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► To cite this version:

Mohamed Ghattassi, Jean Rodolphe Roche, Didier Schmitt, Mohamed Boutayeb. 2D-local existence and uniqueness of a transient state of a coupled radiative-conductive heat transfer problem. 2016. hal-01272997v2

HAL Id: hal-01272997

<https://hal.science/hal-01272997v2>

Preprint submitted on 24 May 2016

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2D-Local Existence and Uniqueness of a Transient State of a Coupled Radiative-Conductive Heat Transfer Problem

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Abstract

This paper deals with local existence and uniqueness results for a transient two-dimensional combined nonlinear radiative-conductive system. This system describes the heat transfer for a grey, semi-transparent and non-scattering medium with homogeneous Dirichlet boundary conditions. We reformulate the full transient state system as a fixed-point problem. The existence and uniqueness proof rests upon the Banach fixed-point Theorem assuming the initial data T_0 is positive and sufficiently small.

Keywords: Nonlinear radiative-conductive heat transfer system, Semi-transparent medium, Local existence-uniqueness, Banach fixed point Theorem.

2010 MSC: 35K51, 35A01, 35B40.

1. Introduction and main results

The aim of this work is to prove the existence and uniqueness of the local solution for a transient two-dimensional combined radiative-conductive system with homogeneous Dirichlet boundary conditions when the initial condition is assumed to be small and positive. The medium is assumed grey, semi-transparent and non-scattering.

Let us consider a bounded, open, connected and convex set $\Omega \subset \mathbb{R}^2$ with C^∞ boundary. Let \mathcal{D} be the unit disk, for the problem considered, $\beta \in \mathcal{D}$, $\mathbf{x} \in \Omega$, $\xi \in [0, \tau]$ for $\tau > 0$, $\mathcal{X} = \Omega \times \mathcal{D}$ and $\mathcal{Q}_\tau = (0, \tau) \times \Omega$. Let \mathbf{n} be the outward unit normal to the boundary $\partial\Omega$. We denote

$$\partial\Omega_- = \{(\mathbf{x}, \beta) \in \partial\Omega \times \mathcal{D} \text{ such that } \beta \cdot \mathbf{n} < 0\}.$$

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The full system of a combined nonlinear radiation-conduction heat transfer is written in dimensionless form,

$$I(\xi, \mathbf{x}, \boldsymbol{\beta}) + \boldsymbol{\beta} \cdot \nabla_{\mathbf{x}} I(\xi, \mathbf{x}, \boldsymbol{\beta}) = T^4(\xi, \mathbf{x}) \quad (\xi, \mathbf{x}, \boldsymbol{\beta}) \in [0, \tau] \times \mathcal{X} \quad (1)$$

$$\partial_{\xi} T(\xi, \mathbf{x}) - \Delta T(\xi, \mathbf{x}) + 4\pi\theta T^4(\xi, \mathbf{x}) = \theta \int_{\mathcal{D}} I(\xi, \mathbf{x}, \boldsymbol{\beta}) \frac{2}{\sqrt{1 - |\boldsymbol{\beta}|^2}} d\boldsymbol{\beta} \quad (\xi, \mathbf{x}) \in (0, \tau) \times \Omega \quad (2)$$

$$T(\xi, \mathbf{x}) = 0 \quad (\xi, \mathbf{x}) \in (0, \tau) \times \partial\Omega \quad (3)$$

$$I(\xi, \mathbf{x}, \boldsymbol{\beta}) = 0 \quad (\xi, \mathbf{x}, \boldsymbol{\beta}) \in [0, \tau] \times \partial\Omega_{-} \quad (4)$$

$$T(0, \mathbf{x}) = T_0(\mathbf{x}) \quad \mathbf{x} \in \Omega \quad (5)$$

where $\partial_{\xi} T = \partial T / \partial \xi$, θ is a positive constant, and T_0 is also positive. In this paper we consider that the mean radiation intensity of the blackbody $I_b(T)$ verifies the Stefan-Boltzmann law which is proportional to T^4 . The radiative transfer equation (RTE) (1) and the conductive equation (NHE) (2) are coupled via the source term.

Radiative-conductive heat transfer problems is the subject of various fields of engineering and science. In the literature, this problem is studied using two different types of model. In the first type, the problem is described using a unique parabolic partial differential equation. In the second type of model, the modeling of the radiation and conduction is given by a coupled system of partial differential equations where each phenomenon is described by an equation.

There is a huge mathematical theory in the first case, see [5, 6, 7, 8, 29, 42, 3, 4, 9, 31, 32, 33, 34]. For example, the paper [5] is devoted to the study of a nonstationary nonlinear nonlocal initial boundary value problem governing radiative conductive heat transfer in opaque bodies with surfaces whose properties depend on the radiation frequency. This paper is a natural extension of the work done in [7], where the corresponding stationary problem was treated. In [28], the authors considered the conductive radiative heat transfer in a scattering and absorbing medium bounded by two reflecting and radiating plane surfaces. The existence and uniqueness of a solution of this problem is established using an iterative procedure.

In [33], M. Laitinen and T. Tiihonen studied the well-posedness of a class of models describing heat transfer by conduction and radiation in the stationary case. The employed theory covers different types of grey materials, that is, both semitransparent and opaque bodies as well as isotropic or non-isotropic scattering/reflection provided that the material properties do not depend on the wavelength of the radiation.

M. M. Porzio and Ó. López Pouso proved in [41] an existence and uniqueness theorem for the non-grey coupled convection-conduction-radiation system in the 3D case by means of accretive operators theory. Leaving aside the space dimension and the grey or non-grey character, the main difference between our problem and the one studied in [41] is that we do not include the transient term in the RTE. This is an interesting point because this term is really negligible in a wide range of applications, and also because the techniques used in [41] do not allow disregarding it.

In this paper, we consider the second type of model where the phenomenon is expressed as a coupled system of

nonlinear partial differential equations. In previous works we can find theoretical results of existence and uniqueness in one-dimensional case. Indeed, in the Kelley's paper [26], the authors considered a steady-state combined radiative-conductive heat transfer. In Asllanaj et al.[12] the authors generalized the Kelley's study and they proved the existence and uniqueness of the 1-D system of coupled radiative conductive in the steady state associated to the non-homogeneous Dirichlet boundary with the black surfaces. The medium is assumed to be a non-grey anisotropic absorbing, emitting, scattering, with axial symmetry and non homogeneous. They considered a nonlinear conduction equation due to the temperature dependence of the thermal conductivity. However, the approach developed by Asllanaj et al. [12] is just adaptable to 1D dimensional geometry. In this paper we prove the existence and uniqueness of local solutions for the nonlinear system (1)-(5) if the initial data T_0 is sufficiently small and positive.

Recently, some attention has been accorded to numerical methods to study the radiative transfer and the nonlinear radiative-conductive heat transfer problem including optimal control problems, for more details see [10, 11, 12, 13, 14, 17, 19, 20, 21, 22, 24, 37, 38, 39, 40, 36, 27, 25, 23]. Asllanaj et al. [13] simulated transient heat transfer by radiation and conduction in two-dimensional complex shaped domains with structured and unstructured triangular meshes working with an absorbing, emitting and non-scattering grey medium.

In order to state the main result, we introduce the following notations

$$L^p(\mathcal{Q}_\tau) = L^p(0, \tau; L^p(\Omega)) \text{ for all } p \in [1, \infty)$$

$$W_2^{2,1}(\mathcal{Q}_\tau) := \{\phi \text{ s.t } \phi, \phi_\xi, \phi_{x_i}, \phi_{x_i x_j} \in L^2(\mathcal{Q}_\tau)\}$$

We will establish that the solution of the problem (1)-(5) is a fixed point of a well posed map \mathbb{H} in the set

$$E_1 = \{T \in L^8(\mathcal{Q}_\tau); \|T\|_{L^8(\mathcal{Q}_\tau)} \leq M\},$$

where M is a positive constant. We assume that the initial data T_0 is positive, belongs to $H_0^1(\Omega)$ and satisfies the following hypothesis

$$\|T_0\|_{L^5(\Omega)} \leq \sqrt[5]{\frac{15\theta\pi M^8}{8}}. \quad (6)$$

The main result of this paper is the following Theorem.

Theorem 1.1. *If the initial data T_0 verifies (6) then there exists $\tau^* > 0$ such that for all $\tau \leq \tau^*$, the system of equation (1)-(5) has a unique local solution (T, I) such that $T \in W_2^{2,1}(\mathcal{Q}_\tau)$ and $I \in L^2(0, \tau; L^2(\mathcal{X}))$. Moreover, there exists $C = C(\Omega, \tau, \theta)$ such that*

$$\|I\|_{L^2(0, \tau; L^2(\mathcal{X}))} \leq \sqrt{\pi} \|T\|_{L^8(\mathcal{Q}_\tau)}^4$$

and

$$\|T\|_{W_2^{2,1}(\mathcal{Q}_\tau)} \leq C(\|G\|_{L^2(\mathcal{Q}_\tau)} + \|T_0\|_{H_0^1(\Omega)}). \quad (7)$$

This paper is organized as follows: In the next section, we introduce the model describing the radiative-conductive heat transfer system in a grey, semi-transparent and non-scattering medium. The section 3 is devoted to construct a contraction mapping \mathbb{H} on a suitable set, whose fixed point gives a solution of the nonlinear coupled radiative conductive heat transfer system (1)-(5).

2. Model problem

Let Ω^* be an open, bounded, connected and convex domain with C^∞ boundary and

$$\partial\Omega_-^* = \{(\mathbf{x}^*, \boldsymbol{\beta}) \in \partial\Omega^* \times \mathcal{D} \text{ such that } \boldsymbol{\beta} \cdot \mathbf{n} < 0\}.$$

The radiative transfer equation (RTE) in a two dimensional gray absorbing and emitting medium is given by, see [13, 24]

$$\boldsymbol{\beta} \cdot \nabla_{\mathbf{x}^*} I^*(t, \mathbf{x}^*, \boldsymbol{\beta}) + \kappa I^*(t, \mathbf{x}^*, \boldsymbol{\beta}) = \kappa n^2 I_b(T^*(t, \mathbf{x}^*)), \quad (t, \mathbf{x}^*, \boldsymbol{\beta}) \in [0, \tau^*] \times \Omega^* \times \mathcal{D}, \quad (8a)$$

$$I^*(t, \mathbf{x}^*, \boldsymbol{\beta}) = I_b(T^*(t, \mathbf{x}^*)), \quad \text{for } (t, \mathbf{x}^*, \boldsymbol{\beta}) \in [0, \tau^*] \times \partial\Omega_-^*, \quad (8b)$$

where I^* is the radiation intensity, κ is the absorption coefficient of the medium and n is the refractive index.

In this work, the refractive index and the absorption coefficient are assumed to be equal to one, $n = 1$, $\kappa = 1 \text{ m}^{-1}$. T^* , the temperature of the medium and $I_b(T^*)$ is the mean radiation intensity of the blackbody which obeys Stefan-Boltzmann's law:

$$I_b(T^*) = \frac{\sigma_B}{\pi} T^{*4},$$

where $\sigma_B = 5.6698 \times 10^{-8} \text{ Wm}^{-2}\text{K}^{-4}$ is the Stefan-Boltzmann constant. In this paper the authors have assumed Dirichlet boundary conditions. Emission and absorption of radiation by the medium lead to a radiative source term in the energy equation of the medium. It is defined by the following relations:

$$S_{rad}^*(t, \mathbf{x}^*) = \kappa \{G^*(t, \mathbf{x}^*) - 4\pi I_b(T^*(t, \mathbf{x}^*))\} \quad (t, \mathbf{x}^*) \in [0, \tau^*] \times \Omega^*,$$

where G^* is the incident radiation intensity,

$$G^*(t, \mathbf{x}^*) = \int_{\mathcal{D}} I^*(t, \mathbf{x}^*, \boldsymbol{\beta}) \frac{2}{\sqrt{1 - |\boldsymbol{\beta}|^2}} d\boldsymbol{\beta} \quad (t, \mathbf{x}^*) \in [0, \tau^*] \times \Omega^*.$$

The overall energy conservation links the three different modes of the heat transfer known as conduction, radiation and convection. In this study, the convection has not been considered, then we have the following conduction equation

$$\rho c_p \frac{\partial T^*}{\partial t}(t, \mathbf{x}^*) - k_c \Delta T^*(t, \mathbf{x}^*) = S_{rad}^*(t, \mathbf{x}^*) \quad (t, \mathbf{x}^*) \in (0, \tau^*] \times \Omega^* \quad (9a)$$

$$T^*(0, \mathbf{x}^*) = T_0^*(\mathbf{x}^*) \quad \mathbf{x}^* \in \Omega^* \quad (9b)$$

$$T^*(t, \mathbf{x}^*) = g^*(\mathbf{x}^*), \quad \text{for } (t, \mathbf{x}^*) \in [0, \tau^*] \times \partial\Omega^*. \quad (9c)$$

The data ρ , c_p , and k_c are the density, the specific heat capacity, and the thermal conductivity of the medium, respectively. In this work, they are assumed to be constant. The NHE (9a) and the RTE (8a) are strongly coupled by the incident radiation intensity G^* and the temperature T^* .

Let $\xi = \frac{\alpha t}{L^2}$ the dimensionless time where $\alpha = \frac{k_c}{\rho c_p}$ is the thermal diffusivity. Let T_{ref} be a reference temperature and L a characteristic length, then we obtain the following dimensionless quantities:

$$\begin{aligned} \mathbf{x} &= \frac{\mathbf{x}^*}{L}, \quad \tau = \frac{\alpha \tau^*}{L^2}, \quad T = \frac{T^*}{T_{ref}}, \\ T_0 &= \frac{T_0^*}{T_{ref}}, \quad G = \frac{G^*}{4\sigma_B T_{ref}^4}, \quad S_{rad} = \frac{S_{rad}^*}{4\sigma_B T_{ref}^4}, \\ I &= \frac{I^*}{I_{ref}} \quad \text{and} \quad g = \frac{g^*}{T_{ref}} \end{aligned}$$

where

$$I_{ref} = \frac{\sigma_B}{\pi} T_{ref}^4$$

is the reference radiative intensity, see [24]. The conduction radiation number which is denoted N_s satisfies the following expression:

$$N_s = \frac{k_c \kappa}{4\sigma_B T_{ref}^3}.$$

and we denote

$$\theta = \frac{\kappa^2 L^2}{N_s}$$

Consequently, we obtain the dimensionless full system of a combined nonlinear radiation-conduction heat transfer system (1)-(5) when the Dirichlet boundary condition g is equal to zero.

3. Local existence and uniqueness of solutions for the coupled system

In this section, we show that the existence of a solution T , and implicitly the existence of a solution I , of the coupled system of equations (1)-(5) is related to the existence of a solution of a fixed point problem. We will apply the fixed point Theorem to a well-chosen map \mathbb{H} . To do so, we must show that this map \mathbb{H} is well defined and completely continuous. At first, we recall the definition of the set E_1

$$E_1 = \{T \in L^8(\mathcal{Q}_\tau) \quad \|T\|_{L^8(\mathcal{Q}_\tau)} \leq M\}$$

and we introduce the following sets

$$E_2 = \{I \in L^2(0, \tau; L^2(\mathcal{X})) \quad \|I\|_{L^2(0, \tau; L^2(\mathcal{X}))} \leq N\},$$

$$E_3 = \{G \in L^2(\mathcal{Q}_\tau) \quad \|G\|_{L^2(\mathcal{Q}_\tau)} \leq \sqrt{\pi}N\},$$

where $N = \sqrt{\pi}M^4$. The map $\mathbb{H} : E_1 \rightarrow E_1$ is a composition of three maps

$$\mathbb{H} = \mathbb{H}_3 \circ \mathbb{H}_2 \circ \mathbb{H}_1.$$

The map $\mathbb{H}_1 : E_1 \rightarrow E_2$ is defined as follows, for $T \in E_1$, $\mathbb{H}_1(T) \in E_2$ is the solution of the radiative transfer equation(1)-(5). The second map $\mathbb{H}_2 : E_2 \rightarrow E_3$ is defined in the following way, for $I \in E_2$, $\mathbb{H}_2(I) = G \in E_3$ where G is given by

$$G(\xi, \mathbf{x}) = \int_{\mathcal{D}} I(\xi, \mathbf{x}, \boldsymbol{\beta}) \frac{2}{\sqrt{1 - |\boldsymbol{\beta}|^2}} d\boldsymbol{\beta} \quad (10)$$

and finally, the map $\mathbb{H}_3 : E_3 \rightarrow E_1$ is defined as follows, for $G \in E_3$, $\mathbb{H}_3(G) \in E_1$ is the solution of NHE (1)-(5).

3.1. The maps \mathbb{H}_1 and \mathbb{H}_2

Now, we focus on the maps \mathbb{H}_1 and \mathbb{H}_2 , we give some properties of the solution of the RTE (1).

Theorem 3.1. *Let us consider $T \in E_1$, the problem (1) has a positive unique solution $\mathbb{H}_1(T) \in L^2(0, \tau; L^2(\mathcal{X}))$. Thus \mathbb{H}_1 is a well-posed map from E_1 to E_2 . Moreover \mathbb{H}_1 is continuous.*

Proof. Let $T \in E_1$ then for all $\xi \in [0, \tau]$, we have $T^4(\xi) \in L^2(\Omega)$. Using a result about the existence and uniqueness of the solution of the transport equation, see[18], the boundary value problem (1)–(4) has a unique solution $I(\xi) \in L^2(\mathcal{X})$ which satisfies the following a priori estimate

$$\|I(\xi)\|_{L^2(\mathcal{X})} \leq \sqrt{\pi} \|T^4(\xi)\|_{L^2(\Omega)}.$$

If we integrate in time between 0 and τ , we obtain

$$\|I\|_{L^2(0, \tau; L^2(\mathcal{X}))} \leq \sqrt{\pi} \|T\|_{L^8(\mathcal{Q}_\tau)}^4$$

hence,

$$\|I\|_{L^2(\mathcal{Q}_\tau)} \leq \sqrt{\pi} M^4 = N.$$

Consequently, $I \in E_2$ and then \mathbb{H}_1 is a well-posed map.

Using the maximum principle [1], this implies that the solution I of (1), (4) is positive.

Now, we show the continuity of the map \mathbb{H}_1 . We consider I_1, I_2 two solutions of (1) associated to T_1, T_2 , respectively. Let $\xi \in [0, \tau]$, we have

$$\|I_1(\xi) - I_2(\xi)\|_{L^2(\Omega \times \mathcal{D})} \leq \sqrt{\pi} \|T_1^4(\xi) - T_2^4(\xi)\|_{L^2(\Omega)}. \quad (11)$$

Using the generalized Hölder's inequality, we have the following inequality

$$\begin{aligned} \|T_1^4(\xi) - T_2^4(\xi)\|_{L^2(\Omega)}^2 &= \int_{\Omega} (T_1(\xi) - T_2(\xi))^2 (T_1(\xi) + T_2(\xi))^2 (T_1^2(\xi) + T_2^2(\xi))^2 dx \\ &\leq \|T_1(\xi) - T_2(\xi)\|_{L^8(\Omega)}^2 \|T_1(\xi) + T_2(\xi)\|_{L^8(\Omega)}^2 \|T_1^2(\xi) + T_2^2(\xi)\|_{L^4(\Omega)}^2. \end{aligned} \quad (12)$$

If we integrate in time, we obtain

$$\|T_1^4 - T_2^4\|_{L^2(\mathcal{Q}_\tau)}^2 \leq \|T_1 - T_2\|_{L^8(\mathcal{Q}_\tau)}^2 \|T_1 + T_2\|_{L^8(\mathcal{Q}_\tau)}^2 \|T_1^2 + T_2^2\|_{L^4(\mathcal{Q}_\tau)}^2.$$

Then we have

$$\begin{aligned} \|T_1^2(\xi) + T_2^2(\xi)\|_{L^4(\Omega)}^4 &\leq \|T_1(\xi)\|_{L^8(\Omega)}^8 + 4\|T_1(\xi)\|_{L^8(\Omega)}^6 \|T_2(\xi)\|_{L^8(\Omega)}^2 + 6\|T_1(\xi)\|_{L^8(\Omega)}^4 \|T_2(\xi)\|_{L^8(\Omega)}^4 \\ &\quad + 4\|T_1(\xi)\|_{L^8(\Omega)}^2 \|T_2(\xi)\|_{L^8(\Omega)}^6 + \|T_2(\xi)\|_{L^8(\Omega)}^8. \end{aligned}$$

Hence

$$\begin{aligned} \|T_1^2 + T_2^2\|_{L^4(\mathcal{Q}_\tau)}^4 &\leq \|T_1\|_{L^8(\mathcal{Q}_\tau)}^8 + 4\|T_1\|_{L^8(\mathcal{Q}_\tau)}^6 \|T_2\|_{L^8(\mathcal{Q}_\tau)}^2 + 6\|T_1\|_{L^8(\mathcal{Q}_\tau)}^4 \|T_2\|_{L^8(\mathcal{Q}_\tau)}^4 \\ &\quad + 4\|T_1\|_{L^8(\mathcal{Q}_\tau)}^2 \|T_2\|_{L^8(\mathcal{Q}_\tau)}^6 + \|T_2\|_{L^8(\mathcal{Q}_\tau)}^8. \end{aligned}$$

Since $T_1, T_2 \in E_1$, then we deduce that

$$\|T_1^2 + T_2^2\|_{L^4(\mathcal{Q}_\tau)} \leq 2M^2. \quad (13)$$

On the other hand, we have

$$\|T_1 + T_2\|_{L^8(\mathcal{Q}_\tau)} \leq 2M, \quad (14)$$

it follows that

$$\|T_1^4 - T_2^4\|_{L^2(\mathcal{Q}_\tau)} \leq 4M^3 \|T_1 - T_2\|_{L^8(\mathcal{Q}_\tau)},$$

From (11), we deduce that

$$\|I_1 - I_2\|_{L^2(0,\tau;L^2(\Omega \times \mathcal{D}))} \leq 4\sqrt{\pi}M^3 \|T_1 - T_2\|_{L^8(\mathcal{Q}_\tau)}. \quad (15)$$

The last inequality shows the continuity of \mathbb{H}_1 . □

Now, we give some properties of the map \mathbb{H}_2 .

Proposition 3.2. \mathbb{H}_2 is a well posed and continuous map from E_2 to E_3 . Moreover, for all I solution (1), $G = \mathbb{H}_2(I)$ is a positive.

Proof. Let us consider $I \in E_2$ and $G = \mathbb{H}_2(I)$, then we have

$$\|G\|_{L^2(\mathcal{Q}_\tau)} \leq \sqrt{\pi} \|I\|_{L^2(0,\tau;L^2(\mathcal{X}))}. \quad (16)$$

Hence, for all $I \in E_2$, $G = \mathbb{H}_2(I)$ belongs to E_3 . Since I is positive then G is positive. Therefore \mathbb{H}_2 is a well posed map. Since \mathbb{H}_2 is a linear function, from the inequality (16), \mathbb{H}_2 is a continuous map. □

3.2. The map \mathbb{H}_3

In this subsection we introduce some properties of the map \mathbb{H}_3 .

Theorem 3.3. *Let us consider $T_0 \in H_0^1(\Omega)$, $T_0 \geq 0$ almost everywhere in Ω and satisfies the hypothesis (6). Let $G \in E_3$ and positive. Then the equation (2) has a positive solution $T = \mathbb{H}_3(G) \in E_1$.*

Proof. For the proof of the existence and uniqueness of the solution of the equation (2), see [2, 30].

First we give a proof of the non-negativity of T .

Let us consider F defined in $(0, \tau) \times \Omega \times \mathbb{R}$ by

$$F(\xi, \mathbf{x}, y) = \theta(G(\xi, \mathbf{x}) - 4\pi y^4)$$

then T is the solution of the following equation

$$\partial_\xi T(\xi, \mathbf{x}) - \Delta T(\xi, \mathbf{x}) = F(\xi, \mathbf{x}, T(\xi, \mathbf{x})) \quad (17)$$

$$T(\xi, \mathbf{x}) = 0 \quad \text{in } \partial\Omega \times (0, \tau)$$

$$T(\mathbf{x}, 0) = T_0(\mathbf{x}) \quad \text{in } \Omega.$$

Now, we define \bar{F} in $(0, \tau) \times \Omega \times \mathbb{R}$ by

$$\bar{F}(\xi, \mathbf{x}, y) = \begin{cases} \theta(G(\xi, \mathbf{x}) - 4\pi y^4) & \text{if } y \geq 0 \\ \theta G(\xi, \mathbf{x}) & \text{if } y < 0. \end{cases}$$

Let us consider \bar{T} the solution of the following equation

$$\partial_\xi \bar{T}(\xi, \mathbf{x}) - \Delta \bar{T}(\xi, \mathbf{x}) = \bar{F}(\xi, \mathbf{x}, \bar{T}(\xi, \mathbf{x})) \quad (18)$$

$$\bar{T}(\xi, \mathbf{x}) = 0 \quad \text{in } \partial\Omega \times (0, \infty)$$

$$\bar{T}(0, \mathbf{x}) = T_0(\mathbf{x}) \quad \text{in } \Omega.$$

Our goal is to prove that the solution \bar{T} of this equation remains positive over the time. In fact, in this case \bar{F} and F coincide, therefore we have by the uniqueness of the solution $T = \bar{T}$ which is positive.

We set $\bar{T}^+ = \max(T, 0)$ and $\bar{T}^- = \max(-T, 0)$, such that $\bar{T} = \bar{T}^+ - \bar{T}^-$.

Multiplying the equation (18) by $(-\bar{T}^-)$ and integrating over Ω , we obtain

$$- \int_{\Omega} \partial_\xi \bar{T}(\xi, \mathbf{x}) \bar{T}^-(\xi, \mathbf{x}) d\mathbf{x} + \int_{\Omega} \Delta \bar{T}(\xi, \mathbf{x}) \bar{T}^-(\xi, \mathbf{x}) d\mathbf{x} = - \int_{\Omega} \bar{F}(\xi, \mathbf{x}, \bar{T}) \bar{T}^-(\xi, \mathbf{x}) d\mathbf{x}.$$

Now, we have

$$- \int_{\Omega} \partial_\xi \bar{T}(\xi, \mathbf{x}) \bar{T}^-(\xi, \mathbf{x}) d\mathbf{x} = \frac{1}{2} \partial_\xi \int_{\Omega} (\bar{T}^-(\xi, \mathbf{x}))^2 d\mathbf{x}$$

$$\begin{aligned} \int_{\Omega} \Delta \bar{T}(\xi, \mathbf{x}) \bar{T}^-(\xi, \mathbf{x}) d\mathbf{x} &= \int_{\Omega} (\nabla \bar{T}^-(\xi, \mathbf{x}))^2 d\mathbf{x} \\ - \int_{\Omega} \bar{F}(\xi, \mathbf{x}, \bar{T}) \bar{T}^-(\xi, \mathbf{x}) d\mathbf{x} &= - \int_{\{\bar{T} < 0\}} \bar{F}(\xi, \mathbf{x}, \bar{T}) \bar{T}^-(\xi, \mathbf{x}) d\mathbf{x} = - \int_{\{\bar{T} < 0\}} \theta G(\xi, \mathbf{x}) \bar{T}^-(\xi, \mathbf{x}) d\mathbf{x} \leq 0. \end{aligned}$$

Consequently,

$$\frac{1}{2} \partial_{\xi} \int_{\Omega} (\bar{T}^-(\xi, \mathbf{x}))^2 d\mathbf{x} \leq 0.$$

As $\bar{T}^-(x, 0) = 0$ for all $x \in \Omega$ because $T_0(x) \geq 0$ for all $x \in \Omega$, we deduce that $\bar{T}^-(x, \xi) = 0$ for all $(x, \xi) \in \Omega \times (0, \infty)$.

It follows that $\bar{T}(x, \xi) \geq 0$ for all $(x, \xi) \in \Omega \times (0, \infty)$, hence

$$\bar{F}(x, \xi, \bar{T}(x, \xi)) = F(x, \xi, \bar{T}(x, \xi)).$$

Then \bar{T} is the solution of (17) and by uniqueness of the solution $T = \bar{T}$ which proves the non-negativity of T .

In the following, we prove that $T \in L^8(\mathcal{Q}_{\tau})$. Let z be the solution of the parabolic problem

$$\begin{cases} \partial_{\xi} z(\xi, \mathbf{x}) - \Delta z(\xi, \mathbf{x}) = \theta G(\xi, \mathbf{x}) & \text{for } (\xi, \mathbf{x}) \in]0, \tau] \times \Omega \\ z(\xi, \mathbf{x}) = 0 & \text{for } (\xi, \mathbf{x}) \in]0, \tau] \times \partial\Omega \\ z(0, \mathbf{x}) = T_0(\mathbf{x}) & \text{for } \mathbf{x} \in \Omega \end{cases} \quad (19)$$

then we have that $T \leq z$.

Since $G \in L^2(\mathcal{Q}_{\tau})$, $T_0 \in L^2(\Omega)$ and thanks to a result on parabolic regularity, see [30], then $z \in W_2^{2,1}(\mathcal{Q}_{\tau})$ and there exists a constant $\tilde{C} > 0$ such that

$$\|z\|_{W_2^{2,1}(\mathcal{Q}_{\tau})} \leq \tilde{C} (\|G\|_{L^2(\mathcal{Q}_{\tau})} + \|T_0\|_{L^2(\Omega)}). \quad (20)$$

Then, we have $z \in L^8(\mathcal{Q}_{\tau})$ (see [30, p.80]) and consequently $T \in L^8(\mathcal{Q}_{\tau})$.

Since G and T^4 belong to $L^2(\mathcal{Q}_{\tau})$, we have $\theta G - 4\pi\theta T^4$ belongs to $L^2(\mathcal{Q}_{\tau})$. Consequently, using the same result on parabolic regularity we obtain $T \in W_2^{2,1}(\mathcal{Q}_{\tau})$.

To prove that $T = \mathbb{H}_3(G) \in E_1$, we need a more precise control of $\|T\|_{L^8(\mathcal{Q}_{\tau})}$.

We multiply the equation (2) by T^4 and we integrate over Ω , so we obtain

$$\frac{1}{5} \frac{d}{d\xi} \int_{\Omega} T^5(\xi) d\mathbf{x} + 4 \int_{\Omega} (\nabla T(\xi))^2 (T(\xi))^3 d\mathbf{x} + 4\pi\theta \int_{\Omega} T^8(\xi) d\mathbf{x} = \theta \int_{\Omega} G T^4(\xi) d\mathbf{x}.$$

As $T \in W_2^{2,1}(\mathcal{Q}_{\tau})$ then T^4 belongs to $L^2(0, \tau; H_0^1(\Omega))$.

Using the Young's inequality, we get

$$\frac{1}{5} \frac{d}{d\xi} \int_{\Omega} (T^{\frac{5}{2}}(\xi))^2 d\mathbf{x} + \frac{16}{25} \int_{\Omega} (\nabla T^{\frac{5}{2}}(\xi))^2 d\mathbf{x} + 4\pi\theta \int_{\Omega} T^8(\xi) d\mathbf{x} \leq \frac{\theta}{2\epsilon} \int_{\Omega} G^2(\xi) d\mathbf{x} + \theta \frac{\epsilon}{2} \int_{\Omega} T^8(\xi) d\mathbf{x}.$$

If we assume $\epsilon = 2\pi$ and we integrate in time, we obtain

$$\frac{1}{5} \sup_{0 \leq \xi \leq \tau} \|T^{\frac{5}{2}}(\xi)\|_{L^2(\Omega)}^2 + 4 \|T^{\frac{5}{2}}\|_{L^2(0, \tau; H_0^1(\Omega))}^2 + 3\pi\theta \|T\|_{L^8(\mathcal{Q}_{\tau})}^8 \leq \frac{\theta}{4\pi} \|G\|_{L^2(\mathcal{Q}_{\tau})}^2 + \frac{1}{5} \|T_0^{\frac{5}{2}}\|_{L^2(\Omega)}^2,$$

hence

$$\|T\|_{L^8(\mathcal{Q}_\tau)}^8 \leq \frac{1}{12\pi^2} \|G\|_{L^2(\mathcal{Q}_\tau)}^2 + \frac{1}{150\pi} \|T_0\|_{L^5(\Omega)}^5.$$

Since $G \in E_1$, we deduce

$$\|T\|_{L^8(\mathcal{Q}_\tau)}^8 \leq \frac{M^8}{4} + \frac{1}{150\pi} \|T_0\|_{L^5(\Omega)}^5.$$

If the initial data satisfies the hypothesis (6), it follows that

$$\|T\|_{L^8(\mathcal{Q}_\tau)}^8 \leq M^8.$$

Thus $T \in E_1$ and the map \mathbb{H}_3 is well-posed. □

Theorem 3.4. *Under the assumptions of Theorem 3.3, \mathbb{H}_3 is a continuous map from E_3 to E_1 .*

Proof. Let $G_1, G_2 \in E_3$, $T_1 = \mathbb{H}_3(G_1)$ and $T_2 = \mathbb{H}_3(G_2)$. Let us set $w = T_1 - T_2$, then w is solution of the following equation

$$\begin{aligned} \partial_\xi w(\xi, \mathbf{x}) - \Delta w(\xi, \mathbf{x}) &= -4\pi\theta(T_1^4 - T_2^4)(\xi, \mathbf{x}) + \theta(G_1 - G_2)(\xi, \mathbf{x}) & \text{in } (0, \tau] \times \Omega, \\ w(\xi, \mathbf{x}) &= 0 & \text{in } (0, \tau] \times \partial\Omega \\ w(0, \mathbf{x}) &= 0 & \text{in } \Omega. \end{aligned}$$

We have

$$w(\xi) = -4\pi\theta \int_0^\xi \mathbb{T}(\xi - s)(T_1^4 - T_2^4)(s) ds + \theta \int_0^\xi \mathbb{T}(\xi - s)(G_1 - G_2)(s) ds$$

where $\mathbb{T}(\xi)$ is a semigroup of contraction in $H_0^1(\Omega)$ generated by the operator \mathbf{A} defined by

$$D(\mathbf{A}) = \{T \in H_0^1(\Omega), \Delta T \in L^2(\Omega)\} \quad \text{and} \quad \mathbf{A}T = \Delta T, \quad \forall T \in D(\mathbf{A}).$$

Now, using the regularizing effects of the heat equation, see [15, proposition 3.5.7, p.44] with $p = 8$ and $q = 2$, we deduce the following inequality

$$\|w(\xi)\|_{L^8(\Omega)} \leq 4\pi\theta \int_0^\xi \frac{1}{(4\pi(\xi - s))^{\frac{1}{2} - \frac{1}{8}}} \|T_1^4(s) - T_2^4(s)\|_{L^2(\Omega)} ds + \theta \int_0^\xi \frac{1}{(4\pi(\xi - s))^{\frac{1}{2} - \frac{1}{8}}} \|G_1(s) - G_2(s)\|_{L^2(\Omega)} ds. \quad (21)$$

In view of the inequality (12) and the Cauchy-Schwarz inequality, then (21) becomes

$$\begin{aligned} \|w(\xi)\|_{L^8(\Omega)} &\leq 4\pi\theta \int_0^\xi \frac{1}{(4\pi(\xi - s))^{\frac{1}{2} - \frac{1}{8}}} \|w(s)\|_{L^8(\Omega)} \|T_1(s) + T_2(s)\|_{L^8(\Omega)} \|(T_1^2 + T_2^2)(s)\|_{L^4(\Omega)} ds \\ &\quad + \theta \left(\int_0^\xi \frac{ds}{(4\pi(\xi - s))^{\frac{3}{4}}} \right)^{\frac{1}{2}} \|G_1 - G_2\|_{L^2(\mathcal{Q}_\tau)}. \end{aligned} \quad (22)$$

We have

$$\left(\int_0^\xi \frac{ds}{(4\pi(\xi - s))^{\frac{3}{4}}} \right)^{\frac{1}{2}} = \left(\frac{4\sqrt[4]{\xi}}{(4\pi)^{\frac{3}{4}}} \right)^{\frac{1}{2}} = \frac{2\sqrt[8]{\xi}}{(4\pi)^{\frac{3}{8}}}. \quad (23)$$

Thanks to the generalized Hölder's inequality, we get

$$\begin{aligned} & \int_0^\xi \frac{1}{(4\pi(\xi-s))^{\frac{1}{2}-\frac{1}{8}}} \|w(s)\|_{L^8(\Omega)} \|(T_1 + T_2)(s)\|_{L^8(\Omega)} \|(T_1^2 + T_2^2)(s)\|_{L^4(\Omega)} ds \\ & \leq \frac{2\sqrt[8]{t}}{(4\pi)^{\frac{3}{8}}} \left(\int_0^\xi \|w(s)\|_{L^8(\Omega)}^8 ds \right)^{\frac{1}{8}} \|T_1 + T_2\|_{L^8(\mathcal{Q}_\tau)} \|T_1^2 + T_2^2\|_{L^4(\mathcal{Q}_\tau)}. \end{aligned} \quad (24)$$

We substitute (23) and (24) into (22), we obtain

$$\|w(\xi)\|_{L^8(\Omega)} \leq 4\pi\theta \frac{2\sqrt[8]{\xi}}{(4\pi)^{\frac{3}{8}}} \left(\int_0^\xi \|w(s)\|_{L^8(\Omega)}^8 ds \right)^{\frac{1}{8}} \|T_1 + T_2\|_{L^8(\mathcal{Q}_\tau)} \|T_1^2 + T_2^2\|_{L^4(\mathcal{Q}_\tau)} + \theta \frac{2\sqrt[8]{\xi}}{(4\pi)^{\frac{3}{8}}} \|G_1 - G_2\|_{L^2(\mathcal{Q}_\tau)}.$$

The estimations (13) and (14) give

$$\begin{aligned} \|w(\xi)\|_{L^8(\Omega)} & \leq 4\pi\theta \frac{2\sqrt[8]{\xi}}{(4\pi)^{\frac{3}{8}}} 4M^3 \left(\int_0^\xi \|w(s)\|_{L^8(\Omega)}^8 ds \right)^{\frac{1}{8}} + \theta \frac{2\sqrt[8]{\xi}}{(4\pi)^{\frac{3}{8}}} \|G_1 - G_2\|_{L^2(\mathcal{Q}_\tau)} \\ & \leq 4\pi\theta \frac{2\sqrt[8]{\tau}}{(4\pi)^{\frac{3}{8}}} 4M^3 \left(\int_0^\xi \|w(s)\|_{L^8(\Omega)}^8 ds \right)^{\frac{1}{8}} + \theta \frac{2\sqrt[8]{\tau}}{(4\pi)^{\frac{3}{8}}} \|G_1 - G_2\|_{L^2(\mathcal{Q}_\tau)}. \end{aligned} \quad (25)$$

Since $(a+b)^8 \leq 128(a^8 + b^8)$ for all $(a, b) \in \mathbf{R}_+^2$, it follows that

$$\|w(\xi)\|_{L^8(\Omega)}^8 \leq \frac{\theta^8}{\pi^3} 2^{25} M^{24} \tau \int_0^\xi \|w(s)\|_{L^8(\Omega)}^8 ds + \frac{\theta^8}{\pi^3} 2^9 \tau \|G_1 - G_2\|_{L^2(\mathcal{Q}_\tau)}^8.$$

Applying the Gronwall's inequality, we deduce

$$\|\mathbb{H}_3(G_1) - \mathbb{H}_3(G_2)\|_{L^8(\mathcal{Q}_\tau)}^8 \leq \frac{\theta^8}{\pi^3} 2^9 \tau^2 e^{\frac{\theta^8}{\pi^3} 2^{25} M^{24} \tau^2} \|G_1 - G_2\|_{L^2(\mathcal{Q}_\tau)}^8. \quad (26)$$

□

3.3. Existence and uniqueness of the solution for the coupled system

Now, we will use an extension of the contraction mapping Theorem for a Banach space (see [35, p. 117] for example) to prove the uniqueness of the local solution for the coupled system (1)-(5).

Theorem 3.5. *If the initial data T_0 satisfies (6) then there exists $\tau^* > 0$ such that for all $\tau \leq \tau^*$, \mathbb{H} is a contraction map in E_1 .*

Proof. $\mathbb{H} = \mathbb{H}_3 \circ \mathbb{H}_2 \circ \mathbb{H}_1$ is a well-posed map because it is composed by a three well-posed maps.

Now, we show that \mathbb{H} is a contraction map from E_1 to E_1 .

Moreover, synthetically we have

$$\begin{array}{ccccccc} (E_1, \|\cdot\|_{L^8(\mathcal{Q}_\tau)}) & \xrightarrow{\mathbb{H}_1} & (E_2, \|\cdot\|_{L^2(0,\tau;L^2(\mathcal{X}))}) & \xrightarrow{\mathbb{H}_2} & (E_3, \|\cdot\|_{L^2(\mathcal{Q}_\tau)}) & \xrightarrow{\mathbb{H}_3} & (E_1, \|\cdot\|_{L^8(\mathcal{Q}_\tau)}) \\ T \mapsto & & I \mapsto & & G \mapsto & & \bar{T}. \end{array}$$

Let us consider $(T_1, T_2, \bar{T}_1, \bar{T}_2) \in E_1^4$, $(I_1, I_2) \in E_2^2$ and $(G_1, G_2) \in E_3^2$ such that

$$\begin{cases} I_1 = \mathbb{H}_1(T_1), & I_2 = \mathbb{H}_1(T_2) \\ G_1 = \mathbb{H}_2(I_1), & G_2 = \mathbb{H}_2(I_2) \\ \bar{T}_1 = \mathbb{H}_3(G_1), & \bar{T}_2 = \mathbb{H}_3(G_2) \\ \tilde{T}_1 = \mathbb{H}(T_1), & \tilde{T}_2 = \mathbb{H}(T_2). \end{cases}$$

$\mathbb{H}_1, \mathbb{H}_2$ and \mathbb{H}_3 are a continuous maps , then from (15), (16) and (26) it follows that

$$\begin{aligned} \|I_1 - I_2\|_{L^2(0,\tau;L^2(\mathbf{Q} \times \mathcal{D}))} &\leq 4\sqrt{\pi}M^3\|T_1 - T_2\|_{L^8(\mathcal{Q}_\tau)}, \\ \|G_1 - G_2\|_{L^2(\mathcal{Q}_\tau)} &\leq \sqrt{\pi}\|I_1 - I_2\|_{L^2(0,\tau;L^2(\mathbf{Q} \times \mathcal{D}))}, \\ \|\bar{T}_1 - \bar{T}_2\|_{L^8(\mathcal{Q}_\tau)}^8 &\leq \frac{\theta^8}{\pi^3}2^9\tau^2e^{\frac{\theta^8}{\pi^3}2^{25}M^{24}\tau^2}\|G_1 - G_2\|_{L^2(\mathcal{Q}_\tau)}^8. \end{aligned}$$

Then we deduce

$$\|\tilde{T}_1 - \tilde{T}_2\|_{L^8(\mathcal{Q}_\tau)}^8 \leq \pi^8\gamma(\tau)e^{\gamma(\tau)}\|T_1 - T_2\|_{L^8(\mathcal{Q}_\tau)}^8$$

where $\gamma(\tau) = \frac{\pi^8}{\pi^3}2^{25}M^{24}\tau^2$, clearly there exists $\tau^* > 0$ such that for all $\tau \leq \tau^*$, we have

$$\pi^8\gamma(\tau)e^{\gamma(\tau)} < 1.$$

Hence, \mathbb{H} is a contraction map. □

Consequently, the main Theorem 1.1 is a corollary of the last result, Theorem 3.5.

Proof of Theorem 1.1. E_1 is a closed subset of the Banach space $L^8(\mathcal{Q}_\tau)$, then using an extend Banach Theorem and Theorem 3.5 there exists $\tau^* > 0$ such that for all $\tau \leq \tau^*$, \mathbb{H} has a unique fixed-point $T \in E_1$. T satisfies the equation (2) therefore, by Theorem 3.1, $I = \mathbb{H}_1(T)$ is also the unique solution of the equation (1). Thus the couple (T, I) is a solution of the equations system (1)-(5) and this solution is unique. So G and T^4 belong to $L^2(\mathcal{Q}_\tau)$ then we have $\theta G - 4\pi\theta T^4$ belongs to $L^2(\mathcal{Q}_\tau)$. Consequently, thanks to the parabolic regularity, see [30], we obtain $T \in W_2^{2,1}(\mathcal{Q}_\tau)$. □

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