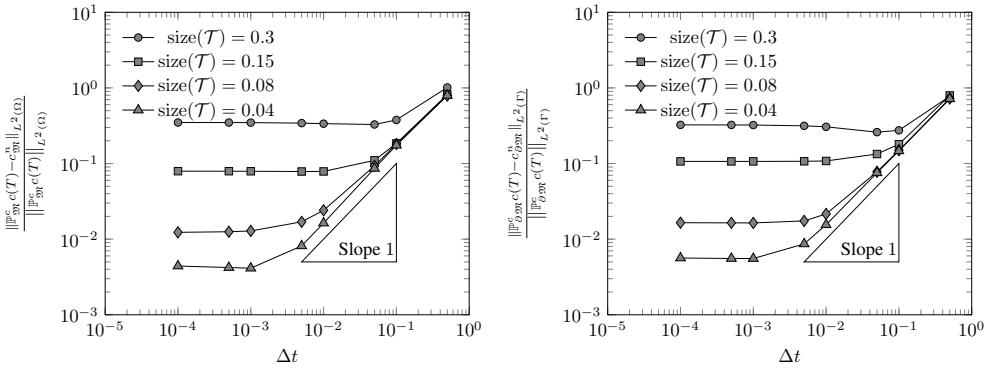
In equation (1.1) we choose the following parameters: the interface thickness $\varepsilon = 0.5$, the mobility and the surface tension $\Gamma_b = \sigma_b = 0.1$, the capillarity coefficient $\sigma_s = 5$ and the relaxation coefficient $\Gamma_s = 10$. As regards the non-linear potentials we set $f_b(c) = f_s(c) = c^2(1-c)^2$ (see Fig. 1.1b) and we use the semi-implicit discretization.

At final time $T = 0.5$ we plot the relative error between the center-value projection of the exact solution $c(T,\cdot)$ and the approximate solution $c_\mathcal{T}^n$, namely $\frac{\|\mathbb{P}_\mathcal{T}^c c(T) - c_\mathcal{T}^n\|}{\|\mathbb{P}_\mathcal{T}^c c(T)\|}$, for the L^2-norm and the H^1-seminorm in Ω and on Γ in two cases:

- when the time step tends to 0 and a fixed mesh size (see Fig. 5.2 and 5.3);
- when the mesh size tends to 0 and a fixed time step (see Fig. 5.4 and 5.5).

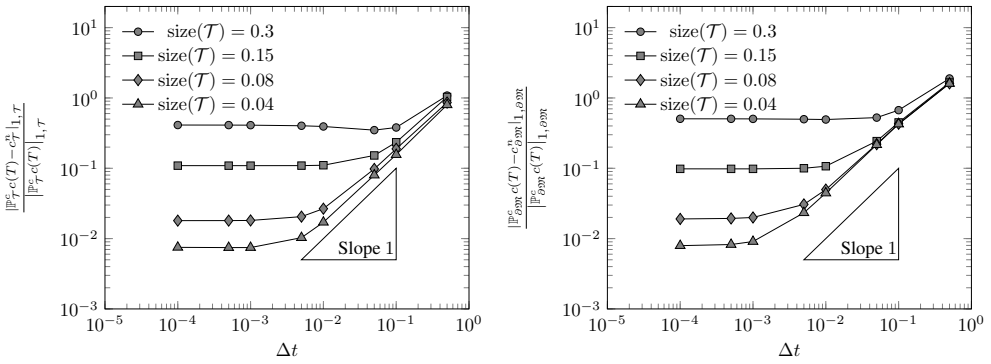
First we observe that, as expected, we obtain a first-order convergence in time in each case (see Fig. 5.2 and 5.3).

For the space convergence we have several behaviours. Noting that when we study the Laplace problem with a finite-volume two-point flux approximation, from a computational point of view we observe a second-order convergence for the L^2-norm while at the theoretical level, we are only able to prove the first-order convergence. We observe here the same super-convergence phenomena for the L^2-norms (see Fig. 5.4). As regards the $H^1(\Omega)$-seminorm (see Fig. 5.5a), although we observe a second-order convergence when the mesh size is coarse, we recover asymptotically the expected first-order convergence. However for the $H^1(\Gamma)$-seminorm (see Fig. 5.5b) we observe a second-order convergence instead of the first-order. This super-convergence phenomena is owed to the symmetric meshing of the boundary Γ.



(a) L^2-error in Ω (b) L^2-error on Γ

Fig. 5.2: First-order convergence in time for the L^2-norms



(a) H^1-error in Ω (b) H^1-error on Γ

Fig. 5.3: First-order convergence in time for the H^1-seminorms

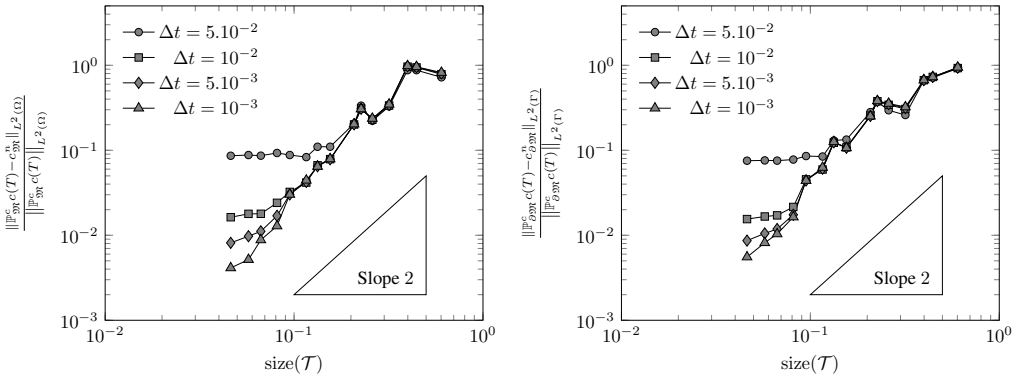
Appendix. Proof of Proposition 4.9.

We consider the following problem: find $u : \Omega \to \mathbb{R}$ such that $\int_\Omega u = \alpha$ and

$$\text{(A.1a)} \qquad \begin{cases} -\Delta u = f & \text{in } \Omega; \\ -\Delta_\Gamma u_{|\Gamma} + \partial_n u = g & \text{on } \Gamma; \end{cases}$$

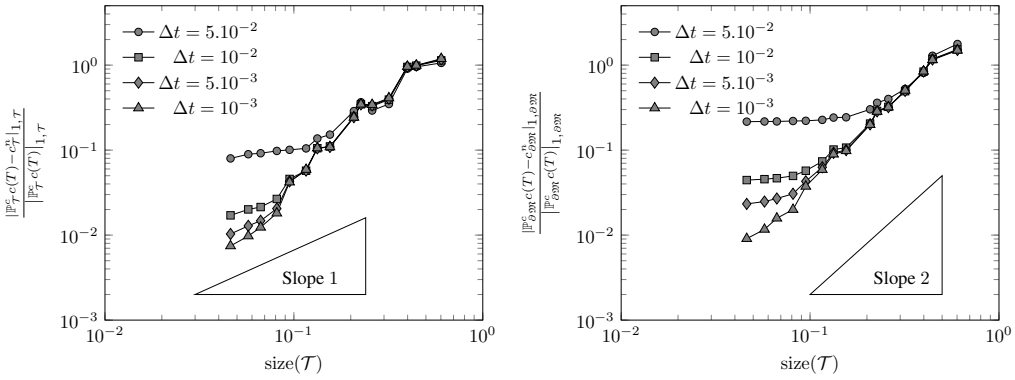
with $f \in L^2(\Omega)$ and $g \in L^2(\Gamma)$.

By integrating equation (A.1a) on all interior control volumes $\kappa \in \mathfrak{M}$ and the boundary condition (A.1b) on all boundary control volumes $\mathcal{L} \in \partial\mathfrak{M}$, the two-point flux approximation of problem (A.1) writes as follows.



(a) L^2-error in Ω (b) L^2-error on Γ

Fig. 5.4: Second-order convergence in space for the L^2-norms



(a) H^1-error in Ω (b) H^1-error on Γ

Fig. 5.5: Convergence in space for the H^1-seminorms

PROBLEM A.1. *Find $u_\mathcal{T} \in \mathbb{R}^\mathcal{T}$ such that $\sum_{\mathcal{K}\in\mathfrak{M}} m_\mathcal{K} u_\mathcal{K} = \alpha$ and,*

$$\begin{cases} \displaystyle\sum_{\sigma\in\mathcal{E}_\mathcal{K}^{int}} m_\sigma \frac{u_\mathcal{K} - u_\mathcal{L}}{d_{\mathcal{K},\mathcal{L}}} + \sum_{\sigma\in\mathcal{E}_\mathcal{K}^{ext}} m_{\mathbf{e}_\mathcal{L}} \frac{u_\mathcal{K} - u_\mathcal{L}}{d_{\mathcal{K},\mathcal{L}}} = m_\mathcal{K} \mathbb{P}_\mathcal{K}^m f, \quad \forall \mathcal{K} \in \mathfrak{M}; & \text{(A.2a)} \\ \displaystyle\sum_{\mathbf{v}\in\mathcal{V}_\mathcal{L}} \frac{u_\mathcal{L} - u_{\mathcal{L}'}}{d_{\mathcal{L},\mathcal{L}'}} + m_{\mathbf{e}_\mathcal{L}} \frac{u_\mathcal{L} - u_\mathcal{K}}{d_{\mathcal{K},\mathcal{L}}} = m_\mathcal{L} \mathbb{P}_\mathcal{L}^m g, \quad \forall \mathcal{L} \in \partial\mathfrak{M}. & \text{(A.2b)} \end{cases}$$

We can prove that this problem admits a unique solution.

Because of the complex geometry of Ω, it is possible to take into account some points $x \in \Omega^c$ in the proof of the error estimate. To ensure that all the quantities used in the proof of Theorem A.4 are well defined, we will use an extension in $\mathbb{R}^2$ of the function u. Since $u \in H^2(\Omega)$, there exists an extension $\widetilde{u} \in H^2(\mathbb{R}^2)$ (that we fix in the sequel) such that:

$$\widetilde{u}(x) = u(x), \ \ \forall x \in \Omega \text{ and } \|\widetilde{u}\|_{H^2(\mathbb{R}^2)} \leq C_{24} \|u\|_{H^2(\Omega)} \tag{A.3}$$

with $C_{24} > 0$ depending only on Ω.

PROPOSITION A.1. *The tangential gradient of $u : \Gamma \to \mathbb{R}$ to the vertex $\mathbf{v} = \mathcal{L}|\mathcal{L}'$ satisfies:*

$$\left| \frac{u(x_{\mathcal{L}'}) - u(x_{\mathcal{L}})}{m_{\gamma_{\mathcal{L}\mathcal{L}'}}} - \nabla_\Gamma u(\mathbf{v}) \cdot \vec{\boldsymbol{\tau}}_{\mathbf{v},\boldsymbol{\mathcal{L}}}(\mathbf{v}) \right| \leq \int_{\gamma_{\mathcal{L}\mathcal{L}'}} \left| (u \circ \varphi)''(\varphi^{-1}(x)) \right| \mathrm{d}\sigma(x),$$

where φ is an arc-length parametrization of the curve Γ and $\vec{\boldsymbol{\tau}}_{\mathbf{v},\boldsymbol{\mathcal{L}}}(\mathbf{v})$ is the unit tangent vector to Γ at point $\mathbf{v} = \mathcal{L}|\mathcal{L}'$ going from $\mathcal{L}$ to $\mathcal{L}'$.

Proof . Let us consider the points $t_{\mathcal{L}}, t_{\mathcal{L}'}, t_{\mathbf{v}} \in \mathbb{R}$ such that $x_{\mathcal{L}} = \varphi(t_{\mathcal{L}})$, $x_{\mathcal{L}'} = \varphi(t_{\mathcal{L}'})$ and $\mathbf{v} = \varphi(t_{\mathbf{v}})$, then the Taylor's formulas give

$$\begin{aligned} u(x_{\mathcal{L}'}) - u(x_{\mathcal{L}}) =& (t_{\mathcal{L}'} - t_{\mathbf{v}})(u \circ \varphi)'(t_{\mathbf{v}}) + \int_{t_{\mathbf{v}}}^{t_{\mathcal{L}'}} (t_{\mathcal{L}'} - s)(u \circ \varphi)''(s)\mathrm{d}s \\ & - (t_{\mathcal{L}} - t_{\mathbf{v}})(u \circ \varphi)'(t_{\mathbf{v}}) - \int_{t_{\mathbf{v}}}^{t_{\mathcal{L}}} (t_{\mathcal{L}} - s)(u \circ \varphi)''(s)\mathrm{d}s. \end{aligned}$$

Noting that $|t_{\mathcal{L}'} - t_{\mathcal{L}}| = m_{\gamma_{\mathcal{L}\mathcal{L}'}}$ we obtain,

$$\begin{aligned} \frac{u(x_{\mathcal{L}'}) - u(x_{\mathcal{L}})}{m_{\gamma_{\mathcal{L}\mathcal{L}'}}} - \nabla_\Gamma u(\mathbf{v}) \cdot \vec{\boldsymbol{\tau}}_{\mathbf{v},\boldsymbol{\mathcal{L}}}(\mathbf{v}) =& \frac{1}{m_{\gamma_{\mathcal{L}\mathcal{L}'}}} \int_{t_{\mathbf{v}}}^{t_{\mathcal{L}'}} (t_{\mathcal{L}'} - s)(u \circ \varphi)''(s)\mathrm{d}s \\ & + \frac{1}{m_{\gamma_{\mathcal{L}\mathcal{L}'}}} \int_{t_{\mathcal{L}}}^{t_{\mathbf{v}}} (t_{\mathcal{L}} - s)(u \circ \varphi)''(s)\mathrm{d}s, \end{aligned}$$

that concludes the proof. □

Thanks to the Taylor's formulas we can prove the two following propositions.

PROPOSITION A.2. *Let $\mathcal{L} \in \partial\mathfrak{M}$ be a boundary control volume and $\mathbf{v}$ be a vertex of $\mathcal{L}$, then the following equality holds,*

$$m_{\gamma_{\mathcal{L}\mathbf{v}}} - d_{\mathcal{L},\mathbf{v}} = \mathcal{O}\left(m_{\mathcal{L}} m_{\gamma_{\mathcal{L}\mathbf{v}}}\right).$$

PROPOSITION A.3. *For any point $x \in \sigma = \mathcal{L} \in \mathcal{E}_{ext}$, the following equality holds*

$$\vec{\mathbf{n}}_{\boldsymbol{\sigma\mathcal{K}}}(x) - \vec{\mathbf{n}}_{\boldsymbol{\mathcal{K}\mathcal{L}}} = \mathcal{O}(m_{\mathcal{L}}),$$

where $\vec{\mathbf{n}}_{\boldsymbol{\sigma\mathcal{K}}}(x)$ is the unit normal vector to σ outward to $\mathcal{K}$ at point x.

THEOREM A.4. *Let us assume that the solution u of the continuous problem* (A.1) *belongs to $H^2_\Gamma(\Omega)$. Let us consider the solution $u_\mathcal{T}$ to discrete problem A.1. Then, there exists $C_{25} > 0$ independent of* $\mathrm{size}(\mathcal{T})$ *such that,*

$$|e_\mathcal{T}|^2_{1,\mathcal{T}} + |e_{\partial\mathfrak{M}}|^2_{1,\partial\mathfrak{M}} \leq C_{25} \mathrm{size}(\mathcal{T})^2 \left\|u\right\|^2_{H^2_\Gamma(\Omega)}, \tag{A.4}$$

with $e_\mathcal{T} = \mathbb{P}^c_\mathcal{T} u - u_\mathcal{T}$.

We decompose the proof of Theorem A.4 into two steps. As a first step, we prove (cf Proposition A.5) that the left hand side of inequality (A.4) is bounded from above by the different consistency errors which intervene in the problem. In a second phase, we have to estimate these different consistency errors.

PROPOSITION A.5. *Let us consider the solution u to problem* (A.1) *and the solution $u_\mathcal{T}$ to discrete Problem A.1. The the following estimate holds,*

$$\begin{aligned} |e_\mathcal{T}|^2_{1,\mathcal{T}} + |e_{\partial\mathfrak{M}}|^2_{1,\partial\mathfrak{M}} \leq & \sum_{\sigma=\mathcal{K}|\mathcal{L}\in\mathcal{E}_{int}} m_\sigma d_{\mathcal{K},\mathcal{L}} (R^{int}_{\sigma,\mathcal{K}})^2 + \sum_{\sigma=\mathcal{L}\in\mathcal{E}_{ext}} m_{\mathbf{e}_\mathcal{L}} d_{\mathcal{K},\mathcal{L}} (R^{ext}_{\sigma,\mathcal{K}})^2 \\ & + \sum_{\mathbf{v}=\mathcal{L}|\mathcal{L}'\in\mathcal{V}} \frac{{R_{\mathbf{v},\mathcal{L}}}^2}{d_{\mathcal{L},\mathcal{L}'}} + \left\|R^u_{\partial\mathfrak{M}}\right\|^2_{L^2(\Gamma)}; \end{aligned} \tag{A.5}$$

where,

$$
\begin{aligned}
R^{int}_{\sigma,\mathcal{K}} &= \frac{1}{m_\sigma}\int_\sigma \nabla u(x)\cdot \vec{\mathbf{n}}_{\boldsymbol{\mathcal{K}\mathcal{L}}}\mathrm{d}x - \frac{u(x_\mathcal{L})-u(x_\mathcal{K})}{d_{\mathcal{K},\mathcal{L}}}, && \forall \sigma=\mathcal{K}|\mathcal{L}\in\mathcal{E}_\mathcal{K}\cap\mathcal{E}_{int};\\
R^{ext}_{\sigma,\mathcal{K}} &= \frac{1}{m_{\mathbf{e}_\mathcal{L}}}\int_\sigma \nabla u(x)\cdot \vec{\mathbf{n}}_{\boldsymbol{\sigma\mathcal{K}}}(x)\mathrm{d}x - \frac{u(x_\mathcal{L})-u(x_\mathcal{K})}{d_{\mathcal{K},\mathcal{L}}}, && \forall \sigma=\mathcal{L}\in\mathcal{E}_\mathcal{K}\cap\mathcal{E}_{ext};\\
R_{\mathbf{v},\mathcal{L}} &= d_{\mathcal{L},\mathcal{L}'}\nabla_\Gamma u_{|\Gamma}(\mathbf{v})\vec{\boldsymbol{\tau}}_{\mathbf{v},\boldsymbol{\mathcal{L}}} - u_{|\Gamma}(x_{\mathcal{L}'}) - u_{|\Gamma}(x_\mathcal{L}), && \forall \mathbf{v}=\mathcal{L}|\mathcal{L}'\in\mathcal{V}.
\end{aligned}
$$

Proof. Let $\kappa\in\mathfrak{M}$, we integrate equation (A.1a) on κ and we subtract the resulting equality with equation (A.2a). Thanks to definitions of $R^{int}_{\sigma,\mathcal{K}}$ and $R^{ext}_{\sigma,\mathcal{K}}$ given in Proposition A.5 imply:

$$
\begin{aligned}
\sum_{\sigma=\mathcal{K}|\mathcal{L}\in\mathcal{E}_{int}}\frac{m_\sigma}{d_{\mathcal{K},\mathcal{L}}}(e_\mathcal{K}-e_\mathcal{L}) + \sum_{\sigma=\mathcal{L}\in\mathcal{E}_{ext}}\frac{m_{\mathbf{e}_\mathcal{L}}}{d_{\mathcal{K},\mathcal{L}}}(e_\mathcal{K}-e_\mathcal{L})&\\
= \sum_{\sigma\in\mathcal{E}^{int}_\mathcal{K}} m_\sigma R^{int}_{\sigma,\mathcal{K}} + \sum_{\sigma\in\mathcal{E}^{ext}_\mathcal{K}} m_{\mathbf{e}_\mathcal{L}} R^{ext}_{\sigma,\mathcal{K}}, \quad \forall \mathcal{K}\in\mathfrak{M}.&
\end{aligned}
\tag{A.6}
$$

In the same way let $\mathcal{L}\in\partial\mathfrak{M}$, we integrate equation (A.1b) on $\mathcal{L}$ and we subtract the resulting equality with equation (A.2b). Then we obtain,

$$
\sum_{\mathbf{v}\in\mathcal{V}_\mathcal{L}}\frac{e_\mathcal{L}-e_{\mathcal{L}'}}{d_{\mathcal{L},\mathcal{L}'}} + m_{\mathbf{e}_\mathcal{L}}\frac{e_\mathcal{L}-e_\mathcal{K}}{d_{\mathcal{K},\mathcal{L}}} = \sum_{\mathbf{v}\in\mathcal{V}_\mathcal{L}}\frac{R_{\mathbf{v},\mathcal{L}}}{d_{\mathcal{L},\mathcal{L}'}} - m_{\mathbf{e}_\mathcal{L}}R^{ext}_{\sigma,\mathcal{K}}, \quad \forall \mathcal{L}\in\partial\mathfrak{M}. \tag{A.7}
$$

Now we multiply equation (A.6) by $e_\mathcal{K}$ and summing up over $\kappa\in\mathfrak{M}$ and we multiply equation (A.7) by $e_\mathcal{L}$ and summing up over $\mathcal{L}\in\partial\mathfrak{M}$. Then, summing the resulting equalities we have,

$$
\begin{aligned}
|e_\mathcal{T}|^2_{1,\mathcal{T}} + |e_{\partial\mathfrak{M}}|^2_{1,\partial\mathfrak{M}} = &\sum_{\sigma=\mathcal{K}|\mathcal{L}\in\mathcal{E}_{int}} m_\sigma(e_\mathcal{K}-e_\mathcal{L})R^{int}_{\sigma,\mathcal{K}} + \sum_{\sigma=\mathcal{L}\in\mathcal{E}_{ext}} m_{\mathbf{e}_\mathcal{L}}(e_\mathcal{K}-e_\mathcal{L})R^{ext}_{\sigma,\mathcal{K}}\\
&+ \sum_{\mathbf{v}=\mathcal{L}|\mathcal{L}'\in\mathcal{V}} R_{\mathbf{v},\mathcal{L}}\frac{e_\mathcal{L}-e_{\mathcal{L}'}}{d_{\mathcal{L},\mathcal{L}'}}.
\end{aligned}
$$

Owing to the Cauchy-Schwarz and the Young inequalities, we obtain estimate (A.5).

□

With this proposition at hand we are now able to prove Theorem A.4 by estimating all the terms of the right hand side of (A.5).

Proof of Theorem A.4.

First, let $\sigma=\kappa|\mathcal{L}\in\mathcal{E}_{int}$ thanks to the Taylor's formulas we have:

$$
\begin{aligned}
R^{int}_{\sigma,\mathcal{K}} =& \frac{1}{m_\sigma d_{\mathcal{K},\mathcal{L}}}\int_\sigma\int_0^1 (1-t)\left\langle D^2u\left((1-t)x+tx_\mathcal{K}\right)(x_\mathcal{K}-x),(x_\mathcal{K}-x)\right\rangle\\
&- \frac{1}{m_\sigma d_{\mathcal{K},\mathcal{L}}}\int_\sigma\int_0^1 (1-t)\left\langle D^2u\left((1-t)x+tx_\mathcal{L}\right)(x_\mathcal{L}-x),(x_\mathcal{L}-x)\right\rangle.
\end{aligned}
$$

Owing to the Jensen inequality and the change of variables $(t,x)\in[0,1]\times\sigma\mapsto y=x+t(x_\mathcal{K}-x)$ (or $(t,x)\in[0,1]\times\sigma\mapsto y=x+t(x_\mathcal{L}-x)$ for the second term) and since $\mathrm{diam}(\kappa)\le \mathrm{reg}(\mathcal{T})d(x_\mathcal{K},\sigma)$ for any $\kappa\in\mathfrak{M}$, for any $\sigma\in\mathcal{E}_\mathcal{K}$ (see definition 2.1) we deduce

$$
(R^{int}_{\sigma,\mathcal{K}})^2 \le C(\mathrm{reg}(\mathcal{T}))\frac{\mathrm{size}(\mathcal{T})^2}{m_\mathcal{D}}\int_\mathcal{D}|D^2u(y)|^2\mathrm{d}y.
$$

Noting that $m_{\mathcal{D}} = \frac{m_\sigma d_{\mathcal{K},\mathcal{L}}}{2}$ we obtain,

$$\sum_{\sigma=\mathcal{K}|\mathcal{L}\in\mathcal{E}_{int}} m_\sigma d_{\mathcal{K},\mathcal{L}}(R^{int}_{\sigma,\mathcal{K}})^2 \leq C(\mathrm{reg}(\mathcal{T}))\mathrm{size}(\mathcal{T})^2 \left\|D^2 u\right\|^2_{L^2(\Omega)}. \tag{A.8}$$

Secondly let $\sigma = \mathcal{L} \in \partial\mathfrak{M}$, thanks to definition (A.3) of $\widetilde{u}$ we have $u(x_{\mathcal{L}}) = \widetilde{u}(x_{\mathcal{L}})$ and $u(x_{\mathcal{K}}) = \widetilde{u}(x_{\mathcal{K}})$, thus since $x_{\mathcal{L}} - x_{\mathcal{K}} = d(x_{\mathcal{K}}, x_{\mathcal{L}})\vec{\mathbf{n}}_{\boldsymbol{\mathcal{K}\mathcal{L}}}$, the definition of $R^{ext}_{\sigma,\mathcal{K}}$, the jensen inequality and the Taylor's formulas imply,

$$\begin{aligned}
m_{\mathbf{e}_{\mathcal{L}}} d_{\mathcal{K},\mathcal{L}}(R^{ext}_{\sigma,\mathcal{K}})^2 \leq & 5 d_{\mathcal{K},\mathcal{L}} \frac{(m_{\mathcal{L}} - m_{\mathbf{e}_{\mathcal{L}}})^2}{m_{\mathbf{e}_{\mathcal{L}}} m_{\mathcal{L}}} \int_{\mathcal{L}} |\nabla\widetilde{u}(x)|^2 \,\mathrm{d}\sigma(x) \\
& + 5\frac{(d_{\mathcal{K},\mathcal{L}} - d(x_{\mathcal{K}}, x_{\mathcal{L}}))^2}{d_{\mathcal{K},\mathcal{L}}} \frac{m_{\mathbf{e}_{\mathcal{L}}}}{m_{\mathcal{L}}} \int_{\mathcal{L}} |\nabla\widetilde{u}(x)|^2 \,\mathrm{d}\sigma(x) \\
& + 5 d_{\mathcal{K},\mathcal{L}} \frac{m_{\mathbf{e}_{\mathcal{L}}}}{m_{\mathcal{L}}} \int_{\mathcal{L}} |\nabla\widetilde{u}(x)|^2 \, |\vec{\mathbf{n}}_{\boldsymbol{\mathcal{K}\mathcal{L}}} - \vec{\mathbf{n}}_{\boldsymbol{\sigma\mathcal{K}}}(x)|^2 \mathrm{d}\sigma(x) \\
& + \frac{5 m_{\mathbf{e}_{\mathcal{L}}}}{d_{\mathcal{K},\mathcal{L}} m_\sigma} \int_\sigma \int_0^1 (1-t)^2 \left|D^2\widetilde{u}\left((1-t)x + t x_{\mathcal{K}}\right)\right|^2 |x_{\mathcal{K}} - x|^4 \mathrm{d}t\mathrm{d}\sigma(x) \\
& + \frac{5 m_{\mathbf{e}_{\mathcal{L}}}}{d_{\mathcal{K},\mathcal{L}} m_\sigma} \int_\sigma \int_0^1 (1-t)^2 \left|D^2\widetilde{u}\left((1-t)x + t x_{\mathcal{L}}\right)\right|^2 |x_{\mathcal{L}} - x|^4 \mathrm{d}t\mathrm{d}\sigma(x).
\end{aligned}$$

Thanks to Propositions 2.2, 2.3 and A.3, there exists $C_{\mathrm{T}} > 0$ independent of $\mathrm{size}(\mathcal{T})$ such that

$$|d(x_{\mathcal{K}}, x_{\mathcal{L}}) - d_{\mathcal{K},\mathcal{L}}| \leq C_{\mathrm{T}} m_{\mathcal{L}} m_{\gamma_{\mathcal{L}\mathbf{v}}}, \quad |m_{\mathbf{e}_{\mathcal{L}}} - m_{\mathcal{L}}| \leq C_{\mathrm{T}} m_{\mathcal{L}}^3 \text{ and } |\vec{\mathbf{n}}_{\boldsymbol{\mathcal{K}\mathcal{L}}} - \vec{\mathbf{n}}_{\boldsymbol{\sigma\mathcal{K}}}(x)| \leq C_{\mathrm{T}} m_{\mathcal{L}}.$$

Thus, thanks to a change of variables in the last two integrals we have:

$$\begin{aligned}
m_{\mathbf{e}_{\mathcal{L}}} d_{\mathcal{K},\mathcal{L}}(R^{ext}_{\sigma,\mathcal{K}})^2 \leq & C_{\mathrm{T}}(\mathrm{reg}(\mathcal{T}))\mathrm{size}(\mathcal{T})^3 \int_{\mathcal{L}} |\nabla\widetilde{u}(x)|^2 \,\mathrm{d}\sigma(x) \\
& + C(\mathrm{reg}(\mathcal{T}))\mathrm{size}(\mathcal{T})^2 \left(\int_{\mathcal{D}_{\mathcal{L}}} \left|D^2\widetilde{u}(y)\right|^2 \mathrm{d}y + \int_{\mathcal{D}} \left|D^2\widetilde{u}(y)\right|^2 \mathrm{d}y \right)
\end{aligned}$$

where $\mathcal{D}_{\mathcal{L}} = \{(1-t)x + t x_{\mathcal{L}} : t \in [0,1], x \in \sigma = \mathcal{L}\}$. Then, owing to (A.3) we obtain:

$$\sum_{\sigma=\mathcal{L}\in\mathcal{E}_{ext}} m_{\mathbf{e}_{\mathcal{L}}} d_{\mathcal{K},\mathcal{L}}(R^{ext}_{\sigma,\mathcal{K}})^2 \leq C_{\mathrm{T}}(\mathrm{reg}(\mathcal{T}))C_{24}\mathrm{size}(\mathcal{T})^2 \|u\|^2_{H^1(\Omega)}. \tag{A.9}$$

Finally, using definition of $R_{\mathbf{v},\mathcal{L}}$ for any $\mathbf{v} = \mathcal{L}|\mathcal{L}' \in \mathcal{V}$ we have,

$$\frac{R_{\mathbf{v},\mathcal{L}}}{d_{\mathcal{L},\mathcal{L}'}} = \left(\nabla_{\mathrm{T}} u(\mathbf{v}) \vec{\boldsymbol{\tau}}_{\mathbf{v},\boldsymbol{\mathcal{L}}} - \frac{u(x_{\mathcal{L}'}) - u(x_{\mathcal{L}})}{m_{\gamma_{\mathcal{L}\mathcal{L}'}}} \right) + (u(x_{\mathcal{L}'}) - u(x_{\mathcal{L}})) \left(\frac{1}{m_{\gamma_{\mathcal{L}\mathcal{L}'}}} - \frac{1}{d_{\mathcal{L},\mathcal{L}'}} \right),$$

that gives,

$$\begin{aligned}
\frac{R_{\mathbf{v},\mathcal{L}}{}^2}{d_{\mathcal{L},\mathcal{L}'}} \leq & 2 d_{\mathcal{L},\mathcal{L}'} \left(\nabla_{\mathrm{T}} u(\mathbf{v}) \vec{\boldsymbol{\tau}}_{\mathbf{v},\boldsymbol{\mathcal{L}}} - \frac{u(x_{\mathcal{L}'}) - u(x_{\mathcal{L}})}{m_{\gamma_{\mathcal{L}\mathcal{L}'}}} \right)^2 \\
& + 2\frac{(u(x_{\mathcal{L}'}) - u(x_{\mathcal{L}}))^2}{m_{\gamma_{\mathcal{L}\mathcal{L}'}}} \frac{(d_{\mathcal{L},\mathcal{L}'} - m_{\gamma_{\mathcal{L}\mathcal{L}'}})^2}{d_{\mathcal{L},\mathcal{L}'} m_{\gamma_{\mathcal{L}\mathcal{L}'}}}.
\end{aligned}$$

Thanks to Proposition A.2, there exists $C_{\Gamma} > 0$ independent of $\mathrm{size}(\mathcal{T})$ such that,

$$\frac{(m_{\gamma_{\mathcal{L}\mathcal{L}'}} - d_{\mathcal{L},\mathcal{L}'})^2}{d_{\mathcal{L},\mathcal{L}'} m_{\gamma_{\mathcal{L}\mathcal{L}'}}} \leq C_{\Gamma} \mathrm{size}(\mathcal{T})^2.$$

Moreover,

$$(u(x_{\mathcal{L}'}) - u(x_{\mathcal{L}}))^2 \leq m_{\gamma_{\mathcal{L}\mathcal{L}'}} \int_{\gamma_{\mathcal{L}\mathcal{L}'}} |\nabla_{\Gamma} u(x)|^2 \, \mathrm{d}\sigma(x),$$

thus owing to Proposition A.1, we can write

$$\sum_{\mathbf{v}=\mathcal{L}|\mathcal{L}'\in\mathcal{V}} \frac{{R_{\mathbf{v},\mathcal{L}}}^2}{d_{\mathcal{L},\mathcal{L}'}} \leq 2C_{\Gamma} \mathrm{size}(\mathcal{T})^2 \left\| u_{\mathrm{I}\Gamma} \right\|^2_{H^2(\Gamma)}. \tag{A.10}$$

Gathering estimates (A.8), (A.9) and (A.10) the claim follows. □

We have obtained an error estimate between the center-value projection of the exact solution $\mathbb{P}^c_{\mathcal{T}} u$ and the approximate solution $u_{\mathcal{T}}$ for the Laplace problem with Ventcell boundary conditions for the H^1-seminorms. However in order to prove Proposition 4.9 we also need to prove an estimate between the exact solution u and the approximate $u_{\mathcal{T}}$ for the L^2-norms. We adopt here the same reasoning as that given in [10, Theorem 3.5] for the Laplace problem with Neumann boundary conditions.

Let $\beta_{\mathcal{T}} \in \mathbb{R}$ such that $\sum_{\kappa\in\mathfrak{M}} m_{\underline{\kappa}} \bar{u}(x_{\kappa}) = \alpha$ with $\bar{u} = u + \beta_{\mathcal{T}}$. Setting $\bar{e}_{\kappa} = \bar{u}(x_{\kappa}) - u_{\kappa}$ for any $\kappa \in \mathfrak{M}$ and $\bar{e}_{\mathcal{L}} = \bar{u}(x_{\mathcal{L}}) - u_{\mathcal{L}}$ for any $\mathcal{L} \in \partial\mathfrak{M}$, estimate (A.4) is also satisfied for $\bar{e}_{\mathcal{T}}$. However, thanks to its definition the error $\bar{e}_{\mathfrak{M}}$ has now a zero average. Thus we can apply the discrete Poincaré inequality (Lemma 2.7) to obtain,

$$\left\| \bar{e}_{\mathfrak{M}} \right\|^2_{L^2(\Omega)} \leq C_4^2 C_{25} \mathrm{size}(\mathcal{T})^2 \left\| u \right\|^2_{H^2_{\Gamma}(\Omega)},$$

and thanks to the trace inequality (Lemma 2.10) we have

$$\left\| \bar{e}_{\partial\mathfrak{M}} \right\|^2_{L^2(\Gamma)} \leq 4C_4^2 C_7^2 C_{25} \mathrm{size}(\mathcal{T})^2 \left\| u \right\|^2_{H^2_{\Gamma}(\Omega)}.$$

To conclude the proof the reasoning is exactly the same than in [10, Theorem 3.5] aside from the fact that the domain is not polygonal. Thus we only detail here the difference. Thanks to the regularity of the function u there exists $C > 0$ such that,

$$\left\| u - u_{\mathcal{T}} \right\|^2_{L^2(\Omega)} \leq 3|\Omega| C \mathrm{size}(\mathcal{T})^2 + 3|\Omega|\beta_{\mathcal{T}}^2 + 3 \left\| \bar{e}_{\mathfrak{M}} \right\|^2_{L^2(\Omega)}.$$

We recall that $\int_{\Omega} u = \sum_{\kappa\in\mathfrak{M}} m_{\underline{\kappa}} \bar{u}(x_{\kappa}) = \alpha$, then by integration

$$|\Omega|\beta_{\mathcal{T}} = \int_{\Omega} \bar{u}(x) \mathrm{d}x - \alpha$$

and

$$\int_{\Omega} \bar{u}(x) \mathrm{d}x = \int_{\Omega} (\bar{u}(x) - \bar{u}(x_{\kappa})) \mathrm{d}x + \sum_{\kappa\in\mathfrak{M}} (m_{\kappa} - m_{\underline{\kappa}}) \bar{u}(x_{\kappa}) + \alpha.$$

Thus thanks to the regularity of u, Proposition 2.1 and the mesh regularity (2.1) we can claim that $|\Omega|\beta_{\mathcal{T}} \leq C\mathrm{size}(\mathcal{T})$ that concludes the proof.

The reasoning is exactly the same for the $L^2(\Gamma)$-norm that concludes the claim.

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