' 2D-Local Existence and Uniqueness of a Transient State of a Coupled Radiative-Conductive Heat Transfer Problem

Mohamed Ghattassi, Jean Rodolphe Roche, Didier Schmitt, Mohamed Boutayeb

‘

2D-Local Existence and Uniqueness of a Transient State of a Coupled Radiative-Conductive Heat Transfer Problem

Mohamed GHATTASSI[a], Jean Rodolphe ROCHE[a], Didier SCHMITT[a], Mohamed BOUTAYEB[b]

[a] *University of Lorraine, IECL UMR CNRS 7502, 54506 Vandoeuvre-lès-Nancy, France.*
[b] *University of Lorraine, CRAN UMR CNRS 7039, 54400 Cosnes et Romain, France.*

Abstract

This paper deals with local existence and uniqueness results for a transient two-dimensional combined nonlinear radiative-conductive system. This system describes the heat transfer for a grey, semi-transparent and non-scattering medium with homogeneous Dirichlet boundary conditions. We reformulate the full transient state system as a fixed-point problem. The existence and uniqueness proof rests upon the Banach fixed-point Theorem assuming the initial data T_0 is non-negative and sufficiently small.

Keywords: Nonlinear radiative-conductive heat transfer system, Semi-transparent medium, Local existence-uniqueness, Banach fixed point Theorem.

2010 MSC: 35K51, 35A01, 35B40.

1. Introduction and main results

The aim of this work is to prove the existence and uniqueness of the local solution for a transient two-dimensional combined radiative-conductive system with homogeneous Dirichlet boundary conditions when the initial condition is assumed to be small and non-negative . The medium is assumed grey, semi-transparent and non-scattering.

Let us consider a bounded, open, connected and convex set $\mathbf{\Omega} \subset \mathbb{R}^2$ with $\mathbb{C}^\infty$ boundary. Let $\mathcal{D}$ be the unit disk, for the problem considered, $\boldsymbol{\beta} \in \mathcal{D}$, $\mathbf{x} \in \mathbf{\Omega}$, $\xi \in [0, \tau]$ for $\tau > 0$, $\mathcal{X} = \mathbf{\Omega} \times \mathcal{D}$ and $\boldsymbol{Q}_\tau = (0, \tau) \times \mathbf{\Omega}$. Let $\mathbf{n}$ be the outward unit normal to the boundary $\partial\mathbf{\Omega}$. We denote

$$\partial\mathbf{\Omega}_- = \{(\mathbf{x}, \boldsymbol{\beta}) \in \partial\mathbf{\Omega} \times \mathcal{D} \ \text{ such that } \ \boldsymbol{\beta}.\mathbf{n} < \mathbf{0}\}.$$

Email addresses: mohamed.ghattassi@univ-lorraine.fr (Mohamed GHATTASSI), jean-rodolphe.roche@univ-lorraine.fr (Jean Rodolphe ROCHE), didier.schmitt@univ-lorraine.fr (Didier SCHMITT), mohamed.boutayeb@univ-lorraine.fr (Mohamed BOUTAYEB)

The full system of a combined nonlinear radiation-conduction heat transfer is written in dimensionless form,

$$I(\xi,\mathbf{x},\boldsymbol{\beta})+\boldsymbol{\beta}.\nabla_{\mathbf{x}}I(\xi,\mathbf{x},\boldsymbol{\beta})=T^4(\xi,\mathbf{x}) \qquad (\xi,\mathbf{x},\boldsymbol{\beta})\in[0,\tau]\times\boldsymbol{\chi} \qquad (1)$$

$$\partial_\xi T(\xi,\mathbf{x})-\Delta T(\xi,\mathbf{x})+4\pi\theta T^4(\xi,\mathbf{x})=\theta\int_{\mathcal{D}}I(\xi,\mathbf{x},\boldsymbol{\beta})d\boldsymbol{\beta} \qquad (\xi,\mathbf{x})\in(0,\tau]\times\boldsymbol{\Omega} \qquad (2)$$

$$T(\xi,\mathbf{x})=0 \qquad (\xi,\mathbf{x})\in(0,\tau]\times\partial\boldsymbol{\Omega} \qquad (3)$$

$$I(\xi,\mathbf{x},\boldsymbol{\beta})=0 \qquad (\xi,\mathbf{x},\boldsymbol{\beta})\in[0,\tau]\times\partial\boldsymbol{\Omega}_- \qquad (4)$$

$$T(0,\mathbf{x})=T_0(\mathbf{x}) \qquad \mathbf{x}\in\boldsymbol{\Omega} \qquad (5)$$

where $\partial_\xi T=\partial T/\partial\xi$, θ is a non-negative constant, and T_0 is also non-negative. In this paper we consider that the mean radiation intensity of the blackbody $I_b(T)$ verifies the Stefan-Bolzmann law which is proportional to T^4. The radiative transfer equation (RTE) (1) and the conductive equation (NHE) (2) are coupled via the source term.

Radiative-conductive heat transfer problems is the subject of various fields of engineering and science.
In the literature, this problem is studied using two different types of model. In the first type, the problem is described using an unique parabolic partial differential equation. In the second type of model, the modeling of the radiation and conduction is given by a coupled system of partial differential equations where each phenomenon is described by an equation.

There is a huge mathematical theory in the first case, see [5, 6, 7, 8, 26, 39, 3, 4, 9, 28, 29, 30, 31]. For example, the paper [5] is devoted to the study of a nonstationary nonlinear nonlocal initial boundary value problem governing radiative conductive heat transfer in opaque bodies with surfaces whose properties depend on the radiation frequency. This paper is a natural extension of the work done in [7], where the corresponding stationary problem was treated. In [25], the authors considered the conductive radiative heat transfer in a scattering and absorbing medium bounded by two reflecting and radiating plane surfaces. The existence and uniqueness of a solution of this problem is established using an iterative procedure.

In [30], M. Laitinen and T. Tiihonen studied the well-posedness of a class of models describing heat transfer by conduction and radiation in the stationary case. The employed theory covers different types of grey materials, that is, both semitransparent and opaque bodies as well as isotropic or non-isotropic scattering/reflection provided that the material properties do not depend on the wavelength of the radiation.

In M. M. Porzio and O. López Pouso [38] have proved the existence of a solution of the coupled radiation conduction system in the 3D case using accretive operators theory. In [38] the authors have assumed that the intensity of radiation emitted from the blackbody in a frequency ν obeys the Planck's law and then they considered a scattering term, that introduces some regularization effects which are not present in our model.

In this paper, we consider the second type of model where the phenomenon is expressed as a coupled system of nonlinear partial differential equations. In previous works we can find theoretical results of existence and uniqueness

in one-dimensional case. Indeed, in the Kelley's paper [24], the authors considered a steady-state combined radiative-conductive heat transfer. In Asllanaj and al.[12] the authors generalized the Kelley's study and they proved the existence and uniqueness of the 1-D system of coupled radiative conductive in the steady state associated to the non-homogeneous Dirichlet boundary with the black surfaces. The medium is assumed to be a non-grey ani-isotopically absorbing, emitting, scattering, with axial symmetry and non homogeneous. They considered a nonlinear conduction equation due to the temperature dependence of the thermal conductivity. However, the approach developed by Asllanaj and al. [12] is just adaptable to 1D dimensional geometry. In this paper we prove the existence and uniqueness of local solutions for the nonlinear system (1)-(5) with the non-negative non-homogeneous Dirichlet boundary conditions if the initial data T_0 is sufficiently small and non-negative .

Recently, the attention has been accorded to numerical methods to study the radiative transfer and the nonlinear radiative-conductive heat transfer problem including optimal control problems, for more details see [10, 11, 12, 13, 14, 17, 19, 20, 21, 22, 23, 34, 35, 36, 37, 33]. Asllanaj and al. [13] simulated transient heat transfer by radiation and conduction in two-dimensional complex shaped domains with structured and unstructured triangular meshes working with an absorbing, emitting and non-scattering grey medium.

In order to state the main result, we introduce the following notations

$$L^p(\boldsymbol{Q}_\tau) = L^p(0,\tau;L^p(\boldsymbol{\Omega})) \text{ for all } \quad p \in [1,\infty)$$
$$W_2^{2,1}(\boldsymbol{\Omega}_\tau) := \{\phi \quad s.t \ \ \phi, \phi_\xi, \phi_{x_i}, \phi_{x_i,x_j} \in L^2(\boldsymbol{\Omega})\}$$

We will establish that the solution of the problem (1)-(5) is a fixed point of a well posed map $\mathbb{H}$ in the set

$$E_1 = \{T \in W_2^{2,1}(\boldsymbol{\Omega}_\tau); \|T\|_{L^8(\boldsymbol{Q}_\tau)} \leqslant M\},$$

where M is a non-negative constant. We assume that the initial data T_0 is non-negative , belongs to $L^5(\boldsymbol{\Omega})$ and satisfies the following hypothesis

$$\|T_0\|_{L^5(\boldsymbol{\Omega})} \leqslant \sqrt[5]{\frac{15\theta\pi M^8}{8}}. \tag{6}$$

Let $N = \sqrt{\pi}M^4$ and

$$E_2 = \{I \in L^2(0,\tau;L^2(\boldsymbol{\mathcal{X}})); \|I\|_{L^2(0,\tau;L^2(\boldsymbol{\mathcal{X}}))} \leqslant N\}.$$

The main result of this paper is the following Theorem.

Theorem 1.1. *If the initial data T_0 verifies (6) then there exists $\tau^\star > 0$ such that for all $\tau \leqslant \tau^\star$, the system of equation (1)-(5) has a unique local solution (T, I) such that $T \in E_1$ and $I \in E_2$. Moreover , there exists $C = C(\boldsymbol{\Omega}, \tau, \theta)$ such that*

$$\|I\|_{L^2(0,\tau;L^2(\mathcal{X}))} \leqslant \sqrt{\pi}\|T\|^4_{L^8(\boldsymbol{Q}_\tau)}$$

and

$$\|T\|_{W_2^{2,1}(\boldsymbol{Q}_\tau)} \leqslant C\Big(\|G\|_{L^2(\boldsymbol{Q}_\tau)} + \|T_0\|_{L^2(\boldsymbol{\Omega})}\Big). \tag{7}$$

This paper is organized as follows: In the next section, we introduce the model describing the radiative-conductive heat transfer system in a grey, semi-transparent and non-scattering medium. The section 3 is devoted to construct a contraction mapping $\mathbb{H}$ on a suitable set, whose fixed point gives a solution of the nonlinear coupled radiative conductive heat transfer system (1)-(5).

2. Model problem

Let $\boldsymbol{\Omega}^\star$ be an open, bounded, connected and convex domain with $\mathbb{C}^\infty$ boundary and

$$\partial\boldsymbol{\Omega}^\star_- = \{(\mathbf{x}^\star, \boldsymbol{\beta}) \in \partial\boldsymbol{\Omega}^\star \times \mathcal{D} \text{ such that } \boldsymbol{\beta}.\mathbf{n} < 0\}.$$

The radiative transfer equation (RTE) in a two dimensional gray absorbing and emitting medium is given by, see [13, 23]

$$\boldsymbol{\beta}.\nabla_{\mathbf{x}^\star} I^\star(t, \mathbf{x}^\star, \boldsymbol{\beta}) + \kappa I^\star(t, \mathbf{x}^\star, \boldsymbol{\beta}) = \kappa n^2 I_b(T^\star(t, \mathbf{x}^\star)), \quad (t, \mathbf{x}^\star, \boldsymbol{\beta}) \in [0, \tau^\star] \times \boldsymbol{\Omega}^\star \times \mathcal{D}, \tag{8a}$$

$$I^\star(t, \mathbf{x}^\star, \boldsymbol{\beta}) = I_b(T^\star(t, \mathbf{x}^\star)), \qquad \text{for } (t, \mathbf{x}^\star, \boldsymbol{\beta}) \in [0, \tau^\star] \times \partial\boldsymbol{\Omega}^\star_-, \tag{8b}$$

where $I^\star$ is the radiation intensity, κ is the absorption coefficient of the medium and n is the refractive index.
In this work, the refractive index and the absorption coefficient are assumed to be equal to one, $n = 1$, $\kappa = 1\, m^{-1}$. $T^\star$, the temperature of the medium and $I_b(T^\star)$ is the mean radiation intensity of the blackbody which obeys Stefan-Boltzmann's law:

$$I_b(T^\star) = \frac{\sigma_B}{\pi} T^{\star 4},$$

where $\sigma_B = 5.6698 \times 10^{-8}\ Wm^{-2}K^{-4}$ is the Stefan-Boltzmann constant. In this paper the authors have assumed Dirichlet boundary conditions. Emission and absorption of radiation by the medium lead to a radiative source term in the energy equation of the medium. It is defined by the following relations:

$$S^\star_{rad}(t, \mathbf{x}^\star) = \kappa\{G^\star(t, \mathbf{x}^\star) - 4\pi I_b(T^\star(t, \mathbf{x}^\star))\} \qquad (t, \mathbf{x}^\star) \in [0, \tau^\star] \times \boldsymbol{\Omega}^\star,$$

where $G^\star$ is the incident radiation intensity,

$$G^\star(t, \mathbf{x}^\star) = \int_{\mathcal{D}} I^\star(t, \mathbf{x}^\star, \boldsymbol{\beta})\, d\boldsymbol{\beta} \qquad (t, \mathbf{x}^\star) \in [0, \tau^\star] \times \boldsymbol{\Omega}^\star.$$

The over all energy conservation links the three different modes of the heat transfer known as conduction, radiation and convection. In this study, the convection has not been considered, then we have the following conduction equation

$$\rho c_p \frac{\partial T^\star}{\partial t}(t,\mathbf{x}^\star) - k_c \Delta T^\star(t,\mathbf{x}^\star) = S^\star_{rad}(t,\mathbf{x}^\star) \qquad (t,\mathbf{x}^\star) \in (0,\tau^\star] \times \mathbf{\Omega}^\star \tag{9a}$$

$$T^\star(0,\mathbf{x}^\star) = T^\star_0(\mathbf{x}^\star) \qquad \mathbf{x}^\star \in \mathbf{\Omega}^\star \tag{9b}$$

$$T^\star(t,\mathbf{x}^\star) = g^\star(\mathbf{x}^\star), \qquad \text{for } (t,\mathbf{x}^\star) \in [0,\tau^\star] \times \partial\mathbf{\Omega}^\star. \tag{9c}$$

The data ρ, c_p, and k_c are the density, the specific heat capacity, and the thermal conductivity of the medium, respectively. In this work, they are assumed to be constant. The NHE (9a) and the RTE (8a) are strongly coupled by the incident radiation intensity $G^\star$ and the temperature $T^\star$.

Let $\xi = \dfrac{\alpha t}{L^2}$ the dimensionless time where $\alpha = \dfrac{k_c}{\rho c_p}$ is the thermal diffusivity. Let T_{ref} be a reference temperature and L a characteristic length, then we obtain the following dimensionless quantities:

$$\mathbf{x} = \frac{\mathbf{x}^\star}{L},\ \tau = \frac{\alpha\tau^\star}{L^2},\ \ T = \frac{T^\star}{T_{ref}},$$
$$T_0 = \frac{T^\star_0}{T_{ref}},\ G = \frac{G^\star}{4\sigma_B T^4_{ref}}\ , S_{rad} = \frac{S^\star_{rad}}{4\sigma_B T^4_{ref}},$$
$$I = \frac{I^\star}{I_{ref}} \ \text{ and } \ g = \frac{g^\star}{T_{ref}}$$

where

$$I_{ref} = \frac{\sigma_B}{\pi} T^4_{ref}$$

is a dimensionless constant, see [23]. The conduction radiation number which is denoted N_s satisfies the following expression:

$$N_s = \frac{k_c \kappa}{4\sigma_B T^3_{ref}}.$$

and we denote

$$\theta = \frac{\kappa^2 L^2}{N_s}$$

Consequently, we obtain the dimensionless full system of a combined nonlinear radiation-conduction heat transfer system (1)-(5) when the Dirichlet boundary condition g is equal to zero.

3. Local existence and uniqueness of solutions for the coupled system

In this section, we show that the existence of a solution T, and implicitly the existence of a solution I, of the coupled system of equations (1)-(5) is related to the existence of a solution of a fixed point problem. We will apply the fixed point Theorem to a well-chosen map $\mathbb{H}$. To do so, we must show that this map $\mathbb{H}$ is well defined and completely continuous. At first, we recall the definition of the set E_1

$$E_1 = \{T \in W^{2,1}_2(\mathbf{\Omega}_\tau); \|T\|_{L^8(Q_\tau)} \leqslant M\}$$

and we introduce the following sets

$$E_2 = \{I \in L^2(0,\tau; L^2(\boldsymbol{\mathcal{X}})); \|I\|_{L^2(0,\tau;L^2(\boldsymbol{\mathcal{X}}))} \leqslant N\},$$
$$E_3 = \{G \in L^2(\boldsymbol{Q}_\tau); \|G\|_{L^2(\boldsymbol{Q}_\tau)} \leqslant \sqrt{\pi}N\},$$

The map $\mathbb{H} : E_1 \longrightarrow E_1$ is a composition of three maps

$$\mathbb{H} = \mathbb{H}_3 \circ \mathbb{H}_2 \circ \mathbb{H}_1.$$

The map $\mathbb{H}_1 : E_1 \longrightarrow E_2$ is defined as follows, for $T \in E_1$, $\mathbb{H}_1(T) \in E_2$ is the solution of the radiative transfer equation (1). The second map $\mathbb{H}_2 : E_2 \longrightarrow E_3$ is defined in the following way, for $I \in E_2$, $\mathbb{H}_2(I) = G \in E_3$ where G is given by

$$G(\xi, \mathbf{x}) = \int_{\mathcal{D}} I(\xi, \mathbf{x}, \boldsymbol{\beta}) d\boldsymbol{\beta} \tag{10}$$

and finally, the map $\mathbb{H}_3 : E_3 \longrightarrow E_1$ is defined as follows, for $G \in E_3$, $\mathbb{H}_3(G) \in E_1$ is the solution of the equation (2).

3.1. The maps $\mathbb{H}_1$ and $\mathbb{H}_2$

Now, we focus on the maps $\mathbb{H}_1$ and $\mathbb{H}_2$, we give some properties of the solution of the RTE (1).

Theorem 3.1. *Let us consider $T \in E_1$, the problem (1) has a non-negative unique solution $\mathbb{H}_1(T) \in L^2(0,\tau; L^2(\mathcal{X}))$. Thus $\mathbb{H}_1$ is a well-posed map from E_1 to E_2. Moreover $\mathbb{H}_1$ is continuous.*

Proof. Let $T \in E_1$ then for all $\xi \in [0,\tau]$, we have $T^4(\xi) \in L^2(\boldsymbol{\Omega})$. Using a result about the existence and uniqueness of the solution of the transport equation, see [18], the boundary value problem (1)−(4) has a unique solution $I(\xi) \in L^2(\mathcal{X})$ which satisfies the following a priori estimate

$$\|I(\xi)\|_{L^2(\mathcal{X})} \leqslant \sqrt{\pi}\|T^4(\xi)\|_{L^2(\boldsymbol{\Omega})}.$$

If we integrate in time between 0 and τ, we obtain

$$\|I\|_{L^2(0,\tau;L^2(\mathcal{X}))} \leqslant \sqrt{\pi}\|T\|^4_{L^8(\boldsymbol{Q}_\tau)}$$

hence,

$$\|I\|_{L^2(\boldsymbol{Q}_\tau)} \leqslant \sqrt{\pi}M^4 = N.$$

Consequently, $I \in E_2$ and then $\mathbb{H}_1$ is a well-posed map.

Using the maximum principle [1], this implies that the solution I of (1) − (4) is non-negative.

Now, we show the continuity of the map $\mathbb{H}_1$. We consider I_1, I_2 two solutions of (1) associated to T_1, T_2, respectively . Let $\xi \in [0,\tau]$, we have

$$\|I_1(\xi) - I_2(\xi)\|_{L^2(\boldsymbol{\Omega}\times\boldsymbol{\mathcal{D}})} \leqslant \sqrt{\pi}\|T_1^4(\xi) - T_2^4(\xi)\|_{L^2(\boldsymbol{\Omega})}. \tag{11}$$

Using the generalized Hölder's inequality, we have the following inequality

$$\begin{aligned}\|T_1^4(\xi) - T_2^4(\xi)\|^2_{L^2(\boldsymbol{\Omega})} &= \int_{\boldsymbol{\Omega}} (T_1(\xi) - T_2(\xi))^2 (T_1(\xi) + T_2(\xi))^2 (T_1^2(\xi) + T_2^2(\xi))^2 dx \\ &\leqslant \|T_1(\xi) - T_2(\xi)\|^2_{L^8(\boldsymbol{\Omega})} \|T_1(\xi) + T_2(\xi)\|^2_{L^8(\boldsymbol{\Omega})} \|T_1^2(\xi) + T_2^2(\xi)\|^2_{L^4(\boldsymbol{\Omega})}. \end{aligned} \tag{12}$$

If we integrate in time, we obtain

$$\|T_1^4 - T_2^4\|^2_{L^2(\boldsymbol{Q}_\tau)} \leqslant \|T_1 - T_2\|^2_{L^8(\boldsymbol{Q}_\tau)} \|T_1 + T_2\|^2_{L^8(\boldsymbol{Q}_\tau)} \|T_1^2 + T_2^2\|^2_{L^4(\boldsymbol{Q}_\tau)}.$$

Then we have

$$\begin{aligned}\|T_1^2(\xi) + T_2^2(\xi)\|^4_{L^4(\boldsymbol{\Omega})} \leqslant \|T_1(\xi)\|^8_{L^8(\boldsymbol{\Omega})} &+ 4\|T_1(\xi)\|^6_{L^8(\boldsymbol{\Omega})} \|T_2(\xi)\|^2_{L^8(\boldsymbol{\Omega})} + 6\|T_1(\xi)\|^4_{L^8(\boldsymbol{\Omega})} \|T_2(\xi)\|^4_{L^8(\boldsymbol{\Omega})} \\ &+ 4\|T_1(\xi)\|^2_{L^8(\boldsymbol{\Omega})} \|T_2(\xi)\|^6_{L^8(\boldsymbol{\Omega})} + \|T_2(\xi)\|^8_{L^8(\boldsymbol{\Omega})}.\end{aligned}$$

Hence

$$\begin{aligned}\|T_1^2 + T_2^2\|^4_{L^4(\boldsymbol{Q}_\tau)} \leqslant \|T_1\|^8_{L^8(\boldsymbol{Q}_\tau)} &+ 4\|T_1\|^6_{L^8(\boldsymbol{Q}_\tau)} \|T_2\|^2_{L^8(\boldsymbol{Q}_\tau)} + 6\|T_1\|^4_{L^8(\boldsymbol{Q}_\tau)} \|T_2\|^4_{L^8(\boldsymbol{Q}_\tau)} \\ &+ 4\|T_1\|^2_{L^8(\boldsymbol{Q}_\tau)} \|T_2\|^6_{L^8(\boldsymbol{Q}_\tau)} + \|T_2\|^8_{L^8(\boldsymbol{Q}_\tau)}.\end{aligned}$$

Since $T_1, T_2 \in E_1$, then we deduce that

$$\|T_1^2 + T_2^2\|_{L^4(\boldsymbol{Q}_\tau)} \leqslant 2M^2. \tag{13}$$

On the other hand, we have

$$\|T_1 + T_2\|_{L^8(\boldsymbol{Q}_\tau)} \leqslant 2M, \tag{14}$$

it follows that

$$\|T_1^4 - T_2^4\|_{L^2(\boldsymbol{Q}_\tau)} \leqslant 4M^3 \|T_1 - T_2\|_{L^8(\boldsymbol{Q}_\tau)},$$

From (11), we deduce that

$$\|I_1 - I_2\|_{L^2(0,\tau;L^2(\boldsymbol{\Omega}\times\boldsymbol{\mathcal{D}}))} \leqslant 4\sqrt{\pi} M^3 \|T_1 - T_2\|_{L^8(\boldsymbol{Q}_\tau)}. \tag{15}$$

The last inequality shows the continuity of $\mathbb{H}_1$. □

Now, we give some properties of the map $\mathbb{H}_2$.

Proposition 3.2. *$\mathbb{H}_2$ is a well posed, monotone and continuous map from E_2 to E_3.*

Proof. Let us consider $I \in E_2$ and $G = \mathbb{H}_2(I)$, then we have

$$\|G\|_{L^2(\boldsymbol{Q}_\tau)} \leqslant \sqrt{\pi} \|I\|_{L^2(0,\tau;L^2(\boldsymbol{\mathcal{X}}))}. \tag{16}$$

Hence , for all $I \in E_2$, $G = \mathbb{H}_2(I)$ belongs to E_3. Since I is non-negative then G is non-negative. Therefore $\mathbb{H}_2$ is a well posed map. Since $\mathbb{H}_2$ is a linear function, from the inequality (16), $\mathbb{H}_2$ is a continuous map. It is easily to deduce that $\mathbb{H}_2$ is a monotone application. □

3.2. The map $\mathbb{H}_3$

In this subsection we introduce some properties of the map $\mathbb{H}_3$.

Theorem 3.3. *Let us consider $T_0 \in L^5(\mathbf{\Omega})$, $T_0 \geqslant 0$ almost everywhere in $\mathbf{\Omega}$ and satisfies the hypothesis (6). Let $G \in E_3$ and non-negative. Then the equation (2) has a non-negative solution $T = \mathbb{H}_3(G) \in E_1$.*

Proof. For the proof of the existence and uniqueness of the solution of the equation (2), see [2, 27, 15].

First we give a proof of the non-negativity of T.

Let us consider F defined in $(0,\tau)\times\mathbf{\Omega}\times\mathbb{R}$ by

$$F(\xi,\mathbf{x},y) = \theta\left(G(\xi,\mathbf{x}) - 4\pi y^4\right)$$

then T is the solution of the following equation

$$\partial_\xi T(\xi,\mathbf{x}) - \Delta T(\xi,\mathbf{x}) = F(\xi,\mathbf{x},T(\xi,\mathbf{x})) \tag{17}$$

$$T(\xi,\mathbf{x}) = 0 \quad \text{in } \partial\mathbf{\Omega}\times(0,\tau)$$

$$T(\mathbf{x},0) = T_0(\mathbf{x}) \quad \text{in } \mathbf{\Omega}.$$

Now, we define $\bar{F}$ in $(0,\tau)\times\mathbf{\Omega}\times\mathbb{R}$ by

$$\bar{F}(\xi,\mathbf{x},y) = \begin{cases} \theta\left(G(\xi,\mathbf{x}) - 4\pi y^4\right) & \text{if} \quad y \geq 0 \\ \theta G(\xi,\mathbf{x}) & \text{if} \quad y < 0. \end{cases}$$

Let us consider $\bar{T}$ the solution of the following equation

$$\partial_\xi \bar{T}(\xi,\mathbf{x}) - \Delta \bar{T}(\xi,\mathbf{x}) = \bar{F}(\xi,\mathbf{x},\bar{T}(\xi,\mathbf{x})) \tag{18}$$

$$\bar{T}(\xi,\mathbf{x}) = 0 \quad \text{in } \partial\mathbf{\Omega}\times(0,\infty)$$

$$\bar{T}(0,\mathbf{x}) = T_0(\mathbf{x}) \quad \text{in } \mathbf{\Omega}.$$

Our goal is to prove that the solution $\bar{T}$ of this equation remains non-negative over the time. In fact, in this case $\bar{F}$ and F coincide, therefore we have by the uniqueness of the solution $T = \bar{T}$ which is non-negative .

We set $\bar{T}^+ = \max(T,0)$ and $\bar{T}^- = \max(-T,0)$, such that $\bar{T} = \bar{T}^+ - \bar{T}^-$.

Multiplying the equation (18) by $(-\bar{T}^-)$ and integrating over $\mathbf{\Omega}$, we obtain

$$-\int_{\mathbf{\Omega}} \partial_\xi \bar{T}(\xi,\mathbf{x})\bar{T}^-(\xi,\mathbf{x})d\mathbf{x} + \int_{\mathbf{\Omega}} \Delta\bar{T}(\xi,\mathbf{x})\bar{T}^-(\xi,\mathbf{x})d\mathbf{x} = -\int_{\mathbf{\Omega}} \bar{F}(\xi,\mathbf{x},\bar{T})\bar{T}^-(\xi,\mathbf{x})d\mathbf{x}.$$

Now, we have

$$-\int_{\mathbf{\Omega}} \partial_\xi \bar{T}(\xi,\mathbf{x})\bar{T}^-(\xi,\mathbf{x})d\mathbf{x} = \frac{1}{2}\partial_\xi \int_{\mathbf{\Omega}} (\bar{T}^-(\xi,\mathbf{x}))^2 d\mathbf{x}$$

$$\int_{\mathbf{\Omega}} \Delta\bar{T}(\xi,\mathbf{x})\bar{T}^{-}(\xi,\mathbf{x})d\mathbf{x} = \int_{\mathbf{\Omega}} (\nabla\bar{T}^{-}(\xi,\mathbf{x}))^2 d\mathbf{x}$$

$$-\int_{\mathbf{\Omega}} \bar{F}(\xi,\mathbf{x},\bar{T})\bar{T}^{-}(\xi,\mathbf{x})d\mathbf{x} = -\int_{\{\bar{T}<0\}} \bar{F}(\xi,\mathbf{x},\bar{T})\bar{T}^{-}(\xi,\mathbf{x})d\mathbf{x} = -\int_{\{\bar{T}<0\}} \theta G(\xi,\mathbf{x})\bar{T}^{-}(\xi,\mathbf{x})d\mathbf{x} \leq 0.$$

Consequently,

$$\frac{1}{2}\partial_\xi \int_{\mathbf{\Omega}} (\bar{T}^{-}(\xi,\mathbf{x}))^2 d\mathbf{x} \leq 0.$$

As $\bar{T}^{-}(x,0) = 0$ for all $x \in \mathbf{\Omega}$ because $T_0(x) \geq 0$ for all $x \in \mathbf{\Omega}$, we deduce that $\bar{T}^{-}(x,\xi) = 0$ for all $(x,\xi) \in \mathbf{\Omega} \times (0,\infty)$. It follows that $\bar{T}(x,\xi) \geq 0$ for all $(x,\xi) \in \mathbf{\Omega} \times (0,\infty)$, hence

$$\bar{F}(x,\xi,\bar{T}(x,\xi)) = F(x,\xi,\bar{T}(x,\xi)).$$

Then $\bar{T}$ is the solution of (17) and by uniqueness of the solution $T = \bar{T}$ which proves the non-negativity of T.

In the following, we prove that $T \in L^8(\boldsymbol{Q}_\tau)$. Let z be the solution of the parabolic problem

$$\begin{cases} \partial_\xi z(\xi,\mathbf{x}) - \Delta z(\xi,\mathbf{x}) = \theta G(\xi,\mathbf{x}) & \text{for } (\xi,\mathbf{x}) \in]0,\tau] \times \mathbf{\Omega} \\ z(\xi,\mathbf{x}) = 0 & \text{for } (\xi,\mathbf{x}) \in]0,\tau] \times \partial\mathbf{\Omega} \\ z(0,\mathbf{x}) = T_0(\mathbf{x}) & \text{for } \mathbf{x} \in \mathbf{\Omega} \end{cases} \tag{19}$$

then we have that $T \leq z$.

Since $G \in L^2(\boldsymbol{Q}_\tau)$, $T_0 \in L^2(\mathbf{\Omega})$ and thanks to a result on parabolic regularity, see [27], then $z \in W_2^{2,1}(\boldsymbol{Q}_\tau)$ and there exists a constant $\tilde{C} > 0$ such that

$$\|z\|_{W_2^{2,1}(\boldsymbol{Q}_\tau)} \leqslant \tilde{C}\Big(\|G\|_{L^2(\boldsymbol{Q}_\tau)} + \|T_0\|_{L^2(\mathbf{\Omega})}\Big). \tag{20}$$

Then, we have $z \in L^8(\boldsymbol{Q}_\tau)$ (see [27, p.80]) and consequently $T \in L^8(\boldsymbol{Q}_\tau)$.

Since G and T^4 belong to $L^2(\boldsymbol{Q}_\tau)$, we have $\theta G - 4\pi T^2$ belongs to $L^2(\boldsymbol{Q}_\tau)$. Consequently, using the same result on parabolic regularity we obtain $T \in W_2^{2,1}(\boldsymbol{Q}_\tau)$.

To prove that $T = \mathbb{H}_3(G) \in E_1$, we need a more precise control of $\|T\|_{L^8(\boldsymbol{Q}_\tau)}$.

We multiply the equation (2) by T^4 and we integrate over $\mathbf{\Omega}$, so we obtain

$$\frac{1}{5}\frac{d}{d\xi}\int_{\mathbf{\Omega}} T^5(\xi)d\mathbf{x} + 4\int_{\mathbf{\Omega}} (\nabla T(\xi))^2 (T(\xi))^3 d\mathbf{x} + 4\pi\theta\int_{\mathbf{\Omega}} T^8(\xi)d\mathbf{x} = \theta\int_{\mathbf{\Omega}} GT^4(\xi)d\mathbf{x}.$$

As $T \in W_2^{2,1}(\boldsymbol{Q}_\tau)$ then T^4 belongs to $L^2(0,\tau;H_0^1(\mathbf{\Omega}))$.

Using the Young's inequality, we get

$$\frac{1}{5}\frac{d}{d\xi}\int_{\mathbf{\Omega}} (T^{\frac{5}{2}}(\xi))^2 d\mathbf{x} + \frac{16}{25}\int_{\mathbf{\Omega}} (\nabla T^{\frac{5}{2}}(\xi))^2 d\mathbf{x} + 4\pi\theta\int_{\mathbf{\Omega}} T^8(\xi)d\mathbf{x} \leqslant \frac{\theta}{2\epsilon}\int_{\mathbf{\Omega}} G^2(\xi)d\mathbf{x} + \theta\frac{\epsilon}{2}\int_{\mathbf{\Omega}} T^8(\xi)d\mathbf{x}.$$

If we assume $\epsilon = 2\pi$ and we integrate in time, we obtain

$$\frac{1}{5}\sup_{0\leqslant\xi\leqslant\tau} \|T^{\frac{5}{2}}(\xi)\|^2_{L^2(\mathbf{\Omega})} + 4\|T^{\frac{5}{2}}\|^2_{L^2(0,\tau;H_0^1(\mathbf{\Omega}))} + 3\pi\theta\|T\|^8_{L^8(\boldsymbol{Q}_\tau)} \leqslant \frac{\theta}{4\pi}\|G\|^2_{L^2(\boldsymbol{Q}_\tau)} + \frac{1}{5}\|T_0^{\frac{5}{2}}\|^2_{L^2(\mathbf{\Omega})},$$

hence

$$\|T\|^8_{L^8(\boldsymbol{Q}_\tau)} \leqslant \frac{1}{12\pi^2}\|G\|^2_{L^2(\boldsymbol{Q}_\tau)} + \frac{1}{15\theta\pi}\|T_0\|^5_{L^5(\boldsymbol{\Omega})}.$$

Since $G \in E_1$, we deduce

$$\|T\|^8_{L^8(\boldsymbol{Q}_\tau)} \leqslant \frac{M^8}{4} + \frac{1}{15\theta\pi}\|T_0\|^5_{L^5(\boldsymbol{\Omega})}.$$

If the initial data satisfies the hypothesis (6), it follows that

$$\|T\|^8_{L^8(\boldsymbol{Q}_\tau)} \leqslant M^8.$$

Thus $T \in E_1$ and the map $\mathbb{H}_3$ is well-posed. □

Theorem 3.4. *Under the assumptions of Theorem 3.3, $\mathbb{H}_3$ is a continuous map from E_3 to E_1.*

Proof. Let $G_1, G_2 \in E_3$, $T_1 = \mathbb{H}_3(G_1)$ and $T_2 = \mathbb{H}_3(G_2)$. Let us set $w = T_1 - T_2$, then w is solution of the following equation

$$\begin{aligned}
\partial_\xi w(\xi,\mathbf{x}) - \Delta w(\xi,\mathbf{x}) &= -4\pi\theta(T_1^4 - T_2^4)(\xi,\mathbf{x}) + \theta(G_1 - G_2)(\xi,\mathbf{x}) \quad && \text{in } (0,\tau]\times\boldsymbol{\Omega},\\
w(\xi,\mathbf{x}) &= 0 && \text{in } (0,\tau]\times\partial\boldsymbol{\Omega}\\
w(0,\mathbf{x}) &= 0 && \text{in } \boldsymbol{\Omega}.
\end{aligned}$$

We have

$$w(\xi) = -4\pi\theta\int_0^\xi \mathbb{T}(\xi - s)(T_1^4 - T_2^4)(s)ds + \theta\int_0^\xi \mathbb{T}(\xi - s)(G_1 - G_2)(s)ds$$

where $\mathbb{T}(\xi)$ a semigroup of contraction in $H_0^1(\boldsymbol{\Omega})$ generated by the operator $\mathbf{A}$ defined by

$$D(\mathbf{A}) = \{T \in H_0^1(\boldsymbol{\Omega}),\ \Delta T \in L^2(\boldsymbol{\Omega})\} \quad \text{and} \quad \mathbf{A}T = \Delta T\ ,\ \ \forall T \in D(\mathbf{A}).$$

Now, using the regularizing effects of the heat equation, see [15, proposition 3.5.7, p.44] with $p = 8$ and $q = 2$, we deduce the following inequality

$$\|w(\xi)\|_{L^8(\boldsymbol{\Omega})} \leqslant 4\pi\theta\int_0^\xi \frac{1}{(4\pi(\xi - s))^{\frac{1}{2}-\frac{1}{8}}}\|T_1^4(s) - T_2^4(s)\|_{L^2(\boldsymbol{\Omega})}ds + \theta\int_0^\xi \frac{1}{(4\pi(\xi - s))^{\frac{1}{2}-\frac{1}{8}}}\|G_1(s) - G_2(s)\|_{L^2(\boldsymbol{\Omega})}ds. \quad (21)$$

In view of the inequality (12) and the Cauchy-Schwarz inequality , then (21) becomes

$$\begin{aligned}
\|w(\xi)\|_{L^8(\boldsymbol{\Omega})} \leqslant & 4\pi\theta\int_0^\xi \frac{1}{(4\pi(\xi - s))^{\frac{1}{2}-\frac{1}{8}}}\|w(s)\|_{L^8(\boldsymbol{\Omega})}\|T_1(s) + T_2(s)\|_{L^8(\boldsymbol{\Omega})}\|(T_1^2 + T_2^2)(s)\|_{L^4(\boldsymbol{\Omega})}ds\\
& + \theta\left(\int_0^\xi \frac{ds}{(4\pi(\xi - s))^{\frac{3}{4}}}\right)^{\frac{1}{2}}\|G_1 - G_2\|_{L^2(\boldsymbol{Q}_\tau)}.
\end{aligned} \quad (22)$$

We have

$$\left(\int_0^\xi \frac{ds}{(4\pi(\xi - s))^{\frac{3}{4}}}\right)^{\frac{1}{2}} = \left(\frac{4\sqrt[4]{\xi}}{(4\pi)^{\frac{3}{4}}}\right)^{\frac{1}{2}} = \frac{2\sqrt[8]{\xi}}{(4\pi)^{\frac{3}{8}}}. \quad (23)$$

Thanks to the generalized Hölder's inequality, we get

$$\int_0^\xi \frac{1}{(4\pi(\xi-s))^{\frac{1}{2}-\frac{1}{8}}}\|w(s)\|_{L^8(\mathbf{\Omega})}\|(T_1+T_2)(s)\|_{L^8(\mathbf{\Omega})}\|(T_1^2+T_2^2)(s)\|_{L^4(\mathbf{\Omega})}ds$$
$$\leqslant \frac{2\sqrt[8]{t}}{(4\pi)^{\frac{3}{8}}}\Big(\int_0^t \|w(s)\|^8_{L^8(\mathbf{\Omega})}\Big)^{\frac{1}{8}}\|T_1+T_2\|_{L^8(\boldsymbol{Q}_\tau)}\|T_1^2+T_2^2\|_{L^4(\boldsymbol{Q}_\tau)}. \tag{24}$$

We substitute (23) and (24) into (22), we obtain

$$\|w(\xi)\|_{L^8(\mathbf{\Omega})} \leqslant 4\pi\theta\frac{2\sqrt[8]{\xi}}{(4\pi)^{\frac{3}{8}}}\Big(\int_0^\xi \|w(s)\|^8_{L^8(\mathbf{\Omega})}ds\Big)^{\frac{1}{8}}\|T_1+T_2\|_{L^8(\boldsymbol{Q}_\tau)}\|T_1^2+T_2^2\|_{L^4(\boldsymbol{Q}_\tau)} + \theta\frac{2\sqrt[8]{\xi}}{(4\pi)^{\frac{3}{8}}}\|G_1-G_2\|_{L^2(\boldsymbol{Q}_\tau)}.$$

The estimations (13) and (14) give

$$\begin{aligned}\|w(\xi)\|_{L^8(\mathbf{\Omega})} &\leqslant 4\pi\theta\frac{2\sqrt[8]{\xi}}{(4\pi)^{\frac{3}{8}}}4M^3\Big(\int_0^\xi \|w(s)\|^8_{L^8(\mathbf{\Omega})}ds\Big)^{\frac{1}{8}} + \theta\frac{2\sqrt[8]{\xi}}{(4\pi)^{\frac{3}{8}}}\|G_1-G_2\|_{L^2(\boldsymbol{Q}_\tau)}\\ &\leqslant 4\pi\theta\frac{2\sqrt[8]{\tau}}{(4\pi)^{\frac{3}{8}}}4M^3\Big(\int_0^\xi \|w(s)\|^8_{L^8(\mathbf{\Omega})}ds\Big)^{\frac{1}{8}} + \theta\frac{2\sqrt[8]{\tau}}{(4\pi)^{\frac{3}{8}}}\|G_1-G_2\|_{L^2(\boldsymbol{Q}_\tau)}.\end{aligned} \tag{25}$$

Since $(a+b)^8 \leqslant 128(a^8+b^8)$ for all $(a,b)\in\mathbf{R}_+^2$, it follows that

$$\|w(\xi)\|^8_{L^8(\mathbf{\Omega})} \leqslant \frac{\theta^8}{\pi^3}2^{25}M^{24}\tau\int_0^\xi \|w(s)\|^8_{L^8(\mathbf{\Omega})}ds + \frac{\theta^8}{\pi^3}2^9\tau\|G_1-G_2\|^8_{L^2(\boldsymbol{Q}_\tau)}.$$

Applying the Gronwall's inequality, we deduce

$$\|\mathbb{H}_3(G_1)-\mathbb{H}_3(G_2)\|^8_{L^8(\boldsymbol{Q}_\tau)} \leqslant \frac{\theta^8}{\pi^3}2^9\tau^2 e^{\frac{\theta^8}{\pi^3}2^{25}M^{24}\tau^2}\|G_1-G_2\|^8_{L^2(\boldsymbol{Q}_\tau)}. \tag{26}$$

□

3.3. Existence and uniqueness of the solution for the coupled system

Now, we will use an extension of the contraction mapping Theorem for a Banach space (see [32, p. 117] for example) to prove the uniqueness of the local solution for the coupled system (1)-(5).

Theorem 3.5. *If the initial data T_0 satisfies (6) then there exists $\tau^\star > 0$ such that for all $\tau \leqslant \tau^\star$, $\mathbb{H}$ is a contraction map in E_1.*

Proof. $\mathbb{H} = \mathbb{H}_3 \circ \mathbb{H}_2 \circ \mathbb{H}_1$ is a well-posed map because it is composed by a three well-posed maps.

Now, we show that $\mathbb{H}$ is a contraction map from E_1 to E_1.

Moreover, synthetically we have

$$\begin{array}{ccccccc}(E_1,\|.\|_{L8(\boldsymbol{Q}_\tau)}) & \xrightarrow{\mathbb{H}_1} & (E_2,\|.\|_{L^2(0,\tau;L^2(\boldsymbol{\chi}))}) & \xrightarrow{\mathbb{H}_2} & (E_3,\|.\|_{L2(\boldsymbol{Q}_\tau)}) & \xrightarrow{\mathbb{H}_3} & (E_1,\|.\|_{L8(\boldsymbol{Q}_\tau)})\\ T & \mapsto & I & \mapsto & G & \mapsto & \bar{T}.\end{array}$$

Let us consider $(T_1, T_2, \bar{T}_1, \bar{T}_2) \in E_1^4$, $(I_1, I_2) \in E_2^2$ and $(G_1, G_2) \in E_3^2$ such that

$$\begin{cases} I_1 = \mathbb{H}_1(T_1), & I_2 = \mathbb{H}_1(T_2) \\ G_1 = \mathbb{H}_2(I_1), & G_2 = \mathbb{H}_2(I_2) \\ \bar{T}_1 = \mathbb{H}_3(G_1), & \bar{T}_2 = \mathbb{H}_3(G_2) \\ \bar{T}_1 = \mathbb{H}(T_1), & \bar{T}_2 = \mathbb{H}(T_2). \end{cases}$$

$\mathbb{H}_1, \mathbb{H}_2$ and $\mathbb{H}_3$ are a continuous maps , then from (15), (16) and (26) it follows that

$$\begin{aligned} \|I_1 - I_2\|_{L^2(0,\tau;L^2(\Omega\times\mathcal{D}))} &\leqslant 4\sqrt{\pi}M^3\|T_1 - T_2\|_{L^8(\boldsymbol{Q}_\tau)}, \\ \|G_1 - G_2\|_{L^2(\boldsymbol{Q}_\tau)} &\leqslant \sqrt{\pi}\|I_1 - I_2\|_{L^2(0,\tau;L^2(\Omega\times\mathcal{D}))}, \\ \|\bar{T}_1 - \bar{T}_2\|^8_{L^8(\boldsymbol{Q}_\tau)} &\leqslant \frac{\theta^8}{\pi^3}2^9\tau^2 e^{\frac{\theta^8}{\pi^3}2^{25}M^{24}\tau^2}\|G_1 - G_2\|^8_{L^2(\boldsymbol{Q}_\tau)}. \end{aligned}$$

Then we deduce

$$\|\bar{T}_1 - \bar{T}_2\|^8_{L^8(\boldsymbol{Q}_\tau)} \leqslant \pi^8\gamma(\tau)e^{\gamma(\tau)}\|T_1 - T_2\|^8_{L^8(\boldsymbol{Q}_\tau)}$$

where $\gamma(\tau) = \frac{\pi^8}{\pi^3}2^{25}M^{24}\tau^2$, clearly there exists $\tau^\star > 0$ such that for all $\tau \leqslant \tau^\star$, we have

$$\pi^8\gamma(\tau)e^{\gamma(\tau)} < 1.$$

Hence, $\mathbb{H}$ is a contraction map. □

Consequently, the main Theorem 1.1 is a corollary of the last result, Theorem 3.5 .

Proof of Theorem 1.1. E_1 is a closed subset of the Banach space $L^8(\boldsymbol{Q}_\tau)$, then using an extend Banach Theorem and Theorem 3.5 there exists $\tau^\star > 0$ such that for all $\tau \leqslant \tau^\star$, $\mathbb{H}$ has a unique fixed-point $T \in E_1$. T satisfies the equation (2) therefore, by Theorem 3.1, $I = \mathbb{H}_1(T)$ is also the unique solution of the equation (1). Thus the couple (T, I) is a solution of the equations system (1)-(5) and this solution is unique. □

References

[1] V. I. Agoshkov, Boundary Value Problems for Transport Equations, Springer, 1998.

[2] H. Amann, Linear and Quasilinear Parabolic Problems,Volume I: Abstract Linear Theory, Springer, 1995.

[3] A. A. Amosov, The solvability of a problem of radiation heat transfer, Sov. Phys. Dokl. 24(4)(1979) 261-262.

[4] A. A. Amosov, The limit connection between two problems of radiation heat Sov. Phys. Dokl. 24(6)(1979) 439-441.

[5] A. A. Amosov, Nonstationary nonlinear nonlocal problem of radiative-conductive heat transfer in a system of opaque bodies with properties depending on the radiation frequency, Journal of Mathematical Sciences 165(1)(2010) 1-41.

[6] A. A. Amosov, Nonstationary radiative-conductive heat transfer problem in a periodic system of grey heat shields, Journal of Mathematical Sciences 169(1)(2010) 1-45.

[7] A. A. Amosov, Stationary nonlinear nonlocal problem of radiativeconductive heat transfer in a system of opaque bodies with properties depending on the radiation frequency, Journal of Mathematical Sciences 164(3)(2010) 309-344.

[8] A. A. Amosov, Semidiscrete and asymptotic approximations for the nonstationary radiative-conductive heat transfer problem in a periodic system of grey heat shields, Journal of Mathematical Sciences 176(3)(2011) 361-408.

[9] A. Amosov. Solvability of a Nonstationary Problem of Radiative - Conductive Heat Transfer in a System of Semitransparent Bodies // Integral Methods in Science and Engineernig. Theoretical and Computational Advances. Constanda Christian, Kirsch Andreas (Eds.), BirkhŁuser basel, pp. 1-13, 2015.

[10] F. Asllanaj, G. Jeandel, J. R. Roche, Numerical solution of radiative transfer equation coupled with nonlinear heat condition equation, Internat. J. Numer. Methods Heat flow 5-6 (11) (2001) 449-472.

[11] F. Asllanaj, G. Jeandel, J. R. Roche, Convergence of a numerical scheme for a nonlinear coupled system of radiative-conductive heat transfer equation, Math. Models Methods Appl. Sc 7 (2004) 943-974.

[12] F. Asllanaj, G. Jeandel, J. R. Roche, D. Schmitt, Existence and Uniqueness of a Steady State Solution of a Coupled Radiative-Conductive Heat Transfer Problem for a Non-grey Anisotropically and Participating Medium, Transport Theory and Statistical Physics 32 (1) (2003) 1-35.

[13] F. Asllanaj, G. Parent, G. Jeandel, Transient Radiation and Conduction Heat Transfer in a Gray Absorbing-Emitting Medium Applied on Two-Dimensional Complex-Shaped Domains, Numerical Heat Transfer, Part B: Fundamentals 52(2)(2007) 179-200.

[14] C. Berthon, J. Dubois, R. Turpault, Numerical approximation of the M1-model. Mathematical models and numerical methods for radiative transfer, Panor. Synthtèses, 28, Soc. Math. France, Paris (2009) 55-86.

[15] T. Cazenave, A. Haraux, An Introduction to Semilinear Evolution Equations, Clarendon Press Oxford, 1998.

[16] M. Cessenat, Théorèmes de trace pour L^p pour des espaces de fonctions de la neutronique, Comptes rendus des séances de l Académie des sciences. Série 1, Mathématique 299 (16) (1984) 831-834.

[17] Y. Coudire, C. Pierre, O. Rousseau, R. Turpault, A 2D/3D discrete duality finite volume scheme. Application to ECG simulation, International Journal on Finite Volumes 6(1)(2009) 24 pages.

[18] R. Dautray, J.-L. Lions, Mathematical Analysis and Numerical Methods for Science and Technology,Springer-Verlag Berlin Heidelberg 2000.

[19] M. Falcone, Ó. López Pouso, Analysis and comparison of two approximation schemes for a radiative transfer system, Math. Models Methods Appl. Sci.13(2)(2003) 159-186.

[20] M. Falcone, Ó. López Pouso, Some remarks on fixed point and Newton techniques for solving the S_2 approximation of transient radiative transfer, Numerical mathematics and advanced applications(2003) 585-591.

[21] M. Frank, K. Axel, R. Pinnau, Optimal control of glass cooling using simplified P_N theory, Transport Theory Statist. Phys. 39(2-4)(2010) 282-311.

[22] M. Ghattassi, Modélisation, observation et commande d'une classe d'équations aux dérivées partielles: Application aux matériaux semi-transparents, Ph.D. thesis, Université Lorraine, 2015.

[23] M. Ghattassi, J.R. Roche, F. Asllanaj, M., Boutayeb, Galerkin method for solving combined radiative and conductive heat transfer, International Journal of Thermal Sciences 102 (2016)122-136.

[24] C. T. Kelley, Existence and uniqueness of solutions of nonlinear systems of conductive-radiative heat transfer equations, Transport Theory and Statistical Physics 25 (2) (1996) 249-260.

[25] A. E. Kovtanyuk, A. Y. Chebotarev, An iterative method for solving a complex heat transfer problem, Applied Mathematics and Computation 219(17)(2013) 9356-9362.

[26] A. E. Kovtanyuk, A. Y. Chebotarev, Steady-state problem of complex heat transfer, Computational Mathematics and Mathematical Physics 54(4)(2014) 719-726.

[27] O. A. Ladyzenskaja, S. Smith, V. A. Solonnikov, N. N. Ural'ceva, Linear and Quasi-linear Equations of Parabolic Types, American Mathe-

matical Soc., 1968.

[28] M. Laitinen, T. Tiihonen, Heat transfer in conducting and radiating bodies, Appl. Math. Lett. 10 (5) (1997) p.58 .

[29] M. T. Laitinen, T. Tiihonen, Heat transfer in conducting, radiating and semitransparent materials, Math. Meth. Appl. Sci. 21(1998) 375-392.

[30] M. Laitinen, T. Tiihonen, Conductive-radiative heat transfer in grey materials, Quart. Appl. Math. 59 (2001)737-768.

[31] M. Laitinen, Asymptotic analysis of conductive-radiative heat transfer, Asympt. Anal. 29 (3-4) (2002) 323-342.

[32] R.H. Martin, Nonlinear Operators and Differential Equations in Banach Spaces: Wiley Interscienc, 1976.

[33] S. C. Mishra, H.K. Roy, Solving transient conduction and radiation heat transfer problems using the lattice Boltzmann method and the finite volume method, Journal of Computational Physics, 223 (1)(2007) 89-107.

[34] S. C. Mishra H. Sahai, Analysis of non-Fourier conduction and radiation in a cylindrical medium using lattice Boltzmann method and finite volume method, International Journal of Heat and Mass Transfer 61 (2013) 41-55.

[35] S.S. Pereverzyev, R. Pinnau, N. Siedow, Initial temperature reconstruction for nonlinear heat equation: Application to a coupled radiative-conductive heat transfer problem, Inverse Prob. Sci. Eng 16(1)(2008) 55-67.

[36] R. Pinnau, A. Schulze, Newton's method for optimal temperature-tracking of glass cooling processes, Inverse Probl. Sci. Eng. 15(4)(2007) 303-323.

[37] R. Pinnau, G. Thommes, Optimal boundary control of glass cooling processes, Math. Methods Appl. Sci. 27(11)(2004) 1261-1281.

[38] M. M.Porzio, Ó. López Pouso, Application of accretive operators theory to evolutive combined conduction, convection and radiation, Rev. Mat. Iberoamericana, 20(1)(2004) 257-275.

[39] M., Thompson, C. Segatto, M.T. De Vilhena, Existence Theory for the Solution of a Stationary Nonlinear Conductive Radiative Heat Transfer Problem in Three Space Dimensions, Transport Theory and Statistical Physics 33(5-7)(2004) 55-67.