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► **To cite this version:**

Alexandre Ern, Jean-Luc Guermond. A converse to Fortin's Lemma in Banach spaces. 2016. hal-01273901v1

**HAL Id: hal-01273901**

**<https://hal.science/hal-01273901v1>**

Preprint submitted on 14 Feb 2016 (v1), last revised 22 Jun 2016 (v2)

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# A converse to Fortin's Lemma in Banach spaces

Alexandre Ern<sup>a</sup> and Jean-Luc Guermond<sup>b</sup>

<sup>a</sup>*Université Paris-Est, CERMICS (ENPC), 77455 Marne la Vallée Cedex 2, France*

<sup>b</sup>*Department of Mathematics, Texas A&M University 3368 TAMU, College Station, TX 77843, USA*

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## Abstract

The converse of Fortin's Lemma in Banach spaces is established in this Note.

**MSC2010 classification** 65N30, 65N12, 46B10

*Key words:* Finite elements, surjective operators, inf-sup condition, duality

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## 1. Introduction

Let  $V$  and  $W$  be two complex Banach spaces equipped with the norms  $\|\cdot\|_V$  and  $\|\cdot\|_W$ , respectively. We adopt the convention that dual spaces are denoted with primes and are composed of antilinear forms; complex conjugates are denoted by an overline. Let  $a$  be a sesquilinear form on  $V \times W$  (linear w.r.t. its first argument and antilinear w.r.t. its second argument). We assume that  $a$  is bounded, i.e.,

$$\|a\| := \sup_{v \in V} \sup_{w \in W} \frac{|a(v, w)|}{\|v\|_V \|w\|_W} < \infty, \quad (1)$$

and that the following inf-sup condition holds:

$$\alpha := \inf_{v \in V} \sup_{w \in W} \frac{|a(v, w)|}{\|v\|_V \|w\|_W} > 0. \quad (2)$$

Here and in what follows, arguments in infima and suprema are implicitly assumed to be nonzero.

Let  $V_h \subset V$  and  $W_h \subset W$  be two finite-dimensional subspaces equipped with the norms of  $V$  and  $W$ , respectively. A question of fundamental importance is to assert the following discrete inf-sup condition:

$$\hat{\alpha} := \inf_{v_h \in V_h} \sup_{w_h \in W_h} \frac{|a(v_h, w_h)|}{\|v_h\|_V \|w_h\|_W} > 0. \quad (3)$$

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*Email addresses:* [ern@cermics.enpc.fr](mailto:ern@cermics.enpc.fr) (Alexandre Ern), [guermond@math.tamu.edu](mailto:guermond@math.tamu.edu) (Jean-Luc Guermond).

The aim of this Note is to prove the following result.

**Theorem 1 (Fortin's Lemma with converse)** *Under the above assumptions, consider the following two statements:*

- (i) *There exists a map  $\Pi_h : W \rightarrow W_h$  and a real number  $\gamma_\Pi > 0$  such that  $a(v_h, \Pi_h w - w) = 0$ , for all  $(v_h, w) \in V_h \times W$ , and  $\gamma_\Pi \|\Pi_h w\|_W \leq \|w\|_W$  for all  $w \in W$ .*
- (ii) *The discrete inf-sup condition (3) holds.*

Then, (i)  $\Rightarrow$  (ii) with  $\hat{\alpha} = \gamma_\Pi \alpha$ . Conversely, (ii)  $\Rightarrow$  (i) with  $\gamma_\Pi = \frac{\hat{\alpha}}{\|a\|}$ , and  $\Pi_h$  can be constructed to be idempotent. Moreover,  $\Pi_h$  can be made linear if  $W$  is a Hilbert space.

The statement (i)  $\Rightarrow$  (ii) in Theorem 1 is classical and is known in the literature as Fortin's Lemma, see [5] and [1, Prop. 5.4.3]. It provides an effective tool to prove the discrete inf-sup condition (3) by constructing explicitly a Fortin operator  $\Pi_h$ . We briefly outline a proof that (i)  $\Rightarrow$  (ii) for completeness. Assuming (i), we have

$$\sup_{w_h \in W_h} \frac{|a(v_h, w_h)|}{\|w_h\|_W} \geq \sup_{w \in W} \frac{|a(v_h, \Pi_h w)|}{\|\Pi_h w\|_W} = \sup_{w \in W} \frac{|a(v_h, w)|}{\|\Pi_h w\|_W} \geq \gamma_\Pi \sup_{w \in W} \frac{|a(v_h, w)|}{\|w\|_W} \geq \gamma_\Pi \alpha \|v_h\|_V,$$

since  $a$  satisfies (2) and  $V_h \subset V$ . This proves (ii) with  $\hat{\alpha} = \gamma_\Pi \alpha$ .

The converse (ii)  $\Rightarrow$  (i) is of independent theoretical interest and is the main object of this Note. This property is useful when it is easier to prove the discrete inf-sup condition directly rather than constructing a Fortin operator. Another application of current interest is the analysis framework for discontinuous Petrov–Galerkin methods (dPG) recently proposed in [3] which includes the existence of a Fortin operator among its key assumptions. Incidentally, we observe that there is a gap in the stability constant  $\gamma_\Pi$  between the direct and converse statements, since the ratio of the two is equal to  $\frac{\|a\|}{\alpha}$  (which is independent of the spaces  $V_h$  and  $W_h$ ).

## 2. Proof of Theorem 1

Assume that the discrete inf-sup condition (3) holds. Let  $A_h : V_h \rightarrow W'_h$  be the operator defined by  $\langle A_h v_h, w_h \rangle_{W'_h, W_h} := a(v_h, w_h)$ . Identifying  $V''_h$  with  $V_h$  and  $W''_h$  with  $W_h$  (since these spaces are finite-dimensional), we consider the linear map  $A_h^* : W_h \rightarrow V'_h$ . Our goal is to construct a right-inverse map  $R_{A_h^*} : V'_h \rightarrow W_h$  (possibly nonlinear) such that, for all  $\theta_h \in V'_h$ ,  $A_h^*(R_{A_h^*}(\theta_h)) = \theta_h$  and  $\hat{\alpha} \|R_{A_h^*}(\theta_h)\|_W \leq \|\theta_h\|_{V'_h}$ . Indeed, if such a map exists, we can consider the linear map  $\Theta : W \rightarrow V'_h$  such that, for all  $w \in W$ ,  $\langle \Theta(w), v_h \rangle_{V'_h, V_h} := \overline{a(v_h, w)}$  for all  $v_h \in V_h$ . Then defining  $\Pi_h = R_{A_h^*} \circ \Theta : W \rightarrow W_h$  yields

$$\begin{aligned} a(v_h, \Pi_h(w)) &= \langle A_h v_h, R_{A_h^*}(\Theta(w)) \rangle_{W'_h, W_h} = \overline{\langle A_h^*(R_{A_h^*}(\Theta(w))), v_h \rangle_{V'_h, V_h}} \\ &= \overline{\langle \Theta(w), v_h \rangle_{V'_h, V_h}} = a(v_h, w), \end{aligned}$$

which proves that  $a(v_h, \Pi_h(w) - w) = 0$  for all  $w \in W$ . Moreover,

$$\hat{\alpha} \|\Pi_h(w)\|_W = \hat{\alpha} \|R_{A_h^*}(\Theta(w))\|_W \leq \|\Theta(w)\|_{V'_h} \leq \|a\| \|w\|_W,$$

which proves that  $\frac{\hat{\alpha}}{\|a\|} \|\Pi_h(w)\|_W \leq \|w\|_W$ . In addition, we observe that

$$\langle \Theta(R_{A_h^*}(\theta_h)), v_h \rangle_{V'_h, V_h} = \overline{\langle A_h v_h, R_{A_h^*}(\theta_h) \rangle_{W'_h, W_h}} = \langle A_h^*(R_{A_h^*}(\theta_h)), v_h \rangle_{V'_h, V_h} = \langle \theta_h, v_h \rangle_{V'_h, V_h},$$

for all  $v_h \in V_h$ , which proves that  $\Theta(R_{A_h^*}(\theta_h)) = \theta_h$  for all  $\theta_h \in V'_h$ . As a result,  $\Pi_h(\Pi_h(w)) = R_{A_h^*}(\Theta \circ R_{A_h^*}(\Theta(w))) = R_{A_h^*}(\Theta(w)) = \Pi_h(w)$ , i.e.,  $\Pi_h$  is idempotent.

It remains to build the right-inverse map  $R_{A_h^*}$  to complete the proof. We can rewrite (3) as follows:

$$\hat{\alpha} = \inf_{v_h \in V_h} \sup_{w_h \in W_h} \frac{|\langle A_h v_h, w_h \rangle_{W_h', W_h}|}{\|v_h\|_V \|w_h\|_W} > 0.$$

Let us assume first that  $W$  is a Hilbert space. Let  $K_h$  be the orthogonal complement of  $\ker(A_h^*)$  in  $W_h$ , i.e.,  $W_h = K_h \oplus \ker(A_h^*)$ . Observing that  $A_h^* : K_h \rightarrow V_h'$  is bijective, we set  $R_{A_h^*} = (A_h^*|_{K_h})^{-1} : V_h' \rightarrow K_h \subset W_h$ . Then  $A_h^* R_{A_h^*} \theta_h = \theta_h$  for all  $\theta_h \in V_h'$ , by definition, and

$$\begin{aligned} \hat{\alpha} &= \inf_{v_h \in V_h} \sup_{w_h \in W_h} \frac{|\langle A_h v_h, w_h \rangle_{W_h', W_h}|}{\|v_h\|_V \|w_h\|_W} = \inf_{w_h \in W_h} \sup_{v_h \in V_h} \frac{|\langle A_h v_h, w_h \rangle_{W_h', W_h}|}{\|v_h\|_V \|w_h\|_W} \\ &\leq \inf_{w_h \in K_h} \sup_{v_h \in V_h} \frac{|\langle A_h v_h, w_h \rangle_{W_h', W_h}|}{\|v_h\|_V \|w_h\|_W} = \inf_{\theta_h \in V_h'} \sup_{v_h \in V_h} \frac{|\langle A_h v_h, R_{A_h^*} \theta_h \rangle_{W_h', W_h}|}{\|v_h\|_V \|R_{A_h^*} \theta_h\|_W} \\ &= \inf_{\theta_h \in V_h'} \sup_{v_h \in V_h} \frac{|\langle A_h^* R_{A_h^*} \theta_h, v_h \rangle_{V_h', V_h}|}{\|v_h\|_V \|R_{A_h^*} \theta_h\|_W} = \inf_{\theta_h \in V_h'} \frac{\|\theta_h\|_{V_h'}}{\|R_{A_h^*} \theta_h\|_W}, \end{aligned}$$

where the first equality results from Lemma 2 below (this lemma provides an abstract counterpart of the fact that the singular values of a square matrix and its transpose coincide; this algebraic result could be invoked here directly). This implies that  $\hat{\alpha} \|R_{A_h^*}\|_W \leq \|\theta_h\|_{V_h'}$  for all  $\theta_h \in V_h'$ . Hence  $R_{A_h^*}$  has the desired properties; note that  $R_{A_h^*}$  is linear.

In the more general setting of Banach spaces, we set  $Y := W_h$ ,  $Z := V_h'$ , and  $B := A_h^*$ . Identifying  $A_h^{**}$  with  $A_h$ , we obtain

$$\hat{\alpha} = \inf_{z' \in Z'} \sup_{y \in Y} \frac{|\langle B^* z', y \rangle_{Y', Y}|}{\|z'\|_{Z'} \|y\|_Y} > 0.$$

We now apply Lemma 4 below and infer that there exists a right-inverse map  $R_{A_h^*} : V_h' \rightarrow W_h$  with the desired properties.

*Remark 1 (Linearity and uniform stability)* Assume that we have at hand a sequence of finite-dimensional subspaces  $\{V_h\}_{h \in \mathcal{H}}$ ,  $\{W_h\}_{h \in \mathcal{H}}$ . Assume the existence of a decomposition  $W_h = \ker(A_h^*) \oplus K_h$  that is uniformly stable with respect to  $h \in \mathcal{H}$ , i.e., there is  $\kappa > 0$ , independent of  $h \in \mathcal{H}$ , such that the induced projector  $\pi_{K_h} : W_h \rightarrow K_h$  satisfies  $\kappa \|\pi_{K_h} w_h\|_W \leq \|w_h\|_W$  for all  $w_h \in W_h$ . This property holds in the Hilbertian setting with  $\kappa = 1$ . Then, even for Banach spaces, one can use the reasoning above for Hilbert spaces to build a linear operator  $\Pi_h$  that is uniformly bounded; the only difference is the bound  $\|R_{A_h^*}(\theta_h)\|_W \leq (\kappa \hat{\alpha})^{-1} \|\theta_h\|_{V_h'}$  leading to  $\gamma_{\Pi} = \frac{\kappa \hat{\alpha}}{\|\alpha\|}$ .

### 3. Operators in Banach spaces

Let  $Y$  and  $Z$  be two complex Banach spaces equipped with the norms  $\|\cdot\|_Y$  and  $\|\cdot\|_Z$ , respectively. Let  $B : Y \rightarrow Z$  be a bounded linear map.

**Lemma 2 (Inf-sup)** Assume that  $B$  is bijective and that  $Y$  is reflexive. Then

$$\inf_{y \in Y} \sup_{z' \in Z'} \frac{|\langle z', B y \rangle_{Z', Z}|}{\|z'\|_{Z'} \|y\|_Y} = \inf_{z' \in Z'} \sup_{y \in Y} \frac{|\langle z', B y \rangle_{Z', Z}|}{\|z'\|_{Z'} \|y\|_Y}. \quad (4)$$

**Proof** Denote by  $l$  and  $r$  the left- and right-hand side of (4), respectively. The left-hand side being equal to  $l$  means that  $l$  is the largest number such that  $\|B y\|_Z \geq l \|y\|_Y$  for all  $y$  in  $Y$ . Let  $z' \in Z'$  and

$z \in Z$ . Since  $B$  is surjective, there is  $y_z \in Y$  so that  $By_z = z$  and the previous statement regarding  $l$  implies that  $l\|y_z\|_Y \leq \|z\|_Z$ . This in turn implies that

$$\begin{aligned} \|z'\|_{Z'} &= \sup_{z \in Z} \frac{|\langle z', z \rangle_{Z', Z}|}{\|w\|_Z} = \sup_{z \in Z} \frac{|\langle w', By_z \rangle_{Z', Z}|}{\|z\|_Z} = \sup_{z \in Z} \frac{|\langle B^*w', y_z \rangle_{Y', Y}|}{\|z\|_Z} \\ &\leq \|B^*w'\|_{Y'} \sup_{z \in Z} \frac{\|y_z\|_Y}{\|z\|_Z} \leq \frac{1}{l} \|B^*w'\|_{Y'}, \end{aligned}$$

which implies  $l \leq r$ . That  $r \leq l$  is proved similarly by working with  $Z'$  in lieu of  $Y$ ,  $Y'$  in lieu of  $Z$  and  $B^*$  in lieu of  $B$  (which is also surjective). We infer that

$$\inf_{z' \in Z'} \sup_{y'' \in Y''} \frac{\langle y'', B^*z' \rangle_{Y'', Y'}}{\|y''\|_{Y''} \|z'\|_{Z'}} \leq \inf_{y'' \in Y''} \sup_{z' \in Z'} \frac{\langle y'', B^*z' \rangle_{Y'', Y'}}{\|y''\|_{Y''} \|z'\|_{Z'}},$$

and we conclude using the reflexivity of  $Y$ .  $\square$

The following result is a consequence of Banach's Open Mapping and Closed Range Theorems, see, e.g., [4, Lem. A.36 & A.40].

**Lemma 3 (Surjectivity)** *The following three statements are equivalent:*

- (i)  $B : Y \rightarrow Z$  is surjective.
- (ii)  $B^* : Z' \rightarrow Y'$  is injective and  $\text{im}(B^*)$  is closed in  $Y'$ .
- (iii) The following holds:

$$\beta := \inf_{z' \in Z'} \sup_{y \in Y} \frac{\langle B^*z', y \rangle_{Y', Y}}{\|z'\|_{Z'} \|y\|_Y} > 0. \quad (5)$$

Assume that  $B$  is surjective, i.e., the inf-sup condition (5) holds. We now show that it is possible to construct a right-inverse map of  $B$  and to control its norm by the inf-sup constant  $\beta$ .

**Lemma 4 (Right inverse)** *Assume that (5) holds and that  $Y$  is reflexive. Then there is a right-inverse map  $R_B : Z \rightarrow Y$  such that*

$$\forall z \in Z, \quad B(R_B(z)) = z \quad \text{and} \quad \beta \|R_B(z)\|_Y \leq \|z\|_Z. \quad (6)$$

Moreover, it is possible to construct a linear map  $R_B$  if  $Y$  is a Hilbert space.

**Proof** This result can be found in [4, Lem. A.42]; for completeness, we present a proof. The inf-sup condition (5) implies that  $B^*$  is injective (see Lemma 3(ii)). Let us set  $\mathcal{H} := \text{im}(B^*) \subset Y'$  equipped with the norm of  $Y'$ . Let  $R_{B^*} : \mathcal{H} \rightarrow Z'$  be such that, for all  $y' \in \mathcal{H}$ ,  $B^*(R_{B^*}(y')) = y'$ .  $B^*$  being injective,  $R_{B^*}$  is uniquely defined and is a linear map; notice also that  $R_{B^*}(B^*(z')) = z'$  for all  $z' \in Z'$  since  $R_{B^*}(B^*(z')) - z'$  is in  $\ker(B^*) = \{0\}$ . Moreover, the inf-sup condition (5) implies that  $\|R_{B^*}(y')\|_{Z'} \leq \beta^{-1} \|y'\|_{Y'}$ . We next define the linear map  $\phi : Z \rightarrow \mathcal{H}'$  such that, for all  $z \in Z$ ,  $\langle \phi(z), y' \rangle_{\mathcal{H}', \mathcal{H}} = \overline{\langle R_{B^*}(y'), z \rangle_{Z', Z}}$  for all  $y' \in \mathcal{H}$ . We infer that

$$|\langle \phi(z), y' \rangle_{\mathcal{H}', \mathcal{H}}| \leq \|R_{B^*}(y')\|_{Z'} \|z\|_Z \leq \beta^{-1} \|y'\|_{Y'} \|z\|_Z.$$

This means that  $\phi(z)$  is bounded on  $\mathcal{H} \subset Y'$  with  $\|\phi(z)\|_{\mathcal{H}'} \leq \beta^{-1} \|z\|_Z$ . Owing to the Hahn–Banach Theorem in complex Banach spaces (see [2, Prop. 11.23]),  $\phi(z)$  can be extended to  $Y'$  with the same norm. Let  $\mathcal{E}(\phi(z)) \in Y''$  be the extension in question with  $\|\mathcal{E}(\phi(z))\|_{Y''} \leq \beta^{-1} \|z\|_Z$ . Since  $Y$  is reflexive, the canonical isometry  $J_Y : Y \rightarrow Y''$  is a linear isomorphism. Let us set  $R_B(z) := J_Y^{-1}(\mathcal{E}(\phi(z)))$ ; notice that  $J_Y(R_B(z)) = \mathcal{E}(\phi(z))$ . Then, the following holds for all  $z' \in Z'$ :

$$\begin{aligned} \langle z', B(R_B(z)) \rangle_{Z', Z} &= \langle B^*z', R_B(z) \rangle_{Y', Y} = \overline{\langle J_Y(R_B(z)), B^*z' \rangle_{Y'', Y'}} = \overline{\langle \mathcal{E}(\phi(z)), B^*z' \rangle_{Y'', Y'}} \\ &= \langle \phi(z), B^*z' \rangle_{\mathcal{H}', \mathcal{H}} = \langle R_{B^*}(B^*z'), z \rangle_{Z', Z} = \langle z', z \rangle_{Z', Z}, \end{aligned}$$

showing that  $B(R_B(z)) = z$ . Moreover,  $\|R_B(z)\|_Y = \|J_Y(R_B(z))\|_{Y''} = \|\mathcal{E}(\phi(z))\|_{Y''} = \|\phi(z)\|_{\mathcal{H}'} \leq \beta^{-1}\|z\|_Z$ , showing that (6) holds. In conclusion,  $R_B = J_Y^{-1} \circ \mathcal{E} \circ \phi$ , where the first and third maps are linear. If  $Y$  is a Hilbert space, then the extension map  $\mathcal{E}$  is also linear; it suffices to extend linear forms on  $\mathcal{H}$  by zero on the orthogonal complement of  $\mathcal{H}$  in  $Y'$ .  $\square$

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