

Quantum thermal transistor : supplemental material

Karl Joulain, Jérémie Drevillon, Younès Ezzahri and Jose Ordonez-Miranda

Institut Pprime, CNRS, Université de Poitiers,

ISAE-ENSMA, F-86962 Futuroscope Chasseneuil, France

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Abstract

In this supplemental material, we derive approximated expressions for the levels populations which are used to determine the thermal currents involved in the simple quantum system for which the only nonzero energy are the coupling between the left and middle TLS and the middle and the right TLS. We also deduce levels decaying rates and justify, from their values, the Born-Markov approximation. We finally discuss conditions for which the thermal transistor effect is observed or not.

**DERIVATION OF THE SIMPLIFIED LEVELS POPULATION EXPRESSIONS
FOR THE CASE $\omega_M = \omega_L = \omega_R = 0$, $\omega_{LM} = \omega_{MR} = \Delta$, $\omega_{RL} = 0$**

As shown in the letter, in these conditions, the system states are degenerated two by two: $|1\rangle = |8\rangle$, $|2\rangle = |7\rangle$, $|3\rangle = |6\rangle$ and $|4\rangle = |5\rangle$. Moreover, $E_1 = E_8 = \Delta$, $E_2 = E_4 = E_5 = E_7 = 0$ and $E_3 = E_6 = -\Delta$, so that there are only three energy levels (Fig. 2 of the manuscript). We rename the states $|1\rangle$ and $|8\rangle$ as $|I\rangle$, $|2\rangle$ and $|7\rangle$ as $|II\rangle$, $|3\rangle$ and $|6\rangle$ as $|III\rangle$, $|4\rangle$ and $|5\rangle$ as $|IV\rangle$. We introduce the new density matrix elements, now reduced to 4, $\rho_I = \rho_{11} + \rho_{88}$, $\rho_{II} = \rho_{22} + \rho_{77}$, $\rho_{III} = \rho_{33} + \rho_{66}$ and $\rho_{IV} = \rho_{44} + \rho_{55}$.

We now analyze the density matrix evolution under the Born-Markov approximation, by solving Eq. (2) of the letter. In steady state, $d\rho/dt = 0$, so that the system is similar to Eqs (3) with the new levels I , II , III and IV .

$$\dot{\rho}_I = 0 = \Gamma_{IV-I}^L + \Gamma_{III-I}^M + \Gamma_{II-I}^R, \quad (1)$$

$$\dot{\rho}_{II} = 0 = \Gamma_{III-II}^L + \Gamma_{IV-II}^M + \Gamma_{I-II}^R, \quad (2)$$

$$\dot{\rho}_{III} = 0 = \Gamma_{II-III}^L + \Gamma_{I-III}^M + \Gamma_{IV-III}^R, \quad (3)$$

$$\dot{\rho}_{IV} = 0 = \Gamma_{I-IV}^L + \Gamma_{II-IV}^M + \Gamma_{III-IV}^R. \quad (4)$$

Note that $\Gamma_{ij} = -\Gamma_{ji}$ and therefore these 4 equations are not independent since their sum is 0. A supplementary equation based on the conservation of levels occupation probability, states that

$$Tr[\rho] = \rho_I + \rho_{II} + \rho_{III} + \rho_{IV} = 1 \quad (5)$$

The decaying rates are function of the populations, as follows

$$\begin{aligned} \Gamma_{I-IV}^L &= \Delta[\rho_I - e^{-\Delta/T_L}\rho_{IV}] \\ \Gamma_{I-III}^M &= 2\Delta[\rho_I - e^{-2\Delta/T_M}\rho_{III}] \\ \Gamma_{I-II}^R &= \Delta[\rho_I - e^{-\Delta/T_R}\rho_{II}] \\ \Gamma_{II-III}^L &= \Delta[\rho_{II} - e^{-\Delta/T_L}\rho_{III}] \\ \Gamma_{II-IV}^M &= T_M[\rho_{II} - \rho_{IV}] \\ \Gamma_{IV-III}^R &= \Delta[\rho_{IV} - e^{-\Delta/T_R}\rho_{III}] \end{aligned} \quad (6)$$

Equations (1-3) and (5) form the system of equations that has to be solved in order to obtain

the 4 populations. This system can be put under the form

$$[M] \begin{bmatrix} \rho_I \\ \rho_{II} \\ \rho_{III} \\ \rho_{IV} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad (7)$$

with

$$M = \begin{bmatrix} 4\Delta & -\Delta e^{-\Delta/T_R} & -2\Delta e^{-2\Delta/T_M} & -\Delta e^{-\Delta/T_L} \\ -\Delta & T_M + \Delta + \Delta e^{-\Delta/T_R} & -\Delta e^{-\Delta/T_L} & -T_M \\ 2\Delta & \Delta & -2\Delta e^{-2\Delta/T_M} - \Delta e^{-\Delta/T_L} - \Delta e^{-\Delta/T_R} & \Delta \\ 1 & 1 & 1 & 1 \end{bmatrix} \quad (8)$$

The temperature T_L is taken so that $e^{-\Delta/T_L} \ll 1$ ($T_L/\Delta \lesssim 4$) whereas $e^{-\Delta/T_R} \ll e^{-\Delta/T_L}$ ($T_R/\Delta \lesssim 8$). Moreover, one has always $T_R \leq T_M \leq T_L$. Under these conditions, M reduces to

$$M \approx \begin{bmatrix} 4\Delta & 0 & -2\Delta e^{-2\Delta/T_M} & -\Delta e^{-\Delta/T_L} \\ -\Delta & T_M + \Delta & -\Delta e^{-\Delta/T_L} & -T_M \\ 2\Delta & \Delta & -\Delta e^{-\Delta/T_L} & \Delta \\ 1 & 1 & 1 & 1 \end{bmatrix} \quad (9)$$

so that the temperature T_R does not show up any more and therefore it will not be involved in the populations expressions.

With this new matrix expression, the system (7) is solved. In particular, the matrix determinant can also be simplified

$$\begin{aligned} \det(M) &= \Delta^3 [-2e^{2\Delta/T_L} + 7e^{\Delta(1/T_L+2/T_M)} + 3e^{2\Delta/T_M} + 4e^{\Delta(1/T_L+1/T_M)}] \\ &\quad + \Delta^2 T_M [-4e^{2\Delta/T_L} + 10e^{\Delta(1/T_L+2/T_M)} + e^{2\Delta/T_M} + 8e^{\Delta(1/T_L+1/T_M)}] \\ &\approx \Delta^2 [4\Delta + 8T_M] e^{2\Delta(1/T_L+1/T_M)}. \end{aligned} \quad (10)$$

Then, the approximated populations expressions derivation is straightforward

$$\rho_I \simeq \frac{e^{-2\Delta/T_M}}{2} + \frac{T_M}{4\Delta + 8T_M} e^{-2\Delta/T_L}, \quad (11)$$

$$\rho_{II} \simeq \frac{\Delta + T_M}{\Delta + 2T_M} e^{-\Delta/T_L}, \quad (12)$$

$$\rho_{III} \simeq 1 - e^{-\Delta/T_L}, \quad (13)$$

$$\rho_{IV} \simeq \frac{T_M}{\Delta + 2T_M} e^{-\Delta/T_L}. \quad (14)$$

Similarly, one obtains the decaying rates

$$\Gamma_{I-IV}^L = \Delta \left[\frac{e^{-2\Delta/T_M}}{2} - \frac{3T_M}{4\Delta + 8T_M} e^{-2\Delta/T_L} \right], \quad (15)$$

$$\Gamma_{I-III}^M = 2\Delta \left[-\frac{e^{-2\Delta/T_M}}{2} + \frac{T_M}{4\Delta + 8T_M} e^{-2\Delta/T_L} \right], \quad (16)$$

$$\Gamma_{I-II}^R = \Delta \left[\frac{e^{-2\Delta/T_M}}{2} + \frac{T_M}{4\Delta + 8T_M} e^{-2\Delta/T_L} \right], \quad (17)$$

$$\Gamma_{II-III}^L = \frac{-\Delta T_M}{\Delta + 2T_M} e^{-\Delta/T_L}, \quad (18)$$

$$\Gamma_{II-IV}^M = \frac{-\Delta T_M}{\Delta + 2T_M} e^{-\Delta/T_L}, \quad (19)$$

$$\Gamma_{IV-III}^R = \frac{\Delta T_M}{\Delta + 2T_M} e^{-\Delta/T_L}. \quad (20)$$

and the thermal currents

$$J_L \simeq \frac{\Delta^2 T_M e^{-\Delta/T_L}}{\Delta + 2T_M}, \quad (21)$$

$$J_M \simeq \Delta^2 \left[-\frac{T_M}{\Delta + 2T_M} e^{-2\Delta/T_L} + 2e^{-2\Delta/T_M} \right], \quad (22)$$

$$J_R \simeq -\frac{\Delta^2 T_M e^{-\Delta/T_L}}{\Delta + 2T_M}. \quad (23)$$

In Fig. 1 we compare the populations values predicted by their exact expressions from Eq. (7-8) with the approximate ones. We note that there is not a visible difference, within a wide range of temperatures T_M . We also report the decaying rates in Fig. 2 and Fig. 3 as well as the comparison of the thermal currents with their approximations formulas, in Fig. 4. The relative error between the exact and asymptotic values is less than 5%. The error increases when T_M approaches T_L .

An important point to note in Fig. 2 and Fig. 3 is that the decaying rates never exceed 10^{-3} , under the conditions of our study. These rates are much smaller than 1, meaning that the system relaxation time is much larger than 1. As the frequency difference involved are around $\Delta = 1$, the conditions for the Born-Markov approximation are fulfilled, given that the inverse frequency difference involved in the problem is small compared to the relaxation time of the system. Actually, the levels decaying rates increase with temperature so that the regime where the Born-Markov approximation is not valid is the regime of high temperatures, which is not considered in the work.

Let us finally note that, if the condition $\omega_M = \omega_L = \omega_R = \omega_{LR} = 0$ is convenient to achieve the simple calculations presented above with only 4 levels, this condition is not

necessary to achieve a thermal transistor. Indeed, a thermal transistor can also be obtained for another simple case when the 3 TLS energies are equal but lower than Δ and if the coupling energies between the 3 TLS are not symmetric. In that case, the three TLS are not degenerated and exhibit two energy levels. The calculations are however more complicated since the system exhibits at least 6 energy levels. As examples, we report in Fig. 5 thermal currents in similar cases to the one studied above except that $\omega_L = \omega_M = \omega_R = 0.1\Delta$ and that the coupling between the left and the right TLS ω_{RL} can be 0, 0.3Δ , Δ or 3Δ . We see that except in the symmetric case where $\omega_{RL} = \Delta$, J_M remains much lower than J_L and J_R in a very similar way to the case of the 3 degenerated TLS so that the transistor effect is still present in these 3 configurations. Note that the case $\omega_{LR} = 0.3\Delta$ provides higher thermal currents and a better switch than the case detailed in the letter. This paves the way to the search of a set of parameters that would optimize the device but which is a task beyond the goal to this work. On the contrary, in the symmetric case the transistor effect disappears.

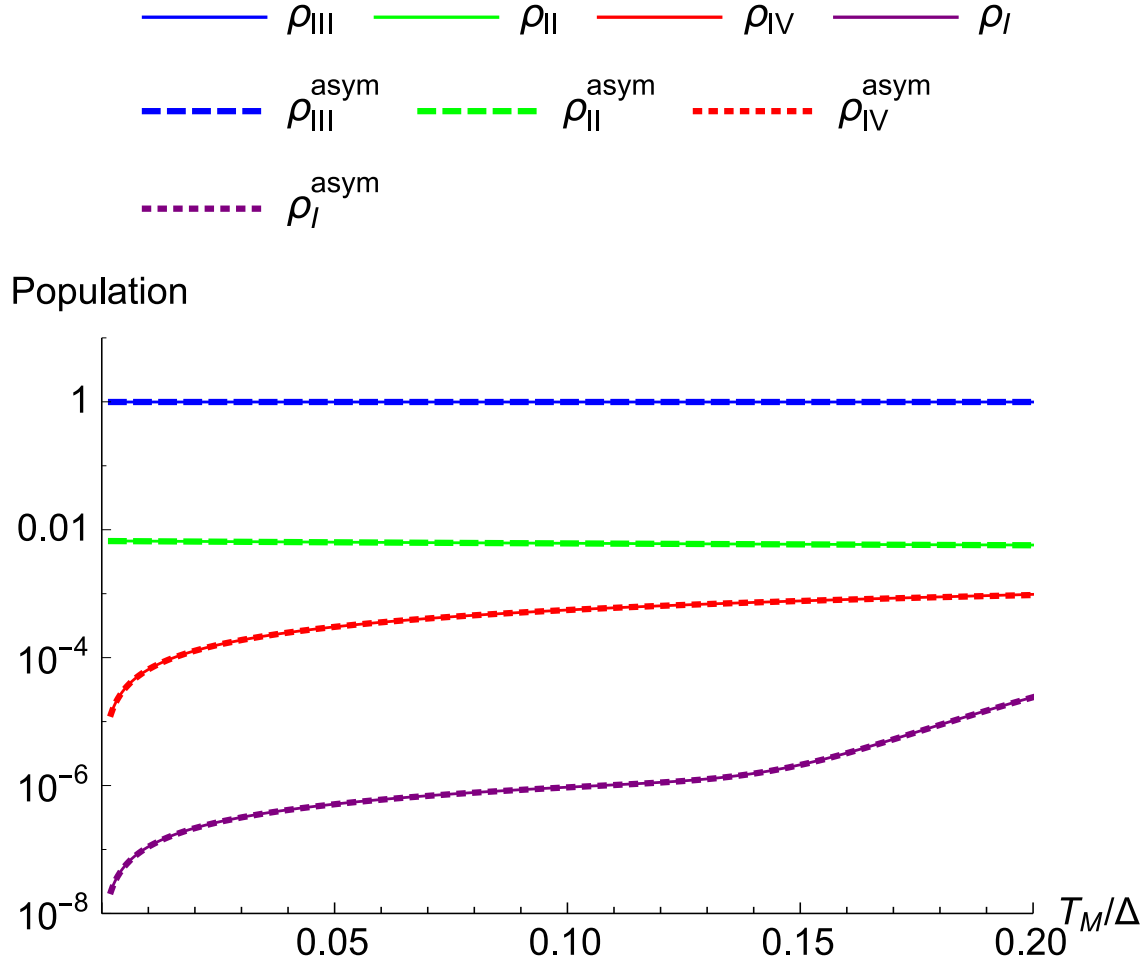


FIG. 1. Populations ρ_I , ρ_{II} , ρ_{III} and ρ_{IV} versus T_M for $\omega_L = \omega_M = \omega_R = 0$, $\omega_{RL} = 0$, $\omega_{LM} = \omega_{MR} = \Delta$, $T_L = 0.2\Delta$ and $T_R = 0.02\Delta$. Comparison is done between populations obtained from the exact calculations and the approximated expressions derived in the asymptotic regime considered here.

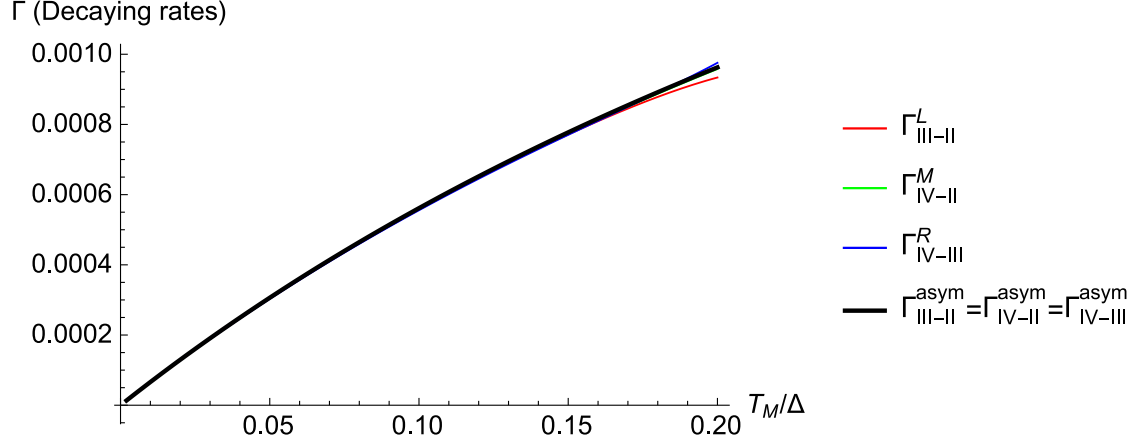


FIG. 2. Decaying rates Γ_{III-II}^L , Γ_{IV-II}^M , Γ_{IV-III}^R versus T_M for $\omega_L = \omega_M = \omega_R = 0$, $\omega_{RL} = 0$, $\omega_{LM} = \omega_{MR} = \Delta$, $T_L = 0.2\Delta$ and $T_R = 0.02\Delta$. Comparison is done between decaying rates obtained from the exact calculations and the approximated expressions obtained in the asymptotic regime considered here.

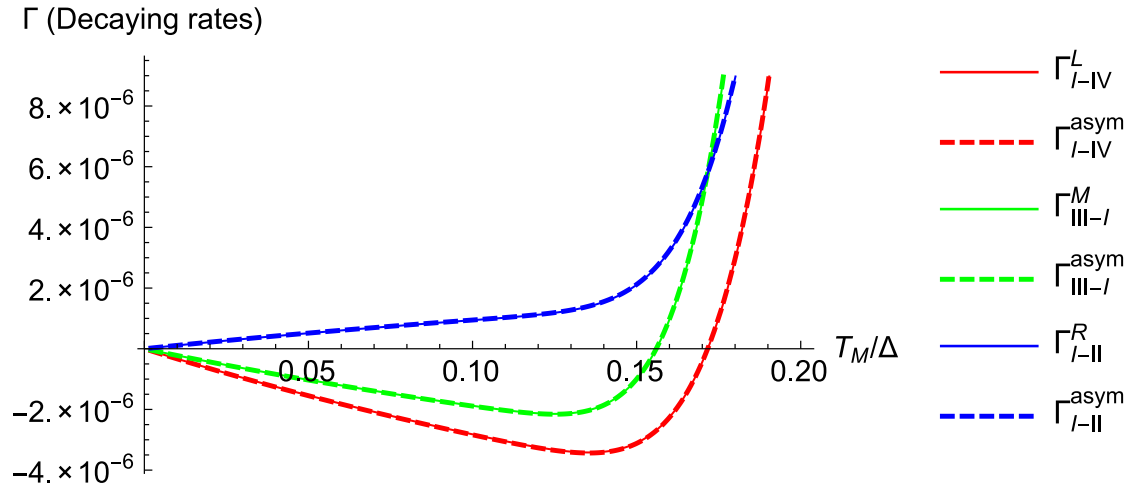


FIG. 3. Decaying rates Γ_{I-IV}^L , Γ_{III-I}^M , Γ_{I-II}^R versus T_M for $\omega_L = \omega_M = \omega_R = 0$, $\omega_{RL} = 0$, $\omega_{LM} = \omega_{MR} = \Delta$, $T_L = 0.2\Delta$ and $T_R = 0.02\Delta$. Comparison is done between decaying rates obtained from the exact calculations and the approximated expressions determined in the asymptotic regime considered here.

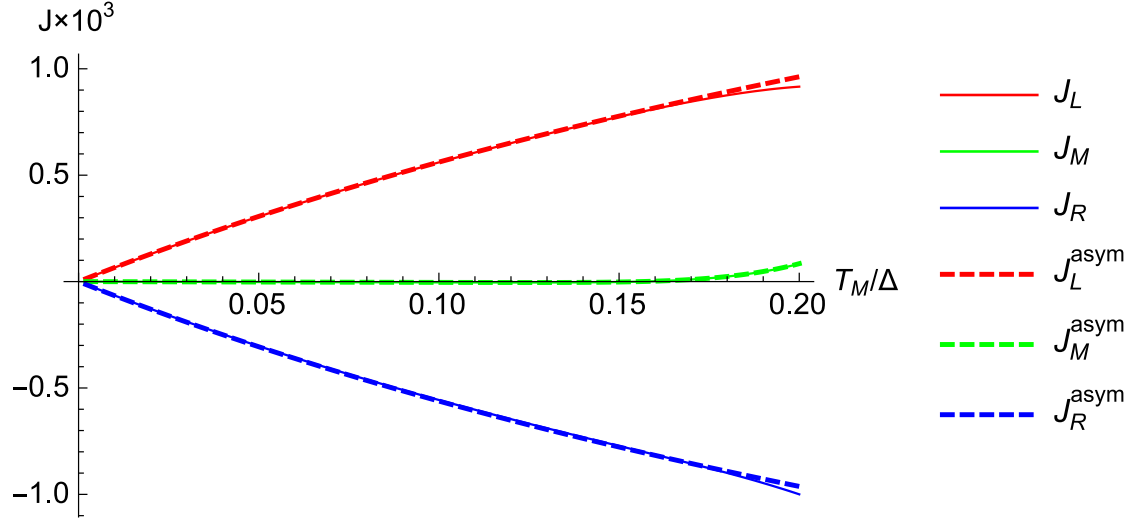


FIG. 4. Thermal currents J_L , J_M , J_R versus T_M for $\omega_L = \omega_M = \omega_R = 0$, $\omega_{RL} = 0$, $\omega_{LM} = \omega_{MR} = \Delta$, $T_L = 0.2\Delta$ and $T_R = 0.02\Delta$. Comparison is done between thermal currents obtained from the exact calculations and the approximated expressions derived in the asymptotic regime considered here.

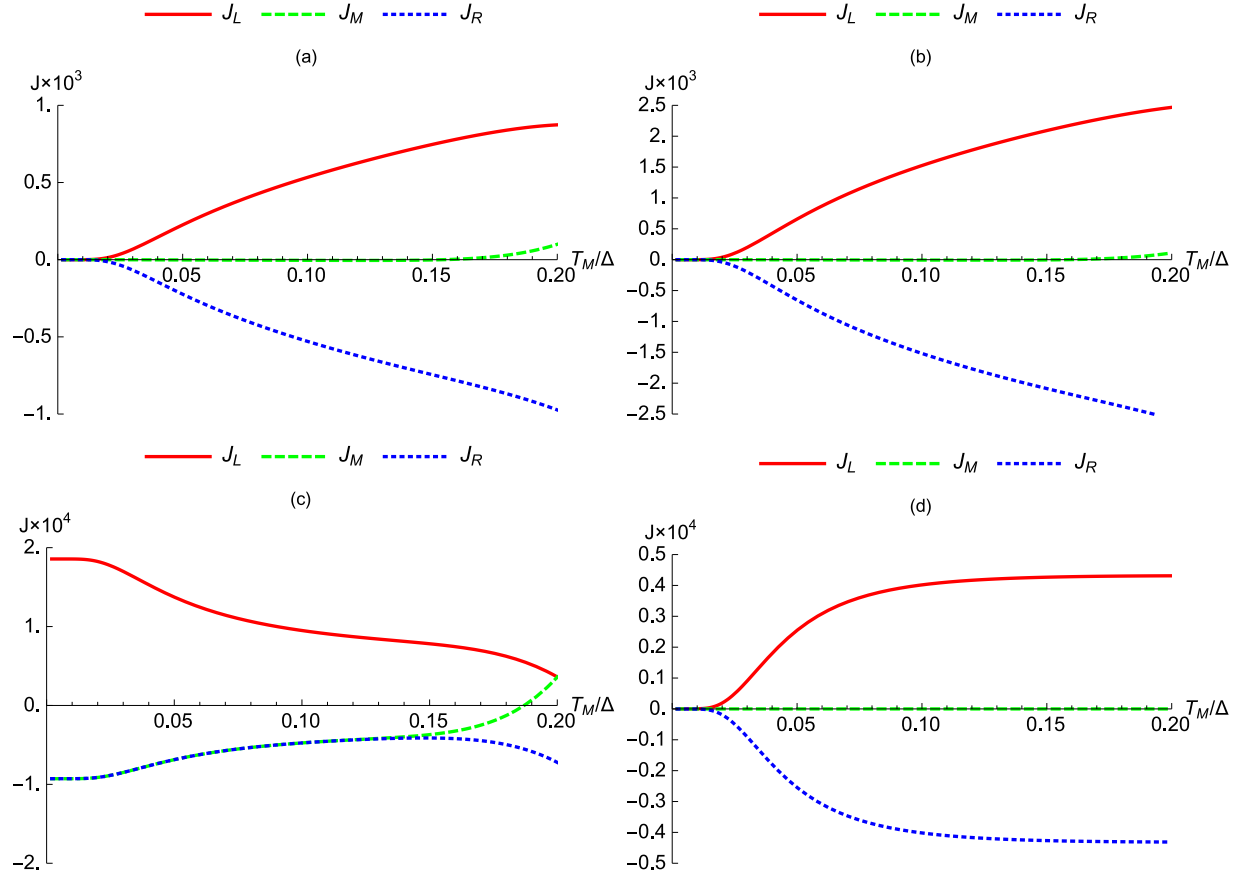


FIG. 5. Thermal currents J_L , J_M , J_R versus T_M for $\omega_L = \omega_M = \omega_R = 0.1\Delta$, $\omega_{LM} = \omega_{MR} = \Delta$, $T_L = 0.2\Delta$ and $T_R = 0.02\Delta$. (a) $\omega_{LR} = 0$. (b) $\omega_{LR} = 0.3\Delta$. (c) $\omega_{LR} = \Delta$. (d) $\omega_{LR} = 3\Delta$.