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A converse to Fortin's Lemma in Banach spaces

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Abstract

The converse of Fortin's Lemma in Banach spaces is established in this Note.

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1. Introduction

Let V and W be two complex Banach spaces equipped with the norms $\|\cdot\|_V$ and $\|\cdot\|_W$, respectively. We adopt the convention that dual spaces are denoted with primes and are composed of antilinear forms; complex conjugates are denoted by an overline. Let a be a sesquilinear form on $V \times W$ (linear w.r.t. its first argument and antilinear w.r.t. its second argument). We assume that a is bounded, i.e.,

$$\|a\| := \sup_{v \in V} \sup_{w \in W} \frac{|a(v, w)|}{\|v\|_V \|w\|_W} < \infty, \quad (1)$$

and that the following inf-sup condition holds:

$$\alpha := \inf_{v \in V} \sup_{w \in W} \frac{|a(v, w)|}{\|v\|_V \|w\|_W} > 0. \quad (2)$$

Here and in what follows, arguments in infima and suprema are implicitly assumed to be nonzero.

Let $V_h \subset V$ and $W_h \subset W$ be two finite-dimensional subspaces equipped with the norms of V and W , respectively. A question of fundamental importance is to assert the following discrete inf-sup condition:

$$\hat{\alpha} := \inf_{v_h \in V_h} \sup_{w_h \in W_h} \frac{|a(v_h, w_h)|}{\|v_h\|_V \|w_h\|_W} > 0. \quad (3)$$

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The aim of this Note is to prove the following result.

Theorem 1 (Fortin's Lemma with converse) *Under the above assumptions, consider the following two statements:*

- (i) *There exists a map $\Pi_h : W \rightarrow W_h$ and a real number $\gamma_\Pi > 0$ such that $a(v_h, \Pi_h w - w) = 0$, for all $(v_h, w) \in V_h \times W$, and $\gamma_\Pi \|\Pi_h w\|_W \leq \|w\|_W$ for all $w \in W$.*
- (ii) *The discrete inf-sup condition (3) holds.*

Then, (i) \Rightarrow (ii) with $\hat{\alpha} = \gamma_\Pi \alpha$. Conversely, (ii) \Rightarrow (i) with $\gamma_\Pi = \frac{\hat{\alpha}}{\|a\|}$, and Π_h can be constructed to be idempotent. Moreover, Π_h can be made linear if W is a Hilbert space.

The statement (i) \Rightarrow (ii) in Theorem 1 is classical and is known in the literature as Fortin's Lemma, see [5] and [1, Prop. 5.4.3]. It provides an effective tool to prove the discrete inf-sup condition (3) by constructing explicitly a Fortin operator Π_h . We briefly outline a proof that (i) \Rightarrow (ii) for completeness. Assuming (i), we have

$$\sup_{w_h \in W_h} \frac{|a(v_h, w_h)|}{\|w_h\|_W} \geq \sup_{w \in W} \frac{|a(v_h, \Pi_h w)|}{\|\Pi_h w\|_W} = \sup_{w \in W} \frac{|a(v_h, w)|}{\|\Pi_h w\|_W} \geq \gamma_\Pi \sup_{w \in W} \frac{|a(v_h, w)|}{\|w\|_W} \geq \gamma_\Pi \alpha \|v_h\|_V,$$

since a satisfies (2) and $V_h \subset V$. This proves (ii) with $\hat{\alpha} = \gamma_\Pi \alpha$.

The converse (ii) \Rightarrow (i) is of independent theoretical interest and is the main object of this Note. This property is useful when it is easier to prove the discrete inf-sup condition directly rather than constructing a Fortin operator. Another application of current interest is the analysis framework for discontinuous Petrov–Galerkin methods (dPG) recently proposed in [3] which includes the existence of a Fortin operator among its key assumptions. Incidentally, we observe that there is a gap in the stability constant γ_Π between the direct and converse statements, since the ratio of the two is equal to $\frac{\|a\|}{\alpha}$ (which is independent of the spaces V_h and W_h).

2. Proof of Theorem 1

Assume that the discrete inf-sup condition (3) holds. Let $A_h : V_h \rightarrow W'_h$ be the operator defined by $\langle A_h v_h, w_h \rangle_{W'_h, W_h} := a(v_h, w_h)$. Identifying V''_h with V_h and W''_h with W_h (since these spaces are finite-dimensional), we consider the linear map $A_h^* : W_h \rightarrow V'_h$. Our goal is to construct a right-inverse map $R_{A_h^*} : V'_h \rightarrow W_h$ (possibly nonlinear) such that, for all $\theta_h \in V'_h$, $A_h^*(R_{A_h^*}(\theta_h)) = \theta_h$ and $\hat{\alpha} \|R_{A_h^*}(\theta_h)\|_W \leq \|\theta_h\|_{V'_h}$. Indeed, if such a map exists, we can consider the linear map $\Theta : W \rightarrow V'_h$ such that, for all $w \in W$, $\langle \Theta(w), v_h \rangle_{V'_h, V_h} := \overline{a(v_h, w)}$ for all $v_h \in V_h$. Then defining $\Pi_h = R_{A_h^*} \circ \Theta : W \rightarrow W_h$ yields

$$\begin{aligned} a(v_h, \Pi_h(w)) &= \langle A_h v_h, R_{A_h^*}(\Theta(w)) \rangle_{W'_h, W_h} = \overline{\langle A_h^*(R_{A_h^*}(\Theta(w))), v_h \rangle_{V'_h, V_h}} \\ &= \overline{\langle \Theta(w), v_h \rangle_{V'_h, V_h}} = a(v_h, w), \end{aligned}$$

which proves that $a(v_h, \Pi_h(w) - w) = 0$ for all $w \in W$. Moreover,

$$\hat{\alpha} \|\Pi_h(w)\|_W = \hat{\alpha} \|R_{A_h^*}(\Theta(w))\|_W \leq \|\Theta(w)\|_{V'_h} \leq \|a\| \|w\|_W,$$

which proves that $\frac{\hat{\alpha}}{\|a\|} \|\Pi_h(w)\|_W \leq \|w\|_W$. In addition, we observe that

$$\langle \Theta(R_{A_h^*}(\theta_h)), v_h \rangle_{V'_h, V_h} = \overline{\langle A_h v_h, R_{A_h^*}(\theta_h) \rangle_{W'_h, W_h}} = \langle A_h^*(R_{A_h^*}(\theta_h)), v_h \rangle_{V'_h, V_h} = \langle \theta_h, v_h \rangle_{V'_h, V_h},$$

for all $v_h \in V_h$, which proves that $\Theta(R_{A_h^*}(\theta_h)) = \theta_h$ for all $\theta_h \in V'_h$. As a result, $\Pi_h(\Pi_h(w)) = R_{A_h^*}(\Theta \circ R_{A_h^*}(\Theta(w))) = R_{A_h^*}(\Theta(w)) = \Pi_h(w)$, i.e., Π_h is idempotent.

It remains to build the right-inverse map $R_{A_h^*}$ to complete the proof. We can rewrite (3) as follows:

$$\hat{\alpha} = \inf_{v_h \in V_h} \sup_{w_h \in W_h} \frac{|\langle A_h v_h, w_h \rangle_{W_h', W_h}|}{\|v_h\|_V \|w_h\|_W} > 0.$$

Let us assume first that W is a Hilbert space. Let K_h be the orthogonal complement of $\ker(A_h^*)$ in W_h , i.e., $W_h = K_h \oplus \ker(A_h^*)$. Observing that $A_h^* : K_h \rightarrow V_h'$ is bijective, we set $R_{A_h^*} = (A_h^*|_{K_h})^{-1} : V_h' \rightarrow K_h \subset W_h$. Then $A_h^* R_{A_h^*} \theta_h = \theta_h$ for all $\theta_h \in V_h'$, by definition, and

$$\begin{aligned} \hat{\alpha} &= \inf_{v_h \in V_h} \sup_{w_h \in W_h} \frac{|\langle A_h v_h, w_h \rangle_{W_h', W_h}|}{\|v_h\|_V \|w_h\|_W} = \inf_{w_h \in W_h} \sup_{v_h \in V_h} \frac{|\langle A_h v_h, w_h \rangle_{W_h', W_h}|}{\|v_h\|_V \|w_h\|_W} \\ &\leq \inf_{w_h \in K_h} \sup_{v_h \in V_h} \frac{|\langle A_h v_h, w_h \rangle_{W_h', W_h}|}{\|v_h\|_V \|w_h\|_W} = \inf_{\theta_h \in V_h'} \sup_{v_h \in V_h} \frac{|\langle A_h v_h, R_{A_h^*} \theta_h \rangle_{W_h', W_h}|}{\|v_h\|_V \|R_{A_h^*} \theta_h\|_W} \\ &= \inf_{\theta_h \in V_h'} \sup_{v_h \in V_h} \frac{|\langle A_h^* R_{A_h^*} \theta_h, v_h \rangle_{V_h', V_h}|}{\|v_h\|_V \|R_{A_h^*} \theta_h\|_W} = \inf_{\theta_h \in V_h'} \frac{\|\theta_h\|_{V_h'}}{\|R_{A_h^*} \theta_h\|_W}, \end{aligned}$$

where the first equality results from Lemma 2 below (this lemma provides an abstract counterpart of the fact that the singular values of a square matrix and its transpose coincide; this algebraic result could be invoked here directly). This implies that $\hat{\alpha} \|R_{A_h^*} \theta_h\|_W \leq \|\theta_h\|_{V_h'}$ for all $\theta_h \in V_h'$. Hence $R_{A_h^*}$ has the desired properties; note that $R_{A_h^*}$ is linear.

In the more general setting of Banach spaces, we set $Y := W_h$, $Z := V_h'$, and $B := A_h^*$. Identifying A_h^{**} with A_h , we obtain

$$\hat{\alpha} = \inf_{z' \in Z'} \sup_{y \in Y} \frac{|\langle B^* z', y \rangle_{Y', Y}|}{\|z'\|_{Z'} \|y\|_Y} > 0.$$

We now apply Lemma 4 below and infer that there exists a right-inverse map $R_{A_h^*} : V_h' \rightarrow W_h$ with the desired properties.

Remark 1 (Linearity and uniform stability) Assume that we have at hand a sequence of finite-dimensional subspaces $\{V_h\}_{h \in \mathcal{H}}$, $\{W_h\}_{h \in \mathcal{H}}$. Assume the existence of a decomposition $W_h = \ker(A_h^*) \oplus K_h$ that is uniformly stable with respect to $h \in \mathcal{H}$, i.e., there is $\kappa > 0$, independent of $h \in \mathcal{H}$, such that the induced projector $\pi_{K_h} : W_h \rightarrow K_h$ satisfies $\kappa \|\pi_{K_h} w_h\|_W \leq \|w_h\|_W$ for all $w_h \in W_h$. This property holds in the Hilbertian setting with $\kappa = 1$. Then, even for Banach spaces, one can use the reasoning above for Hilbert spaces to build a linear operator Π_h that is uniformly bounded; the only difference is the bound $\|R_{A_h^*}(\theta_h)\|_W \leq (\kappa \hat{\alpha})^{-1} \|\theta_h\|_{V_h'}$ leading to $\gamma_{\Pi} = \frac{\kappa \hat{\alpha}}{\|\alpha\|}$.

3. Operators in Banach spaces

Let Y and Z be two complex Banach spaces equipped with the norms $\|\cdot\|_Y$ and $\|\cdot\|_Z$, respectively. Let $B : Y \rightarrow Z$ be a bounded linear map.

Lemma 2 (Inf-sup) Assume that B is bijective and that Y is reflexive. Then

$$\inf_{y \in Y} \sup_{z' \in Z'} \frac{|\langle z', B y \rangle_{Z', Z}|}{\|z'\|_{Z'} \|y\|_Y} = \inf_{z' \in Z'} \sup_{y \in Y} \frac{|\langle z', B y \rangle_{Z', Z}|}{\|z'\|_{Z'} \|y\|_Y}. \quad (4)$$

Proof Denote by l and r the left- and right-hand side of (4), respectively. The left-hand side being equal to l means that l is the largest number such that $\|B y\|_Z \geq l \|y\|_Y$ for all y in Y . Let $z' \in Z'$ and

$z \in Z$. Since B is surjective, there is $y_z \in Y$ so that $By_z = z$ and the previous statement regarding l implies that $l\|y_z\|_Y \leq \|z\|_Z$. This in turn implies that

$$\begin{aligned} \|z'\|_{Z'} &= \sup_{z \in Z} \frac{|\langle z', z \rangle_{Z', Z}|}{\|w\|_Z} = \sup_{z \in Z} \frac{|\langle w', By_z \rangle_{Z', Z}|}{\|z\|_Z} = \sup_{z \in Z} \frac{|\langle B^*w', y_z \rangle_{Y', Y}|}{\|z\|_Z} \\ &\leq \|B^*w'\|_{Y'} \sup_{z \in Z} \frac{\|y_z\|_Y}{\|z\|_Z} \leq \frac{1}{l} \|B^*w'\|_{Y'}, \end{aligned}$$

which implies $l \leq r$. That $r \leq l$ is proved similarly by working with Z' in lieu of Y , Y' in lieu of Z and B^* in lieu of B (which is also surjective). We infer that

$$\inf_{z' \in Z'} \sup_{y'' \in Y''} \frac{\langle y'', B^*z' \rangle_{Y'', Y'}}{\|y''\|_{Y''} \|z'\|_{Z'}} \leq \inf_{y'' \in Y''} \sup_{z' \in Z'} \frac{\langle y'', B^*z' \rangle_{Y'', Y'}}{\|y''\|_{Y''} \|z'\|_{Z'}},$$

and we conclude using the reflexivity of Y . \square

The following result is a consequence of Banach's Open Mapping and Closed Range Theorems, see, e.g., [4, Lem. A.36 & A.40].

Lemma 3 (Surjectivity) *The following three statements are equivalent:*

- (i) $B : Y \rightarrow Z$ is surjective.
- (ii) $B^* : Z' \rightarrow Y'$ is injective and $\text{im}(B^*)$ is closed in Y' .
- (iii) The following holds:

$$\beta := \inf_{z' \in Z'} \sup_{y \in Y} \frac{\langle B^*z', y \rangle_{Y', Y}}{\|z'\|_{Z'} \|y\|_Y} > 0. \quad (5)$$

Assume that B is surjective, i.e., the inf-sup condition (5) holds. We now show that it is possible to construct a right-inverse map of B and to control its norm by the inf-sup constant β .

Lemma 4 (Right inverse) *Assume that (5) holds and that Y is reflexive. Then there is a right-inverse map $R_B : Z \rightarrow Y$ such that*

$$\forall z \in Z, \quad B(R_B(z)) = z \quad \text{and} \quad \beta \|R_B(z)\|_Y \leq \|z\|_Z. \quad (6)$$

Moreover, it is possible to construct a linear map R_B if Y is a Hilbert space.

Proof This result can be found in [4, Lem. A.42]; for completeness, we present a proof. The inf-sup condition (5) implies that B^* is injective (see Lemma 3(ii)). Let us set $\mathcal{H} := \text{im}(B^*) \subset Y'$ equipped with the norm of Y' . Let $R_{B^*} : \mathcal{H} \rightarrow Z'$ be such that, for all $y' \in \mathcal{H}$, $B^*(R_{B^*}(y')) = y'$. B^* being injective, R_{B^*} is uniquely defined and is a linear map; notice also that $R_{B^*}(B^*(z')) = z'$ for all $z' \in Z'$ since $R_{B^*}(B^*(z')) - z'$ is in $\ker(B^*) = \{0\}$. Moreover, the inf-sup condition (5) implies that $\|R_{B^*}(y')\|_{Z'} \leq \beta^{-1} \|y'\|_{Y'}$. We next define the linear map $\phi : Z \rightarrow \mathcal{H}'$ such that, for all $z \in Z$, $\langle \phi(z), y' \rangle_{\mathcal{H}', \mathcal{H}} = \overline{\langle R_{B^*}(y'), z \rangle_{Z', Z}}$ for all $y' \in \mathcal{H}$. We infer that

$$|\langle \phi(z), y' \rangle_{\mathcal{H}', \mathcal{H}}| \leq \|R_{B^*}(y')\|_{Z'} \|z\|_Z \leq \beta^{-1} \|y'\|_{Y'} \|z\|_Z.$$

This means that $\phi(z)$ is bounded on $\mathcal{H} \subset Y'$ with $\|\phi(z)\|_{\mathcal{H}'} \leq \beta^{-1} \|z\|_Z$. Owing to the Hahn–Banach Theorem in complex Banach spaces (see [2, Prop. 11.23]), $\phi(z)$ can be extended to Y' with the same norm. Let $\mathcal{E}(\phi(z)) \in Y''$ be the extension in question with $\|\mathcal{E}(\phi(z))\|_{Y''} \leq \beta^{-1} \|z\|_Z$. Since Y is reflexive, the canonical isometry $J_Y : Y \rightarrow Y''$ is a linear isomorphism. Let us set $R_B(z) := J_Y^{-1}(\mathcal{E}(\phi(z)))$; notice that $J_Y(R_B(z)) = \mathcal{E}(\phi(z))$. Then, the following holds for all $z' \in Z'$:

$$\begin{aligned} \langle z', B(R_B(z)) \rangle_{Z', Z} &= \langle B^*z', R_B(z) \rangle_{Y', Y} = \overline{\langle J_Y(R_B(z)), B^*z' \rangle_{Y'', Y'}} = \overline{\langle \mathcal{E}(\phi(z)), B^*z' \rangle_{Y'', Y'}} \\ &= \langle \phi(z), B^*z' \rangle_{\mathcal{H}', \mathcal{H}} = \langle R_{B^*}(B^*z'), z \rangle_{Z', Z} = \langle z', z \rangle_{Z', Z}, \end{aligned}$$

showing that $B(R_B(z)) = z$. Moreover, $\|R_B(z)\|_Y = \|J_Y(R_B(z))\|_{Y''} = \|\mathcal{E}(\phi(z))\|_{Y''} = \|\phi(z)\|_{\mathcal{H}'} \leq \beta^{-1}\|z\|_Z$, showing that (6) holds. In conclusion, $R_B = J_Y^{-1} \circ \mathcal{E} \circ \phi$, where the first and third maps are linear. If Y is a Hilbert space, then the extension map \mathcal{E} is also linear; it suffices to extend linear forms on \mathcal{H} by zero on the orthogonal complement of \mathcal{H} in Y' . \square

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