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A converse to Fortin's Lemma in Banach spaces

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Abstract

We establish the converse of Fortin's Lemma in Banach spaces. This result is useful to assert the existence of a Fortin operator once a discrete inf-sup condition has been proved. The proof uses a specific construction of a right-inverse of a surjective operator in Banach spaces. The key issue is the sharp determination of the stability constants.

1 Introduction

Let V and W be two complex Banach spaces equipped with the norms $\|\cdot\|_V$ and $\|\cdot\|_W$, respectively. We adopt the convention that dual spaces are denoted with primes and are composed of antilinear forms; complex conjugates are denoted by an overline. Let a be a sesquilinear form on $V \times W$ (linear w.r.t. its first argument and antilinear w.r.t. its second argument). We assume that a is bounded, i.e.,

$$\|a\| := \sup_{v \in V} \sup_{w \in W} \frac{|a(v, w)|}{\|v\|_V \|w\|_W} < \infty, \quad (1)$$

and that the following inf-sup condition holds:

$$\alpha := \inf_{v \in V} \sup_{w \in W} \frac{|a(v, w)|}{\|v\|_V \|w\|_W} > 0. \quad (2)$$

Here and in what follows, arguments in infima and suprema are implicitly assumed to be nonzero.

Assume that we have at hand two sequences of finite-dimensional subspaces $\{V_h\}_{h \in \mathcal{H}}$ and $\{W_h\}_{h \in \mathcal{H}}$ with $V_h \subset V$ and $W_h \subset W$ for all $h \in \mathcal{H}$, where the parameter h typically refers to a family of underlying meshes. The spaces V_h

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and W_h are equipped with the norms of V and W , respectively. A question of fundamental importance is to assert the following discrete inf-sup condition:

$$\hat{\alpha}_h := \inf_{v_h \in V_h} \sup_{w_h \in W_h} \frac{|a(v_h, w_h)|}{\|v_h\|_V \|w_h\|_W} > 0. \quad (3)$$

The aim of this Note is to prove the following result.

Theorem 1 (Fortin’s Lemma with converse) *Under the above assumptions, consider the following two statements:*

- (i) *There exists a map $\Pi_h : W \rightarrow W_h$ and a real number $\gamma_{\Pi_h} > 0$ such that $a(v_h, \Pi_h w - w) = 0$, for all $(v_h, w) \in V_h \times W$, and $\gamma_{\Pi_h} \|\Pi_h w\|_W \leq \|w\|_W$ for all $w \in W$.*
- (ii) *The discrete inf-sup condition (3) holds.*

Then, (i) \Rightarrow (ii) with $\hat{\alpha}_h \geq \gamma_{\Pi_h} \alpha$. Conversely, (ii) \Rightarrow (i) with $\gamma_{\Pi_h} = \frac{\hat{\alpha}_h}{\|a\|}$, and Π_h can be constructed to be idempotent. Moreover, Π_h can be made linear if W is a Hilbert space.

The statement (i) \Rightarrow (ii) in Theorem 1 is classical and is known in the literature as Fortin’s Lemma, see [5] and [1, Prop. 5.4.3]. It provides an effective tool to prove the discrete inf-sup condition (3) by constructing explicitly a Fortin operator Π_h . We briefly outline a proof that (i) \Rightarrow (ii) for completeness. Assuming (i), we have

$$\begin{aligned} \sup_{w_h \in W_h} \frac{|a(v_h, w_h)|}{\|w_h\|_W} &\geq \sup_{w \in W} \frac{|a(v_h, \Pi_h w)|}{\|\Pi_h w\|_W} = \sup_{w \in W} \frac{|a(v_h, w)|}{\|\Pi_h w\|_W} \\ &\geq \gamma_{\Pi_h} \sup_{w \in W} \frac{|a(v_h, w)|}{\|w\|_W} \geq \gamma_{\Pi_h} \alpha \|v_h\|_V, \end{aligned}$$

since a satisfies (2) and $V_h \subset V$. This proves (ii) with $\hat{\alpha}_h \geq \gamma_{\Pi_h} \alpha$.

The proof of the converse (ii) \Rightarrow (i) is the main object of this Note. This property is useful when it is easier to prove the discrete inf-sup condition directly rather than constructing a Fortin operator. Another application of current interest is the analysis framework for discontinuous Petrov–Galerkin methods (dPG) recently proposed in [3] which includes the existence of a Fortin operator among its key assumptions. The proof of the converse is not so straightforward if one wishes to establish a sharp stability bound for Π_h , i.e., that indeed one can take $\gamma_{\Pi_h} = \frac{\hat{\alpha}_h}{\|a\|}$. Incidentally, we observe that there is a gap in the stability constant γ_{Π_h} between the direct and converse statements, since the ratio of the two is equal to $\frac{\|a\|}{\alpha}$ (which is independent of the discrete setting).

This Note is organized as follows. In Section 2, we establish a sharp bound on the stability of the right-inverse of surjective operators in Banach spaces. Since this result can be of independent theoretical interest, we present it in an infinite-dimensional setting. Then in Section 3, we prove the converse of Fortin’s Lemma. The proof is relatively simple once the sharp stability estimate from Section 2 is available.

2 Right-inverse of surjective Banach operators

Let Y and Z be two complex Banach spaces equipped with the norms $\|\cdot\|_Y$ and $\|\cdot\|_Z$, respectively. Let $B : Y \rightarrow Z$ be a bounded linear map. The following result is a well-known consequence of Banach's Open Mapping and Closed Range Theorems, see, e.g., [2, Thm. 2.20] or [4, Lem. A.36 & A.40].

Lemma 2 (Surjectivity) *The following three statements are equivalent:*

- (i) $B : Y \rightarrow Z$ is surjective.
- (ii) $B^* : Z' \rightarrow Y'$ is injective and $\text{im}(B^*)$ is closed in Y' .
- (iii) The following holds:

$$\inf_{z' \in Z'} \frac{\|B^* z'\|_{Y'}}{\|z'\|_{Z'}} = \inf_{z' \in Z'} \sup_{y \in Y} \frac{|\langle B^* z', y \rangle_{Y', Y}|}{\|z'\|_{Z'} \|y\|_Y} =: \beta > 0. \quad (4)$$

Let us now turn to the main result of this section. To motivate the result, assume that (4) holds; then B is surjective and thus admits a bounded right-inverse. The crucial question is whether the stability of this right-inverse can be formulated using precisely the constant $\beta > 0$ from (4).

Lemma 3 (Right inverse) *Assume that (4) holds and that Y is reflexive. Then there is a right-inverse map $B^\dagger : Z \rightarrow Y$ such that*

$$\forall z \in Z, \quad (B \circ B^\dagger)(z) = z \quad \text{and} \quad \beta \|B^\dagger z\|_Y \leq \|z\|_Z. \quad (5)$$

Moreover, this right-inverse map B^\dagger is linear if Y is a Hilbert space.

Proof Parts of this result can be found in [4, Lem. A.42]; for completeness, we present a proof. Owing to Lemma 2, B^* is injective and $R := \text{im}(B^*)$ is closed in Y' . Since the operator B^* is injective, it admits a left-inverse linear map $B^{*\dagger} : R \rightarrow Z'$ such that $(B^{*\dagger} \circ B^*)(z') = z'$ for all $z' \in R$. Moreover, the inf-sup condition (4) implies that $\|B^{*\dagger} y'\|_{Z'} \leq \beta^{-1} \|y'\|_{Y'}$ for all $y' \in R$. Consider now the adjoint $B^{*\dagger*} : Z'' \rightarrow R'$. Let $E_{R'Y''}^{\text{HB}}$ be the Hahn–Banach extension operator that extends antilinear forms over $R \subset Y'$ into antilinear forms over Y' (see [2, Prop. 11.23]); $E_{R'Y''}^{\text{HB}}$ maps from R' to Y'' . Let J_Y (resp., J_Z) be the canonical isometry from Y to Y'' (resp., Z to Z''), and observe that J_Y is an isomorphism since Y is assumed to be reflexive. Let us set

$$B^\dagger := J_Y^{-1} \circ E_{R'Y''}^{\text{HB}} \circ B^{*\dagger*} \circ J_Z : Z \rightarrow Y, \quad (6)$$

and let us verify that B^\dagger satisfies the expected properties. We have, for all $(z', z) \in Z' \times Z$,

$$\begin{aligned} \langle z', B(B^\dagger(z)) \rangle_{Z', Z} &= \langle B^* z', B^\dagger(z) \rangle_{Y', Y} = \overline{\langle J_Y(B^\dagger(z)), B^* z' \rangle_{Y'', Y'}} \\ &= \overline{\langle E_{R'Y''}^{\text{HB}}(B^{*\dagger*}(J_Z z)), B^* z' \rangle_{Y'', Y'}} = \overline{\langle B^{*\dagger*}(J_Z z), B^* z' \rangle_{R', R}} \\ &= \overline{\langle J_Z z, B^{*\dagger} B^* z' \rangle_{Z'', Z'}} = \overline{\langle J_Z z, z' \rangle_{Z'', Z'}} = \langle z', z \rangle_{Z', Z}, \end{aligned}$$

where we have used that $B^*z' \in R$ to pass from the first to the second line. This shows that $(B \circ B^\dagger)(z) = z$. Moreover, since J_Y is an isometry and the extension operator $E_{R'Y''}^{\text{HB}}$ preserves the norm, we observe that, for all $z \in Z$,

$$\begin{aligned} \|B^\dagger z\|_Y &= \|B^{*\dagger}(J_Z z)\|_{R'} = \sup_{z' \in Z'} \frac{|\langle B^{*\dagger}(J_Z z), B^*z' \rangle_{R',R}|}{\|B^*z'\|_{Y'}} \\ &= \sup_{z' \in Z'} \frac{|\langle J_Z z, z' \rangle_{Z'',Z'}|}{\|B^*z'\|_{Y'}} \leq \sup_{z' \in Z'} \frac{\|z'\|_{Z'}}{\|B^*z'\|_{Y'}} \|z\|_Z. \end{aligned}$$

We conclude from (4) that $\beta\|B^\dagger z\|_Y \leq \|z\|_Z$. Finally, if Y is a Hilbert space, we can consider the orthogonal complement of R in Y' (recall that R is a closed subspace of Y') and write $Y' = R \oplus R^\perp$. Then, the Hahn–Banach extension operator $E_{R'Y''}^{\text{HB}}$ in (6) can be replaced by the linear map $E_{R'Y''}^\perp$ such that, for all $\phi \in R'$, $\langle E_{R'Y''}^\perp \phi, y' \rangle_{Y'',Y'} = \langle \phi, r \rangle_{R',R}$ for all $y' \in Y'$ with $y' = r + r^\perp$, $r \in R$, $r^\perp \in R^\perp$. \square

3 Proof of the converse in Theorem 1

Let $A_h : V_h \rightarrow W'_h$ be the operator defined by $\langle A_h v_h, w_h \rangle_{W'_h, W_h} := a(v_h, w_h)$ for all $(v_h, w_h) \in V_h \times W_h$. We identify V''_h with V_h and W''_h with W_h (since these spaces are finite-dimensional). We consider the adjoint operator $A_h^* : W_h \rightarrow V'_h$, and identify A_h^{**} with A_h . We apply Lemma 3 to $Y := W_h$, $Z := V'_h$, and $B := A_h^*$. Owing to the discrete inf-sup condition (3), we infer that (4) holds with $\beta = \hat{\alpha}_h$. Therefore, there exists a right-inverse map $A_h^{*\dagger} : V'_h \rightarrow W_h$ such that, for all $\theta_h \in V'_h$, $(A_h^* \circ A_h^{*\dagger})(\theta_h) = \theta_h$ and $\hat{\alpha}_h \|A_h^{*\dagger} \theta_h\|_W \leq \|\theta_h\|_{V'_h}$. Let us now set

$$\Pi_h := A_h^{*\dagger} \circ \Theta : W \rightarrow W_h, \quad (7)$$

with the linear map $\Theta : W \rightarrow V'_h$ such that, for all $w \in W$, $\langle \Theta(w), v_h \rangle_{V'_h, V_h} := \overline{a(v_h, w)}$ for all $v_h \in V_h$. We then infer that

$$\begin{aligned} a(v_h, \Pi_h(w)) &= \langle A_h v_h, A_h^{*\dagger}(\Theta(w)) \rangle_{W'_h, W_h} = \overline{\langle A_h^*(A_h^{*\dagger}(\Theta(w))), v_h \rangle_{V'_h, V_h}} \\ &= \overline{\langle \Theta(w), v_h \rangle_{V'_h, V_h}} = a(v_h, w), \end{aligned}$$

which establishes that $a(v_h, \Pi_h(w) - w) = 0$ for all $w \in W$. Moreover,

$$\hat{\alpha}_h \|\Pi_h(w)\|_W = \hat{\alpha}_h \|A_h^{*\dagger}(\Theta(w))\|_W \leq \|\Theta(w)\|_{V'_h} \leq \|a\| \|w\|_W,$$

which proves that $\frac{\hat{\alpha}_h}{\|a\|} \|\Pi_h(w)\|_W \leq \|w\|_W$. In addition, we observe that

$$\begin{aligned} \langle \Theta(A_h^{*\dagger}(\theta_h)), v_h \rangle_{V'_h, V_h} &= \overline{\langle A_h v_h, A_h^{*\dagger}(\theta_h) \rangle_{W'_h, W_h}} \\ &= \langle A_h^*(A_h^{*\dagger}(\theta_h)), v_h \rangle_{V'_h, V_h} = \langle \theta_h, v_h \rangle_{V'_h, V_h}, \end{aligned}$$

for all $v_h \in V_h$, which proves that $\Theta(A_h^{*\dagger}(\theta_h)) = \theta_h$ for all $\theta_h \in V'_h$. As a result, $\Pi_h(\Pi_h(w)) = A_h^{*\dagger}(\Theta \circ A_h^{*\dagger}(\Theta(w))) = A_h^{*\dagger}(\Theta(w)) = \Pi_h(w)$, i.e., Π_h is

idempotent. Finally, if W is a Hilbert space, the right-inverse map $A_h^{*\dagger}$ is linear by Lemma 3, and so is the operator Π_h defined from (7).

Remark 1 (Value of γ_{Π_h}) Without the use of Lemma 3, one only knows that A_h^* has a stable right-inverse, but a stability bound for this right-inverse is not available. In particular, if the discrete inf-sup condition (3) holds uniformly with respect to h , i.e., if there is $\hat{\alpha}_0 > 0$ such that $\hat{\alpha}_h \geq \hat{\alpha}_0$ for all $h \in \mathcal{H}$, then a uniform stability bound for Π_h is $\gamma_{\Pi_h} \geq \gamma_{\Pi_0} = \frac{\hat{\alpha}_0}{\|a\|}$ for all $h \in \mathcal{H}$.

Remark 2 (Linearity) Even in the case of Banach spaces, the linearity of the map Π_h can be asserted if one has at hand a stable decomposition $W_h = \ker(A_h^*) \oplus K_h$ such that there is $\kappa_h > 0$ such that the induced projector $\pi_{K_h} : W_h \rightarrow K_h$ satisfies $\kappa_h \|\pi_{K_h} w_h\|_W \leq \|w_h\|_W$ for all $w_h \in W_h$ (this property holds in the Hilbertian setting with $\kappa_h = 1$). Then, one can adapt the reasoning at the end of the proof of Lemma 3 to build a stable, linear right-inverse map $A_h^{*\dagger}$. The mild price to be paid is that the stability constant of Π_h now becomes $\gamma_{\Pi_h} = \frac{\kappa_h \hat{\alpha}_h}{\|a\|}$.

References

- [1] D. Boffi, F. Brezzi, and M. Fortin. *Mixed finite element methods and applications*, volume 44 of *Springer Series in Computational Mathematics*. Springer, Heidelberg, 2013.
- [2] H. Brezis. *Functional analysis, Sobolev spaces and partial differential equations*. Universitext. Springer, New York, 2011.
- [3] C. Carstensen, L. Demkowicz, and J. Gopalakrishnan. A posteriori error control for DPG methods. *SIAM J. Numer. Anal.*, 52(3):1335–1353, 2014.
- [4] A. Ern and J.-L. Guermond. *Theory and practice of finite elements*, volume 159 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 2004.
- [5] M. Fortin. An analysis of the convergence of mixed finite element methods. *RAIRO, Anal. Num.*, 11:341–354, 1977.