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SPINORIAL REPRESENTATION OF SUBMANIFOLDS IN RIEMANNIAN SPACE FORMS

PIERRE BAYARD, MARIE-AMÉLIE LAWN AND JULIEN ROTH

Abstract. In this paper we give a spinorial representation of submanifolds of any dimension and codimension into Riemannian space forms in terms of the existence of so called generalized Killing spinors. We then discuss several applications, among them a new and concise proof of the fundamental theorem of submanifold theory. We also recover results of T. Friedrich, B. Morel and the authors in dimension 2 and 3.

keywords: Spin geometry, Isometric Immersions, Weierstrass Representation.

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1. Introduction

One of the fundamental problems in submanifold theory deals with the existence of isometric immersions from a Riemannian manifold M^n into another fixed Riemannian manifold N^{n+p}. If the ambient manifold is a space form $\mathbb{R}^{n+p}$, $\mathbb{S}^{n+p}$ or $\mathbb{H}^{n+p}$, the fundamental theorem of submanifold theory states that the Gauß, Ricci and Codazzi equations, also called structure equations, are necessary and sufficient conditions.

In the case of surfaces, another approach is given by the study of Weierstrass representations. Historically, these representations are describing a conformal minimal immersion of a Riemann surface M into the three-dimensional Euclidean space $\mathbb{R}^3$. Precisely, given a pair (h, g) consisting of a holomorphic and a meromorphic function, the formula

$$f(x, y) = \Re e \int \big((1 - g^2(z))h(z), (1 + g^2(z))h(z), 2g(z)h(z)\big)\, dz,$$

with $z = x + iy$ some complex coordinate, gives a local parametrization of a minimal surface in Euclidean three-space. Conversely every minimal surface can be parametrized in this way with respect to isothermal coordinates. However, relaxing the condition of holomorphicity on the pair (h, g), this representation is much more general and can actually describe all surfaces in $\mathbb{R}^3$ as shown in [9].

At the end of the 1990s, following an idea of U. Abresch and D. Sullivan, R. Kusner and N. Schmidt reformulated this approach in a more concise and simpler way in terms of spinor fields (see [12]). These so called spinorial Weierstrass representations were studied extensively by B.G. Konopelchenko, I. Taimanov and many others (see [10, 11, 23, 24] and the references there).

However these formulae were given in local coordinates and remained purely computational until Friedrich gave in [8] an elegant and geometrically invariant description using spinor bundles. We point out that the equivalence between the two approaches was recently showed in [21]. The main idea is to use the identification

between the ambient spinor bundle restricted to the surface and the spinor bundle of the surface. Note that the condition to be a spin manifold is not restrictive here since any oriented surface is also spin. More generally, the restriction φ of a parallel spinor field on $\mathbb{R}^{n+1}$ to an oriented Riemannian hypersurface M^n is a solution of a *generalized Killing equation*

$$\nabla^{\Sigma M}_X \varphi = A(X) \cdot \varphi,$$

where $\nabla^{\Sigma M}$ and $\cdot$ are respectively the spin connection and the Clifford multiplication on M, and A is the shape operator of the immersion. Conversely, Friedrich showed that in the case where M is a surface, if there exists a particular spinor field φ satisfying the generalized Killing equation, where A is an arbitrary field of symmetric endomorphisms of the tangent bundle, then there exists an isometric immersion of M into $\mathbb{R}^3$ with shape operator A. Moreover φ is the restriction to M of a parallel spinor of $\mathbb{R}^3$. The proof consists in showing that A indeed satisfies the structure equations. This result was generalized to surfaces into other three-dimensional ambient spaces [17, 20, 22], to three-dimensional manifolds into four-dimensional space forms [14, 20] and also to the two-dimensional pseudo-Riemannian setting [15]. However the question whether in general a manifold of arbitrary dimension carrying a generalized Killing spinor can be immersed isometrically into some Euclidean space remained until now unanswered. Some of the few achievement in this direction were obtained in [1] for real analytic manifolds and in [3, 19] when A is a Codazzi tensor, showing the existence of an immersion into a Ricci flat manifold admitting a parallel spinor which restricts to φ.

Similarly, in higher codimension, very little is known. In [5], we extended the approach to the case of surfaces in four-dimensional space forms. The key point was to use the remark due to Bär [2] that an ambient spinor restricted to an immersed submanifold M can be identified with a section of the spinor bundle of the submanifold twisted with the spin bundle of the normal bundle. This was then extended to the pseudo-Riemannian setting in [4, 6].

Following the same idea, we use in this paper a particular twisted spin bundle over a spin manifold of arbitrary dimension to give a geometrically invariant spinorial representation of submanifolds of Euclidean spaces in any codimension. Note that our proof does not use the structure equations but merely the existence of a generalized Killing spinor on the manifold. We later show that one indeed recovers the previously mentioned result of Friedrich [8] in the case of surfaces in $\mathbb{R}^3$, as well as the one of Lawn-Roth [14] for three-dimensional hypersurfaces and of Bayard-Lawn-Roth [5] for surfaces in $\mathbb{R}^4$ (section 7). It is worth pointing out that the study of generalized Killing spinors has revealed very interesting applications. Moroianu and Semmelmann were for instance able to construct new examples of Lagrangian submanifolds of the nearly Kähler $\mathbb{S}^3 \times \mathbb{S}^3$ using the existence of such spinors on the sphere $\mathbb{S}^3$ [18]. Moreover it is well-known that there is a close relationship to G-structures: for instance a generalized Killing spinor defines a cocalibrated G2-structure on the manifold in dimension 7 and a half-flat SU(3)-structure in dimension 6 (see for example [7]). However the existence of such spinors is a non-trivial problem: our construction is therefore of particular interest.

Besides the above-mentioned, we discuss several other applications. A notable achievement is a new and concise proof of the fundamental theorem of submanifold theory. In the special case of surfaces, we show that our approach is equivalent to the spinorial Weierstrass representations, i.e., we obtain explicit formulas in terms

of functions involving the components of the spinor field which are holomorphic if the surface is minimal. Our result can thus be seing as a generalization of most of the concrete Weierstrass representation formulae existing in the literature: it provides a general framework to understand formulae appearing in a variety of contexts. Moreover, since the basic ideas and constructions behind our representation are fairly simple, we hope that our result will be useful to obtain new concrete Weierstrass representation formulae, once some geometric context is specified: this is especially interesting for surfaces, in low-dimensional pseudo-Riemannian space forms, under some curvature assumptions.

Finally, in the last section, we extend our result to submanifolds immersed into the other space forms S^n and $\mathbb{H}^n$, and recover the results of Morel [17] if $n = 3$.

2. Preliminaries

2.1. **The spin representation.** Let us denote by Cl_n the real Clifford algebra on $\mathbb{R}^n$ with its standard scalar product. We consider the representation

$$\begin{array}{rccc} \rho: & Cl_n & \to & End(Cl_n) \\ & a & \mapsto & \xi \mapsto a\xi \end{array}$$

and its restriction to the group $Spin(n)$

$$\begin{array}{rccc} \rho_{|Spin(n)}: & Spin(n) & \to & GL(Cl_n) \\ & a & \mapsto & \xi \mapsto a\xi. \end{array}$$

Note that this is not the adjoint representation of the spin group on the Clifford algebra, but rather the representation given by left multiplication.
Moreover we want to point out that we are not taking as usual the restriction of an irreducible representation of the Clifford algebra to the spin group, but that we consider instead the restriction of the entire real Clifford algebra. This real representation splits into a sum of 2^k copies of spinor spaces of dimension 2^{n-k}, where the number k depends on the dimension n and can be computed using the Radon-Hurwitz numbers (we refer to [16] for further details).
If $p + q = n$, we have a natural map

$$Spin(p) \times Spin(q) \to Spin(n)$$

associated to the splitting $\mathbb{R}^n = \mathbb{R}^p \oplus \mathbb{R}^q$ and to the corresponding isomorphism of Clifford algebras

$$Cl_n = Cl_p \hat{\otimes} Cl_q,$$

where $\hat{\otimes}$ denotes the $\mathbb{Z}_2-$graded tensor product.
We get thus the following representation, still denoted by ρ,

$$\begin{array}{rccc} \rho: & Spin(p) \times Spin(q) & \to & GL(Cl_n) \\ & a & \mapsto & \xi \mapsto a\xi. \end{array} \tag{1}$$

2.2. **The twisted spinor bundle Σ.** We consider M a p-dimensional Riemannian manifold, $E \to M$ a bundle of rank q, with a fibre metric and a compatible connection. We assume that E and TM are oriented and spin, with given spin structures

$$\tilde{Q}_M \overset{2:1}{\to} Q_M \quad \text{and} \quad \tilde{Q}_E \overset{2:1}{\to} Q_E$$

where Q_M and Q_E are the bundles of positively oriented orthonormal frames of TM and E, and we set

$$\tilde{Q} := \tilde{Q}_M \times_M \tilde{Q}_E;$$

this is a $Spin(p) \times Spin(q)$-principal bundle. We define the associated bundle

$$\Sigma := \tilde{Q} \times_\rho Cl_n$$

and its restriction

$$U\Sigma := \tilde{Q} \times_\rho Spin(n) \quad \subset \ \Sigma \tag{2}$$

where ρ is the representation (1) given by left multiplication. We remark that if we used the adjoint representation instead, we would just get the Clifford algebra bundle. Again we point out that our spinor bundle Σ is a real vector bundle with fiber the entire Clifford algebra and not, as usual, an irreducible complex Clifford module.
The vector bundle Σ is equipped with the covariant derivative ∇ naturally associated to the spinorial connections on $\tilde{Q}_M$ and $\tilde{Q}_E$.

Remark 1. *The bundle Σ is a spinor bundle on TM twisted by a spinor bundle on E : indeed, let us consider the representations*

$$\rho_1 : \ Spin(p) \to GL(Cl_p) \qquad \text{and} \qquad \rho_2 : \ Spin(q) \to GL(Cl_q)$$

given by left multiplication, and the associated bundles

$$\Sigma_1 := \tilde{Q}_M \times_{\rho_1} Cl_p \qquad \text{and} \qquad \Sigma_2 := \tilde{Q}_E \times_{\rho_2} Cl_q$$

equipped with their natural connections ∇^1 and ∇^2; then

$$\Sigma_1 \otimes \Sigma_2 \ \simeq \ \Sigma$$

and

$$\nabla^1 \otimes id_{\Sigma_2} \ \oplus \ id_{\Sigma_1} \otimes \nabla^2 \ \simeq \ \nabla.$$

This is a consequence of the fact that the natural isomorphism $i : Cl_p \otimes Cl_q \simeq Cl_n$ is an equivalence of representations of $Spin(p) \times Spin(q)$, i.e., for $g_1 \in Spin(p)$ and $g_2 \in Spin(q)$,

$$i \circ \ \rho_1(g_1) \otimes \rho_2(g_2) = \rho(g_1, g_2) \circ i;$$

indeed, if $\xi_1 \in Cl_p$ and $\xi_2 \in Cl_q$,

$$i(\rho_1(g_1) \otimes \rho_2(g_2)(\xi_1 \otimes \xi_2)) = i(g_1\xi_1 \otimes g_2\xi_2) = g_1\xi_1 g_2\xi_2 = g_1 g_2 \xi_1\xi_2 = \rho(g_1, g_2)(i(\xi_1 \otimes \xi_2)),$$

where the products in the third and fourth terms are products in Cl_n (note that ξ_1 and g_2 commute since ξ_1 belongs to Cl_p and g_2 is a product of an even number of vectors belonging to $\mathbb{R}^q$).

As in the usual construction in spin geometry, the spin bundle Σ is endowed with a natural action of the Clifford bundle $Cl(TM \oplus E)$: indeed, the Clifford product

$$\begin{array}{rcl} Cl(\mathbb{R}^p \oplus \mathbb{R}^q) \times Cl_n & \to & Cl_n \\ (\eta, \xi) & \mapsto & \eta \cdot \xi \end{array}$$

is $Spin(p) \times Spin(q)$ equivariant, if the action of $Spin(p) \times Spin(q)$ on $Cl(\mathbb{R}^p \oplus \mathbb{R}^q)$ is the adjoint action, and the action on Cl_n is the left multiplication: we obviously have, for $(g_1, g_2) \in Spin(p) \times Spin(q)$ and $g = g_1 g_2 \in Spin(n)$,

$$(g\xi g^{-1}) \cdot (g\eta) = g \cdot (\xi\eta)$$

for $\xi, \eta \in Cl_n$.

2.3. **A Cl_n-valued bilinear map on Σ.** Let us denote by $\tau : Cl_n \to Cl_n$ the anti-automorphism of Cl_n such that

$$\tau(x_1 \cdot x_2 \cdots x_k) = x_k \cdots x_2 \cdot x_1$$

for all $x_1, x_2, \ldots, x_k \in \mathbb{R}^n$, where '$\cdot$' denotes as usual the Clifford multiplication, and set

$$\langle\langle .,. \rangle\rangle : \quad \begin{array}{rcl} Cl_n \times Cl_n & \to & Cl_n \\ (\xi, \xi') & \mapsto & \tau(\xi')\xi. \end{array} \tag{3}$$

This map is $Spin(n)-$invariant: for all $\xi, \xi' \in Cl_n$ and $g \in Spin(n)$ we have

$$\langle\langle g\xi, g\xi' \rangle\rangle = \tau(g\xi')g\xi = \tau(\xi')\tau(g)g\xi = \tau(\xi')\xi = \langle\langle \xi, \xi' \rangle\rangle,$$

since $Spin(n) \subset \{g \in Cl_n^0 : \ \tau(g)g = 1\}$; this map thus induces a Cl_n-valued map

$$\langle\langle .,. \rangle\rangle : \quad \begin{array}{rcl} \Sigma \times \Sigma & \to & Cl_n \\ (\varphi, \varphi') & \mapsto & \langle\langle [\varphi], [\varphi'] \rangle\rangle \end{array} \tag{4}$$

where $[\varphi]$ and $[\varphi'] \in Cl_n$ represent φ and φ' in some spinorial frame $\tilde{s} \in \tilde{Q}$.

Lemma 2.1. *The map $\langle\langle .,. \rangle\rangle : \Sigma \times \Sigma \to Cl_n$ satisfies the following properties: for all $\varphi, \psi \in \Gamma(\Sigma)$ and $X \in \Gamma(TM)$,*

$$\langle\langle \varphi, \psi \rangle\rangle = \tau\langle\langle \psi, \varphi \rangle\rangle \tag{5}$$

and

$$\langle\langle X \cdot \varphi, \psi \rangle\rangle = \langle\langle \varphi, X \cdot \psi \rangle\rangle. \tag{6}$$

Proof. We have

$$\langle\langle \varphi, \psi \rangle\rangle = \tau[\psi] \ [\varphi] = \tau(\tau[\varphi] \ [\psi]) = \tau\langle\langle \psi, \varphi \rangle\rangle$$

and

$$\langle\langle X \cdot \varphi, \psi \rangle\rangle = \tau[\psi] \ [X][\varphi] = \tau([X][\psi])[\varphi] = \langle\langle \varphi, X \cdot \psi \rangle\rangle$$

where $[\varphi]$, $[\psi]$ and $[X] \ \in Cl_n$ represent φ, ψ and X in some given frame $\tilde{s} \in \tilde{Q}$. □

Lemma 2.2. *The connection ∇ is compatible with the product $\langle\langle .,. \rangle\rangle$:*

$$\partial_X \langle\langle \varphi, \varphi' \rangle\rangle = \langle\langle \nabla_X \varphi, \varphi' \rangle\rangle + \langle\langle \varphi, \nabla_X \varphi' \rangle\rangle$$

for all $\varphi, \varphi' \in \Gamma(\Sigma)$ and $X \in \Gamma(TM)$.

Proof. If $\varphi = [\tilde{s}, [\varphi]]$ is a section of $\Sigma = \tilde{Q} \times_\rho Cl_n$, we have

$$\nabla_X \varphi = [\tilde{s}, \partial_X[\varphi] + \rho_*(\tilde{s}^*\alpha(X))([\varphi])], \qquad \forall X \in \ TM,$$

where ρ is the representation (1) and α is the connection form on $\tilde{Q}$; the term $\rho_*(\tilde{s}^*\alpha(X))$ is an endomorphism of Cl_n given by the multiplication on the left by an element belonging to $\Lambda^2\mathbb{R}^n \subset Cl_n$, still denoted by $\rho_*(\tilde{s}^*\alpha(X))$. Such an element satisfies

$$\tau\left(\rho_*(\tilde{s}^*\alpha(X))\right) = -\rho_*(\tilde{s}^*\alpha(X)),$$

and we have

$$\begin{array}{rcl} \langle\langle \nabla_X \varphi, \varphi' \rangle\rangle + \langle\langle \varphi, \nabla_X \varphi' \rangle\rangle & = & \tau\{[\varphi']\}\left(\partial_X[\varphi] + \rho_*(\tilde{s}^*\alpha(X))[\varphi]\right) \\ & & +\tau\left\{\partial_X[\varphi'] + \rho_*(\tilde{s}^*\alpha(X))[\varphi']\right\}[\varphi] \\ & = & \tau\{[\varphi']\}\partial_X[\varphi] + \tau\left\{\partial_X[\varphi']\right\}[\varphi] \\ & = & \partial_X\langle\langle \varphi, \varphi' \rangle\rangle. \end{array}$$

□

3. The spin geometry of a submanifold in $\mathbb{R}^n$

We keep the notation of the previous section, assuming moreover here that M is a submanifold of $\mathbb{R}^n$ and that $E \to M$ is its normal bundle. Let as before $\tilde{Q}_M \overset{2:1}{\to} Q_M$ be a spin structure of M. Our goal is to construct $\tilde{Q}$ such that we obtain an identification

$$\Sigma = \tilde{Q} \times_\rho Cl_n \simeq M \times Cl_n.$$

Although this type of result is used in several places in the literature, we could not find a complete statement or proof. Therefore we will give a detailed proof, which we believe may be useful in its own right.
Let $(e_1, \dots e_p)$ resp. $(e_{p+1}, \dots e_{p+q})$ be orthonormal frames of TM resp. E and $Q_{\mathbb{R}^n}$ the bundle of positively oriented orthonormal frames of $\mathbb{R}^n$. We can consider the map

$$\begin{array}{rcl} \iota : Q_M \times_M Q_E & \to & Q_{\mathbb{R}^n} \\ ((e_1, \dots e_p), (e_{p+1}, \dots e_{p+q})) & \mapsto & (e_1, e_2, \dots e_{p+q}) \end{array}$$

given by the concatenation of frames.
The map

$$\tilde{Q}_M \times_M Q_E \to Q_M \times_M Q_E$$

is obviously a two-to-one covering of $Q_M \times_M Q_E$.
Let now $\tilde{Q}_{\mathbb{R}^n} \overset{2:1}{\to} Q_{\mathbb{R}^n}$ be the (unique) spin structure of $\mathbb{R}^n$.
Then the bundle

$$\tilde{Q} := (\tilde{Q}_M \times_M Q_E) \times_{Q_{\mathbb{R}^n}} \tilde{Q}_{\mathbb{R}^n}$$

is a $\mathrm{Spin}(p) \times \mathrm{Spin}(q)$-principal bundle over M and a four-to-one covering of $Q_M \times_M Q_E$. Observe that $\tilde{Q} = \tilde{Q}_M \times_M \tilde{Q}_E$, where $\tilde{Q}_E := \tilde{Q}/\mathrm{Spin}(p)$ (and the projection $\tilde{Q}/\mathrm{Spin}(q) \to \tilde{Q}_M$ is a map of principal $\mathrm{Spin}(p)$-bundles, hence an isomorphism). Moreover, $\tilde{Q}_E$ is a spin structure on E, canonically associated to the spin structures on M and $\mathbb{R}^n$.

Claim. *Consider the bundle*

$$\tilde{Q} \times_c \mathrm{Spin}(p+q) := (\tilde{Q} \times \mathrm{Spin}(p+q))/(\mathrm{Spin}(p) \times \mathrm{Spin}(q)),$$

where

$$c : Spin(p) \times Spin(q) \to Spin(p+q)$$

is the map corresponding to the isomorphism of Clifford algebras $Cl_p \hat{\otimes} Cl_q \cong Cl_{p+q}$. *Then there is a canonical isomorphism of* $\mathrm{Spin}(n)$*-principal bundles,*

$$\tilde{Q} \times_c \mathrm{Spin}(p+q) \cong \tilde{Q}_{\mathbb{R}^n}|_M.$$

Proof. Consider the projection $\pi : \tilde{Q} \to \tilde{Q}_{\mathbb{R}^n}$ to the last factor. Then the map

$$\begin{array}{rcl} \tilde{\pi} : \tilde{Q} \times \mathrm{Spin}(p+q) & \to & \tilde{Q}_{\mathbb{R}^n} \\ (\tilde{q}, s) & \mapsto & s\pi(\tilde{q}) \end{array}$$

satisfies $\tilde{\pi}(s_0\tilde{q}, ss_0^{-1}) = \tilde{\pi}(\tilde{q}, s)$ for any $s_0 \in \mathrm{Spin}(p) \times \mathrm{Spin}(q)$, so $\tilde{\pi}$ descends to a map $\tilde{Q} \times_c \mathrm{Spin}(p+q) \to \tilde{Q}_{\mathbb{R}^n}$. The source is clearly a $\mathrm{Spin}(p+q)$-principal bundle on M, as is the target, and the map is $\mathrm{Spin}(p+q)$-equivariant and the identity over M. Hence it is an isomorphism of principal bundles. □

Corollary 1. *If now $\rho : Spin(p) \times Spin(q) \to GL(Cl_n)$ is the map given by $\tilde{\rho} \circ c$, where $\tilde{\rho} : \mathrm{Spin}(n) \to GL(Cl_n)$ is the representation induced by left multiplication, we get*

$$\tilde{Q} \times_\rho Cl_n \cong \tilde{Q}_{\mathbb{R}^n}|_M \times_{\tilde{\rho}} Cl_n \cong M \times Cl_n.$$

Proof. The first isomorphism is immediate from the claim, and the second follows since $\tilde{Q}_{\mathbb{R}^n}$ is trivial. □

Two connections are thus defined on Σ, the connection ∇ introduced in the previous section and the trivial connection ∂; they satisfy the following Gauss formula:

$$\partial_X \varphi = \nabla_X \varphi + \frac{1}{2} \sum_{j=1}^{p} e_j \cdot B(X, e_j) \cdot \varphi \tag{7}$$

for all $\varphi \in \Gamma(\Sigma)$ and all $X \in \Gamma(TM)$, where $B : TM \times TM \to E$ is the second fundamental form of M into $\mathbb{R}^n$. We refer to [2] for the proof (in a slightly different context).

4. Spinorial representation of submanifolds in $\mathbb{R}^n$

We state the main result of the paper. Let M be a p-dimensional Riemannian manifold and $E \to M$ a bundle of rank q, with a fiber metric and a compatible connection; we assume that E and TM are oriented and spin, with given spin structures. We keep the notation of Section 2.

Theorem 1. *We moreover assume that M is simply connected, and suppose that $B : TM \times TM \to E$ is bilinear and symmetric. The following statements are equivalent:*

(1) There exists a section $\varphi \in \Gamma(U\Sigma)$ such that

$$\nabla_X \varphi = -\frac{1}{2} \sum_{j=1}^{p} e_j \cdot B(X, e_j) \cdot \varphi \tag{8}$$

for all $X \in TM$.

(2) There exists an isometric immersion $F : M \to \mathbb{R}^n$ with normal bundle E and second fundamental form B.
Moreover, $F = \int \xi$ where ξ is the $\mathbb{R}^n-$valued 1-form defined by

$$\xi(X) := \langle\langle X \cdot \varphi, \varphi \rangle\rangle \tag{9}$$

for all $X \in TM$.

The representation formula (9) generalizes the classical Weierstrass representation formula; see Section 7.

Remark 2. *Taking the trace of (8) we get*

$$D\varphi = \frac{p}{2} \vec{H} \cdot \varphi$$

where $D\varphi = \sum_{j=1}^{p} e_j \cdot \nabla_{e_j} \varphi$, and $\vec{H} = \frac{1}{p} \sum_{j=1}^{p} B(e_j, e_j)$ is the mean curvature vector of M in $\mathbb{R}^n$. This Dirac equation is known to be equivalent to (8) only for $p = 2$ or 3 (see e.g. [8, 14, 15, 5]).

Proof. 2) $\Rightarrow$ 1) is a direct consequence of the Gauss formula (7) for a submanifold of $\mathbb{R}^n$: the restriction of parallel spinor fields of the ambient space $\mathbb{R}^n$ to the submanifold M are obviously solutions of equation (8) (recall that in the paper the spinors are constructed with the whole Clifford algebra). The immersion takes the form $F = \int \xi$ where ξ is given by (9) for the special choice $\varphi = 1_{Cl_n|M}$, since, in that case, for all $X \in TM$,

$$\xi(X) = \tau[\varphi]\ [X]\ [\varphi] = [X] \simeq X$$

where $[\varphi] = \pm 1_{Cl_n}$ and $[X] \in \mathbb{R}^n$ represent φ and X in one of the two spinorial frames of $\mathbb{R}^n$ above the canonical basis.
1) $\Rightarrow$ 2): We will prove that the 1-form ξ defined in (9) gives us indeed an immersion preserving the metric, the second fundamental form and the normal connection. This follows directly from Propositions 4.1 and 4.2 below. □

Proposition 4.1. *Assume that $\varphi \in \Gamma(U\Sigma)$ is a solution of (8) and define ξ by (9). Then*

(1) ξ takes its values in $\mathbb{R}^n \subset Cl_n$;
(2) ξ is a closed 1-form: $d\xi = 0$.

Proof. 1- By the very definition of ξ, we have

$$\xi(X) = \tau[\varphi][X][\varphi]$$

for all $X \in TM$, where $[X]$ and $[\varphi]$ represent X and φ in a given frame $\tilde{s}$ of $\tilde{Q}$. Since $[X]$ belongs to $\mathbb{R}^n \subset Cl_n$ and $[\varphi]$ is an element of $Spin(n)$, $\xi(X)$ belongs to $\mathbb{R}^n$.
2- We compute, for $X, Y \in \Gamma(TM)$ such that $\nabla X = \nabla Y = 0$ at some point x_0,

$$\begin{aligned} \partial_X\ \xi(Y) &= \langle\langle Y \cdot \nabla_X \varphi, \varphi\rangle\rangle + \langle\langle Y \cdot \varphi, \nabla_X \varphi\rangle\rangle \\ &= (id + \tau)\langle\langle Y \cdot \varphi, \nabla_X \varphi\rangle\rangle \\ &= (id + \tau)\langle\langle \varphi, -\frac{1}{2}\sum_{j=1}^{p} Y \cdot e_j \cdot B(X, e_j) \cdot \varphi\rangle\rangle. \end{aligned}$$

Hence

$$\begin{aligned} d\xi(X, Y) &= \partial_X\ \xi(Y) - \partial_Y\ \xi(X) \\ &= (id + \tau)\langle\langle \varphi, \mathcal{C} \cdot \varphi\rangle\rangle \end{aligned}$$

with

$$\mathcal{C} := -\frac{1}{2}\sum_{j=1}^{p} \{Y \cdot e_j \cdot B(X, e_j) - X \cdot e_j \cdot B(Y, e_j)\}.$$

Now, for $X = \sum_{1 \le k \le p} x_k e_k$ and $Y = \sum_{1 \le k \le p} y_k e_k$,

$$\sum_{j=1}^{p} X \cdot e_j \cdot B(Y, e_j) = -B(Y, X) + \sum_{j=1}^{p}\sum_{k \neq j} x_k e_k \cdot e_j \cdot B(Y, e_j)$$

and

$$\sum_{j=1}^{p} Y \cdot e_j \cdot B(X, e_j) = -B(X, Y) + \sum_{j=1}^{p}\sum_{k \neq j} y_k e_k \cdot e_j \cdot B(X, e_j),$$

which yields the formula

$$\mathcal{C} = -\frac{1}{2}\sum_{j=1}^{p}\sum_{k\neq j} e_k\cdot e_j\cdot(y_k B(X,e_j) - x_k B(Y,e_j)).$$

This shows that $\tau[\mathcal{C}] = -[\mathcal{C}]$, which implies that

$$\tau\langle\langle\varphi,\mathcal{C}\cdot\varphi\rangle\rangle = \tau(\tau[\varphi]\tau[\mathcal{C}][\varphi]) = -\tau[\varphi]\tau[\mathcal{C}][\varphi] = -\langle\langle\varphi,\mathcal{C}\cdot\varphi\rangle\rangle.$$

Thus

$$d\xi(X,Y) = (id+\tau)\langle\langle\varphi,\mathcal{C}\cdot\varphi\rangle\rangle = 0.$$

□

We keep the notation of Proposition 4.1, and moreover assume that M is simply connected; since ξ is closed by Proposition 4.1 we can consider

$$F : M \to \mathbb{R}^n$$

such that $dF = \xi$. The next proposition follows from the properties of the Clifford product:

Proposition 4.2. *1. The map $F : M \to \mathbb{R}^n$ is an isometry.*
2. The map

$$\begin{array}{rccc} \Phi_E : & E & \to & M\times\mathbb{R}^n \\ & X\in E_m & \mapsto & (F(m),\xi(X)) \end{array}$$

is an isometry between E and the normal bundle of $F(M)$ into $\mathbb{R}^n$, preserving connections and second fundamental forms. Here, for $X\in E$, $\xi(X)$ still stands for the quantity $\langle\langle X\cdot\varphi,\varphi\rangle\rangle$.

Proof. For $X,Y\in\Gamma(TM\oplus E)$, we have

$$\begin{aligned}\langle\xi(X),\xi(Y)\rangle &= -\frac{1}{2}\left(\xi(X)\xi(Y)+\xi(Y)\xi(X)\right)\\ &= -\frac{1}{2}\left(\tau[\varphi][X][\varphi]\tau[\varphi][Y][\varphi]+\tau[\varphi][Y][\varphi]\tau[\varphi][X][\varphi]\right)\\ &= -\frac{1}{2}\tau[\varphi]\left([X][Y]+[Y][X]\right)[\varphi]\\ &= \langle X,Y\rangle,\end{aligned}$$

since $[X][Y]+[Y][X] = -2\langle[X],[Y]\rangle = -2\langle X,Y\rangle$. This implies that F is an isometry, and that Φ_E is a bundle map between E and the normal bundle of $F(M)$ into $\mathbb{R}^n$ which preserves the metrics of the fibers. Let us denote by B_F and ∇'^F the second fundamental form and the normal connection of the immersion F; we want to show that

$$\xi(B(X,Y)) = B_F(\xi(X),\xi(Y)) \quad\text{and}\quad \xi(\nabla'_X N) = \nabla'^F_{\xi(X)}\xi(N) \tag{10}$$

for $X,Y\in\Gamma(TM)$ and $N\in\Gamma(E)$. First,

$$B_F(\xi(X),\xi(Y)) = \{\partial_X\ \xi(Y)\}^N$$

where the superscript N means that we consider the component of the vector which is normal to the immersion. We showed in the proof of Proposition 4.1 that fixing

a point $x_0 \in M$, and assuming that $\nabla Y = 0$ at x_0 we have

$$\partial_X\ \xi(Y) = -\frac{1}{2}(id+\tau)\langle\langle \varphi, Y \cdot \sum_{j=1}^{p} e_j \cdot B(X,e_j)\cdot \varphi\rangle\rangle.$$

and that moreover

$$Y \cdot \sum_{j=1}^{p} e_j \cdot B(X,e_j) = -B(X,Y) + \mathcal{D}$$

where $\mathcal{D}$ is a term which satisfies $\tau\mathcal{D} = -\mathcal{D}$. This implies that

$$\begin{aligned} B_F(\xi(X),\xi(Y)) &= \left\{\frac{1}{2}(id+\tau)\langle\langle \varphi, B(X,Y)\cdot\varphi\rangle\rangle\right\}^N \\ &= \xi(B(X,Y)), \end{aligned}$$

where the last equality holds since $\tau[B(X,Y)] = [B(X,Y)]$ and $\xi(B(X,Y))$ is normal to the immersion. We finally show the second identity in (10): we have

$$\begin{aligned} \nabla'^F_{\xi(X)}\xi(N) &= (\partial_X\ \xi(N))^N \\ &= \langle\langle \nabla'_X N\cdot\varphi,\varphi\rangle\rangle^N + \langle\langle N\cdot\nabla_X\varphi,\varphi\rangle\rangle^N + \langle\langle N\cdot\varphi,\nabla_X\varphi\rangle\rangle^N. \end{aligned}$$

The first term in the right hand side is $\xi(\nabla'_X N)$, and we only need to show that

$$\langle\langle N\cdot\nabla_X\varphi,\varphi\rangle\rangle^N + \langle\langle N\cdot\varphi,\nabla_X\varphi\rangle\rangle^N = 0. \tag{11}$$

We have

$$\begin{aligned} \langle\langle N\cdot\nabla_X\varphi,\varphi\rangle\rangle + \langle\langle N\cdot\varphi,\nabla_X\varphi\rangle\rangle &= (id+\tau)\langle\langle N\cdot\nabla_X\varphi,\varphi\rangle\rangle \\ &= \frac{1}{2}(id+\tau)\langle\langle \sum_{j=1}^{p} e_j\cdot N\cdot B(X,e_j)\cdot\varphi,\varphi\rangle\rangle, \end{aligned}$$

and the identity (11) will thus be proved if we show that this vector is tangent to the immersion. We have

$$\begin{aligned} \sum_{j=1}^{p} e_j\cdot N\cdot B(X,e_j) &= -\sum_{j=1}^{p} e_j\cdot B(X,e_j)\cdot N - 2\sum_{j=1}^{p}\langle B(X,e_j),N\rangle\ e_j \\ &= -\sum_{j=1}^{p} B(X,e_j)\cdot N\cdot e_j - 2B^*(X,N) \\ &= -\tau\left(\sum_{j=1}^{p} e_j\cdot N\cdot B(X,e_j)\right) - 2B^*(X,N) \end{aligned}$$

where we have set $B^*(X,N) = \sum_{j=1}^{p}\langle B(X,e_j),N\rangle\ e_j$; thus

$$\frac{1}{2}(id+\tau)\langle\langle \sum_{j=1}^{p} e_j\cdot N\cdot B(X,e_j)\cdot\varphi,\varphi\rangle\rangle = -\langle\langle B^*(X,N)\cdot\varphi,\varphi\rangle\rangle,$$

which is a vector tangent to the immersion since $B^*(X,N)$ belongs to TM; (11) follows, which finishes the proof. □

Remark 3. *The group $Spin(n)$ naturally acts on $U\Sigma$ by multiplication on the right, and if $\varphi \in \Gamma(U\Sigma)$ is a solution of (8) and g_0 belongs to $Spin(n)$, then $\varphi \cdot g_0$ is also a solution of (8); in fact, $\varphi \cdot g_0$ defines an immersion which is congruent to the immersion defined by φ : indeed, for all $X \in \Gamma(TM)$,*

$$\begin{aligned} \xi_{\varphi\cdot g_0}(X) &= \tau[\varphi \cdot g_0][X][\varphi \cdot g_0] \\ &= \tau(g_0)\tau[\varphi][X][\varphi]g_0 \\ &= \tau(g_0)\xi_\varphi(X)g_0, \end{aligned}$$

i.e.

$$\xi_{\varphi.g_0} = Ad({g_0}^{-1}) \circ \xi_\varphi;$$

the linear part of the rigid motion between the immersions defined by φ and $\varphi \cdot g_0$ is thus $Ad({g_0}^{-1}) \in SO(n)$.

5. An application: the Fundamental Theorem of Submanifold Theory

We first recall the equations of Gauss, Ricci and Codazzi for the symmetric bilinear form B. Let R^T and R^N stand respectively for the curvature tensors of the connections on TM and on E. Further, let $B^* : TM \times E \to TM$ be the bilinear map such that for all $X, Y \in \Gamma(TM)$ and $N \in \Gamma(E)$

$$\langle B(X,Y), N\rangle = \langle Y, B^*(X,N)\rangle,$$

then we have, for all $X, Y, Z \in \Gamma(TM)$ and $N \in \Gamma(E)$,

(1) the Gauss equation

$$R^T(X,Y)Z = B^*(X, B(Y,Z)) - B^*(Y, B(X,Z)),$$

(2) the Ricci equation

$$R^N(X,Y)N = B(X, B^*(Y,N)) - B(Y, B^*(X,N)),$$

(3) the Codazzi equation

$$\tilde{\nabla}_X B(Y,Z) = \tilde{\nabla}_Y B(X,Z);$$

in the last equation, $\tilde{\nabla}$ denotes the natural connection on $T^*M \otimes T^*M \otimes E$.

Proposition 5.1. *The equations of Gauss, Ricci and Codazzi on B are the integrability conditions of (8).*

Proof. We assume that $\varphi \in \Gamma(U\Sigma)$ is a solution of (8) and compute the curvature

$$R(X,Y)\varphi = \nabla_X\nabla_Y\varphi - \nabla_Y\nabla_X\varphi - \nabla_{[X,Y]}\varphi.$$

We fix a point $x_0 \in M$, and assume that $\nabla X = \nabla Y = 0$ at x_0. We have

$$\begin{aligned} \nabla_X\nabla_Y\varphi &= -\frac{1}{2}\sum_{j=1}^p e_j \cdot \Big(\tilde{\nabla}_X B(Y,e_j)\cdot\varphi + B(Y,e_j)\cdot\nabla_X\varphi\Big) \\ &= -\frac{1}{2}\sum_{j=1}^p e_j \cdot \tilde{\nabla}_X B(Y,e_j)\cdot\varphi - \frac{1}{4}\sum_{j,k=1}^p e_j\cdot e_k \cdot B(Y,e_j)\cdot B(X,e_k). \end{aligned}$$

Thus

$$\begin{aligned} R(X,Y)\varphi &= -\frac{1}{2}\sum_{j=1}^{p} e_j\cdot\left(\tilde{\nabla}_X B(Y,e_j)-\tilde{\nabla}_Y B(X,e_j)\right)\cdot\varphi \\ &\quad +\frac{1}{4}\sum_{j\neq k} e_j\cdot e_k\cdot\left(B(X,e_j)\cdot B(Y,e_k)-B(Y,e_j)\cdot B(X,e_k)\right)\cdot\varphi \\ &\quad -\frac{1}{4}\sum_{j=1}^{p}\left(B(X,e_j)\cdot B(Y,e_j)-B(Y,e_j)\cdot B(X,e_j)\right)\cdot\varphi. \end{aligned} \tag{12}$$

We compute the last two terms in the following lemma:

Lemma 5.2. *Let us set*

$$\mathcal{A} := \frac{1}{4}\sum_{j\neq k} e_j\cdot e_k\cdot\left(B(X,e_j)\cdot B(Y,e_k)-B(Y,e_j)\cdot B(X,e_k)\right)$$

and

$$\mathcal{B} := -\frac{1}{4}\sum_{j=1}^{p}\left(B(X,e_j)\cdot B(Y,e_j)-B(Y,e_j)\cdot B(X,e_j)\right).$$

We have

$$\mathcal{A} = \frac{1}{2}\sum_{j<k}\left\{\langle B^*(X,B(Y,e_j)),e_k\rangle-\langle B^*(Y,B(X,e_j)),e_k\rangle\right\} e_j\cdot e_k$$

and

$$\mathcal{B} = \frac{1}{2}\sum_{k<l}\langle B(X,B^*(Y,n_k))-B(Y,B^*(X,n_k)),n_l\rangle\, n_k\cdot n_l.$$

Here $e_1,\dots,e_p$ and $n_1,\dots,n_q$ are orthonormal basis of $T_{x_o}M$ and E_{x_o}, respectively.

Proof. For the computation of $\mathcal{A}$, we notice that

$$\sum_{j\neq k} e_j\cdot e_k\cdot B(Y,e_j)\cdot B(X,e_k) = -\sum_{j\neq k} e_j\cdot e_k\cdot B(Y,e_k)\cdot B(X,e_j),$$

and get

$$\begin{aligned} \mathcal{A} &= \frac{1}{4}\sum_{j\neq k} e_j\cdot e_k\cdot\left(B(X,e_j)\cdot B(Y,e_k)+B(Y,e_k)\cdot B(X,e_j)\right) \\ &= -\frac{1}{2}\sum_{j\neq k}\langle B(X,e_j),B(Y,e_k)\rangle e_j\cdot e_k \\ &= -\frac{1}{2}\sum_{j<k}\left\{\langle B(X,e_j),B(Y,e_k)\rangle-\langle B(Y,e_j),B(X,e_k)\rangle\right\} e_j\cdot e_k \\ &= -\frac{1}{2}\sum_{j<k}\left\{\langle B^*(Y,B(X,e_j)),e_k\rangle-\langle B^*(X,B(Y,e_j)),e_k\rangle\right\} e_j\cdot e_k. \end{aligned}$$

For the computation of $\mathcal{B}$, we write

$$B(Y,e_j) = \sum_k \langle B(Y,e_j),n_k\rangle n_k \quad \text{and} \quad B(X,e_j) = \sum_l \langle B(X,e_j),n_l\rangle n_l$$

and get

$$\begin{aligned}\sum_j B(Y,e_j)\cdot B(X,e_j) &= \sum_{kl}\sum_j \langle B(Y,e_j),n_k\rangle\langle B(X,e_j),n_l\rangle n_k\cdot n_l\\ &= \sum_{kl}\sum_j \langle e_j,B^*(Y,n_k)\rangle\langle e_j,B^*(X,n_l)\rangle n_k\cdot n_l\\ &= \sum_{kl}\langle B^*(Y,n_k),B^*(X,n_l)\rangle n_k\cdot n_l\\ &= \sum_{kl}\langle B(X,B^*(Y,n_k)),n_l\rangle n_k\cdot n_l;\end{aligned}$$

thus

$$\begin{aligned}\mathcal{B} &= \frac{1}{4}\sum_{kl}\left\langle B(X,B^*(Y,n_k))-B(Y,B^*(X,n_k)),n_l\right\rangle n_k\cdot n_l\\ &= \frac{1}{2}\sum_{k<l}\left\langle B(X,B^*(Y,n_k))-B(Y,B^*(X,n_k)),n_l\right\rangle n_k\cdot n_l.\end{aligned}$$

□

On the other hand, the curvature of the spinorial connection is given by

$$\begin{aligned}R(X,Y)\varphi &= \frac{1}{2}\Bigg(\sum_{1\le j<k\le p}\langle R^T(X,Y)(e_j),e_k\rangle\ e_j\cdot e_k\\ &\quad+\sum_{1\le k<l\le q}\langle R^N(X,Y)(n_k),n_l\rangle\ n_k\cdot n_l\Bigg)\cdot\varphi.\end{aligned}\tag{13}$$

We now compare the expressions (12) and (13) using the calculations in Lemma 5.2: since in a given frame $\tilde{s}$ belonging to $\tilde{Q}$, φ is represented by an element which is invertible in Cl_n (it is in fact represented by an element belonging to $Spin(n)$), we may identify the coefficients and get

$$\langle R^T(X,Y)(ej),e_k\rangle = \langle B^*(X,B(Y,e_j)),e_k\rangle - \langle B^*(Y,B(X,e_j)),e_k\rangle,$$

$$\langle R^N(X,Y)(n_k),n_l\rangle = \langle B(X,B^*(Y,n_k)),n_l\rangle - \langle B(Y,B^*(X,n_k)),n_l\rangle$$

and

$$\tilde{\nabla}_X B(Y,e_j) - \tilde{\nabla}_Y B(X,e_j) = 0$$

for all the indices. These equations are the equations of Gauss, Ricci and Codazzi.

We finally show that the equations of Gauss, Codazzi and Ricci are also sufficient to get a solution of (8): by the computation above, the connection on Σ defined by

$$\nabla'_X\varphi := \nabla_X\varphi + \frac{1}{2}\sum_{j=1}^p e_j\cdot B(X,e_j)\cdot\varphi\tag{14}$$

for all $\varphi\in\Gamma(\Sigma)$ and $X\in\Gamma(TM)$ is then a flat connection. Moreover, this connection may be regarded as a connection on the principal bundle $U\Sigma$ (with the group $Spin(n)$ acting from the right): indeed, ∇ defines such a connection (since it comes

from a connection on $\tilde{Q}$ and by (2)), and the right hand side term in (14) defines a linear map

$$\begin{array}{rcl} TM & \to & \chi_V^{inv}(U\Sigma) \\ X & \mapsto & \varphi \mapsto \frac{1}{2}\sum_{j=1}^{p} e_j \cdot B(X, e_j) \cdot \varphi \end{array}$$

from TM to the vector fields $\chi_V^{inv}(U\Sigma)$ on $U\Sigma$ which are vertical and invariant under the action of the group (these vector fields are indeed of the form $\varphi \mapsto \eta \cdot \varphi$, $\eta \in \Lambda^2(TM \oplus E) \subset Cl(TM \oplus E)$). Since a flat connection on a principal bundle admits a local parallel section, there exists a local section $\varphi \in \Gamma(U\Sigma)$ such that $\nabla'\varphi = 0$, and thus a solution of (8). □

As a consequence of Theorem 1 and Proposition 5.1 we therefore get immediately

Corollary 2 (Fundamental Theorem of Submanifold Theory)**.** *We keep the hypotheses and notation of Section 2, and moreover assume that M is simply connected and that $B : TM \times TM \to E$ is bilinear, symmetric and satisfies the equations of Gauss, Codazzi and Ricci. Then there exists an isometric immersion of M into $\mathbb{R}^n$ with normal bundle E and second fundamental form B. The immersion is unique up to a rigid motion in $\mathbb{R}^n$.*

Proof. As proved in Proposition 5.1, the equations of Gauss, Codazzi and Ricci are exactly the integrability conditions of (8). By Theorem 1, with a solution $\varphi \in \Gamma(U\Sigma)$ of equation (8) at hand, $F = \int \xi$, where ξ is the 1-form defined in (9) is the immersion. Finally, a solution of (8) is unique up to the multiplication on the right by an element of $Spin(n)$ (since this is a parallel section of the $Spin(n)$ principal bundle $U\Sigma$, see the proof of Proposition 5.1); the multiplication on the right of φ by an element of $Spin(n)$ and the adding of a constant vector in $\mathbb{R}^n$ in the last integration give an immersion which is congruent to the immersion defined by φ (see Remark 3). □

6. Relation to the Gauss map

We show here that the spinor field representing the immersion is an horizontal lift of the Gauss map. Let us consider the Grassmannian $Gr_{p,n} \subset \Lambda^p(\mathbb{R}^n)$ of the oriented p-dimensional linear spaces in $\mathbb{R}^n$. Using the natural isomorphism of vector spaces between the exterior algebra over $\mathbb{R}^n$ and Cl_n, it identifies with the set

$$\mathcal{Q}_o = \{e_1 \cdot e_2 \cdots e_p \in Cl_n,\ e_i \in \mathbb{R}^n,\ |e_i| = 1,\ e_i \perp e_j,\ i, j = 1, \ldots, p,\ i \neq j\}.$$

We recall that for an oriented p-dimensional submanifold $F : M \to \mathbb{R}^n$ the Gauss map is defined as the map which assigns each point $x \in M$ to the oriented tangent space $dF(T_xM)$ considered as a vector subspace of $\mathbb{R}^n$. It can hence be seen as the map into the Grassmannian

$$\begin{array}{rrcl} G: & M & \to & \mathcal{Q}_o \\ & x & \mapsto & dF(e_1) \cdot dF(e_2) \cdots dF(e_p), \end{array}$$

where $e_1, e_2, \ldots, e_p$ is a positively oriented orthonormal basis of T_xM.

We assume that the immersion F of M into $\mathbb{R}^n$ is given by a spinor field φ, as in Theorem 1.

Proposition 6.1. *The spinor field φ, which is a section of $U\Sigma$, is a lift of the Gauss map: the diagram*

$$\begin{tikzcd} & U\Sigma \arrow[d, "\chi"] \\ M \arrow[ur, bend left, "\varphi"] \arrow[r, "\overline{G}"'] & M \times \mathcal{Q}_o \end{tikzcd}$$

commutes, where $\overline{G}(x) = (x, G(x))$ and the projection $U\Sigma \to M \times \mathcal{Q}_o$ is given by

$$\chi : \quad \varphi \in U\Sigma_x \quad \mapsto \quad (x, \langle\langle \omega \cdot \varphi, \varphi \rangle\rangle), \tag{15}$$

where ω is the volume form in $Cl(T_xM)$ (the product of the elements of a positively oriented orthonormal basis of T_xM).

It is moreover parallel with respect to the connection

$$\nabla'_X \varphi := \nabla_X \varphi + \frac{1}{2} \sum_{j=1}^{p} e_j \cdot B(X, e_j) \cdot \varphi$$

on $U\Sigma$.

Proof. We first explain why the map χ as defined indeed has target $M \times \mathcal{Q}_o$. Consider the map

$$\begin{aligned} \Xi : \Sigma \times_M Cl(TM) &\to M \times Cl_n \\ (\psi, c) &\mapsto \langle\langle c \cdot \psi, \psi \rangle\rangle =: \Xi_\psi(c). \end{aligned}$$

Suppose $\psi \in U\Sigma$ and $c = e_1 \cdots e_k$ for k orthonormal vectors $e_1, \ldots, e_k \in T_xM$. Then, we can rewrite $\Xi_\psi(c) = \langle\langle c \cdot \psi, \psi \rangle\rangle$ in any spinorial frame at x as

$$\tau[\psi][e_1] \cdots [e_k][\psi] = (\tau[\psi][e_1][\psi])(\tau[\psi][e_2][\psi]) \cdots (\tau[\psi][e_k][\psi]). \tag{16}$$

The k vectors on the right-hand side are still orthonormal, so $\Xi_\psi(c)$ lies in the corresponding Grassmannian $Gr_{k,n}$. Consequently $\chi(\psi) = \Xi_\psi(\omega)$ lies in $M \times \mathcal{Q}_o$.

We next verify the formula for the Gauss map. Recall that the immersion is given by $F = \int \xi$, where ξ is the 1-form defined by $\xi(X) = \langle\langle X \cdot \varphi, \varphi \rangle\rangle$ for all $X \in TM$. Thus, $dF = \xi$. We fix a positively oriented and orthonormal frame $(e_1, \ldots, e_p)$ of TM, and a spinorial frame $\tilde{s} \in \tilde{Q}$ which is above $(e_1, \ldots, e_p)$. Then, $\omega = e_1 \cdots e_p$. In any spinorial frame, $\tau[\varphi][v][\varphi] = \xi(v)$ for all $v \in T_xM$. Therefore (16) yields that $\chi(\varphi) = \xi(e_1)\xi(e_2) \cdots \xi(e_p) = G(x)$. This proves the first part of the proposition.

Finally, φ is horizontal with respect to the connection ∇' since it is a solution of (8). □

7. Special cases: minimal surfaces, hypersurfaces, and surfaces in $\mathbb{R}^4$

7.1. **Minimal surfaces in $\mathbb{R}^n$.** If J denotes the natural complex structure on M, the 1-form

$$\tilde{\xi}(X) := \xi(X) - i\xi(JX), \qquad X \in TM,$$

is $\mathbb{C}$-linear, with values in the complexified Clifford algebra $\tilde{Cl}_n = Cl_n \oplus iCl_n$; in general

$$F = \int \Re e\, \tilde{\xi} = \int \Re e \left(\tilde{f}(z) dz \right)$$

where z is a complex parameter of M and $\tilde{f}$ is a smooth function. Note that $\tilde{\xi}$ and $\tilde{f}$ take in fact their values in $\mathbb{C}^n := \mathbb{R}^n \oplus i\mathbb{R}^n \subset \tilde{Cl}_n$.

Proposition 7.1. *The form $\tilde{\xi}$ is closed (and thus holomorphic) if and only if $\vec{H}=0$. In that case, we have*

$$F=\Re e\int \tilde{f}(z)dz$$

where $\tilde{f}$ is a holomorphic function.

Proof. We assume that (e_1,e_2) is a local orthonormal frame on M, positively oriented, such that $\nabla e_1=\nabla e_2=0$ at a point x_0. We thus have

$$d\tilde{\xi}(e_1,e_2)=\partial_{e_1}\left(\xi(e_2)+i\xi(e_1)\right)-\partial_{e_2}\left(\xi(e_1)-i\xi(e_2)\right).$$

Noticing that, for $j,k\in\{1,2\}$,

$$\begin{aligned}\partial_{e_j}\left(\xi(e_k)\right) &= \partial_{e_j}\langle\langle e_k\cdot\varphi,\varphi\rangle\rangle\\ &= \langle\langle e_k\cdot\nabla_{e_j}\varphi,\varphi\rangle\rangle+\langle\langle e_k\cdot\varphi,\nabla_{e_j}\varphi\rangle\rangle\\ &= (id+\tau)\langle\langle e_k\cdot\nabla_{e_j}\varphi,\varphi\rangle\rangle,\end{aligned}$$

we obtain

$$\begin{aligned}d\tilde{\xi}(e_1,e_2) &= i\ (id+\tau)\langle\langle e_1\cdot\nabla_{e_1}\varphi+e_2\cdot\nabla_{e_2}\varphi,\varphi\rangle\rangle\\ &\quad+(id+\tau)\langle\langle e_2\cdot\nabla_{e_1}\varphi-e_1\cdot\nabla_{e_2}\varphi,\varphi\rangle\rangle.\end{aligned}$$

The first term on the right hand side is

$$\begin{aligned}i\ (id+\tau)\langle\langle e_1\cdot\nabla_{e_1}\varphi+e_2\cdot\nabla_{e_2}\varphi,\varphi\rangle\rangle &= i\ (id+\tau)\langle\langle\vec{H}\cdot\varphi,\varphi\rangle\rangle\\ &= 2i\ \langle\langle\vec{H}\cdot\varphi,\varphi\rangle\rangle\end{aligned}$$

since, by (8),

$$D\varphi:=e_1\cdot\nabla_{e_1}\varphi+e_2\cdot\nabla_{e_2}\varphi=\vec{H}\cdot\varphi$$

and $\tau[\vec{H}]=[\vec{H}]$. The second term is

$$\begin{aligned}(id+\tau)\langle\langle e_2\cdot\nabla_{e_1}\varphi-e_1\cdot\nabla_{e_2}\varphi,\varphi\rangle\rangle &= -(id+\tau)\langle\langle e_1\cdot\nabla_{e_1}\varphi+e_2\cdot\nabla_{e_2}\varphi,e_1\cdot e_2\cdot\varphi\rangle\rangle\\ &= -(id+\tau)\langle\langle\vec{H}\cdot\varphi,e_1\cdot e_2\cdot\varphi\rangle\rangle\\ &= 0,\end{aligned}$$

using again that $D\varphi=\vec{H}\cdot\varphi$ and since $\tau\left([\vec{H}][e_1][e_2]\right)=-[\vec{H}][e_1][e_2]$. We thus obtain the formula

$$d\tilde{\xi}(e_1,e_2)=2i\ \langle\langle\vec{H}\cdot\varphi,\varphi\rangle\rangle$$

which may be written in the form

$$d\tilde{\xi}=-\mu^2\ \langle\langle\vec{H}\cdot\varphi,\varphi\rangle\rangle\ dz\wedge d\overline{z} \tag{17}$$

where μ is such that the metric is $\mu^2dzd\overline{z}$. This gives the first part of the lemma. Assuming that $\vec{H}=0$, the 1-form $\tilde{\xi}$ is closed, and the $\mathbb{C}^n-$valued function $\tilde{f}$ such that $\tilde{\xi}=\tilde{f}dz$ is holomorphic; the result follows. □

The aim now is to obtain explicit formulas in terms of holomorphic functions involving the components of the spinor field. We first note the following expression of $\tilde{f}$ in terms of the spinor field φ :

Lemma 7.2. *We have*

$$\tilde{f} = \mu \left\{ \tau[\varphi] e_1^o[\varphi] - i\tau[\varphi] e_2^o[\varphi] \right\}$$

where the real function μ is such that the metric is

$$\mu^2 (dx^2 + dy^2)$$

in $z = x+iy$, $[\varphi]$ represents the spinor field φ in a spinorial frame above $(\frac{1}{\mu}\partial_x, \frac{1}{\mu}\partial_y)$, and e_1^o, e_2^o are the first two vectors of the canonical basis of $\mathbb{R}^n \subset Cl_n$.

Proof. We have

$$\tilde{f} = \tilde{\xi}(\partial_x) = \tau[\varphi][\partial_x][\varphi] - i\tau[\varphi][\partial_y][\varphi]$$

and the result follows since $[\frac{1}{\mu}\partial_x] = e_1^o$ and $[\frac{1}{\mu}\partial_y] = e_2^o$ in such a spinorial frame. □

7.1.1. *Minimal surfaces in $\mathbb{R}^3$.* Assuming that $n = 3$ and $H = 0$, we easily get by a computation using Lemma 7.2 that

$$F = \int \Re e(\tilde{f}(z)dz) = \Re e(\int \tilde{f}(z)dz)$$

where $\tilde{f} = \left(i\frac{f}{2}(1+g^2), \frac{f}{2}(1-g^2), fg \right)$ with

$$f = 2\mu z_1^2, \qquad g = -i\frac{\overline{z_2}}{z_1};$$

the complex functions z_1, z_2 are the components of φ in a spinorial frame above $(\frac{1}{\mu}\partial_x, \frac{1}{\mu}\partial_y)$, and the functions f and g are holomorphic, since so are $\sqrt{\mu}z_1$ and $\sqrt{\mu}\overline{z_2}$ (this is a consequence of the Dirac equation $D\varphi = 0$, in $z = x + iy$). This is the classical Weierstrass representation of minimal surfaces in $\mathbb{R}^3$.

7.1.2. *Minimal surfaces in $\mathbb{R}^4$.* In the case of a surface in $\mathbb{R}^4$, we may also recover the explicit formulas of Konopelchenko and Landolfi [11] expressing a general immersion in terms of 4 complex functions, which are solutions of first order PDE's; the functions are holomorphic if $\vec{H} = 0$. We do not include the calculations, since the general representation in Theorem 1 easily reduces to the spinor representation given in [5] if $p = 2$ and $n = 4$ (see Section 7.3), and the equivalence of this representation with the Konopelchenko-Landolfi representation is proved in [21].

Remark 4. *For surfaces in $\mathbb{R}^n$, $n \geq 5$, it is still possible to obtain an explicit representation in terms of the components of the spinor field which represents the surface, with holomorphic datas if $\vec{H} = 0$, if the bundle E is assumed to be flat. We do not know if such a representation is possible without this additional assumption.*

7.2. **Hypersurfaces in $\mathbb{R}^n$.** We set $p = n - 1$, and assume that M is a p-dimensional Riemannian manifold and E is the trivial line bundle on M, oriented by a unit section $\nu \in \Gamma(E)$. We moreover suppose that M is simply connected and that $h : TM \times TM \to \mathbb{R}$ is a given symmetric bilinear form. According to Theorem 1, an isometric immersion of M into $\mathbb{R}^{p+1}$ with normal bundle E and second fundamental form $B = h\nu$ is equivalent to a section φ of $\Gamma(U\Sigma)$ solution of the Killing equation (8). Note that $Q_E \simeq M$ and the double covering

$$\tilde{Q}_E \to Q_E$$

is trivial, since M is assumed to be simply connected. Fixing a section $\tilde{s}_E$ of $\tilde{Q}_E$ we get an injective map

$$\begin{array}{rcl} \tilde{Q}_M & \to & \tilde{Q}_M \times_M \tilde{Q}_E =: \tilde{Q} \\ \tilde{s}_M & \mapsto & (\tilde{s}_M, \tilde{s}_E). \end{array}$$

Using

$$Cl_p \simeq Cl^0_{p+1} \subset Cl_{p+1}$$

(induced by the Clifford map $\mathbb{R}^p \to Cl_{p+1}$, $X \mapsto X \cdot e_{p+1}$), we deduce a bundle isomorphism

$$\begin{array}{rcll} \tilde{Q}_M \times_\rho Cl_p & \to & \tilde{Q} \times_\rho Cl^0_{p+1} & \subset \Sigma \\ \psi & \mapsto & \psi^*. & \end{array} \tag{18}$$

It satisfies the following properties: for all $X \in TM$ and $\psi \in \tilde{Q}_M \times_\rho Cl_p$,

$$(X \cdot_M \psi)^* = X \cdot \nu \cdot \psi^*$$

and

$$\nabla_X(\psi^*) = (\nabla_X \psi)^*.$$

The section $\varphi \in \Gamma(U\Sigma)$ solution of (8) thus identifies to a section ψ of $\tilde{Q}_M \times_\rho Cl_p$ solution of

$$\nabla_X \psi = -\frac{1}{2}\sum_{j=1}^{p} h(X, e_j) e_j \cdot_M \psi = -\frac{1}{2} T(X) \cdot_M \psi$$

for all $X \in TM$, where $T : TM \to TM$ is the symmetric operator associated to h. We deduce the following result:

Theorem 2. *Let $T : TM \to TM$ be a symmetric operator. The following two statements are equivalent:*

(1) there exists an isometric immersion of M into $\mathbb{R}^{p+1}$ with shape operator T;
(2) there exists a normalized spinor field $\psi \in \Gamma(\tilde{Q}_M \times_\rho Cl_p)$ solution of

$$\nabla_X \psi = -\frac{1}{2} T(X) \cdot_M \psi \tag{19}$$

for all $X \in TM$.

Here, a spinor field $\psi \in \Gamma(\tilde{Q}_M \times_\rho Cl_p)$ is said to be normalized if it is represented in some frame $\tilde{s} \in \tilde{Q}_M$ by an element $[\psi] \in Cl_p \simeq Cl^0_{p+1}$ belonging to $Spin(p+1)$.

We will see below explicit representation formulas in the cases of the dimensions 3 and 4.

7.2.1. *Surfaces in $\mathbb{R}^3$.* Since $Cl_2 \simeq \Sigma_2$ we have

$$\tilde{Q}_M \times_\rho Cl_2 \simeq \Sigma M,$$

and φ is equivalent to a normalized spinor field $\psi \in \Gamma(\Sigma M)$ solution of

$$\nabla_X \psi = -\frac{1}{2} T(X) \cdot_M \psi$$

for all $X \in TM$; this equation is also equivalent to the equation $D\psi = H\psi$. This is the result obtained by Friedrich in [8].

We now write the representation formula (9) using a special model for Cl_3, and indicate how to recover the representation formula given in [8]. We first consider the Clifford map

$$(x_1, x_2, x_3) \in \mathbb{R}^3 \mapsto \begin{pmatrix} x & 0 \\ 0 & -x \end{pmatrix} \in \mathbb{H}(2),$$

where $x = -ix_3 + j(x_1 + ix_2)$, which identifies Cl_3 to the set

$$\left\{ \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix},\ x, y \in \mathbb{H} \right\}$$

and $\mathbb{R}^3 \subset Cl_3$ to the set of the imaginary quaternions; we also consider the ideal of Cl_3

$$\Sigma_3 = \left\{ \begin{pmatrix} y & 0 \\ 0 & 0 \end{pmatrix},\ y \in \mathbb{H} \right\} \subset Cl_3, \tag{20}$$

which is a model of the spin representation. Now φ, section of $U\Sigma = \tilde{Q} \times_\rho Spin(3)$, is equivalent to a unit spinor field $\varphi' \in \Gamma(\tilde{Q} \times_\rho \Sigma_3)$ (obtained by projection) and a direct computation yields

$$\langle\langle X \cdot \varphi, \varphi \rangle\rangle = i\, \mathcal{I}m\, \langle X \cdot \varphi', \varphi' \rangle + j \langle X \cdot \varphi', \alpha(\varphi') \rangle \tag{21}$$

for all $X \in TM$, where the brackets $\langle ., . \rangle$ stand for the natural hermitian product on Σ_3 and $\alpha : \Sigma_3 \to \Sigma_3$ is the natural quaternionic structure. The representation formula given by the right hand side term of (21) appears in [8]. Finally, the identification (18) for the dimension $p = 2$

$$\begin{array}{rcll} \tilde{Q}_M \times_\rho Cl_2 & \to & \tilde{Q} \times_\rho Cl_3^0 & \subset \Sigma \\ \psi & \mapsto & \psi^* & \end{array}$$

identifies $\varphi \in \Gamma(U\Sigma)$ to a unit spinor field $\psi \in \Gamma(\Sigma M)$, and it may be proved by a computation that

$$\langle\langle X \cdot \varphi, \varphi \rangle\rangle = i2\mathcal{R}e \langle X \cdot \psi^+, \psi^- \rangle + j \left(\langle X \cdot \psi^+, \alpha(\psi^+) \rangle - \langle X \cdot \psi^-, \alpha(\psi^-) \rangle \right)$$

where the brackets $\langle ., . \rangle$ stand here for the natural hermitian product on Σ_2 and $\alpha : \Sigma_2 \to \Sigma_2$ is the natural quaternionic structure; this is the explicit formula of the immersion in terms of ψ given in [8].

7.2.2. *Hypersurfaces in* $\mathbb{R}^4$. Since $Cl_3 \simeq \Sigma_3 \oplus \Sigma_3'$ where Σ_3 and Σ_3' are the two (non-equivalent) irreducible representations of Cl_3, we get two unit spinor fields $\psi_1 \in \Gamma(\Sigma M)$, $\psi_2 \in \Gamma(\Sigma' M)$ solutions of (19). Noting finally that there is a natural identification

$$i : \Sigma' M \to \Sigma M$$

satisfying

$$i(X \cdot \psi) = -X \cdot i(\psi)$$

for all $X \in TM$ and $\psi \in \Sigma' M$, the spinor fields ψ_1 and $i(\psi_2) \in \Gamma(\Sigma M)$ satisfy

$$\nabla_X \psi_1 = -\frac{1}{2} T(X) \cdot_M \psi_1 \quad \text{and} \quad \nabla_X i(\psi_2) = \frac{1}{2} T(X) \cdot_M i(\psi_2). \tag{22}$$

We thus recover a result of [14]: the immersion is equivalent to two spinor fields on the hypersurface which are solutions of (22). We may also obtain a new explicit representation formula. On one hand, we note that

$$\langle\langle X\cdot\varphi,\varphi\rangle\rangle = \begin{pmatrix} 0 & \overline{\xi_1}x\xi_2 \\ \overline{\xi_2}x\xi_1 & 0 \end{pmatrix} \tag{23}$$

in Cl_4^0, where $\varphi\in\Gamma(U\Sigma)$ and $X\in TM$ are respectively represented in Cl_4^0 by

$$\begin{pmatrix} \xi_1 & 0 \\ 0 & \xi_2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & x \\ x & 0 \end{pmatrix},$$

with $\xi_1,\xi_2\in\mathbb{H}$ and $x\in\Im m\,\mathbb{H}$. On the other hand, Σ_3 naturally identifies to $\mathbb{H}$ (see (20)) and the bilinear map

$$\begin{array}{rcl} \Sigma_3\times\Sigma_3 & \to & \mathbb{H} \\ (\xi,\xi') & \mapsto & \overline{\xi'}\xi \end{array}$$

induces a pairing

$$\langle\langle .,.\rangle\rangle_{\Sigma M}: \qquad \Sigma M\times\Sigma M\to\mathbb{H}$$

on $\Sigma M=\tilde{Q}_M\times_\rho\Sigma_3$. If

$$\psi=\psi_1+\psi_2, \qquad \psi_1\in\Sigma M,\ \psi_2\in\Sigma' M$$

is such that $\varphi=\psi^*$ (by (18), with $p=3$), the spinor fields ψ_1 and $i(\psi_2)\in\Sigma M$ are respectively represented by ξ_1 and ξ_2, and we readily get

$$\langle\langle X\cdot_M i(\psi_2),\psi_1\rangle\rangle_{\Sigma M} = \overline{\xi_1}x\xi_2. \tag{24}$$

The identities (23) and (24) identify

$$\langle\langle X\cdot\varphi,\varphi\rangle\rangle \simeq \langle\langle X\cdot_M i(\psi_2),\psi_1\rangle\rangle_{\Sigma M};$$

this gives an explicit representation of the immersion into $\mathbb{R}^4$ in terms of the two spinor fields ψ_1 and $i(\psi_2)$ of ΣM introduced in [14].

7.3. **Surfaces in $\mathbb{R}^4$.** For a surface in $\mathbb{R}^4$, Theorem 1 with $p=2$ and $n=4$ reduces to the result obtained in [5], since the bundle Σ naturally identifies to the bundle $\Sigma M\otimes\Sigma E$ in that case (see Remark 1, observing that the representation of $Spin(2)$ on Cl_2 by left multiplication is also the usual complex spin representation Σ_2). Note that we may similarly recover the main results in [4, 6] concerning immersions in $\mathbb{R}^{3,1}$ and $\mathbb{R}^{2,2}$, if we consider in our constructions the Clifford algebras $Cl_{3,1}$ and $Cl_{2,2}$ instead of Cl_4.

8. Spinorial representation of submanifolds in S^n and $\mathbb{H}^n$

We extend here Theorem 1 to the other space forms.

8.1. **Submanifolds of S^n.** Let M be a Riemannian manifold of dimension p, and E be a bundle on M of rank $q=n-p$, with a fibre metric and a compatible connection; we assume that TM and E are spin, and consider

$$\Sigma := \tilde{Q}\times_\rho Cl_{n+1}$$

where $\tilde{Q}=\tilde{Q}_M\times_M\tilde{Q}_E$ is the $Spin(p)\times Spin(q)$ principal bundle given by the two spin structures and $\rho: Spin(p)\times Spin(q)\to Aut(Cl_{n+1})$ is the representation obtained by the composition of the maps

$$Spin(p)\times Spin(q)\to Spin(n)\subset Spin(n+1) \tag{25}$$

and

$$Spin(n+1) \to Aut(Cl_{n+1}). \tag{26}$$

The maps in (25) correspond to the decompositions

$$\mathbb{R}^p \oplus \mathbb{R}^q =: \mathbb{R}^n \qquad \subset \mathbb{R}^n \oplus \mathbb{R}e_{n+1} =: \mathbb{R}^{n+1},$$

and in (26) the action of $Spin(n+1)$ on Cl_{n+1} is the multiplication on the left. We also define

$$U\Sigma = \tilde{Q} \times_\rho Spin(n+1) \ \subset \Sigma.$$

Let us denote by ν the element of the Clifford bundle $\tilde{Q} \times_{Ad} Cl_{n+1}$ such that its component in an arbitrary frame $\tilde{s} \in \tilde{Q}$ is the constant vector e_{n+1} (note that for all $g \in Spin(p) \times Spin(q) \subset Spin(n) \subset Spin(n+1)$, $Ad(g)(e_{n+1}) = e_{n+1}$).

Theorem 3. *Let $B : TM \times TM \to E$ be a symmetric and bilinear map. The following two statements are equivalent:*

(1) There exists an isometric immersion F of M into S^n with normal bundle E and second fundamental form B.

(2) There exists a spinor field $\varphi \in \Gamma(U\Sigma)$ satisfying

$$\nabla_X \varphi = -\frac{1}{2}\sum_{j=1}^{p} e_j \cdot B(X, e_j) \cdot \varphi + \frac{1}{2} X \cdot \nu \cdot \varphi \tag{27}$$

for all $X \in TM$.

Moreover we have the representation formula

$$F = \langle\langle \nu \cdot \varphi, \varphi \rangle\rangle \qquad \in S^n \subset \mathbb{R}^{n+1}, \tag{28}$$

where the brackets $\langle\langle .,. \rangle\rangle$ are defined as in (3)-(4).

Proof. We only prove that (2) implies (1), using the explicit formula (28). Setting $F = \langle\langle \nu \cdot \varphi, \varphi \rangle\rangle$, we have

$$F = [\varphi]^{-1} e_{n+1} [\varphi] = Ad([\varphi]^{-1})(e_{n+1})$$

where $[\varphi] \in Spin(n+1)$ represents φ in some frame $\tilde{s} \in \tilde{Q}$ and $Ad : Spin(n+1) \to SO(n+1)$ is the natural double covering; thus F belongs to S^n. We will need the following

Lemma 8.1. *If $\varphi \in \Gamma(U\Sigma)$ is a solution of (27) then $F = \langle\langle \nu \cdot \varphi, \varphi \rangle\rangle$ is such that, for all $X \in TM$,*

$$dF(X) = \langle\langle X \cdot \varphi, \varphi \rangle\rangle. \tag{29}$$

Proof. We first observe that $\nabla \nu = 0$: if α is the connection form on $\tilde{Q}$ and $\tilde{s} \in \Gamma(\tilde{Q})$ is a local frame, then $\nu = [\tilde{s}, e_{n+1}]$ and

$$\nabla_X \nu = [\tilde{s}, \partial_X e_{n+1} + Ad_*(\alpha(\tilde{s}_*(X)))(e_{n+1})] = 0$$

for all $X \in TM$, since e_{n+1} is constant and α takes values in $\Lambda^2 \mathbb{R}^n \subset Cl_n$. Thus, for all $X \in TM$,

$$\begin{aligned} dF(X) &= \langle\langle \nu \cdot \nabla_X \varphi, \varphi \rangle\rangle + \langle\langle \nu \cdot \varphi, \nabla_X \varphi \rangle\rangle \\ &= (id + \tau)\langle\langle \nu \cdot \nabla_X \varphi, \varphi \rangle\rangle \\ &= -\frac{1}{2}(id + \tau)\sum_{j=1}^{p} \langle\langle \nu \cdot e_j \cdot B(X, e_j) \cdot \varphi, \varphi \rangle\rangle + \frac{1}{2}(id + \tau)\langle\langle X \cdot \varphi, \varphi \rangle\rangle. \end{aligned}$$

But

$$\begin{aligned}\tau\langle\langle\nu\cdot e_j\cdot B(X,e_j)\cdot\varphi,\varphi\rangle\rangle &= \langle\langle\varphi,\nu\cdot e_j\cdot B(X,e_j)\cdot\varphi\rangle\rangle\\ &= \langle\langle B(X,e_j)\cdot e_j\cdot\nu\cdot\varphi,\varphi\rangle\rangle\\ &= -\langle\langle\nu\cdot e_j\cdot B(X,e_j)\cdot\varphi,\varphi\rangle\rangle\end{aligned}$$

since the three vectors $B(X,e_j)$, e_j and ν are mutually orthogonal, and

$$\tau\langle\langle X\cdot\varphi,\varphi\rangle\rangle = \langle\langle\varphi,X\cdot\varphi\rangle\rangle = \langle\langle X\cdot\varphi,\varphi\rangle\rangle.$$

Thus (29) follows. □

By the lemma and the properties of the Clifford product, F is an isometric immersion, and the map

$$\begin{aligned}E &\to TS^n\\ X\in E_m &\mapsto (F(m),\langle\langle X\cdot\varphi,\varphi\rangle\rangle)\end{aligned}$$

identifies E with the normal bundle of $F(M)$ into S^n; it moreover identifies the connection on E with the normal connection of $F(M)$ in S^n and B with the second fundamental form. We omit the proof since it is very similar to the proof of Lemma 4.2. □

Remark 5. *Taking the trace of (27) we get*

$$D\varphi = \frac{p}{2}\left(\vec{H}-\nu\right)\cdot\varphi \tag{30}$$

where $\vec{H}=\frac{1}{p}\sum_{j=1}^p B(e_j,e_j)$ *is the mean curvature vector of* M *in* S^n.

Remark 6. *We may also obtain a proof using spinors of the fundamental theorem of submanifold theory in* S^n*, showing, as in Section 5, that the equations of Gauss, Codazzi and Ricci in a space of constant sectional curvature 1 are exactly the integrability conditions of (27).*

We finally show how to recover the spinorial characterization of a surface in S^3 given by Morel in [17]. In the model $Cl_4\simeq\mathbb{H}(2)$ we have

$$\varphi=\begin{pmatrix}[\varphi^+] & 0\\ 0 & [\varphi^-]\end{pmatrix},\qquad F=\begin{pmatrix}0 & \overline{[\varphi^+]}[\nu][\varphi^-]\\ -\overline{[\varphi^-]}[\nu][\varphi^+] & 0\end{pmatrix}$$

and

$$\xi(X)=\begin{pmatrix}0 & \overline{[\varphi^+]}[X][\varphi^-]\\ -\overline{[\varphi^-]}[X][\varphi^+] & 0\end{pmatrix}$$

where $[\varphi^+]$, $[\varphi^-]$, $[\nu]$ and $[X]\in\mathbb{H}$ represent φ^+, φ^-, ν and X in some spinor frame adapted to the immersion in S^3; thus Lemma 8.1 gives

$$F\simeq\overline{[\varphi^+]}[\nu][\varphi^-]\quad\text{ and }\quad dF(X)\simeq\overline{[\varphi^+]}[X][\varphi^-].$$

If $[\varphi^+]$ is given, this system has a solution $[\varphi^-]$, unique up to the multiplication by S^3 on the right. The spinor field φ is thus essentially determined by its component φ^+, which may be identified with a spinor field $\psi\in\Gamma(\Sigma M)$ solution of

$$D\psi = H\psi - i\overline{\psi},\qquad |\psi|=1;$$

details are given in [5]. This is the spinor characterization of an immersion in S^3 given in [17].

8.2. **Submanifolds of $\mathbb{H}^n$.** We now consider the n-dimensional hyperbolic space $\mathbb{H}^n$ as a hypersurface of the Minkowski space $\mathbb{R}^{n,1}$. Since the constructions of the paper may be also carried out in a linear space with a semi-riemannian metric, we obtain a spinor representation of a submanifold in $\mathbb{H}^n$ exactly as we did for a submanifold in S^n. We thus only state the results here, and refer to the previous section for the proofs. Let M be a riemannian manifold of dimension p, and E be a bundle on M of rank $q = n-p$, with a Riemannian fibre metric and a compatible connection; we assume that TM and E are spin, and consider

$$\Sigma := \tilde{Q} \times_\rho Cl_{n,1}$$

where $\tilde{Q} = \tilde{Q}_M \times_M \tilde{Q}_E$ is the $Spin(p) \times Spin(q)$ principal bundle given by the two spin structures and $\rho : Spin(p) \times Spin(q) \to Aut(Cl_{n,1})$ is the representation obtained by the composition of the maps

$$Spin(p) \times Spin(q) \to Spin(n) \subset Spin(n,1) \tag{31}$$

and

$$Spin(n,1) \to Aut(Cl_{n,1}). \tag{32}$$

The maps in (31) correspond to the decompositions

$$\mathbb{R}^p \oplus \mathbb{R}^q =: \mathbb{R}^n \qquad \subset \mathbb{R}^n \oplus \mathbb{R}e_{n+1} =: \mathbb{R}^{n,1},$$

and in (32) the action of $Spin(n,1)$ on $Cl_{n,1}$ is the multiplication on the left; here e_{n+1} is a vector with negative norm -1. We also define

$$U\Sigma = \tilde{Q} \times_\rho Spin(n,1) \ \subset \Sigma.$$

Let us denote by ν the element of the Clifford bundle $\tilde{Q} \times_{Ad} Cl_{n,1}$ such that its component in an arbitrary frame $\tilde{s} \in \tilde{Q}$ is the constant vector e_{n+1}.

Theorem 4. *Let $B : TM \times TM \to E$ be a symmetric and bilinear map. The following two statements are equivalent:*

(1) There exists an isometric immersion F of M into $\mathbb{H}^n$ with normal bundle E and second fundamental form B.
(2) There exists a spinor field $\varphi \in \Gamma(U\Sigma)$ satisfying

$$\nabla_X \varphi = -\frac{1}{2}\sum_{j=1}^{p} e_j \cdot B(X, e_j) \cdot \varphi - \frac{1}{2} X \cdot \nu \cdot \varphi \tag{33}$$

for all $X \in TM$.

Moreover we have the representation formula

$$F = \langle\langle \nu \cdot \varphi, \varphi \rangle\rangle \qquad \in \mathbb{H}^n \subset \mathbb{R}^{n,1}, \tag{34}$$

where the brackets $\langle\langle .,. \rangle\rangle$ are defined as in (3)-(4).

We may also recover the spinor characterization of an immersion of a surface in $\mathbb{H}^3$ given by Morel in [17]: if M is a surface and (e_1, e_2) is an orthonormal basis of TM, setting $\vec{H}_{\mathbb{H}^3} := \frac{1}{2}\left(B(e_1,e_1) + B(e_2,e_2)\right)$ we see that (33) is equivalent to

$$D\varphi = (\vec{H}_{\mathbb{H}^3} + \nu) \cdot \varphi$$

where φ is a spinor field which is represented in a frame $\tilde{s} \in \tilde{Q}$ by $[\varphi]$ belonging to $Spin(3,1)$. This is exactly the spinor representation of an immersion in $\mathbb{H}^3$ as described in [4] Section 5, where it is moreover proved that it is equivalent to the

spinor characterization given in [17].

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