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# A converse to Fortin's Lemma in Banach spaces

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## Abstract

We establish the converse of Fortin's Lemma in Banach spaces. This result is useful to assert the existence of a Fortin operator once a discrete inf-sup condition has been proved. The proof uses a specific construction of a right-inverse of a surjective operator in Banach spaces. The key issue is the sharp determination of the stability constants.

## 1 Introduction

Let  $V$  and  $W$  be two complex Banach spaces equipped with the norms  $\|\cdot\|_V$  and  $\|\cdot\|_W$ , respectively. We adopt the convention that dual spaces are denoted with primes and are composed of antilinear forms; complex conjugates are denoted by an overline. Let  $a$  be a sesquilinear form on  $V \times W$  (linear w.r.t. its first argument and antilinear w.r.t. its second argument). We assume that  $a$  is bounded, i.e.,

$$\|a\| := \sup_{v \in V} \sup_{w \in W} \frac{|a(v, w)|}{\|v\|_V \|w\|_W} < \infty, \quad (1)$$

and that the following inf-sup condition holds:

$$\alpha := \inf_{v \in V} \sup_{w \in W} \frac{|a(v, w)|}{\|v\|_V \|w\|_W} > 0. \quad (2)$$

Here and in what follows, arguments in infima and suprema are implicitly assumed to be nonzero.

Assume that we have at hand two sequences of finite-dimensional subspaces  $\{V_h\}_{h \in \mathcal{H}}$  and  $\{W_h\}_{h \in \mathcal{H}}$  with  $V_h \subset V$  and  $W_h \subset W$  for all  $h \in \mathcal{H}$ , where the parameter  $h$  typically refers to a family of underlying meshes. The spaces  $V_h$

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and  $W_h$  are equipped with the norms of  $V$  and  $W$ , respectively. A question of fundamental importance is to assert the following discrete inf-sup condition:

$$\hat{\alpha}_h := \inf_{v_h \in V_h} \sup_{w_h \in W_h} \frac{|a(v_h, w_h)|}{\|v_h\|_V \|w_h\|_W} > 0. \quad (3)$$

The aim of this Note is to prove the following result.

**Theorem 1 (Fortin’s Lemma with converse)** *Under the above assumptions, consider the following two statements:*

- (i) *There exists a map  $\Pi_h : W \rightarrow W_h$  and a real number  $\gamma_{\Pi_h} > 0$  such that  $a(v_h, \Pi_h w - w) = 0$ , for all  $(v_h, w) \in V_h \times W$ , and  $\gamma_{\Pi_h} \|\Pi_h w\|_W \leq \|w\|_W$  for all  $w \in W$ .*
- (ii) *The discrete inf-sup condition (3) holds.*

Then, (i)  $\Rightarrow$  (ii) with  $\hat{\alpha}_h \geq \gamma_{\Pi_h} \alpha$ . Conversely, (ii)  $\Rightarrow$  (i) with  $\gamma_{\Pi_h} = \frac{\hat{\alpha}_h}{\|a\|}$ , and  $\Pi_h$  can be constructed to be idempotent. Moreover,  $\Pi_h$  can be made linear if  $W$  is a Hilbert space.

The statement (i)  $\Rightarrow$  (ii) in Theorem 1 is classical and is known in the literature as Fortin’s Lemma, see [5] and [1, Prop. 5.4.3]. It provides an effective tool to prove the discrete inf-sup condition (3) by constructing explicitly a Fortin operator  $\Pi_h$ . We briefly outline a proof that (i)  $\Rightarrow$  (ii) for completeness. Assuming (i), we have

$$\begin{aligned} \sup_{w_h \in W_h} \frac{|a(v_h, w_h)|}{\|w_h\|_W} &\geq \sup_{w \in W} \frac{|a(v_h, \Pi_h w)|}{\|\Pi_h w\|_W} = \sup_{w \in W} \frac{|a(v_h, w)|}{\|\Pi_h w\|_W} \\ &\geq \gamma_{\Pi_h} \sup_{w \in W} \frac{|a(v_h, w)|}{\|w\|_W} \geq \gamma_{\Pi_h} \alpha \|v_h\|_V, \end{aligned}$$

since  $a$  satisfies (2) and  $V_h \subset V$ . This proves (ii) with  $\hat{\alpha}_h \geq \gamma_{\Pi_h} \alpha$ .

The proof of the converse (ii)  $\Rightarrow$  (i) is the main object of this Note. This property is useful when it is easier to prove the discrete inf-sup condition directly rather than constructing a Fortin operator. Another application of current interest is the analysis framework for discontinuous Petrov–Galerkin methods (dPG) recently proposed in [3] which includes the existence of a Fortin operator among its key assumptions. The proof of the converse is not so straightforward if one wishes to establish a sharp stability bound for  $\Pi_h$ , i.e., that indeed one can take  $\gamma_{\Pi_h} = \frac{\hat{\alpha}_h}{\|a\|}$ . Incidentally, we observe that there is a gap in the stability constant  $\gamma_{\Pi_h}$  between the direct and converse statements, since the ratio of the two is equal to  $\frac{\|a\|}{\alpha}$  (which is independent of the discrete setting).

This Note is organized as follows. In Section 2, we establish a sharp bound on the stability of the right-inverse of surjective operators in Banach spaces. Since this result can be of independent theoretical interest, we present it in an infinite-dimensional setting. Then in Section 3, we prove the converse of Fortin’s Lemma. The proof is relatively simple once the sharp stability estimate from Section 2 is available.

## 2 Right-inverse of surjective Banach operators

Let  $Y$  and  $Z$  be two complex Banach spaces equipped with the norms  $\|\cdot\|_Y$  and  $\|\cdot\|_Z$ , respectively. Let  $B : Y \rightarrow Z$  be a bounded linear map. The following result is a well-known consequence of Banach's Open Mapping and Closed Range Theorems, see, e.g., [2, Thm. 2.20] or [4, Lem. A.36 & A.40].

**Lemma 2 (Surjectivity)** *The following three statements are equivalent:*

- (i)  $B : Y \rightarrow Z$  is surjective.
- (ii)  $B^* : Z' \rightarrow Y'$  is injective and  $\text{im}(B^*)$  is closed in  $Y'$ .
- (iii) The following holds:

$$\inf_{z' \in Z'} \frac{\|B^* z'\|_{Y'}}{\|z'\|_{Z'}} = \inf_{z' \in Z'} \sup_{y \in Y} \frac{|\langle B^* z', y \rangle_{Y', Y}|}{\|z'\|_{Z'} \|y\|_Y} =: \beta > 0. \quad (4)$$

Let us now turn to the main result of this section. To motivate the result, assume that (4) holds; then  $B$  is surjective and thus admits a bounded right-inverse. The crucial question is whether the stability of this right-inverse can be formulated using precisely the constant  $\beta > 0$  from (4).

**Lemma 3 (Right inverse)** *Assume that (4) holds and that  $Y$  is reflexive. Then there is a right-inverse map  $B^\dagger : Z \rightarrow Y$  such that*

$$\forall z \in Z, \quad (B \circ B^\dagger)(z) = z \quad \text{and} \quad \beta \|B^\dagger z\|_Y \leq \|z\|_Z. \quad (5)$$

Moreover, this right-inverse map  $B^\dagger$  is linear if  $Y$  is a Hilbert space.

**Proof** Parts of this result can be found in [4, Lem. A.42]; for completeness, we present a proof. Owing to Lemma 2,  $B^*$  is injective and  $R := \text{im}(B^*)$  is closed in  $Y'$ . Since the operator  $B^*$  is injective, it admits a left-inverse linear map  $B^{*\dagger} : R \rightarrow Z'$  such that  $(B^{*\dagger} \circ B^*)(z') = z'$  for all  $z' \in Z'$ . Moreover, the inf-sup condition (4) implies that  $\|B^{*\dagger} y'\|_{Z'} \leq \beta^{-1} \|y'\|_{Y'}$  for all  $y' \in R$ . Consider now the adjoint  $B^{*\dagger*} : Z'' \rightarrow R'$ . Let  $E_{R'Y''}^{\text{HB}}$  be the Hahn–Banach extension operator that extends antilinear forms over  $R \subset Y'$  into antilinear forms over  $Y'$  (see [2, Prop. 11.23]);  $E_{R'Y''}^{\text{HB}}$  maps from  $R'$  to  $Y''$ . Let  $J_Y$  (resp.,  $J_Z$ ) be the canonical isometry from  $Y$  to  $Y''$  (resp.,  $Z$  to  $Z''$ ), and observe that  $J_Y$  is an isomorphism since  $Y$  is assumed to be reflexive. Let us set

$$B^\dagger := J_Y^{-1} \circ E_{R'Y''}^{\text{HB}} \circ B^{*\dagger*} \circ J_Z : Z \rightarrow Y, \quad (6)$$

and let us verify that  $B^\dagger$  satisfies the expected properties. We have, for all  $(z', z) \in Z' \times Z$ ,

$$\begin{aligned} \langle z', B(B^\dagger(z)) \rangle_{Z', Z} &= \langle B^* z', B^\dagger(z) \rangle_{Y', Y} = \overline{\langle J_Y(B^\dagger(z)), B^* z' \rangle_{Y'', Y'}} \\ &= \overline{\langle E_{R'Y''}^{\text{HB}}(B^{*\dagger*}(J_Z z)), B^* z' \rangle_{Y'', Y'}} = \overline{\langle B^{*\dagger*}(J_Z z), B^* z' \rangle_{R', R}} \\ &= \overline{\langle J_Z z, B^{*\dagger} B^* z' \rangle_{Z'', Z'}} = \overline{\langle J_Z z, z' \rangle_{Z'', Z'}} = \langle z', z \rangle_{Z', Z}, \end{aligned}$$

where we have used that  $B^*z' \in R$  to pass from the first to the second line. This shows that  $(B \circ B^\dagger)(z) = z$ . Moreover, since  $J_Y$  is an isometry and the extension operator  $E_{R'Y''}^{\text{HB}}$  preserves the norm, we observe that, for all  $z \in Z$ ,

$$\begin{aligned} \|B^\dagger z\|_Y &= \|B^{*\dagger}(J_Z z)\|_{R'} = \sup_{z' \in Z'} \frac{|\langle B^{*\dagger}(J_Z z), B^*z' \rangle_{R',R}|}{\|B^*z'\|_{Y'}} \\ &= \sup_{z' \in Z'} \frac{|\langle J_Z z, z' \rangle_{Z'',Z'}|}{\|B^*z'\|_{Y'}} \leq \sup_{z' \in Z'} \frac{\|z'\|_{Z'}}{\|B^*z'\|_{Y'}} \|z\|_Z. \end{aligned}$$

We conclude from (4) that  $\beta \|B^\dagger z\|_Y \leq \|z\|_Z$ . Finally, if  $Y$  is a Hilbert space, we can consider the orthogonal complement of  $R$  in  $Y'$  (recall that  $R$  is a closed subspace of  $Y'$ ) and write  $Y' = R \oplus R^\perp$ . Then, the Hahn–Banach extension operator  $E_{R'Y''}^{\text{HB}}$  in (6) can be replaced by the linear map  $E_{R'Y''}^\perp$  such that, for all  $\phi \in R'$ ,  $\langle E_{R'Y''}^\perp \phi, y' \rangle_{Y'',Y'} = \langle \phi, r \rangle_{R',R}$  for all  $y' \in Y'$  with  $y' = r + r^\perp$ ,  $r \in R$ ,  $r^\perp \in R^\perp$ .  $\square$

### 3 Proof of the converse in Theorem 1

Let  $A_h : V_h \rightarrow W'_h$  be the operator defined by  $\langle A_h v_h, w_h \rangle_{W'_h, W_h} := a(v_h, w_h)$  for all  $(v_h, w_h) \in V_h \times W_h$ . We identify  $V''_h$  with  $V_h$  and  $W''_h$  with  $W_h$  (since these spaces are finite-dimensional). We consider the adjoint operator  $A_h^* : W_h \rightarrow V'_h$ , and identify  $A_h^{**}$  with  $A_h$ . We apply Lemma 3 to  $Y := W_h$ ,  $Z := V'_h$ , and  $B := A_h^*$ . Owing to the discrete inf-sup condition (3), we infer that (4) holds with  $\beta = \hat{\alpha}_h$ . Therefore, there exists a right-inverse map  $A_h^{*\dagger} : V'_h \rightarrow W_h$  such that, for all  $\theta_h \in V'_h$ ,  $(A_h^* \circ A_h^{*\dagger})(\theta_h) = \theta_h$  and  $\hat{\alpha}_h \|A_h^{*\dagger} \theta_h\|_W \leq \|\theta_h\|_{V'_h}$ . Let us now set

$$\Pi_h := A_h^{*\dagger} \circ \Theta : W \rightarrow W_h, \quad (7)$$

with the linear map  $\Theta : W \rightarrow V'_h$  such that, for all  $w \in W$ ,  $\langle \Theta(w), v_h \rangle_{V'_h, V_h} := \overline{a(v_h, w)}$  for all  $v_h \in V_h$ . We then infer that

$$\begin{aligned} a(v_h, \Pi_h(w)) &= \langle A_h v_h, A_h^{*\dagger}(\Theta(w)) \rangle_{W'_h, W_h} = \overline{\langle A_h^*(A_h^{*\dagger}(\Theta(w))), v_h \rangle_{V'_h, V_h}} \\ &= \overline{\langle \Theta(w), v_h \rangle_{V'_h, V_h}} = a(v_h, w), \end{aligned}$$

which establishes that  $a(v_h, \Pi_h(w) - w) = 0$  for all  $w \in W$ . Moreover,

$$\hat{\alpha}_h \|\Pi_h(w)\|_W = \hat{\alpha}_h \|A_h^{*\dagger}(\Theta(w))\|_W \leq \|\Theta(w)\|_{V'_h} \leq \|a\| \|w\|_W,$$

which proves that  $\frac{\hat{\alpha}_h}{\|a\|} \|\Pi_h(w)\|_W \leq \|w\|_W$ . In addition, we observe that

$$\begin{aligned} \langle \Theta(A_h^{*\dagger}(\theta_h)), v_h \rangle_{V'_h, V_h} &= \overline{\langle A_h v_h, A_h^{*\dagger}(\theta_h) \rangle_{W'_h, W_h}} \\ &= \langle A_h^*(A_h^{*\dagger}(\theta_h)), v_h \rangle_{V'_h, V_h} = \langle \theta_h, v_h \rangle_{V'_h, V_h}, \end{aligned}$$

for all  $v_h \in V_h$ , which proves that  $\Theta(A_h^{*\dagger}(\theta_h)) = \theta_h$  for all  $\theta_h \in V'_h$ . As a result,  $\Pi_h(\Pi_h(w)) = A_h^{*\dagger}(\Theta \circ A_h^{*\dagger}(\Theta(w))) = A_h^{*\dagger}(\Theta(w)) = \Pi_h(w)$ , i.e.,  $\Pi_h$  is

idempotent. Finally, if  $W$  is a Hilbert space, the right-inverse map  $A_h^{*\dagger}$  is linear by Lemma 3, and so is the operator  $\Pi_h$  defined from (7).

*Remark 1 (Value of  $\gamma_{\Pi_h}$ )* Without the use of Lemma 3, one only knows that  $A_h^*$  has a stable right-inverse, but a stability bound for this right-inverse is not available. In particular, if the discrete inf-sup condition (3) holds uniformly with respect to  $h$ , i.e., if there is  $\hat{\alpha}_0 > 0$  such that  $\hat{\alpha}_h \geq \hat{\alpha}_0$  for all  $h \in \mathcal{H}$ , then a uniform stability bound for  $\Pi_h$  is  $\gamma_{\Pi_h} \geq \gamma_{\Pi_0} = \frac{\hat{\alpha}_0}{\|a\|}$  for all  $h \in \mathcal{H}$ .

*Remark 2 (Linearity)* Even in the case of Banach spaces, the linearity of the map  $\Pi_h$  can be asserted if one has at hand a stable decomposition  $W_h = \ker(A_h^*) \oplus K_h$  such that there is  $\kappa_h > 0$  such that the induced projector  $\pi_{K_h} : W_h \rightarrow K_h$  satisfies  $\kappa_h \|\pi_{K_h} w_h\|_W \leq \|w_h\|_W$  for all  $w_h \in W_h$  (this property holds in the Hilbertian setting with  $\kappa_h = 1$ ). Then, one can adapt the reasoning at the end of the proof of Lemma 3 to build a stable, linear right-inverse map  $A_h^{*\dagger}$ . The mild price to be paid is that the stability constant of  $\Pi_h$  now becomes  $\gamma_{\Pi_h} = \frac{\kappa_h \hat{\alpha}_h}{\|a\|}$ .

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