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ABSTRACT. With a basic knowledge of cohomology theory, the background necessary to understand Hodge theory and polarization, Deligne's mixed Hodge structure on cohomology of complex algebraic varieties is described.

INTRODUCTION

We assume that the reader is familiar with the basic theory of manifolds, basic algebraic geometry as well as cohomology theory. For instance, we will use freely the notes [7, 20] in this book and we recommend to the reader books like e.g. [50], [3], or [45], and the beginning of [33] and, for complementary reading on complex algebraic and analytic geometry, [48], [44], [52].

According to Deligne, the cohomology space $H^n(X, \mathbb{C})$ of a complex algebraic variety X carries two finite filtrations by complex subvector spaces, the rationally weight filtration W and the Hodge filtration F defining a mixed Hodge structure (MHS) (see [11] and [12]).

For a non-singular compact complex algebraic variety, the weight filtration W is trivial, while the Hodge filtration F and its conjugate $\overline{F}$ with respect to the rational cohomology, define a Hodge decomposition. In this case the structure in linear algebra defined on a complex vector space by such decomposition is called a Hodge structure (HS) [38], [52].

On a non-singular variety X, the weight filtration W and the Hodge filtration F reflect properties of the normal crossing divisor (NCD) at infinity of an adequate completion of the variety obtained by resolution of singularities (see [35]), but are independent of the choice of the normal crossing divisor.

Inspired by the properties of étale cohomology of varieties over fields with positive characteristic, constructed by A. Grothendieck and M. Artin, P. Deligne established the existence of a MHS on the cohomology of complex algebraic varieties, depending naturally on algebraic morphisms but not on continuous maps. The theory has been fundamental in the study of topological properties of complex algebraic varieties. At the end of this introduction, we recall the topological background of homology theory and Poincaré duality, then we refer to chapter I in this volume [7] to state the Hodge decomposition on the cohomology of Kähler manifolds.

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The theory involves a significant change in the method and technique from the construction of harmonic forms, which is dependent on the metric, while the results here are stated on cohomology which is a topological invariant and the cohomology subspaces of type (p, q). The course is organized as follows:

§1. The abstract category of Hodge structures is defined in the first section and spectral sequences are introduced. The decomposition on the cohomology of Kähler manifolds is used to prove the degeneration at rank one of the spectral sequence defined by the filtration F on the de Rham complex in the projective non-singular case. An important result here is the degeneration at rank one of the spectral sequence, since this result extends to all complex algebraic varieties. The section ends with the proof of the existence of a Hodge structure on the cohomology of a non-singular compact complex algebraic variety.

§2. In the second section, we introduce an abstract definition of MHS as an object of interest in linear algebra, following closely Deligne [11]. The algebraic properties of MHS are developed on objects with three opposite filtrations in an abelian category. Then we prove Deligne's lemma on the two filtrations and explain how it can be applied to construct a MHS on the cohomology of a normal crossing divisor.

§3. In section three, we need to develop algebraic homology techniques on filtered complexes up to filtered quasi-isomorphisms of complexes. Since a MHS involves rational cohomology constructed via different techniques than the ones used in de Rham cohomology, when we define a morphism of mixed Hodge structures, we ask for compatibility with the filtrations as well as the identification of cohomology via the various constructions. For this reason it is convenient to work constantly at the level of complexes in the language of filtered and bi-filtered derived categories to ensure that the MHS constructed on cohomology are canonical and do not depend on particular resolutions used in the definition of cohomology. The main contribution by Deligne was to fix the axioms of a mixed Hodge complex (MHC) and to prove the existence of a MHS on its cohomology. This central result is applied in section four.

§4. We give here the construction of the MHS on any algebraic variety. On non-compact non-singular algebraic variety X, we need to introduce Deligne's logarithmic de Rham complex to take into account the properties at infinity, that is the properties of the NCD, which is the complement of the variety in an adequate compactification of X. If X is singular, we introduce a smooth simplicial covering to construct the MHC. We mention also an alternative construction.

As applications let us mention deformations of non-singular proper analytic families which define a linear invariant called Variation of Hodge structure (VHS) introduced by P. Griffiths (see chapter 3 [8]) , and limit MHS adapted to study the degeneration of Variation of Hodge structure. Variations of mixed Hodge structure, which arise in the case of any algebraic deformation, are the topic of the lectures in chapters 3 and 11 of this book [8], [4].

Finally we stress that we only introduce the necessary language to express the statements and their implications in Hodge theory, but we encourage mathematicians to look into the foundational work of Deligne in references [11] and [12] to discover his dense and unique style, since intricate proofs in Hodge theory and spectral sequences are often just surveyed here.

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Topological background. The theory of homology and cohomology traces its origin to the work of Henri Poincaré in the late nineteenth century. There are actually many different theories, for example, simplicial and singular (see e.g. [34]). In 1931, Georges de Rham proved a conjecture of Poincaré on a relationship between cycles and differential forms that establishes for a compact orientable manifold an isomorphism between singular cohomology with real coefficients and what is known now as de Rham cohomology. The result has had a great impact on the theory since it shows the diversity of applications of cohomology theory.
A basic result that we also mention is the isomorphism established by William Hodge between the space of harmonic forms on a Riemannian compact manifold and cohomology. While the space of harmonic forms depends on the choice of the metric, the cohomology space appears as an invariant of the topology. These results on de Rham cohomology as well Čech cohomology, shifted the attention to the construction of cohomology by sheaf theory.
However, our subject starts with the Hodge decomposition on the cohomology of a compact Kähler manifold. For an accessible introduction to the subject covering the fundamental statements that we need here, see the notes of E. Cattani in the

chapter 1 of this volume ([7]). For an excellent full account one may also refer to [52].

There are various applications of Hodge decomposition. On one hand, the work of Phillip Griffiths on the variation of Hodge structures [24] depends on the variation of the analytic (or algebraic) structure of a non-singular projective variety in a family (by comparison, cohomology can only detect the underlying topological structure and it is locally constant with respect to the parameter space of the family). On the other hand, we cite the work of André Weil [51] and his conjecture, which motivated Alexander Grothendieck and his theory of motives and the work of Pierre Deligne (who solved the conjecture) on mixed Hodge structures described here.

Fundamental class. The idea that homology should represent the classes of topological subspaces has been probably at the origin of homology theory, although the effective construction of homology groups is different and more elaborate. The simplest way to construct homology groups (in fact assumed with compact support) is to use singular simplexes, but the definition of homology groups $H_j(X, \mathbb{Z})$ of a triangulated space is the most natural. Indeed, the sum of oriented triangles of highest dimensions of an oriented triangulated topological space X of real dimension n, defines a homology class $[X] \in H_n(X, \mathbb{Z})$ ([25], Chapter 0, paragraph 4).

We take as granted here that a compact complex algebraic variety X of dimension m can be triangulated ([36]) such that the sum of its oriented triangles of highest dimension defines a class in the homology group $H_{2m}(X, \mathbb{Z})$.

Cap product. Cohomology groups of a topological space $H^i(X, \mathbb{Z})$ are dually defined, and there exists a topological operation on homology and cohomology, called the cap product:

$$\cap : H^q(X, \mathbb{Z}) \otimes H_p(X, \mathbb{Z}) \to H_{p-q}(X, \mathbb{Z})$$

We can now state the duality theorem of Poincaré, frequently used in geometry.

Theorem (Poincaré isomorphism). *Let X be a compact oriented topological manifold of dimension n. The cap product with the fundamental class $[X] \in H_n(X, \mathbb{Z})$:*

$$D_X : H^j(X, \mathbb{Z}) \xrightarrow{\cap [X]} H_{n-j}(X, \mathbb{Z})$$

defines an isomorphism, for all j, $0 \leq j \leq n$.

Intersection product in topology and geometry. On a triangulated space, cycles are defined as a sum of triangles of the same dimension with boundary zero, and homology is defined by classes of cycles modulo boundaries. It is always possible to represent two homology classes of degree $n - p$ and $n - q$ by two cycles of codimension p and q in "transversal position" on an oriented topological manifold, so that their intersection is defined as a cycle of codimension $p + q$. Moreover, for two representations by transversal cycles, the intersection cycles are homologous (see e.g. [23] or [19] 2.8). Then, on an oriented topological manifold a theory of intersection product on homology can be deduced:

$$H_{n-p}(X, \mathbb{Z}) \otimes H_{n-q}(X, \mathbb{Z}) \xrightarrow{\cap} H_{n-p-q}(X, \mathbb{Z})$$

In geometry, two closed submanifolds V_1 and V_2 of a compact oriented manifold M can be isotopically deformed into a transversal position so that their intersection can be defined as a submanifold W with a sign depending on the orientation ([30], chapter 2), then the homology class $[W]$ of W is up to sign $[V_1] \cap [V_2]$.

Poincaré duality in homology. (see [25], p. 53) On an oriented manifold X, the intersection pairing:

$$H_j(X,\mathbb{Z}) \otimes H_{n-j}(X,\mathbb{Z}) \xrightarrow{\cap} H_0(X,\mathbb{Z}) \xrightarrow{degree} \mathbb{Z}$$

for $0 \leq j \leq n$ is unimodular: the induced morphism

$$H_j(X,\mathbb{Z}) \to Hom(H_{n-j}(X,\mathbb{Z}),\mathbb{Z})$$

is surjective and its kernel is the torsion subgroup of $H_j(X,\mathbb{Z})$.

Cup product. The cup product is a topological product defined on cohomology of a topological space with coefficients in $\mathbb{Z}$ [34]. In de Rham cohomology of a differentiable manifold, the cup product is defined by the exterior product of differential forms.
On an oriented topological compact manifold X of dimension n, a trace map $Tr : H^n(X,\mathbb{Z}) \to \mathbb{Z}$ is defined. In de Rham cohomology, the trace map is defined by integrating a differential form of degree n on the oriented differentiable manifold X.

Poincaré duality in cohomology. On an oriented manifold X, the composition of the trace with the cup product:

$$H^j(X,\mathbb{Z}) \otimes H^{n-j}(X,\mathbb{Z}) \xrightarrow{\cup} H^n(X,\mathbb{Z}) \xrightarrow{Tr} \mathbb{Z}$$

defines a unimodular pairing for $0 \leq j \leq n$ inducing an isomorphism:

$$H^j(X,\mathbb{Q}) \xrightarrow{\sim} Hom(H^{n-j}(X,\mathbb{Q}),\mathbb{Q}).$$

Poincaré isomorphism transforms the intersection pairing into the cup product. The following result is proved in [25] (p. 59) in the case $k' = n-k$:
Let σ be a $k-$cycle on an oriented manifold X of real dimension n and τ an $k'-$cycle on X with Poincaré duals $\eta_\sigma \in H^{n-k}(X)$ and $\eta_\tau \in H^{n-k'}(X)$, then:

$$\eta_\sigma \cup \eta_\tau = \eta_{\sigma\cap\tau} \in H^{n-k-k'}(X)$$

Remark 0.1 (Trace). Generalizations of the trace morphism will be present in various forms, including a definition on the level of complexes [47] (2.3.4) and [32] $(III.10)$, $(VI,4)$.

Hodge decomposition. Our starting point in this chapter is the Hodge decomposition explained in chapter I, theorem (5.2) ([7]). Let $\mathcal{E}^*(X)$ denotes the de Rham complex of differential forms with complex values on a complex manifold X and $\mathcal{E}^{p,q}(X)$ the differentiable forms with complex coefficients of type (p,q), i.e. of the form:

$$\omega = \sum c_{i_1,\dots,i_p,j_1,\dots,j_q}(z,\bar{z})dz_{i_1} \wedge \ldots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \ldots \wedge d\bar{z}_{j_q}$$

(see chapter I of this book, formula 2.4 and example A.5 in the appendix [7]). The cohomology subspaces of type (p,q) are defined as the space of cohomology classes represented by a form of type (p,q):

$$H^{p,q}(X) = \frac{Z_d^{p,q}(X)}{d\mathcal{E}^*(X) \cap Z_d^{p,q}(X)} \quad \text{where} \quad Z_d^{p,q}(X) = Ker\, d \cap \mathcal{E}^{p,q}(X)$$

Theorem (Hodge decomposition). *Let X be a compact Kähler manifold. There exists a decomposition of the complex cohomology spaces into a direct sum of complex subspaces of type (p, q):*

$$H^i(X, \mathbb{C}) = \oplus_{p+q=i} H^{p,q}(X), \quad \text{satisfying} \quad H^{p,q}(X) = \overline{H^{q,p}(X)}. \tag{0.1}$$

See chapter I, theorem 5.2 [7].

Since a complex non-singular projective variety is Kähler (see chapter I, example 3.7 and theorem 3.9 [7]), we deduce:

Corollary. There exists a Hodge decomposition on the cohomology of a complex non-singular projective variety.

We remark that the decomposition is canonical on the cohomology level, although it is obtained from the decomposition of the space of harmonic forms which depends on the metric. The above Hodge decomposition theorem uses successive fundamental concepts from Hermitian geometry including the definition and the properties of the *Laplacian* on the Kähler manifold with its associated *fundamental closed* $(1, 1)$ *form*, analytic geometry on the underlying complex analytic manifold such as the type of a form and Dolbeault cohomology, and Riemannian geometry on the underlying Riemannian manifold such as harmonic forms. Besides chapter I [7], an extended exposition of the theory with full proofs, including the subtle linear algebra of Hermitian metrics needed here, can be found in [52], see also ([44] volume 2, chapter IX) and for original articles see [38]. The aim of the next section is to define a structure that extends to algebraic geometry.

1. Hodge Structure on a smooth compact complex variety

In this section we shift our attention from harmonic forms to a new structure defined on the cohomology of non-singular compact complex algebraic varieties with underlying analytic structure not necessarily Kähler. In this setting the Hodge filtration F on cohomology plays an important role since it is obtained directly from a filtration on the de Rham complex. In this context, it is natural to introduce the spectral sequence defined by F, then the proof will consist in its degeneration at rank 1, although the ultimate argument will be a reduction to the decomposition on a Kähler manifold.

1.1. **Hodge structure (HS).**

It is rewarding to introduce the Hodge decomposition as a formal structure in linear algebra without any reference to its construction.

Definition 1.1 (HS1). *A Hodge structure of weight n is defined by the data:*
i) A finitely generated abelian group $H_{\mathbb{Z}}$;
ii) A decomposition by complex subspaces:

$$H_{\mathbb{C}} := H_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C} = \oplus_{p+q=n} H^{p,q} \quad \textit{satisfying} \quad H^{p,q} = \overline{H^{q,p}}.$$

The conjugation on $H_{\mathbb{C}}$ makes sense with respect to $H_{\mathbb{Z}}$.
A subspace $V \subset H_{\mathbb{C}} := H_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}$ satisfying $\overline{V} = V$ has a real structure, that is $V = (V \cap H_{\mathbb{R}}) \otimes_{\mathbb{R}} \mathbb{C}$. In particular $H^{p,p} = (H^{p,p} \cap H_{\mathbb{R}}) \otimes_{\mathbb{R}} \mathbb{C}$. We may suppose that $H_{\mathbb{Z}}$ is a free abelian group (the lattice), if we are interested only in its image in $H_{\mathbb{Q}} := H_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$.

With such an abstract definition we can perform linear algebra operations on Hodge structures and define morphisms of HS.

We can replace the data i) above by "a finite dimensional vector space $H_{\mathbb{Q}}$ over the rational numbers $\mathbb{Q}$". Equivalently we may consider a finite dimensional vector space over the real numbers $\mathbb{R}$. In these cases we speak of rational Hodge structure or real Hodge structure. The main point is to have a conjugation on $H_{\mathbb{C}}$ with respect to $H_{\mathbb{Q}}$ (or $H_{\mathbb{R}}$).

1.1.1. *The Hodge filtration.* To study variations of HS, Griffiths introduced the Hodge filtration which varies holomorphically with parameters. Given a Hodge decomposition $(H_{\mathbb{Z}}, H^{p,q})$ of weight n, we define a decreasing filtration F by subspaces of $H_{\mathbb{C}}$:

$$F^pH_{\mathbb{C}} := \oplus_{r\geq p} H^{r,n-r}.$$

Then, the following decomposition is satisfied:

$$H_{\mathbb{C}} = F^pH_{\mathbb{C}} \oplus \overline{F^{n-p+1}H_{\mathbb{C}}}.$$

since $\overline{F^{n-p+1}H_{\mathbb{C}}} = \oplus_{r\geq n-p+1}\overline{H^{r,n-r}} = \oplus_{i\leq p-1}H^{i,n-i}$ while $F^pH_{\mathbb{C}} = \oplus_{i\geq p}H^{i,n-i}$. The Hodge decomposition may be recovered from the filtration by the formula:

$$H^{p,q} = F^pH_{\mathbb{C}} \cap \overline{F^qH_{\mathbb{C}}}, \quad \text{for } p+q=n.$$

Hence, we obtain an equivalent definition of Hodge decompositions which play an important role in the development of Hodge theory, since the Hodge filtration exists naturally on the cohomology of a smooth compact complex algebraic variety X and it is defined by a natural filtration on the algebraic de Rham complex.

Definition 1.2 (HS2). *A Hodge structure of weight n is defined equivalently by the data:*
i) A finitely generated abelian group $H_{\mathbb{Z}}$;
ii) A filtration F by complex subspaces $F^pH_{\mathbb{C}}$ of $H_{\mathbb{C}}$ satisfying

$$H_{\mathbb{C}} = F^pH_{\mathbb{C}} \oplus \overline{F^{n-p+1}H_{\mathbb{C}}}.$$

then

$$H^{p,q} = F^pH_{\mathbb{C}} \cap \overline{F^qH_{\mathbb{C}}}, \quad \text{for}\, p+q=n.$$

1.1.2. *Linear algebra operations on Hodge structures.* Classical linear algebra operations may be carried on the above abstract definition of HS.

Definition 1.3. *A morphism $f : H = (H_{\mathbb{Z}}, H^{p,q}) \to H' = (H'_{\mathbb{Z}}, H'^{p,q})$ of Hodge structures of same weight n, is a homomorphism of abelian groups $f\colon H_{\mathbb{Z}} \to H'_{\mathbb{Z}}$ such that $f_{\mathbb{C}}\colon H_{\mathbb{C}} \to H'_{\mathbb{C}}$ is compatible with the decompositions, i.e, for any p, q, the $\mathbb{C}$-linear map $f_{\mathbb{C}}$ induces a $\mathbb{C}$-linear map from $H^{p,q}$ into $H'^{p,q}$.*

We have the following important result:

Proposition 1.4. *The Hodge structures of same weight n form an abelian category.*

In particular, the decomposition on the kernel of a morphism $\varphi : H \to H'$ is induced by the decomposition of H while on the cokernel it is the image of the decomposition of H'.

Remark 1.5. 1) A morphism of HS of distinct weights is torsion. Therefore, HS form an abelian category whose objects are HS and morphisms are morphisms of HS.
2) On a general subvector space V of H endowed with a HS, the induced subspaces $H^{p,q} \cap V$ do not define a HS decomposition on V.

1.1.3. *Tensor product and Hom.* Let H and H' be two HS of weight n and n'.

1) A Hodge structure on $H \otimes H'$ of weight $n + n'$ is defined as follows:
i) $(H \otimes H')_{\mathbb{Z}} = H_{\mathbb{Z}} \otimes H'_{\mathbb{Z}}$
ii) the bigrading of $(H \otimes H')_{\mathbb{C}} = H_{\mathbb{C}} \otimes H'_{\mathbb{C}}$ is the tensor product of the bigradings of $H_{\mathbb{C}}$ and $H'_{\mathbb{C}}$:

$$(H \otimes H')^{a,b} := \oplus_{p+p'=a,q+q'=b} H^{p,q} \otimes H'^{p',q'}.$$

2) A Hodge structure on $Hom(H, H')$ of weight $n' - n$ is defined as follows:
i) $Hom(H, H')_{\mathbb{Z}} := Hom_{\mathbb{Z}}(H_{\mathbb{Z}}, H'_{\mathbb{Z}})$;
ii) the components of the decomposition of $Hom(H, H')_{\mathbb{C}} := Hom_{\mathbb{Z}}(H_{\mathbb{Z}}, H'_{\mathbb{Z}}) \otimes \mathbb{C} \simeq Hom_{\mathbb{C}}(H_{\mathbb{C}}, H'_{\mathbb{C}})$ are defined by:

$$Hom(H, H')^{a,b} := \oplus_{p'-p=a,q'-q=b} Hom_{\mathbb{C}}(H^{p,q}, H^{p',q'}),$$

In particular the dual H^* to H is a HS of weight $-n$.

Example 1.6. Tate Hodge structure $\mathbb{Z}(1)$ is a HS of weight -2 defined by:

$$H_{\mathbb{Z}} = 2i\pi\mathbb{Z} \subset \mathbb{C}, \quad H_{\mathbb{C}} = H^{-1,-1}$$

It is purely bigraded of type $(-1, -1)$ of rank 1. The $m-$tensor product $\mathbb{Z}(1) \otimes \cdots \otimes \mathbb{Z}(1)$ of $\mathbb{Z}(1)$ is a HS of weight $-2m$ denoted by $\mathbb{Z}(m)$:

$$H_{\mathbb{Z}} = (2i\pi)^m\mathbb{Z} \subset \mathbb{C}, \quad H_{\mathbb{C}} = H^{-m,-m}$$

Let $H = (H_{\mathbb{Z}}, \oplus_{p+q=n} H^{p,q})$ be a HS of weight n, its $m-$twist is a Hodge Structure of weight $n - 2m$ denoted $H(m)$ and defined by

$$H(m)_{\mathbb{Z}} := H_{\mathbb{Z}} \otimes (2i\pi)^m\mathbb{Z}, \quad H(m)^{p,q} := H^{p+m,q+m}$$

Remark 1.7. The group of morphisms of HS is called the internal morphism group of the category of HS and is denoted by $Hom_{HS}(H, H')$; it is the subgroup of $Hom_{\mathbb{Z}}(H_{\mathbb{Z}}, H'_{\mathbb{Z}})$ of elements of type $(0, 0)$ in the HS on $Hom(H, H')$.
A homomorphism of type (r, r) is a morphism of the HS: $H \to H'(-r)$.

1.1.4. *Equivalent definition of HS.* Let $\mathbb{S}(\mathbb{R})$ denotes the subgroup:

$$\mathbb{S}(\mathbb{R}) = \left\{ M(u, v) = \begin{pmatrix} u & -v \\ v & u \end{pmatrix} \in GL(2, \mathbb{R}), \quad u, v \in \mathbb{R} \right\}.$$

It is isomorphic to $\mathbb{C}^*$ via the group homomorphism $M(u, v) \mapsto z = u + iv$.
The interest in this isomorphism is to give a structure of a real algebraic group on $\mathbb{C}^*$; indeed the set $\mathbb{S}(\mathbb{R})$ of matrices $M(u, v)$ is a real algebraic subgroup of $GL(2, \mathbb{R})$.

Definition 1.8. *A rational Hodge structure of weight $m \in \mathbb{Z}$, is defined by a $\mathbb{Q}-$vector space H and a representation of real algebraic groups $\varphi : \mathbb{S}(\mathbb{R}) \to GL(H_{\mathbb{R}})$ such that for $t \in \mathbb{R}^*$, $\varphi(t)(v) = t^m v$ for all $v \in H_{\mathbb{R}}$.*

See the proof of the equivalence with the action of the group $\mathbb{S}(\mathbb{R})$ in the appendix to chapter I ([7], formula A.21 and definition A.7). It is based on the lemma:

Lemma 1.9. *Let* $(H_{\mathbb{R}}, F)$ *be a real HS of weight* m*, defined by the decomposition* $H_{\mathbb{C}} = \oplus_{p+q=m} H^{p,q}$*, then the action of* $\mathbb{S}(\mathbb{R}) = \mathbb{C}^*$ *on* $H_{\mathbb{C}}$*, defined by:*

$$(z, v = \sum_{p+q=m} v_{p,q}) \mapsto \sum_{p+q=m} z^p \overline{z}^q v_{p,q},$$

corresponds to a real representation $\varphi : \mathbb{S}(\mathbb{R}) \to GL(H_{\mathbb{R}})$ *satisfying* $\varphi(t)(v) = t^m v$ *for* $t \in \mathbb{R}^*$*.*

Remark 1.10. i) The complex points of:

$$\mathbb{S}(\mathbb{C}) = \left\{ M(u,v) = \begin{pmatrix} u & -v \\ v & u \end{pmatrix} \in GL(2,\mathbb{C}), \quad u, v \in \mathbb{C} \right\}$$

are the matrices with complex coefficients $u, v \in \mathbb{C}$ with determinant $u^2 + v^2 \neq 0$. Let $z = u + iv, z' = u - iv$, then $zz' = u^2 + v^2 \neq 0$ such that $\mathbb{S}(\mathbb{C}) \xrightarrow{\sim} \mathbb{C}^* \times \mathbb{C}^*$: $(u, v) \mapsto (z, z')$ is an isomorphism satisfying $z' = \overline{z}$, for $u, v \in \mathbb{R}$; in particular $\mathbb{R}^* \hookrightarrow \mathbb{C}^* \times \mathbb{C}^* : t \mapsto (t, t)$.
ii) We can write $\mathbb{C}^* \simeq S^1 \times \mathbb{R}^*$ as the product of the real points of the unitary subgroup $U(1)$ of $\mathbb{S}(\mathbb{C})$ defined by $u^2 + v^2 = 1$ and of the multiplicative subgroup $\mathbb{G}_m(\mathbb{R})$ defined by $v = 0$ in $\mathbb{S}(\mathbb{R})$, and $\mathbb{S}(\mathbb{R})$ is the semi-product of S^1 and $\mathbb{R}^*$. Then the representation gives rise to a representation of $\mathbb{R}^*$ called the scaling since it fixes the weight, and of $U(1)$ which fixes the decomposition into $H^{p,q}$ and on which $\varphi(z)$ acts as multiplication by z^{p-q}.

1.1.5. *Polarized Hodge structure.* We add the additional structure of polarization since it will be satisfied by the cohomology of non-singular complex projective varieties (see 1.25).

Definition 1.11 (Polarization of HS)**.** *A Hodge structure* $(H_{\mathbb{Z}}, (H_{\mathbb{C}} \simeq \oplus_{p+q=n} H^{p,q}))$ *of weight* n *is polarized if a non-degenerate scalar product* Q *is defined on* $H_{\mathbb{Z}}$*, alternating if* n *is odd and symmetric if* n *is even, such that different terms* $H^{p,q}$ *of the Hodge decomposition are orthogonal to each other relative to the Hermitian form* F *on* $H_{\mathbb{C}}$ *defined as* $F(\alpha, \beta) := i^{p-q} Q(\alpha, \overline{\beta})$ *for* $\alpha, \beta \in H^{p,q}$ *and* $F(\alpha, \alpha)$ *is positive definite on the component of type* (p,q)*, i.e., it satisfies the Hodge-Riemann bilinear relations.*

1.2. **Spectral sequence of a filtered complex.** The techniques used in the construction of the Mixed Hodge structure on the cohomology of algebraic variety are based on the use of spectral sequences. Since we need to prove later an important result in this setting (the lemma on two filtrations), it is necessary to review here the theory using Deligne's notations.

1.2.1. *Spectral sequence defined by a filtered complex* (K, F)*.* Let $\mathbb{A}$ be an abelian category (for a first reading, we may suppose $\mathbb{A}$ is the category of vector spaces over a field). A complex K is defined by a family $(K^j)_{j \in \mathbb{Z}}$ of objects of $\mathbb{A}$ and morphisms $d_j : K^j \to K^{j+1}$ in $\mathbb{A}$ satisfying $d_{j+1} \circ d_j = 0$. A filtration F is defined by a family of subobjects $F^i \subset K^j$ satisfying $d_j(F^i K^j) \subset F^i K^{j+1}$. We consider decreasing filtrations $F^{i+1} \subset F^i$. In the case of an increasing filtration W_i, we obtain the terms of the spectral sequence from the decreasing case by a change of the sign of the indices of the filtration which transforms the increasing filtration into a decreasing one F with $F^i = W_{-i}$.

Definition 1.12. *Let K be a complex of objects of an abelian category $\mathbb{A}$, with a decreasing filtration by subcomplexes F. It induces a filtration F on the cohomology $H^*(K)$, defined by:*

$$F^iH^j(K) = Im(H^j(F^iK) \to H^j(K)), \quad \forall i,j \in \mathbb{Z}.$$

Let F^iK/F^jK for $i < j$ denotes the complex $(F^iK^r/F^jK^r, d_r)_{r\in\mathbb{Z}}$ with induced filtrations; in particular we write $Gr^p_F(K)$ for $F^pK/F^{p+1}K$ and the graded object $Gr^*_F(K) = \oplus_{p\in\mathbb{Z}} Gr^p_F(K)$. We define similarly $Gr^i_F H^j(K)$ and

$$Gr^*_F H^*(K) := \oplus_{j\in\mathbb{Z}} F^jH^*(K)/F^{j+1}H^*(K)$$

The spectral sequence defined by the filtered complex (K,F) gives a method to compute the graded object $Gr^*_F H^*(K)$ out of the cohomology $H^*(F^iK/F^jK)$ of the indices $i > j$ of the filtration. The spectral sequence $E^{p,q}_r(K,F)$ associated to F ([6], [11]) leads for large r and under mild conditions, to such graded cohomology defined by the filtration. It consists of indexed objects of $\mathbb{A}$ endowed with differentials (see below explicit definitions):

(1) terms $E^{p,q}_r$ for $r > 0, p,q \in \mathbb{Z}$,
(2) differentials $d_r : E^{p,q}_r \to E^{p+r,q-r+1}_r$ such that $d_r \circ d_r = 0$,
(3) isomorphisms:

$$E^{p,q}_{r+1} \simeq H(E^{p-r,q+r-1}_r \xrightarrow{d_r} E^{p,q}_r \xrightarrow{d_r} E^{p+r,q-r+1}_r)$$

of the (p,q)-term of index $r+1$ with the corresponding cohomology of the sequence with index r. To avoid ambiguity we may write ${}_FE^{p,q}_r$ or $E^{p,q}_r(K,F)$. The first term is defined as:

$$E^{p,q}_1 = H^{p+q}(Gr^p_F(K)).$$

The aim of the spectral sequence is to compute the term:

$$E^{p,q}_\infty := Gr^p_F(H^{p+q}(K))$$

These terms $Gr^p_F(H^{p+q}(K))$ are called the limit of the spectral sequence.
The spectral sequence is said to degenerate if:

$$\forall\, p,q,\ \exists\ r_0(p,q) \ \text{ such that } \ \forall r \geq r_0, \quad E^{p,q}_r \simeq E^{p,q}_\infty := Gr^p_F H^{p+q}(K).$$

1.2.2. *Formulas for the terms of the spectral sequence.*
It will be convenient to set for $r = 0$, $E^{p,q}_0 = Gr^p_F(K^{p+q})$.
To define the spectral terms $E^{p,q}_r(K,F)$ or ${}_FE^{p,q}_r$ or simply $E^{p,q}_r$ with respect to F for $r > 1$, we put for $r > 0$, $p,q \in \mathbb{Z}$:

$$Z^{p,q}_r = Ker(d : F^pK^{p+q} \to K^{p+q+1}/F^{p+r}K^{p+q+1})$$
$$B^{p,q}_r = F^{p+1}K^{p+q} + d(F^{p-r+1}K^{p+q-1})$$

Such formula still makes sense for $r = \infty$ if we set, for a filtered object (A,F), $F^{-\infty}(A) = A$ and $F^\infty(A) = 0$:

$$Z^{p,q}_\infty = Ker(d : F^pK^{p+q} \to K^{p+q+1})$$
$$B^{p,q}_\infty = F^{p+1}K^{p+q} + d(K^{p+q+1})$$

We set by definition:

$$E^{p,q}_r = Z^{p,q}_r/(B^{p,q}_r \cap Z^{p,q}_r) \quad E^{p,q}_\infty = Z^{p,q}_\infty/B^{p,q}_\infty \cap Z^{p,q}_\infty$$

The notations are similar to [11] but different from [22].

Remark 1.13. Given a statement on spectral sequences, we may deduce a dual statement if we use the following definition of B_r^{pq} and $Z_\infty^{p,q}$, dual to the definition of $Z_r^{p,q}$ and $Z_\infty^{p,q}$:

$$K^{p+q}/B_r^{p,q} = coker(d : F^{p-r+1}K^{p+q-1} \to K^{p+q}/F^{p+1}(K^{p+q}))$$
$$K^{p+q}/B_\infty^{p,q} = coker(d : K^{p+q+1} \to K^{p+q}/F^{p+1}K^{p+q})$$
$$E_r^{p,q} = Im(Z_r^{p,q} \to K^{p+q}/B_r^{p,q}) = Ker(K^{p+q}/B_r^{p,q} \to K^{p+q}/(Z_r^{p,q} + B_r^{p,q})).$$

Lemma 1.14. *For each r, there exists a differential d_r on the terms $E_r^{p,q}$ with the property that the cohomology is isomorphic to $E_{r+1}^{p,q}$:*

$$E_{r+1}^{p,q} \simeq H(E_r^{p-r,q+r-1} \xrightarrow{d_r} E_r^{p,q} \xrightarrow{d_r} E_r^{p+r,q-r+1}).$$

where d_r is induced by the differential $d : Z_r^{p,q} \to Z_r^{p+r,q-r+1}$.

The first term may be written as:

$$E_1^{p,q} = H^{p+q}(Gr_F^p(K))$$

so that the differentials d_1 are obtained as connecting morphisms defined by the short exact sequences of complexes

$$0 \to Gr_F^{p+1}K \to F^pK/F^{p+2}K \to Gr_F^pK \to 0.$$

Definition 1.15. *The decreasing filtration on the complex $K^\bullet$ is biregular if it induces a finite filtration on K^n for each degree n.*

Then, for each (p,q), there exists an integer $r_0 = r_0(p,q)$ such that for all $r \geq r_0$: $Z_r^{p,q} = Ker(d : F^pK^{p+q} \to K^{p+q+1})$, $\quad B_r^{p,q} = F^{p+1}K^{p+q} + dK^{p+q-1}$, hence the spectral sequence degenerates:

$$Z_r^{p,q} = Z_\infty^{p,q}, \quad B_r^{p,q} = B_\infty^{p,q}, \quad E_r^{p,q} = E_\infty^{p,q}, \quad \forall r \geq r_0$$

The spectral sequence degenerates at rank r (independent of (p,q)) if the differentials d_i of E_i^{pq} vanish for $i \geq r$ for a fixed r.
Most known applications are in the case of degenerate spectral sequences [25]. There is no easy general construction for (E_r, d_r) for $r > 0$.
Note that in some cases, it is not satisfactory to get only the graded cohomology and this is one motivation to be not happy with spectral sequences and to prefer to keep the complex as we shall do in the later section on derived category.

We emphasize that in Deligne-Hodge theory, the spectral sequences of the next section degenerates at rank 1 or 2: we will impose sufficient conditions to imply specifically in our case that $d_r = 0$ for $r > 1$, hence the terms $E_r^{p,q}$ are identical for all $r > 1$.

1.2.3. *The simple complex associated to a double complex and its filtrations.* A double complex is a bigraded object of an abelian category $\mathbb{A}$ with two differentials

$$K^{\bullet,\bullet} := (K^{i,j})_{i,j\in\mathbb{N}} \in \mathbb{A}, d' : K^{i,j} \to K^{i+1,j}, d'' : K^{i,j} \to K^{i,j+1}$$

satisfying $d' \circ d' = 0, d'' \circ d'' = 0$, and $d'' \circ d' + d' \circ d'' = 0$, The associated complex is defined as

$$(s(K^{\bullet,\bullet}), d) : s(K^{\bullet,\bullet})^n := \oplus_{i+j=n}K^{i,j}, d = d' + d''$$

There exist two natural decreasing filtrations F' and F'' on $s(K^{\bullet,\bullet})$ defined by:

$$(F')^p s(K^{\bullet,\bullet})^n := \oplus_{i+j=n, i\geq p}K^{i,j}, \quad (F'')^p s(K^{\bullet,\bullet})^n := \oplus_{i+j=n, j\geq p}K^{i,j}$$

The terms of the spectral sequences are:
${}'E_1^{p,q} := H^{p+q}(K^{p,\bullet}[-p], d'') = H^q(K^{p,\bullet}, d'')$, ${}'E_2^{p,q} := H^p[H^q(K^{\bullet,\bullet}, d''), d']$
${}''E_1^{p,q} := H^{p+q}(K^{\bullet,p}[-p], d') = H^q(K^{\bullet,p}), d')$, ${}''E_2^{p,q} := H^p[H^q(K^{\bullet,\bullet}, d'), d'']$
where q and p are the partial degrees with opposition to the total degree.

1.2.4. *Morphisms of spectral sequences.* A morphism of filtered complexes of objects of an abelian category $\mathbb{A}$:

$$f : (K, F) \to (K', F')$$

compatible with the filtrations: $f(F^i(K) \subset F^i(K')$ induces a morphism of the corresponding spectral sequences.

Definition 1.16. *i) A filtration F on a complex K is called biregular if it is finite in each degree of K.*
ii) A morphism $f : K \xrightarrow{\approx} K'$ of complexes of objects of $\mathbb{A}$ is a quasi-isomorphism denoted by $\approx$ if the induced morphisms on cohomology $H^*(f) : H^*(K) \xrightarrow{\sim} H^*(K')$ are isomorphisms for all degrees.
iii) A morphism $f : (K, F) \xrightarrow{\approx} (K', F)$ of complexes with biregular filtrations is a filtered quasi-isomorphism if it is compatible with the filtrations and induces a quasi-isomorphism on the graded object $Gr_F^*(f) : Gr_F^*(K) \xrightarrow{\approx} Gr_F^*(K')$.

In the case iii) we call (K', F) a filtered resolution of (K, F), while in ii) it is just a resolution.

Proposition 1.17. *Let $f : (K, F) \to (K', F')$ be a filtered morphism with biregular filtrations, then the following assertions are equivalent:*
i) f is a filtered quasi-isomorphism.
ii) $E_1^{p,q}(f) : E_1^{p,q}(K, F) \to E_1^{p,q}(K', F')$ is an isomorphism for all p, q.
iii) $E_r^{p,q}(f) : E_r^{p,q}(K, F) \to E_r^{p,q}(K', F')$ is an isomorphism for all $r \geq 1$ and all p, q.

By definition of the terms $E_1^{p,q}$, (ii) is equivalent to (i). We deduce iii) from ii) by induction: if we suppose the isomorphism in iii) satisfied for $r \leq r_0$, the isomorphism for $r_0 + 1$ follows since $E_{r_0}^{p,q}(f)$ is compatible with d_{r_0}.

1.3. **Hodge structure on the cohomology of non-singular compact complex algebraic varieties.** Here we consider the case of projective varieties which are Kähler, but also algebraic varieties which may be not Kähler. The technique of proof is based on the spectral sequence defined by the trivial filtration F on the de Rham complex of sheaves of analytic differential forms Ω_X^*. The new idea here is to observe the degeneracy of the spectral sequence of the filtered complex (Ω_X^*, F) at rank 1 and deduce the definition of the Hodge filtration on cohomology from the trivial filtration F on the complex without any reference to harmonic forms, although the proof of the decomposition is given via a reduction to the case of a projective variety, hence a compact Kähler manifold and the results on harmonic forms in this case.

Theorem 1.18 (Deligne [10]). *Let X be a smooth compact complex algebraic variety, then the filtration F by subcomplexes of the de Rham complex:*

$$F^p\Omega_X^* := \Omega_X^{*\geq p} = 0 \to 0 \cdots 0 \to \Omega_X^p \to \Omega_X^{p+1} \to \cdots \to \Omega_X^n \to 0$$

induces a Hodge filtration of a Hodge structure on the cohomology of X.

Definition 1.19. *The Hodge filtration F is defined on de Rham cohomology as follows:*

$$F^pH^i(X,\mathbb{C}) = F^p\mathbb{H}^i(X,\Omega_X^*) := Im(\mathbb{H}^i(X,F^p\Omega_X^*) \to \mathbb{H}^i(X,\Omega_X^*)),$$

where the first isomorphism is defined by holomorphic Poincaré lemma on the resolution of the constant sheaf $\mathbb{C}$ by the analytic de Rham complex Ω_X^*.

The proof is based on the degeneration at rank one of the spectral sequence with respect to F defined as follows:

$${}_FE_1^{p,q} := \mathbb{H}^{p+q}(X, Gr_F^p\Omega_X^*) \simeq H^q(X,\Omega_X^p) \Rightarrow Gr_F^p\mathbb{H}^{p+q}(X,\Omega_X^*).$$

We distinguish the projective case which follows from the underlying structure of Kähler variety from which the general case is deduced.

i) For X projective, the Dolbeault resolution (chapter I, 2.3 [7]) of the subcomplex $F^p\Omega_X^*$ define a double complex $(\mathcal{E}_X^{r,*})_{r\geq p}$ with the differentials: ∂ acting on the first degree r and $\overline{\partial}$ acting on the second degree $*$. A simple complex denoted $s(\mathcal{E}_X^{*,*})$ is defined in degree m by summing over the terms in degrees $p+q=m$, on which the filtration F is defined as follows (see 1.2.3):

$$(\mathcal{E}_X^*, d) = s(\mathcal{E}_X^{*,*}, \overline{\partial}, \partial), \quad (F^p\mathcal{E}_X^*, d) = s(\mathcal{E}_X^{r,*}, \overline{\partial}, \partial)_{r\geq p}$$

is a fine resolution of de Rham complex:

$$\mathbb{H}^n(X, F^p\Omega_X^*) \simeq H^n(X, F^p\mathcal{E}^*).$$

For $p=0$, the complex $(\mathcal{E}_X^*, d)$ contains the subcomplex of harmonic forms on which the differentials ∂ and $\overline{\partial}$ vanish. On the Kähler manifold, the terms $E_1^{p,q} = H^q(X,\Omega_X^p)$ are isomorphic to the vector subspace $\mathcal{H}^{p,q}(X)$ of $\mathbb{H}^{p+q}(X,\Omega_X^*) \simeq H^{p+q}(X,\mathcal{E}_X^*) \simeq H^{p+q}(X,\mathbb{C})$ consisting of harmonic forms of type (p,q). Since the spectral sequence $({}_FE_r^{p,q}, d_r)$ degenerates to the space $Gr_F^p\mathbb{H}^{p+q}(X,\Omega_X^*) \simeq \mathcal{H}^{p,q}(X)$, and since already $E_1^{p,q} = Gr_F^p\mathbb{H}^{p+q}(X,\Omega_X^*)$ have the same dimension, we deduce for all $r\geq 1$: $\dim_F E_1^{p,q} = \dim_F E_r^{p,q} = \dim\, \mathcal{H}^{p,q}(X)$. Hence $d_r = 0$ for $r\geq 1$, otherwise the dimension would drop, which means that the spectral sequence $({}_FE_r^{p,q}, d_r)$ degenerates at rank 1, and the HS is defined by the subspaces $H^{p,q}$ lifting the subspaces $Gr_F^p\mathbb{H}^{p+q}(X,\Omega_X^*)$ into $\mathbb{H}^{p+q}(X,\Omega_X^*)$.

ii) If X is not projective, there exists a projective variety and a projective birational morphism $f: X' \to X$, by Chow's lemma (see [44] p.69). By Hironaka's desingularization ([35]) we can suppose X' smooth projective, hence X' is a Kähler manifold. We continue to write f for the associated analytic morphism defined by the algebraic map f. By the general duality theory [47], there exists for all integers p a trace map $Tr(f): Rf_*\Omega_{X'}^p \to \Omega_X^p$ inducing a map on cohomology $Tr(f): H^q(X',\Omega_{X'}^p) \to H^q(X,\Omega_X^p)$, since $\mathbb{H}^q(X, Rf_*\Omega_{X'}^p) \simeq H^q(X',\Omega_{X'}^p)$ ([10] §4, [32] VI,4). In our case, since f is birational, the trace map is defined on the level of de Rham complexes as follows. Let U be an open subset of X and $V\subset U$ an open subset of U such that f induces an isomorphism: $f^{-1}(V) \xrightarrow{\sim} V$. A differential form $\omega' \in \Gamma(f^{-1}(U), \Omega_{X'}^p)$ induces a form ω on V which extends to a unique holomorphic form on U, then we define $Trf(\omega') := \omega \in \Gamma(U,\Omega_X^p)$ ([17], II, 2.1), from which we deduce that the composition morphism with the canonical reciprocal morphism f^* is the identity:

$$Tr(f)\circ f^* = Id : H^q(X,\Omega_X^p) \xrightarrow{f^*} H^q(X',\Omega_{X'}^p) \xrightarrow{Tr(f)} H^q(X,\Omega_X^p)$$

In particular f^* is injective. Since the map f^* is compatible with the filtrations F on de Rham complexes on X and X', we deduce a map of spectral sequences:

$$E_1^{p,q} = H^q(X, \Omega_X^p) \xrightarrow{f^*} E_1'^{p,q} = H^q(X', \Omega_{X'}^p), \quad f^* : E_r^{p,q}(X) \hookrightarrow E_r^{p,q}(X')$$

which commutes with the differentials d_r and is injective on all terms. The differential d_1 vanishes on X', hence it must vanish on X, then the terms for $r = 2$ coincide with the terms for $r = 1$. The proof continue by induction on r as we can repeat the argument for all r.

The degeneration of the Hodge spectral sequence on X at rank 1 follows, and it is equivalent to the isomorphism:

$$\mathbb{H}^n(X, F^p\Omega_X^*) \xrightarrow{\sim} F^p\mathbb{H}^n(X, \Omega_X^*).$$

Equivalently, the dimension of the hypercohomology of the de Rham complex $\mathbb{H}^j(X, \Omega_X^*)$ is equal to the dimension of Hodge cohomology $\oplus_{p+q=j} H^q(X, \Omega_X^p)$.

However, we still need to lift Dolbeault cohomology $H^q(X, \Omega_X^p)$ into subspaces of $\mathbb{H}^{p+q}(X, \Omega_X^*)$. Using conjugation, we deduce from the Hodge filtration, the definition of the following subspaces:

$$H^{p,q}(X) := F^pH^n(X, \mathbb{C}) \cap \overline{F^qH^n(X, \mathbb{C})}, \quad \text{for}\, p + q = n$$

satisfying $H^{p,q}(X) = \overline{H^{q,p}(X)}$. Now we check the decomposition:
$H^n(X, \mathbb{C}) = \oplus_{p+q=n} H^{p,q}(X)$ and deduce: $H^{p,q}(X) \simeq H^q(X, \Omega_X^p)$. Since f^* is injective we have:

$$F^pH^n(X) \cap \overline{F^{n-p+1}H^n(X)} \subset F^pH^n(X') \cap \overline{F^{n-p+1}H^n(X')} = 0.$$

This shows that $F^pH^n(X) + \overline{F^{n-p+1}H^n(X)}$ is a direct sum. We want to prove that this sum equals $H^n(X)$.

Let $h^{p,q} = \dim H^q(X, \Omega_X^p)$; since the spectral sequence degenerates at rank 1, we have:

$$\dim F^pH^n(X) = \sum_{i \geq p} h^{i,n-i}, \dim \overline{F^{n-p+1}H^n(X)} = \sum_{i \geq n-p+1} h^{i,n-i},$$

then:

$$\sum_{i \geq p} h^{i,n-i} + \sum_{i \geq n-p+1} h^{i,n-i} \leq \dim H^n(X) = \sum_i h^{i,n-i}$$

from which we deduce the inequality: $\sum_{i \geq p} h^{i,n-i} \leq \sum_{i \leq n-p} h^{i,n-i}$.

By Serre duality ([33], corollary 7.13) on X of dimension N, we have $H^j(X, \Omega_X^i)^* \simeq H^{N-j}(X, \Omega_X^{N-i})$ hence: $h^{i,j} = h^{N-i,N-j}$, which transforms the inequality into:

$$\sum_{N-i \leq N-p} h^{N-i,N-n+i} \leq \sum_{N-i \geq N-n+p} h^{N-i,N-n+i}$$

from which, by setting $j = N - i, q = N - n + p$, we deduce the opposite inequality on $H^m(X)$ for $m = 2N - n$:

$$\sum_{j \geq q} h^{j,m-j} \geq \sum_{j \leq m-q} h^{j,m-j},$$

for all q and m. In particular, setting $j = i$ and $m = n$:

$$\sum_{i \geq p} h^{i,n-i} \geq \sum_{i \leq n-p} h^{i,n-i} \quad \text{hence} \quad \sum_{i \geq p} h^{i,n-i} = \sum_{i \leq n-p} h^{i,n-i}.$$

This implies $\dim F^p + \dim \overline{F^{n-p+1}} = \dim H^n(X)$. Hence:

$$H^n(X) = F^p H^n(X) \oplus \overline{F^{n-p+1}} H^n(X)$$

which, in particular, induces a decomposition:

$$F^{n-1} H^n(X) = F^p H^n(X) \oplus H^{n-1,n-p+1}(X).$$

Corollary 1.20 ([11] corollary 5.4). *On a non-singular complex algebraic variety:*
i) A cohomology class a is of type (p,q) ($a \in H^{p,q}(X)$), if and only if it can be represented by a closed form α of type (p,q).
ii) A cohomology class a may be represented by a form α satisfying $\overline{\partial}(\alpha) = 0$ and $\partial\alpha = 0$.
iii) If a form α satisfy $\overline{\partial}(\alpha) = 0$ and $\partial\alpha = 0$, then the following four conditions are equivalent:
1) there exists β such that $\alpha = d\beta$, 2) there exists β such that $\alpha = \partial\beta$,
3) there exists β such that $\alpha = \overline{\partial}\beta$, 4) there exists β such that $\alpha = \partial\overline{\partial}\beta$.

Remark 1.21. 1) Classically, we use distinct notations for X with Zariski topology and X^{an} for the analytic associated manifold, then the filtration F is defined on the algebraic de Rham hypercohomology groups and the comparison theorem (see [28]) is compatible with the filtrations: $\mathbb{H}^i(X, F^p\Omega^*_X) \simeq \mathbb{H}^i(X^{an}, F^p\Omega^*_{X^{an}})$.

1.3.1. *Compatibility of Poincaré duality with HS.* On compact oriented differentiable manifolds, the wedge product of differential forms defines the cup-product on de Rham cohomology [25] and the integration of form of maximal degree defines the trace, so that we can deduce the compatibility of Poincaré duality with Hodge structure.

The Trace map. On a compact oriented manifold X of dimension n, we already mentioned that the cup product on de Rham cohomology is defined by the wedge product on the level of differential forms:

$$H^i_{deRham}(X) \otimes H^j_{deRham}(X) \xrightarrow{\cup} H^{i+j}_{deRham}(X)$$

By Stokes theorem, the integral over X of differential forms ω of highest degree n depends only on the class of ω modulo boundaries, hence it defines a map called the trace:

$$Tr : H^n(X, \mathbb{C}) \to \mathbb{C} \quad [\omega] \mapsto \int_X \omega.$$

It is convenient to extend the trace to a map on de Rham complex inducing zero in degree different than n:

$$Tr : \Omega^*_X[n] \to \mathbb{C} \quad [\omega] \mapsto \int_X \omega.$$

which is a special case of the definition of the trace on the dualizing complex which is needed for a general duality theory. On de Rham cohomology, we re-state:

Theorem 1.22 (Poincaré duality). *Let X be a compact oriented manifold of dimension n. The cup-product and the trace map:*

$$H^j(X, \mathbb{C}) \otimes H^{n-j}(X, \mathbb{C}) \xrightarrow{\cup} H^n(X, \mathbb{C}) \xrightarrow{Tr} \mathbb{C}$$

define an isomorphism:

$$H^j(X, \mathbb{C}) \xrightarrow{\sim} Hom(H^{n-j}(X, \mathbb{C}), \mathbb{C}).$$

Then we define the trace as a map of HS:

$$H^{2n}(X,\mathbb{C}) \xrightarrow{\sim} \mathbb{C}(-n), \quad \omega \to \frac{1}{(2i\pi)^n}\int_X \omega$$

such that Poincaré duality is compatible with HS:

$$H^{n-i}(X,\mathbb{C}) \simeq Hom(H^{n+i}(X,\mathbb{C}),\mathbb{C}(-n))$$

where the duality between $H^{p,q}$ and $H^{n-p,n-q}$ corresponds to Serre duality [33], corollary 7.13. The HS on homology is defined by duality:

$$(H_i(X,\mathbb{C}),F) \simeq Hom((H^i(X,\mathbb{C}),F),\mathbb{C})$$

where $\mathbb{C}$ is a HS of weight 0, hence $H_i(X,\mathbb{Z})$ is of weight $-i$. Then, Poincaré duality becomes an isomorphism of HS: $H^{n+i}(X,\mathbb{C}) \simeq H_{n-i}(X,\mathbb{C})(-n)$.

1.3.2. *Gysin morphism.* Let $f : X \to Y$ be an algebraic morphism of non-singular compact algebraic varieties with $\dim X = n$ and $\dim Y = m$, since $f^* : H^i(Y,\mathbb{Q}) \to H^i(X,\mathbb{Q})$ is compatible with HS, its Poincaré dual:

$$Gysin(f) : H^j(X,\mathbb{Q}) \to H^{j+2(m-n)}(Y,\mathbb{Q})(m-n)$$

is compatible with HS after a shift by $-2(m-n)$ on the right term.

1.4. **Lefschetz decomposition and Polarized Hodge structure.** We define one more specific structure on cohomology of compact Kähler manifold, namely the Lefschetz decomposition and Riemann bilinear relations on the primitive cohomology subspaces, which leads to the abstract definition of polarized Hodge Structures.

1.4.1. *Lefschetz decomposition and primitive cohomology.* The class of the fundamental form $[\omega] \in H^2(X,\mathbb{R})$ of Hodge type $(1,1)$ defined by the Hermitian metric on the underlying Kähler structure on X acts on cohomology by repeated cup-product with $[\omega]$ and defines morphisms:

$$L : H^q(X,\mathbb{R}) \to H^{q+2}(X,\mathbb{R}), \quad L : H^q(X,\mathbb{C}) \to H^{q+2}(X,\mathbb{C}) : [\varphi] \mapsto [\omega] \wedge [\varphi]$$

Referring to de Rham cohomology, the action of L is represented on the level of forms as $\varphi \mapsto \omega \wedge \varphi$ (since ω is closed, the image of a closed form (resp. a boundary) is closed (resp. a boundary)) and L depends on $[\omega]$.

Definition 1.23. *Let $n = \dim X$. The primitive cohomology subspaces are defined for $i \geq 0$ as:*

$$H^{n-i}_{prim}(X,\mathbb{R}) := Ker(L^{i+1} : H^{n-i}(X,\mathbb{R}) \to H^{n+i+2}(X,\mathbb{R}))$$

and similarly for complex coefficients $H^{n-i}_{\mathrm{prim}}(X,\mathbb{C}) \simeq H^{n-i}_{\mathrm{prim}}(X,\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$.

The operator L is compatible with Hodge decomposition since it sends the subspace $H^{p,q}$ to $H^{p+1,q+1}$. We say that the morphism L is of Hodge type $(1,1)$. Hence, the action $L^{i+1} : H^{n-i}(X,\mathbb{C}) \to H^{n+i+2}(X,\mathbb{C})$ is a morphism of Hodge type $(i+1,i+1)$, and the kernel is endowed with an induced Hodge decomposition which is a strong condition on the primitive subspaces :

$$H^{p,q}_{\mathrm{prim}} := H^{p+q}_{\mathrm{prim}} \cap H^{p,q}(X,\mathbb{C}), \quad H^i_{\mathrm{prim}}(X,\mathbb{C}) = \oplus_{p+q=i} H^{p,q}_{prim}.$$

The following isomorphism, referred to as Hard Lefschetz Theorem, puts a strong condition on the cohomology of projective, and more generally compact Kähler, manifolds and gives rise to a decomposition of the cohomology in terms of primitive subspaces:

Theorem 1.24. *Let X be a compact Kähler manifold.*
i) Hard Lefschetz Theorem. *The iterated linear operator L induces isomorphisms for each i:*

$$L^i : H^{n-i}(X,\mathbb{R}) \xrightarrow{\sim} H^{n+i}(X,\mathbb{R}),\ L^i : H^{n-i}(X,\mathbb{C}) \xrightarrow{\sim} H^{n+i}(X,\mathbb{C})$$

ii) Lefschetz Decomposition. *The cohomology decomposes into a direct sum of image of primitive subspaces by L^r, $r \geq 0$:*

$$H^q(X,\mathbb{R}) = \oplus_{r\geq 0} L^r H^{q-2r}_{\text{prim}}(\mathbb{R}), \quad H^q(X,\mathbb{C}) = \oplus_{r\geq 0} L^r H^{q-2r}_{\text{prim}}(\mathbb{C})$$

The Lefschetz decomposition is compatible with Hodge decomposition.
iii) If X is moreover projective, then the action of L is defined with rational coefficients and the decomposition applies to rational cohomology.

We refer to the proof in (chapter I, 5.2 [7]) and for more details to [52].

1.4.2. *Hermitian product on cohomology.* We deduce from the isomorphism in the hard Lefschetz theorem and Poincaré duality, a scalar product on cohomology of smooth complex projective varieties compatible with HS and satisfying relations known as Hodge Riemann relations leading to a polarization of the primitive cohomology which is an additional highly rich structure characteristic of such varieties. Representing cohomology classes by differential forms, we define a bilinear form:

$$\mathrm{Q}(\alpha,\beta) = (-1)^{\frac{j(j-1)}{2}} \int_X \alpha\wedge\beta\wedge\omega^{n-j}, \quad \forall[\alpha],[\beta]\in H^j(X,\mathbb{C})$$

where ω is the Kähler class, the product of α with ω^{n-j} represents the action of L^{n-j} and the integral of the product with β represents Poincaré duality.

Properties of the product. The above product $\mathrm{Q}(\alpha,\beta)$ depends on j and only on the class of α and β. The following properties are satisfied:
i) the product Q is real (it takes real values on real forms) since ω is real, in other terms the matrix of Q is real, skew-symmetric if j is odd and symmetric if j is even;
ii) It is non-degenerate (by Lefschetz isomorphism and Poincaré duality);
iii) By consideration of type, the Hodge and Lefschetz decompositions satisfy, with respect to Q, the relations :

$$\mathrm{Q}(H^{p,q}, H^{p',q'}) = 0, \quad \text{unless } p = q', q = p'.$$

On projective varieties the Kähler class is in the integral lattice defined by cohomology with coefficients in $\mathbb{Z}$, hence the product is defined on rational cohomology and preserves the integral lattice. In this case we have more precise positivity relations in terms of the primitive component $H^{p,q}_{prim}(X,\mathbb{C})$ of the cohomology $H^{p+q}(X,\mathbb{C})$.

Proposition 1.25 (Hodge-Riemann bilinear relations)**.** *Let X be smooth projective, then the product $i^{p-q}Q(\alpha,\overline{\alpha})$ is positive definite on the primitive component $H^{p,q}_{prim}$:*

$$i^{p-q}Q(\alpha,\overline{\alpha}) > 0, \quad \forall\alpha\in H^{p,q}_{prim},\ \alpha\neq 0$$

We refer to the proof in (chapter I, 5.3 [7]).
This result suggests to introduce the Weil operator C on cohomology:

$$C(\alpha) = i^{p-q}\alpha, \quad \forall\alpha\in H^{p,q}$$

Notice that C is a real operator since for a real vector $v = \sum_{p+q=j} v^{p,q}$, $v^{q,p} = \overline{v^{p,q}}$, hence $\overline{Cv} = \sum \overline{i^{p-q}v^{p,q}} = \sum \bar{i}^{p-q} v^{q,p} = \sum i^{q-p} v^{q,p} = C\overline{v}$, as $\bar{i}^{p-q} = i^{q-p}$. It depends on the decomposition; in particular for a varying Hodge structure H_t^{p-q} with parameters t, the action $C = C_t$ depends on t. We deduce from Q a non-degenerate Hermitian product:

$$\mathrm{F}(\alpha,\beta) = Q(C(\alpha),\overline{\beta}),\ \mathrm{F}(\beta,\alpha) = \overline{\mathrm{F}(\alpha,\beta)} \quad \forall[\alpha],[\beta] \in H^j(X,\mathbb{C})$$

We use $\overline{Q(\alpha,\beta)} = Q(\overline{\alpha},\overline{\beta})$ since Q is real, to check for $\alpha,\beta \in H^{p,q}$:
$\overline{\mathrm{F}(\alpha,\beta)} = \overline{Q(i^{p-q}\alpha,\overline{\beta})} = \overline{i}^{p-q}Q(\overline{\alpha},\beta) = (-1)^j i^{p-q}Q(\overline{\alpha},\beta) = (-1)^{2j} i^{p-q}Q(\beta,\overline{\alpha}) = \mathrm{F}(\beta,\alpha)$.

Remark 1.26. When the class $[\omega] \in H^j(X,\mathbb{Z})$ is integral, which is the case for projective varieties, the product Q is integral, i.e. with integral value on integral classes.

1.4.3. *Projective case.* On a smooth projective variety $X \subset \mathbb{P}^n_{\mathbb{C}}$, we can choose the first Chern class $c_1(\mathcal{L})$ of the line bundle $\mathcal{L}$ defined by the restriction of the hyperplane line bundle $\mathcal{O}(1)$ on $\mathbb{P}^n_{\mathbb{C}}$ to represent the Kähler class $[\omega]$ defined by the Hermitian metric on the underlying Kähler structure on X: $c_1(\mathcal{L}) = [\omega]$ see (chapter I, 3.2 [7]). Hence we have an integral representative of the class $[\omega]$ in the image of $H^2(X,\mathbb{Z}) \to H^2(X,\mathbb{C})$, which has as consequence the action of the operator L on rational cohomology $L : H^q(X,\mathbb{Q}) \to H^{q+2}(X,\mathbb{Q})$. This fact characterizes projective varieties among compact Kähler manifolds since a theorem of Kodaira ([52] chapter VI) states that a Kähler variety with an integral class $[\omega]$ is projective, i.e. it can be embedded as a closed analytic subvariety in a projective space, hence by Chow lemma it is necessarily a projective subvariety.

Remark 1.27 (Topological interpretation). In the projective case, the class $[\omega]$ corresponds to the homology class of an hyperplane section $[H] \in H_{2n-2}(X,\mathbb{Z})$, so that the operator L corresponds to the intersection with $[H]$ in X and the result is an isomorphism:

$$H_{n+k}(X) \xrightarrow{(\cap[H])^k} H_{n-k}(X)$$

The primitive cohomology $H^{n-k}_{prim}(X)$ corresponds to the image of:

$$H_{n-k}(X - H,\mathbb{Q}) \to H_{n-k}(X,\mathbb{Q}).$$

Corollary 1.28. *The cohomology of a projective complex smooth variety carries a polarized Hodge structure defined by its Hodge decomposition and the above positive definite Hermitian product.*

1.5. **Examples.** We list now some known examples of HS mainly on tori.

1.5.1. *Cohomology of projective spaces.* The HS on cohomology is polarized by the first Chern class of the canonical line bundle $H = c_1(\mathcal{O}_{\mathbb{P}^n}(1))$ dual to the homology class of a hyperplane (chapter I, 3.2, formula 3.13 [7]).

Proposition 1.29. $H^i(\mathbb{P}^n,\mathbb{Z}) = 0$ *for i odd and* $H^i(\mathbb{P}^n,\mathbb{Z}) = \mathbb{Z}$ *for i even with generator $[H]^i$ equal to the cup product to the power i of the cohomology class of an hyperplane $[H]$, hence:* $H^{2r}(\mathbb{P}^n,\mathbb{Z}) = \mathbb{Z}(-r)$ *as HS.*

1.5.2. *Hodge decomposition on complex tori.* Let T_Λ be a complex torus of dimension r defined by a lattice $\Lambda \subset \mathbb{C}^r$ as in (chapter I, example 1.17 [7]). The cohomology of a complex torus T_Λ is easy to compute by Künneth formula, since it is diffeomorphic to a product of circles: $T_\Lambda \simeq (S^1)^{2r}$. Hence $H^1(T_\Lambda, \mathbb{Z}) \simeq \mathbb{Z}^{2r}$ and $H^j(T_\Lambda, \mathbb{Z}) \simeq \wedge^j H^1(T_\Lambda, \mathbb{Z})$.

The cohomology with complex coefficients may be computed by de Rham cohomology. In this case, since the complex tangent space is trivial, we have natural cohomology elements given by the translation-invariant forms, hence with constant coefficients. The finite complex vector space T_0^* of constant holomorphic $1-$forms is isomorphic to $\mathbb{C}^r$ and generated by $dz_j, j \in [1, r]$ and the Hodge decomposition reduces to prove $H^j(X, \mathbb{C}) \simeq \oplus_{p+q=j} \wedge^p T_0^* \otimes \wedge^q \overline{T_0^*}$, $p \geq 0, q \geq 0$ which is here a consequence of the above computation of the cohomology.

1.5.3. *Moduli space of complex tori.* We may parameterize all lattices as follows:
- the group $GL(2r, \mathbb{R})$ acts transitively on the set of all lattices of $\mathbb{C}^r$.
We choose a basis $\tau = (\tau_1, \ldots, \tau_{2i-1}, \tau_{2i} \ldots \tau_{2r})$, $i \in [1, r]$, of a lattice L, then it defines a basis of $\mathbb{R}^{2r}$ over $\mathbb{R}$. An element φ of $GL(2r, \mathbb{R})$ is given by the linear transformation which sends τ into the basis $\varphi(\tau) = \tau'$ of $\mathbb{R}^{2r}$ over $\mathbb{R}$. The element φ of $GL(2r, \mathbb{R})$ is carrying the lattice L onto the lattice L' defined by the basis τ'.
- The isotopy group of this action is $GL(2r, \mathbb{Z})$, since τ and τ' define the same lattice if and only if $\varphi \in GL(2r, \mathbb{Z})$.
Hence the space of lattices is the quotient group $GL(2r, \mathbb{R})/GL(2r, \mathbb{Z})$.
- Two tori defined by the lattice L and L' are analytically isomorphic if and only if there is an element of $GL(r, \mathbb{C})$ which transforms the lattice L into the lattice L' (see chapter 1, example (1.17) [7]).

It follows that the parameter space of complex tori is the quotient:

$$GL(2r, \mathbb{Z}) \backslash GL(2r, \mathbb{R})/GL(r, \mathbb{C})$$

where $GL(r, \mathbb{C})$ is embedded naturally in $GL(2r, \mathbb{R})$ as a complex linear map is $\mathbb{R}-$linear.

For $r = 1$, the quotient $GL(2, \mathbb{R})/GL(1, \mathbb{C})$ is isomorphic to $\mathbb{C} - \mathbb{R}$, since, up to complex isomorphisms, a lattice is generated by $1, \tau \in \mathbb{C}$ independent over $\mathbb{R}$, hence completely determined by $\tau \in \mathbb{C} - \mathbb{R}$. The moduli space is the orbit space of the action of $GL(2, \mathbb{Z})$ on the space $GL(2, \mathbb{R})/GL(1, \mathbb{C}) = \mathbb{C} - \mathbb{R}$. Since $GL(2, \mathbb{Z})$ is the disjoint union of $SL(2, \mathbb{Z})$ and the integral 2×2-matrices of determinant equal to -1, that orbit space is the one of the action of $SL(2, \mathbb{Z})$ on the upper half plane:

$$(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, z) \mapsto \frac{az + b}{cz + d}.$$

The Hodge structures of the various complex tori define a variation of Hodge structures on the moduli space of all complex tori.

1.5.4. *Polarized Hodge structures of dimension 2 and weight 1.* Let H be a real vector space of dimension 2 endowed with a skew symmetric quadratic form, (e_1, e_2) a basis in which the matrix of Q is

$$Q = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad Q(u, v) = {}^t v Q u, \quad u, v \in \mathbb{Q}^2$$

then Hodge decomposition $H_\mathbb{C} = H^{1,0} \oplus H^{0,1}$ is defined by the one dimensional subspace $H^{1,0}$ with generator v of coordinates $(v_1, v_2) \in \mathbb{C}^2$. While $Q(v, v) = 0$

since Q is skew-symmetric, the Hodge-Riemann positivity condition is written as $iQ(v,\overline{v}) = -i(v_1\overline{v}_2 - \overline{v}_1 v_2) > 0$, hence $v_2 \neq 0$, so we divide by v_2 to get a unique representative of $H^{1,0}$ by a vector of the form $v = (\tau, 1)$ with $\mathrm{Im}(\tau) > 0$. Hence the Poincaré half-plane $\{z \in \mathbb{C} : \mathrm{Im} z > 0\}$ is a classifying space for polarized Hodge Structures of dimension 2. This will apply to the cohomology $H := H^1(T, \mathbb{R})$ of a complex torus of dim.1. Note that the torus is a projective variety. Indeed, a Weierstrass elliptic function $\mathcal{P}(z)$ is defined as a sum of a series over the elements of the non-degenerate lattice in $\mathbb{C}$; it defines with its derivative a map $(\mathcal{P}(z), \mathcal{P}'(z)) : T \to \mathbb{C}^2 \subset \mathbb{P}^2_C$ in the affine space, which is completed into an embedding of the torus onto a smooth elliptic curve in the projective space ([33], chapter 4, §4).

1.5.5. *Polarized Hodge structures of weight 1 and abelian varieties.* Given a Hodge structure $(H_{\mathbb{Z}}, H^{1,0} \oplus H^{0,1})$, the projection on $H^{0,1}$ induces an isomorphism of $H_{\mathbb{R}}$ onto $H^{0,1}$ as real vector spaces:

$$H_{\mathbb{R}} \to H_{\mathbb{C}} = H^{1,0} \oplus H^{0,1} \to H^{0,1}$$

since $\overline{H^{0,1}} = H^{1,0}$, hence $H_{\mathbb{R}} \cap H^{0,1} = 0$. Then we deduce that $H_{\mathbb{Z}}$ is a lattice in the complex space $H^{0,1}$, and the quotient $T := H^{0,1}/H_{\mathbb{Z}}$ is a complex torus.
In the case of a complex manifold X, the exact sequence of sheaves defined by $f \mapsto e^{2i\pi f}$:

$$0 \to \mathbb{Z} \to \mathcal{O}_X \to \mathcal{O}_X^* \to 1$$

where 1 at right is the neutral element of the multiplicative group structure on $\mathcal{O}_X^*$, and its associated long exact sequence:

$$\to H^1(X, \mathbb{Z}) \to H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_X^*) \to H^2(X, \mathbb{Z}) \to$$

have the following geometric interpretation. When the space $H^1(X, \mathcal{O}_X^*)$ is identified with isomorphisms classes of line bundles on X, the last morphism defines the Chern class of the line bundle. We deduce the isomorphism

$$T := \frac{H^1(X, \mathcal{O}_X)}{ImH^1(X, \mathbb{Z})} \simeq Pic^0(X) := Ker(H^1(X, \mathcal{O}_X^*) \xrightarrow{c_1} H^2(X, \mathbb{Z}))$$

of the torus T with the Picard variety $Pic^0(X)$ parameterizing the holomorphic line bundles $\mathcal{L}$ on X with first Chern class equal to zero: $c_1(\mathcal{L}) = 0$. The Picard variety of a smooth projective variety is an abelian variety (define a Kähler form with integral class on $Pic^0(X)$).

1.5.6. *Polarized Hodge structures of weight 2.*

$$(H_{\mathbb{Z}},\, H^{2,0} \oplus H^{1,1} \oplus H^{0,2};\, H^{0,2} = \overline{H^{2,0}},\, H^{1,1} = \overline{H^{1,1}};\, Q)$$

the intersection form Q is symmetric and $\mathrm{F}(\alpha, \beta) = Q(\alpha, \overline{\beta})$ is Hermitian. The decomposition is orthogonal for F with F positive definite on $H^{1,1}$, negative definite on $H^{2,0}$ and $H^{0,2}$. The subspace $H^{2,0} \subset H_{\mathbb{C}}$ is totally isotropic for Q and the Hermitian product F is negative definite on $H^{2,0}$. Then we define $H^{0,2}$ by conjugation, and by orthogonality: $H^{1,1} = (H^{2,0} \oplus H^{0,2})^{\perp}$. The signature of Q is $(h^{1,1}, 2h^{2,0})$.

1.6. **Cohomology class of a subvariety and Hodge conjecture.** An oriented compact topological variety V of dimension n has a fundamental class in its homology group $H_{2n}(V,\mathbb{Z})$, which plays an important role in Poincaré duality. By duality, the class corresponds to a class in the cohomology group $H^0(V,\mathbb{Z})$. In algebraic and analytic geometry, the existence of the fundamental class extends to all subvarieties since their singular subset has codimension two. We construct here the class of a closed complex algebraic subvariety (resp. complex analytic subspace) of codimension p in a smooth complex projective variety V (resp. compact Kähler manifold) in de Rham cohomology $H^{2p}_{DR}(V)$ and we show it is rational of Hodge type (p,p), then we state the Hodge conjecture.

Lemma 1.30. *Let X be a complex manifold and Z a compact complex analytic subspace of dimension m in X. The integral on the smooth open subset of Z of the restriction of a form ω on X, is convergent and defines a linear map on forms of degree $2m$. It induces a linear map on cohomology*

$$cl(Z): H^{2m}(X,\mathbb{C}) \to \mathbb{C}, \quad [\omega] \mapsto \int_{Z_{smooth}} \omega_{|Z}$$

where Z_{smooth} is the non-singular part of Z. Moreover, if X is compact Kähler, $cl(Z)$ vanish on all components of the Hodge decomposition which are distinct from $H^{m,m}$.

If Z is compact and smooth, the integral is well defined on the class $[\omega]$, since for $\omega = d\eta$ the integral vanishes by Stokes theorem, as the integral of η on the boundary $\partial Z = \emptyset$ of Z vanish.

If Z is not smooth, the easiest proof is to use a desingularisation $\pi : Z' \to Z$ inducing an isomorphism $Z'_{smooth} \simeq Z_{smooth}$ (see [35]) which implies the convergence of the integral of the restriction $\omega_{|Z}$ and the equality of the integrals of $\pi^*(\omega_{|Z})$ and $\omega_{|Z}$. In particular the integral is independent of the choice of Z'. The restriction $\omega_{|Z}$ of degree $2m$ vanish unless it is of type (m,m) since Z_{smooth} is an analytic manifold of dimension m.

If the compact complex analytic space Z is of codimension p in the smooth compact complex manifold X, its class $cl(Z) \in H^{2n-2p}(X,\mathbb{C})^*$ corresponds by Poincaré duality on X, to a fundamental cohomology class $[\eta_Z] \in H^{2p}(X,\mathbb{C})$. Then we have by definition:

Lemma and Definition 1.31. *For a compact complex manifold X, the fundamental cohomology class $[\eta_Z] \in H^{p,p}(X,\mathbb{C})$ of a closed complex submanifold Z of codimension p satisfies the following relation:*

$$\int_X \varphi \wedge \eta_Z = \int_Z \varphi_{|Z}, \quad \forall \varphi \in \mathcal{E}^{n-p,n-p}(X).$$

Lemma 1.32. *For a compact Kähler manifold X:*

$$H^{p,p}(X,\mathbb{C}) \neq 0 \quad \textit{for } 0 \leq p \leq \dim X.$$

In fact, the integral of the volume form $\int_X \omega^n > 0$. It follows that the cohomology class $\omega^n \neq 0 \in H^{2n}(X,\mathbb{C})$, hence the cohomology class $\omega^p \neq 0 \in H^{p,p}(X,\mathbb{C})$ since its cup product with ω^{n-p} is not zero.

Lemma 1.33. *For a compact Kähler manifold X, the cohomology class of a compact complex analytic closed submanifold Z of codimension p is a non-zero element $[\eta_Z] \neq 0 \in H^{p,p}(X,\mathbb{C})$, for $0 \leq p \leq$ dim X.*

Proof. For a compact Kähler manifold X, let ω be a Kähler form, then the integral on Z of the restriction $\omega_{|Z}$ is positive since it is a Kähler form on Z, hence $[\eta_Z] \neq 0$:

$$\int_X (\wedge^{n-p}\omega) \wedge \eta_Z = \int_Z \wedge^{n-p}(\omega_{|Z}) > 0.$$

□

The dual vector space $Hom_{\mathbb{C}}(H^{2m}(X,\mathbb{C}),\mathbb{C})$ is naturally isomorphic to the homology vector space which corresponds to fundamental class naturally defined in homology.

Indeed, the homology class $i_*[Z]$ of a compact complex analytic subspace $i : Z \to X$ of codimension p in the smooth compact complex manifold X of dimension n corresponds by the inverse of Poincaré duality isomorphism D_X, to a fundamental cohomology class:

$$[\eta_Z]^{top} \in H^{2p}(X,\mathbb{Z}).$$

Lemma 1.34. *The canonical morphism $H_{2n-2p}(X,\mathbb{Z}) \to H^{2n-2p}(X,\mathbb{C})^*$ carries the topological class $[Z]$ of an analytic subspace Z of codimension p in X into the fundamental class $cl(Z)$.*

Respectively. the morphism $H^{2p}(X,\mathbb{Z}) \to H^{2p}(X,\mathbb{C})$ carries the topological class $[\eta_Z]^{top}$ to $[\eta_Z]$.

1.6.1. A natural question is to find what conditions can be made on cohomology classes representing classes of algebraic subvarieties, including the classes in cohomology with coefficients in $\mathbb{Z}$. The Hodge type (p,p) of the fundamental class of analytic compact submanifold of codimension p is an analytic condition. The search for properties characterizing classes of cycles has been motivated by Hodge conjecture:

Definition 1.35 (Hodge classes)**.** *For each integer $p \in \mathbb{N}$, let $H^{p,p}(X)$ denotes the subspace of $H^{2p}(X,\mathbb{C})$ of type (p,p), the group of rational (p,p) cycles*

$$H^{p,p}(X,\mathbb{Q}) := H^{2p}(X,\mathbb{Q}) \cap H^{p,p}(X) = H^{2p}(X,\mathbb{Q}) \cap F^pH^{2p}(X,\mathbb{C})\}$$

is called the group of rational Hodge classes of type (p,p).

Definition 1.36. *An $r-$cycle of an algebraic variety X is a formal finite linear combination $\sum_{i\in[1,h]} m_iZ_i$ of closed irreducible subvarieties Z_i of dimension r with integer coefficients m_i. The group of $r-$cycles is denoted by $\mathcal{Z}_r(X)$.*

For a compact complex algebraic manifold, the class of closed irreducible subvarieties of codimension p extends into a linear morphism:

$$cl_{\mathbb{Q}} : \mathcal{Z}_p(X) \otimes \mathbb{Q} \to H^{p,p}(X,\mathbb{Q}) : \sum_{i\in[1,h]} m_iZ_i \mapsto \sum_{i\in[1,h]} m_i\eta_{Z_i}, \forall m_i \in \mathbb{Q}$$

The elements of the image of $cl_{\mathbb{Q}}$ are called rational algebraic Hodge classes of type (p,p).

Hodge conjecture [37]. On a non-singular complex projective variety, any rational Hodge class of type (p,p) is algebraic, i.e in the image of $cl_{\mathbb{Q}}$.

Originally, the Hodge conjecture was stated with $\mathbb{Z}$-coefficients: let $\varphi : H^{2p}(X,\mathbb{Z}) \to H^{2p}(X,\mathbb{C})$ denotes the canonical map and define the group of integral Hodge classes of type (p,p) as:

$$H^{p,p}(X,\mathbb{Z}) := \{x \in H^{2p}(X,\mathbb{Z}) : \varphi(x) \in H^{p,p}(X,\mathbb{C})\}$$

then: is any integral Hodge class of type (p, p) algebraic ?
However, there are torsion elements which cannot be represented by algebraic cycles [34]. Also, there exists compact Kähler complex manifolds not containing enough analytic subspaces to represent all Hodge cycles [49]. See [15] for a summary of results related to Hodge conjecture.

Remark 1.37 (Absolute Hodge cycle). Deligne added another property for algebraic cycles by introducing the notion of absolute Hodge cycle (see [13], chapter 10 [9]). An algebraic cycle Z is necessarily defined over a field extension K of finite type over $\mathbb{Q}$. Then its cohomology class in the de Rham cohomology of X over the field K defines, for each embedding $\sigma : K \to \mathbb{C}$, a cohomology class $[Z]_\sigma$ of type (p, p) in the cohomology of the complex manifold X^{an}_σ. We refer to chapter 10 of this volume [9] for a comprehensive study of this theory.

Remark 1.38 (Grothendieck construction of the fundamental class). For an algebraic subvariety Z of codimension p in a variety X of dimension n, the fundamental class of Z can be defined as an element of the group $Ext^p(\mathcal{O}_Z, \Omega^p_X)$ (see [27], [32]). Let U be an affine open subset and suppose that $Z \cap U$ is defined as a complete intersection by p regular functions f_i, if we use the Koszul resolution of $\mathcal{O}_{Z\cap U}$ defined by the elements f_i to compute the extension groups, then the fundamental class is defined by a symbol:

$$\begin{bmatrix} df_1 \wedge \cdots \wedge df_p \\ f_1 \cdots f_p \end{bmatrix} \in Ext^p(\mathcal{O}_{Z\cap U}, \Omega^p_U).$$

This symbol is independent of the choice of generators of $\mathcal{O}_{Z\cap U}$, and it is the restriction of a unique class defined over Z which defines the cohomology class of Z in the de Rham cohomology group $\mathbb{H}^{2p}_Z(X, \Omega^*_X)$ with support in Z ([16], [17] IV, proposition 5). The extension groups and cohomology groups with support in closed algebraic subvarieties form the basic idea to construct the dualizing complex ω_X of X as part of Grothendieck duality theory (see [32]).

2. Mixed Hodge Structures (MHS)

Motivated by conjectured properties of cohomology of varieties in positive characteristic as stated by A. Weil, Deligne imagined the correct structure to put on cohomology of any complex algebraic variety, possibly open or singular (later Deligne solved Weil's conjecture).
Since the knowledge of the linear algebra structure underlying MHS is supposed to help the reader before being confronted with their construction, we first introduce the category of mixed Hodge structures (MHS) consisting of vector spaces endowed with weight W and Hodge F filtrations by subvector spaces satisfying adequate axioms [11]. The striking result to remember is that morphisms of MHS are *strict* (see 2.1.3) for both filtrations W and F.
Only, linear algebra is needed for the proofs in the abstract setting. The corresponding theory in an abelian category is developed for objects with opposite filtrations.
- We start with a formal study of filtrations needed in the definitions of MHS. Since we are essentially concerned by filtrations of vector spaces, it is not more difficult to describe this notion in the terminology of abelian categories.
- Next we define MHS and prove that they form an abelian category with morphims which are strict with respect to W and F (see 2.20). The proof is based on a canonical decomposition of a MHS (see 2.2.3).

- We end this section with a result on spectral sequences which is essential for the construction of MHS (Deligne's lemma on two filtrations see 2.3.1).

2.1. **Filtrations.** Given a morphism in an additive category, the isomorphism between the image and co-image is one of the conditions to define an abelian category. In the additive category of filtered vector spaces endowed with finite filtrations by vector subspaces, given a morphism $f : (H, F) \to (H', F')$ compatible with the filtrations: $f(F^j) \subset F'^j$, the filtration obtained from F on the co-image does not coincide in general with the filtration induced by F' on the image. The morphism is called strict if they coincide (see 2.1.3).
This kind of problem will occur for various repeated restrictions of filtrations. Here we need to define with precision the properties of induced filtrations, since this is at the heart of the subject of MHS.

On a sub-quotient B/C of a filtered vector space A, and in general of a filtered object of an abelian category, there are two ways to restrict the filtration, first on the sub-object B, then on the quotient B/C, or on the quotient A/C then on the sub-object B/C. On an object A with two filtrations W and F, we can restrict F on each Gr^n_W, or W on each Gr^m_F and we may get different objects if we start with W then F or we inverse the order. We need to know precise relations between the induced filtrations in these various ways, and to know the precise behavior of a linear morphism with respect to such induced filtrations.

As a main application, we will indicate (see 2.3.1 below), three different ways to induce a filtration on the terms of a spectral sequence. A central result for MHS is to give conditions on the complex such that the three induced filtrations coincide.

We recall first preliminaries on filtrations in an abelian category $\mathbb{A}$.

Definition 2.1. *A decreasing (resp. increasing) filtration F of an object A of $\mathbb{A}$ is a family of sub-objects of A, satisfying*

$$\forall n, m, \qquad n \leq m \Longrightarrow F^m(A) \subset F^n(A) \quad (resp.\, n \leq m \Longrightarrow F_n(A) \subset F_m(A))$$

The pair (A, F) will be called a filtered object.

If F is a decreasing filtration (resp. W an increasing filtration), a shifted filtration $F[n]$ by an integer n is defined as

$$(F[n])^p(A) = F^{p+n}(A), \quad (W[n])_p(A) = W_{p-n}(A).$$

Decreasing filtrations F are considered for a general study. Statements for increasing filtrations W are deduced by the change of indices $W_n(A) = F^{-n}(A)$. A filtration is finite if there exist integers n and m such that $F^n(A) = A$ and $F^m(A) = 0$.

2.1.1. *Induced filtration.* A filtered object (A, F) induces a filtration on a sub-object $i : B \to A$ of A defined by $F^n(B) = B \cap F^n(A)$. Dually, the quotient filtration on A/B with canonical projection $p : A \to A/B$, is defined by:

$$F^n(A/B) = p(F^n(A)) = (B + F^n(A))/B \simeq F^n(A)/(B \cap F^n(A)).$$

Definition 2.2. *The graded object associated to (A, F) is defined as:*

$$Gr_F(A) = \oplus_{n \in \mathbb{Z}} Gr^n_F(A) \quad \textit{where} \quad Gr^n_F(A) = F^n(A)/F^{n+1}(A).$$

2.1.2. The cohomology of a sequence of filtered morphisms:

$$(A, F) \xrightarrow{f} (B, F) \xrightarrow{g} (C, F)$$

satisfying $g \circ f = 0$ is defined as $H = Ker\, g/Im\, f$; it is *filtered* and endowed with the quotient filtration of the induced filtration on $Ker\, g$. It is equal to the induced filtration on H by the quotient filtration on $B/Im\, f$ where $H \subset (B/Im\, f)$.

A morphism of filtered objects $(A, F) \xrightarrow{f} (B, F)$ is a morphism $A \xrightarrow{f} B$ satisfying $f(F^n(A)) \subset F^n(B)$ for all $n \in \mathbb{Z}$. Filtered objects (resp. of finite filtration) form an additive category with existence of *kernel*, *cokernel*, *image* and *coimage* of a morphism with natural induced filtrations. However, the image and co-image will not be necessarily filtered-isomorphic.
This is the main obstruction to obtain an abelian category. To get around this obstruction, we are lead to define the notion of strictness for compatible morphims.

2.1.3. *Strictness.* For filtered modules over a ring, a morphism of filtered objects $f : (A, F) \to (B, F)$ is called strict if the relation:

$$f(F^n(A)) = f(A) \cap F^n(B)$$

is satisfied; that is, any element $b \in F^n(B) \cap ImA$ is already in $Im\, F^n(A)$. Next, we consider an additive category where we suppose the existence of a subobject of B, still denoted $f(A) \cap F^n(B)$, containing all common subobjects of $f(A)$ and $F^n(B)$.

Definition 2.3. *A filtered morphism in an additive category $f : (A, F) \to (B, F)$ is called strict, or strictly compatible with the filtrations, if it induces a filtered isomorphism $(Coim(f), F) \to (Im(f), F)$ from the coimage to the image of f with their induced filtrations. Equivalently: $f(F^n(A)) = f(A) \cap F^n(B)$ for all n.*

This concept is basic to the theory, so we mention the following criteria:

Proposition 2.4. *i) A filtered morphism $f : (A, F) \to (B, F)$ of objects with finite filtrations is strict if and only if the following exact sequence of graded objects is exact:*

$$0 \to Gr_F(Kerf) \to Gr_F(A) \to Gr_F(B) \to Gr_F(Cokerf) \to 0$$

ii) Let $S :\ (A, F) \xrightarrow{f} (B, F) \xrightarrow{g} (C, F)$ be a sequence S of strict morphisms such that $g \circ f = 0$, then the cohomology H with induced filtration satisfies:

$$H(Gr_F(S)) \simeq Gr_F(H(S)).$$

2.1.4. *Degeneration of a spectral sequence and strictness.* The following proposition gives a remarkable criteria of degeneration at rank 1 of the spectral sequence with respect to the filtration F.

Proposition 2.5. *(Prop. 1.3.2* [11]*) Let K be a complex with a biregular filtration F. The following conditions are equivalent:*
i) The spectral sequence defined by F degenerates at rank 1 ($E_1 = E_\infty$)
ii) The morphism $H^i(F^p(K)) \to F^pH^i(K)$ is an isomorphism for all p.
iii) The differentials $d : K^i \to K^{i+1}$ are strictly compatible with the filtrations.

2.1.5. *Two filtrations.* Let A be an object of $\mathbb{A}$ with two filtrations F and G. By definition, $Gr^n_F(A)$ is a quotient of a sub-object of A, and as such, it is endowed with an induced filtration by G. Its associated graded object defines a bigraded object $Gr^n_G Gr^m_F(A)_{n,m\in\mathbb{Z}}$. We refer to [11] for:

Lemma 2.6 (Zassenhaus' lemma)**.** *The objects $Gr^n_G Gr^m_F(A)$ and $Gr^m_F Gr^n_G(A)$ are isomorphic.*

Remark 2.7. Let H be a third filtration on A. It induces a filtration on $Gr_F(A)$, hence on $Gr_G Gr_F(A)$. It induces also a filtration on $Gr_F Gr_G(A)$. These filtrations do not correspond in general under the above isomorphism. In the formula $Gr_H Gr_G Gr_F(A)$, G et H have symmetric role, but not F and G.

2.1.6. *Hom and tensor functors.* If A and B are two filtered objects of $\mathbb{A}$, we define a filtration on the left exact functor Hom:

$$F^k Hom(A,B) = \{f : A \to B : \forall n, f(F^n(A)) \subset F^{n+k}(B)\}$$

Hence:

$$Hom((A,F),(B,F)) = F^0(Hom(A,B)).$$

If A and B are modules on some ring, we define:

$$F^k(A \otimes B) = \sum_m Im(F^m(A) \otimes F^{k-m}(B) \to A \otimes B)$$

2.1.7. *Multifunctor.* In general if $H : \mathbb{A}_1 \times \ldots \times \mathbb{A}_n \to \mathbb{B}$ is a right exact multi-additive functor, we define:

$$F^k(H(A_1,...,A_n)) = \sum_{\sum k_i = k} Im(H(F^{k_1}A_1,...,F^{k_n}A_n) \to H(A_1,...,A_n))$$

and dually if H is left exact:

$$F^k(H(A_1,...,A_n) = \bigcap_{\sum k_i = k} Ker(H(A_1,...,A_n) \to H(A_1/F^{k_1}A_1,...,A_n/F^{k_n}A_n))$$

If H is exact, both definitions are equivalent.

Example 2.8 (Spectral sequence for increasing filtrations)**.** To illustrate the notations of induced filtrations, we give the formulas of the spectral sequence with respect to an increasing filtration W on a complex K, where the precedent formulas (see 1.2.1) are applied to the decreasing filtration F deduced from W by the change of indices: $F^i = W_{-i}$.
We set for all j, $n \leq m$ and $n \leq i \leq m$:

$$W_i H^j(W_m K/W_n K) = Im(H^j(W_i K/W_n K) \to H^j(W_m K/W_n K)$$

then the terms for all $r \geq 1, p$ and q are written as follows:

Lemma 2.9. *The terms of the spectral sequence for (K,W) are equal to:*

$$E^{p,q}_r(K,W)) = Gr^W_{-p} H^{p+q}(W_{-p+r-1}K/W_{-p-r}K).$$

Proof. Let (K^p_r, W) denotes the quotient complex $K^p_r := W_{-p+r-1}K/W_{-p-r}K$ with the induced filtration by subcomplexes W; we put:

$Z^{p,q}_\infty(K^p_r,W) := Ker\,(d : (W_{-p}K^{p+q}/W_{-p-r}K^{p+q}) \to (W_{-p+r-1}K^{p+q+1}/W_{-p-r}K^{p+q+1}))$

$B^{p,q}_\infty(K^p_r,W) := (W_{-p-1}K^{p+q} + dW_{-p+r-1}K^{p+q-1})/W_{-p-r}K^{p+q}$

which coincide, *up to the quotient by* $W_{-p-r}K^{p+q}$, with $Z_r^{p,q}(K,W)$ (resp. $B_r^{p,q}(K,W)$) with:

$$Z_r^{p,q} := Ker\,(d : W_{-p}K^{p+q} \to K^{p+q+1}/W_{-p-r}K^{p+q+1})$$
$$B_r^{p,q} := W_{-p-1}K^{p+q} + dW_{-p+r-1}K^{p+q-1}$$

then, we define:

$$E_\infty^{p,q}(K_r^p, W) = \frac{Z_\infty^{p,q}(K_r^p, W)}{B_\infty^{p,q}(K_r^p, W) \cap Z_\infty^{p,q}(K_r^p, W)} = Gr_{-p}^W H^{p+q}(W_{-p+r-1}K/W_{-p-r}K)$$

and find:

$$E_r^{p,q}(K,W) = Z_r^{p,q}/(B_r^{p,q} \cap Z_r^{pq}) = Z_\infty^{p,q}(K_r^p, W)/(B_\infty^{p,q}(K_r^p, W) \cap Z_\infty^{p,q}(K_r^p, W))$$
$$= E_\infty^{p,q}(K_r^p, W) = Gr_{-p}^W H^{pq}(W_{-p+r-1}K/W_{-p-r}K)$$

□

To define the differential d_r, we consider the exact sequence:

$$0 \to W_{-p-r}K/W_{-p-2r}K \to W_{-p+r-1}K/W_{-p-2r}K \to W_{-p+r-1}K/W_{-p-r}K \to 0$$

and the connecting morphism:

$$H^{p+q}(W_{-p+r-1}K/W_{-p-r}K) \xrightarrow{\partial} H^{p+q+1}(W_{-p-r}K/W_{-p-2r}K)$$

the injection $W_{-p-r}K \to W_{-p-1}K$ induces a morphism:

$$\varphi : H^{p+q+1}(W_{-p-r}K/W_{-p-2r}K) \to W_{-p-r}H^{p+q+1}(W_{-p-1}K/W_{-p-2r}K).$$

Let π denote the projection on the right term below, equal to $E_r^{p+r,q-r+1}$:

$$W_{-p-r}H^{p+q+1}(W_{-p-1}K/W_{-p-2r}K) \xrightarrow{\pi} Gr_{-p-r}^W H^{p+q+1}(W_{-p-1}K/W_{-p-2r}K)$$

the composition of morphisms $\pi\circ\varphi\circ\partial$ restricted to $W_{-p}H^{p+q}(W_{-p+r-1}K/W_{-p-r}K)$ induces the differential:

$$d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$$

while the injection $W_{-p+r-1} \to W_{-p+r}K$ induces the isomorphism:

$$H(E_r^{p,q}, d_r) \xrightarrow{\sim} E_{r+1}^{pq} = Gr_{-p}^W H^{p+q}(W_{-p+r}K/W_{-p-r-1}K).$$

2.1.8. *$n-$opposite filtrations.* The linear algebra of HS applies to an abelian category $\mathbb{A}$ if we use the following definition where no conjugation map appears.

Definition 2.10 (n-opposite filtrations)**.** *Two finite filtrations F and G on an object A of an abelian category $\mathbb{A}$ are n-opposite if:*

$$Gr_F^p Gr_G^q(A) = 0 \quad \textit{for } p+q \neq n.$$

Hence, the Hodge filtration F on a Hodge structure of weight n is n-opposite to its conjugate $\overline{F}$.

Example 2.11. Let $A^{p,q}$ be a bigraded object of $\mathbb{A}$ such that: $A^{p,q} = 0$ for $p+q \neq n$ and $A^{p,q} = 0$ for all but a finite number of pairs (p,q), then we define two n-opposite filtrations F and G on $A := \bigoplus_{p,q} A^{p,q}$ by:

$$F^p(A) = \bigoplus_{p' \geq p} A^{p',q'}, \quad G^q(A) = \bigoplus_{q' \geq q} A^{p',q'}$$

We have $Gr_F^p Gr_G^q(A) = A^{p,q}$.

Proposition 2.12. *i) Two finite filtrations F and G on an object A are n-opposite, if and only if:*

$$\forall\, p, q, \quad p + q = n + 1 \Rightarrow F^p(A) \oplus G^q(A) \simeq A.$$

ii) If F and G are n-opposite, and if we put $A^{p,q} = F^p(A) \cap G^q(A)$, for $p + q = n$, $A^{p,q} = 0$ for $p + q \neq n$, then A is a direct sum of $A^{p,q}$.

The above constructions define an equivalence of categories between objects of $\mathbb{A}$ with two n-opposite filtrations and bigraded objects of $\mathbb{A}$ of the type described in the example, moreover F and G can be deduced from the bigraded object $A^{p,q}$ of A by the above procedure.

2.1.9. *Opposite filtrations on a HS.* The previous definitions of HS may be stated in terms of induced filtrations. For any A-module H_A where A is a subring of $\mathbb{R}$, the complex conjugation extends to a conjugation on the space $H_\mathbb{C} = H_A \otimes_A \mathbb{C}$. A filtration F on $H_\mathbb{C}$ has a conjugate filtration $\overline{F}$ such that $(\overline{F})^j H_\mathbb{C} = \overline{F^j H_\mathbb{C}}$.

Definition 2.13 (HS3)**.** *An A-Hodge structure H of weight n consists of:*
i) an A-module of finite type H_A,
ii) a finite filtration F on $H_\mathbb{C}$ (the Hodge filtration) such that F and its conjugate $\overline{F}$ satisfy the relation:

$$Gr^p_F Gr^q_{\overline{F}}(H_\mathbb{C}) = 0, \quad \textit{for } p + q \neq n$$

equivalently F is opposite to its conjugate $\overline{F}$.

The HS is called real when $A = \mathbb{R}$, rational when $A = \mathbb{Q}$ and integral when $A = \mathbb{Z}$, then the module H_A or its image in $H_\mathbb{Q}$ is called the lattice.

2.1.10. *Complex Hodge structure.* For some arguments in analysis, we don't need a real substructure.

Definition 2.14. *A complex HS of weight n on a complex vector space H is given by a pair of $n-$opposite filtrations F and $\overline{F}$, hence a decomposition into a direct sum of subspaces:*

$$H = \oplus_{p+q=n} H^{p,q}, \quad \textit{where } H^{p,q} = F^p \cap \overline{F}^q$$

The two $n-$opposite filtrations F and $\overline{F}$ on a complex HS of weight n can be recovered from the decomposition by the formula:

$$F^p = \oplus_{i \geq p} H^{i,n-i} \quad \overline{F}^q = \oplus_{i \leq n-q} H^{i,n-i}$$

Here we do not assume the existence of conjugation although we keep the notation $\overline{F}$. An A-Hodge Structure of weight n defines a complex HS on $H = H_\mathbb{C}$.

2.1.11. *Polarized complex Hodge structure.* To define polarization, we recall that the conjugate space $\overline{H}$ of a complex vector space H, is the same group H with a different complex structure. The identity map on the group H defines a real linear map $\sigma : H \to \overline{H}$ and the product by scalars satisfying the relation:

$$\forall \lambda \in \mathbb{C},\, v \in H : \, \lambda \times_{\overline{H}} \sigma(v) := \sigma(\overline{\lambda} \times_H v),$$

where $\lambda \times_{\overline{H}} \sigma(v)$ (resp. $(\overline{\lambda} \times_H v)$) is the product with a scalar in $\overline{H}$ (resp. H). Then, the complex structure on $\overline{H}$ is unique. On the other hand a complex linear morphism $f : V \to V'$ defines a complex linear conjugate morphism $\overline{f} : \overline{V} \to \overline{V}'$ satisfying $\overline{f}(\sigma(v)) = \sigma(f(v))$.

Definition 2.15. *A polarization of a complex HS of weight n is a bilinear morphism $S : H \otimes \overline{H} \to \mathbb{C}$ such that:*

$$S(x,\sigma(y)) = (-1)^n \overline{S(y,\sigma(x))} \ \textit{ for } x,y \in L \textit{ and } S(F^p, \sigma(\overline{F}^q)) = 0 \text{ for } p+q > n.$$

and moreover $S(C(H)u, \sigma(v))$ is a positive definite Hermitian form on H where $C(H)$ denotes the Weil action on H ($C(H)u := i^{p-q}u$ for $u \in H^{p,q}$).

Example 2.16. A complex HS of weight 0 on a complex vector space H is given by a decomposition into a direct sum of subspaces $H = \oplus_{p\in\mathbb{Z}} H^p$ with $F^p = \oplus_{i\geq p} H^p$ and $\overline{F}^q = \oplus_{i\leq -q} H^i$, hence $F^p \cap \overline{F}^{-p} = H^p$.

A polarization is an Hermitian form on H for which the decomposition is orthogonal and whose restriction to H^p is definite for p even and negative definite for odd p.

2.2. **Mixed Hodge Structure (MHS).** Let $A = \mathbb{Z}, \mathbb{Q}$ or $\mathbb{R}$, and define $A \otimes \mathbb{Q}$ as $\mathbb{Q}$ if $A = \mathbb{Z}$ or $\mathbb{Q}$ and $\mathbb{R}$ if $A = \mathbb{R}$ (according to ([12], III.0.3) we can suppose A a Noetherian subring of $\mathbb{R}$ such that $A \otimes \mathbb{R}$ is a field). For an A-module H_A of finite type, we write $H_{A\otimes\mathbb{Q}}$ for the $(A\otimes\mathbb{Q})$-vector space $(H_A)\otimes_A (A\otimes\mathbb{Q})$. It is a rational space if $A = \mathbb{Z}$ or $\mathbb{Q}$ and a real space if $A = \mathbb{R}$.

Definition 2.17 (MHS)**.** *An A-Mixed Hodge Structure H consists of:*
1) an A-module of finite type H_A,
2) a finite increasing filtration W of the $A\otimes\mathbb{Q}$-vector space $H_{A\otimes\mathbb{Q}}$ called the weight filtration,
3) a finite decreasing filtration F of the $\mathbb{C}$-vector space $H_\mathbb{C} = H_A \otimes_A \mathbb{C}$, called the Hodge filtration, and $W^\mathbb{C} := W \otimes \mathbb{C}$, such that the systems:

$$Gr_n^W H = (Gr_n^W(H_{A\otimes\mathbb{Q}}), Gr_n^W(H_{A\otimes\mathbb{Q}})\otimes\mathbb{C} \simeq Gr_n^{W^\mathbb{C}} H_\mathbb{C}, (Gr_n^{W^\mathbb{C}} H_\mathbb{C}, F))$$

are $A\otimes\mathbb{Q}$-HS of weight n.

Recall the definition of the induced filtration:
$F^p Gr_n^{W^\mathbb{C}} H_\mathbb{C} := ((F^p \cap W_n^\mathbb{C}) + W_{n-1}^\mathbb{C})/W_{n-1}^\mathbb{C} \subset W_n^\mathbb{C}/W_{n-1}^\mathbb{C}$.The MHS is called real, if $A = \mathbb{R}$, rational, if $A = \mathbb{Q}$, and integral, if $A = \mathbb{Z}$.

2.2.1. *Three opposite filtrations.* Most of the proofs on the algebraic structure of MHS may be carried for three filtrations in an abelian category defined as follows

Definition 2.18 (Opposite filtrations)**.** *Three finite filtrations W (increasing), F and G on an object A of $\mathbb{A}$ are opposite if*

$$Gr_F^p Gr_G^q Gr_n^W(A) = 0 \quad \textit{for } p+q \neq n.$$

This condition is symmetric in F and G. It means that F and G induce on $W_n(A)/W_{n-1}(A)$ two n-opposite filtrations, then $Gr_n^W(A)$ is bigraded

$$W_n(A)/W_{n-1}(A) = \oplus_{p+q=n} A^{p,q} \quad \text{where } A^{p,q} = Gr_F^p Gr_G^q Gr_{p+q}^W(A)$$

Example 2.19. i) A bigraded object $A = \oplus A^{p,q}$ of finite bigrading has the following three opposite filtrations

$$W_n = \oplus_{p+q\leq n} A^{p,q}, \quad F^p = \oplus_{p'\geq p} A^{p',q'}, \quad G^q = \oplus_{q'\geq q} A^{p',q'}$$

ii) In the definition of an A-MHS, the filtration $W_\mathbb{C}$ on $H_\mathbb{C}$ obtained from W by scalar extension, the filtration F and its complex conjugate, form a system $(W_\mathbb{C}, F, \overline{F})$ of three opposite filtrations.

2.2.2. *Morphism of mixed Hodge structures.* A morphism $f : H \to H'$ of MHS is a morphism $f : H_A \to H'_A$ whose extension to $H_{\mathbb{Q}}$ (resp. $H_{\mathbb{C}}$) is compatible with the filtration W, i.e. $f(W_j H_A) \subset W_j H'_A$ (resp. F, i.e. $f(F^j H_A) \subset F^j H'_A$), which implies that it is also compatible with $\overline{F}$).
These definitions allow us to speak of the category of MHS. The main result of this chapter is:

Theorem 2.20 (Deligne)**.** *The category of mixed Hodge structures is abelian.*

The proof relies on the following decomposition.

2.2.3. *Canonical decomposition of the weight filtration.* While there is an equivalence for a HS, between the Hodge filtration and the Hodge decomposition, there is no such result for the weight filtration of a MHS. In the category of MHS, the short exact sequence $0 \to Gr^W_{n-1} \to W_n/W_{n-2} \to Gr^W_n \to 0$ is a non-split extension of the two Hodge structures Gr^W_n and Gr^W_{n-1}. The construction relies on the existence, for each pair of integers (p,q), of the following canonical subspaces of $H_{\mathbb{C}}$:

$$I^{p,q} = (F^p \cap W_{p+q}) \cap (\overline{F^q} \cap W_{p+q} + \overline{F^{q-1}} \cap W_{p+q-2} + \overline{F^{q-2}} \cap W_{p+q-3} + \cdots) \tag{2.1}$$

By construction they are related for $p+q = n$ to the components $H^{p,q}$ of the Hodge decomposition $Gr^W_n H \simeq \oplus_{n=p+q} H^{p,q}$:

Proposition 2.21. *The projection, for each (p,q):*

$$\varphi : W_{p+q} \to Gr^W_{p+q} H \simeq \oplus_{p'+q'=p+q} H^{p',q'}$$

induces an isomorphism $I^{p,q} \xrightarrow{\sim} H^{p,q}$. Moreover:

$$W_n = \oplus_{p+q \leq n} I^{p,q},\ F^p = \oplus_{p' \geq p} I^{p',q'}.$$

Remark 2.22. The proof by induction is based on the formula for $i > 0$

$$F^{p_i} H \oplus \overline{F^{q_i}} H \simeq Gr^W_{n-i} H, \quad \forall p_i, q_i : p_i + q_i = n - i + 1,$$

used to construct for $i > 0$ a decreasing family (p_i, q_i) starting with $p_0 = p, q_0 = q : p+q = n$. We choose here a sequence of the following type $(p_0, q_0) = (p,q)$, and for $i > 0$, $p_i = p, q_i = q - i + 1$ which explains the asymmetry in the formula defining $I^{p,q}$.
ii) In general $I^{p,q} \neq \overline{I^{q,p}}$, we have only $I^{p,q} \equiv \overline{I^{q,p}}$ modulo W_{p+q-2}.
iii) A morphism of MHS is necessarily compatible with this decomposition. This fact is the main ingredient to prove later the strictness (see 2.1.3) with respect to W and F.

2.2.4. *Proof of the proposition.* The restriction of φ to $I^{p,q}$ is an isomorphism:
i) *Injectivity of φ.* Let $n = p+q$. We deduce from the formula modulo W_{n-1} for $I^{p,q}$: $\varphi(I^{p,q}) \subset H^{p,q} = (F^p \cap \overline{F^q})(Gr^W_n H)$. Let $v \in I^{p,q}$ such that $\varphi(v) = 0$, then $v \in F^p \cap W_{n-1}$ and, since the class $cl(v) \in (F^p \cap \overline{F^q})(Gr^W_{n-1} H) = 0$ as $p+q > n-1$, $cl(v)$ must vanish; so we deduce that $v \in F^p \cap W_{n-2}$. This is a step in an inductive argument based on the formula $F^p \oplus \overline{F^{q-r+1}} \simeq Gr^W_{n-r} H$. We want to prove $v \in F^p \cap W_{n-r}$ for all $r > 0$. We just proved this for $r = 2$. We write

$$v \in \overline{F^q} \cap W_n + \sum_{r-1 \geq i \geq 1} \overline{F^{q-i}} \cap W_{n-i-1} + \sum_{i > r-1 \geq 1} \overline{F^{q-i}} \cap W_{n-i-1}.$$

Since $W_{n-i-1} \subset W_{n-r-1}$ for $i > r-1$, the right term is in W_{n-r-1}, and since $\overline{F}$ is decreasing, we deduce: $v \in F^p \cap \overline{F^{q-r+1}} \cap W_{n-r}$ modulo W_{n-r-1}.
As $(F^p \cap \overline{F^{q-r+1}})Gr^W_{n-r}H = 0$ for $r > 0$, the class $cl(v) = 0 \in Gr^W_{n-r}H$, then $v \in F^p \cap W_{n-r-1}$. Finally, as $W_{n-r} = 0$ for large r, we deduce $v = 0$.
ii) *Surjectivity of* φ. Let $\alpha \in H^{p,q}$; there exists $v_0 \in F^p \cap W_n$ and $u_0 \in F^q \cap W_n$ such that $\varphi(v_0) = \alpha = \varphi(\overline{u}_0)$, hence $v_0 = \overline{u}_0 + w_0$ with $w_0 \in W_{n-1}$. Applying the formula $F^p \oplus \overline{F^q} \simeq Gr^W_{n-1}H$, the class of w_0 decomposes as $cl(w_0) = cl(v') + cl(\overline{u}')$ with $v' \in F^p \cap W_{n-1}$ and $u' \in F^q \cap W_{n-1}$; hence there exists $w_1 \in W_{n-2}$ such that $v_0 = \overline{u}_0 + v' + \overline{u}' + w_1$. Let $v_1 := v_0 - v'$ and $u_1 = u_0 + u'$, then

$$v_1 = \overline{u}_1 + w_1, \quad \text{where } u_1 \in F^q \cap W_n,\ v_1 \in F^p \cap W_n,\ w_1 \in W_{n-2}.$$

By an inductive argument on k, we apply the formula: $F^p \oplus \overline{F^{q-k+1}} \simeq Gr^W_{n-k}H$ to find vectors v_k, u_k, w_k satisfying:

$$\begin{aligned} &v_k \in F^p \cap W_n,\ w_k \in W_{n-1-k},\ \varphi(v_k) = \alpha,\ v_k = \overline{u}_k + w_k \\ &u_k \in F^q \cap W_n + F^{q-1} \cap W_{n-2} + F^{q-2} \cap W_{n-3} + \dots + F^{q+1-k} \cap W_{n-k} \end{aligned}$$

then, we decompose the class of w_k in $Gr^W_{n-k-1}H$ in the inductive step as above. For large k, $W_{n-1-k} = 0$, hence we find: $v_k = \overline{u}_k \in I^{p,q}$ and $\varphi(v_k) = \alpha$.
Moreover $W_n = W_{n-1} \oplus (\oplus_{p+q=n} I^{p,q})$, hence, by induction on n, W_n is a direct sum of $I^{p,q}$ for $p + q \leq n$.
Next we suppose, by induction, the formula for F^p satisfied for all $v \in W_{n-1} \cap F^p$. The image of an element $v \in F^p \cap W_n$ in $Gr^W_n H$ decomposes into Hodge components of type $(i, n-i)$ with $i \geq p$ since $v \in F^p \cap W_n$. Hence, the decomposition of v may be writen as $v = v_1 + v_2$ with $v_1 \in \oplus_{i<p} I^{i,n-i}$ and $v_2 \in \oplus_{i \geq p} I^{i,n-i}$ with $v_1 \in W_{n-1}$ since its image vanishes in $Gr^W_n H$. The formula for F^p follows by induction.

2.2.5. *Proof of the theorem: Abelianness of the category of MHS and strictness.* The definition of MHS has a surprising strong property, since any morphism of MHS is necessarily strict for each filtration W and F. In consequence, the category is abelian.

Lemma 2.23. *The kernel (resp. cokernel) of a morphism f of mixed Hodge structure: $H \to H'$ is a mixed Hodge structure K with underlying module K_A equal to the kernel (resp. cokernel) of $f_A : H_A \to H'_A$; moreover $K_{A\otimes\mathbb{Q}}$ and $K_{A\otimes\mathbb{C}}$ are endowed with induced filtrations (resp. quotient filtrations) by W on $H_{A\otimes\mathbb{Q}}$ (resp. $H'_{A\otimes\mathbb{Q}}$) and F on $H_{\mathbb{C}}$ (resp. $H'_{\mathbb{C}}$).*

Proof. A morphism compatible with the filtrations is necessarily compatible with the canonical decomposition of the MHS into $\oplus I^{p,q}$. It is enough to check the statement on $K_{\mathbb{C}}$, hence we drop the index $\mathbb{C}$ in the notation. We consider on $K = Ker(f)$ the induced filtrations from H. The morphism $Gr^W K \to Gr^W H$ is injective, since it is injective on the corresponding terms $I^{p,q}$; moreover, the filtration F (resp. $\overline{F}$) of K induces on $Gr^W K$ the inverse image of the filtration F (resp. $\overline{F}$) on $Gr^W H$:

$$Gr^W K = \oplus_{p,q}(Gr^W K) \cap H^{p,q}(Gr^W H) \text{ and } H^{p,q}(Gr^W K) = (Gr^W K) \cap H^{p,q}(Gr^W H)$$

Hence the filtrations W, F on K define a MHS on K which is a kernel of f in the category of MHS. The statement on the cokernel follows by duality. $\square$

We still need to prove that for a morphism f of MHS, the canonical morphism $Coim(f) \to Im(f)$ is an isomorphism of MHS. Since by the above lemma $Coim(f)$ and $Im(f)$ are endowed with natural MHS, the result follows from the following statement:
A morphism of MHS which induces an isomorphism on the lattices, is an isomorphism of MHS.

Corollary 2.24. *i) Each morphism $f : H \to H'$ is strictly compatible with the filtrations W on $H_{A\otimes\mathbb{Q}}$ and $H'_{A\otimes\mathbb{Q}}$ as well the filtrations F on $H_\mathbb{C}$ and $H'_\mathbb{C}$. It induces morphisms $Gr^W_n(f) : Gr^W_n H_{A\otimes\mathbb{Q}} \to Gr^W_n H'_{A\otimes\mathbb{Q}}$ compatible with the $A\otimes\mathbb{Q}$-HS, and morphisms $Gr^p_F(f) : Gr^p_F(H_\mathbb{C}) \to Gr^p_F(H'_\mathbb{C})$ strictly compatible with the induced filtrations by $W^\mathbb{C}$.*
ii) The functor Gr^W_n from the category of MHS to the category $A\otimes\mathbb{Q}$-HS of weight n is exact and the functor Gr^p_F is also exact.

Remark 2.25. The above result shows that any exact sequence of MHS gives rise to various exact sequences which, in the case of MHS on cohomology of algebraic varieties that we are going to construct, have in general interesting geometrical interpretation, since we deduce from any long exact sequence of MHS:

$$H'^i \to H^i \to H''^i \to H'^{i+1}$$

various exact sequences:

$$Gr^W_n H'^i \to Gr^W_n H^i \to Gr^W_n H''^i \to Gr^W_n H'^{i+1}$$

for $\mathbb{Q}$ (resp. $\mathbb{C}$) coefficients, and:

$$Gr^n_F H'^i \to Gr^n_F H^i \to Gr^n_F H''^i \to Gr^n_F H'^{i+1}$$

$$Gr^m_F Gr^W_n H'^i \to Gr^m_F Gr^W_n H^i \to Gr^m_F Gr^W_n H''^i \to Gr^m_F Gr^W_n H'^{i+1}.$$

2.2.6. *Hodge numbers.* Let H be a MHS and set:

$$H^{pq} = Gr^p_F Gr^q_{\overline{F}} Gr^W_{p+q} H_\mathbb{C} = (Gr^W_{p+q} H_\mathbb{C})^{p,q}.$$

The Hodge numbers of H are the integers $h^{pq} = \dim_\mathbb{C} H^{pq}$, the Hodge numbers of the Hodge Structure $Gr^W_{p+q} H$.

In fact the proof by Deligne is in terms of opposite filtrations in an abelian category:

Theorem 2.26 (Deligne [11])**.** *Let $\mathbb{A}$ be an abelian category and $\mathbb{A}'$ the category of objects of $\mathbb{A}$ endowed with three opposite filtrations W (increasing), F and G. The morphisms of $\mathbb{A}'$ are morphisms in $\mathbb{A}$ compatible with the three filtrations.*
i) $\mathbb{A}'$ is an abelian category.
ii) The kernel (resp. cokernel) of a morphism $f : A \to B$ in $\mathbb{A}'$ is the kernel (resp. cokernel) of f in $\mathbb{A}$, endowed with the three induced filtrations from A (resp. quotient of the filtrations on B).
iii) Any morphism $f : A \to B$ in $\mathbb{A}'$ is strictly compatible with the filtrations W, F and G; the morphism $Gr^W(f)$ is compatible with the bigradings of $Gr^W(A)$ and $Gr^W(B)$; the morphisms $Gr_F(f)$ and $Gr_G(f)$ are strictly compatible with the induced filtration by W.
iv) The forget-filtration functors, as well Gr^W, Gr_F, Gr_G, $Gr^W Gr_F \simeq Gr_F Gr^W$, $Gr_F Gr_G Gr^W$ and $Gr_G Gr^W \simeq Gr^W Gr_G$ of $\mathbb{A}'$ in $\mathbb{A}$ are exact.

Example 2.27. 1) A Hodge structure H of weight n, is a MHS with a trivial weight filtration:

$$W_i(H_{\mathbb{Q}}) = 0 \quad \text{for} \quad i < n \qquad \text{and} \quad W_i(H_{\mathbb{Q}}) = H_{\mathbb{Q}} \quad \text{for} \quad i \geq n.$$

this MHS is called pure of weight n.
2) Let (H^i, F_i) be a finite family of A-HS of weight $i \in \mathbb{Z}$; then $H = \oplus_i H^i$ is endowed with the following MHS:

$$H_A = \oplus_i H^i_A, \quad W_n = \oplus_{i \leq n} H^i_A \otimes \mathbb{Q}, \quad F^p = \oplus_i F^p_i.$$

3) Let $H_{\mathbb{Z}} = i\mathbb{Z}^n \subset \mathbb{C}^n$, then we consider the isomorphism $H_{\mathbb{Z}} \otimes \mathbb{C} \simeq \mathbb{C}^n$ defined with respect to the canonical basis e_j of $\mathbb{Z}^n$ by:

$$H_{\mathbb{C}} \xrightarrow{\sim} \mathbb{C}^n : ie_j \otimes (a_j + ib_j) \mapsto i(a_j + ib_j)e_j = (-b_j + ia_j)e_j$$

hence, the conjugation $\sigma(ie_j \otimes (a_j + ib_j)) = ie_j \otimes (a_j - ib_j)$ on $H_{\mathbb{C}}$, corresponds to the following conjugation on $\mathbb{C}^n$: $\sigma(-b_j + ia_j)e_j = (b_j + ia_j)e_j$.
4) Let $H = (H_{\mathbb{Z}}, F, W)$ be a MHS; its m-twist is a MHS denoted by $H(m)$ and defined by:

$$H(m)_{\mathbb{Z}} := H_{\mathbb{Z}} \otimes (2i\pi)^m \mathbb{Z},\ W_r H(m) := (W_{r+2m} H_{\mathbb{Q}}) \otimes (2i\pi)^m \mathbb{Q},\ F^r H(m) := F^{r+m} H_{\mathbb{C}}.$$

2.2.7. *Tensor product and Hom.* Let H and H' be two MHS.

1) The MHS tensor product $H \otimes H'$ of the two MHS is defined by applying the general rules of filtrations, as follows:
i) $(H \otimes H')_{\mathbb{Z}} = H_{\mathbb{Z}} \otimes H'_{\mathbb{Z}}$
ii) $W_r(H \otimes H')_{\mathbb{Q}} := \sum_{p+p'=r} W_p H_{\mathbb{Q}} \otimes W_{p'} H'_{\mathbb{Q}}$
iii) $F^r(H \otimes H')_{\mathbb{C}} := \sum_{p+p'=r} F^p H_{\mathbb{C}} \otimes F^{p'} H'_{\mathbb{C}}$.

2) The MHS: $Hom(H, H')$ called internal Hom of the two MHS (to distinguish from the group of morphisms of the two MHS) is defined as follows:
i) $Hom(H, H')_{\mathbb{Z}} := Hom_{\mathbb{Z}}(H_{\mathbb{Z}}, H'_{\mathbb{Z}})$
ii) $W_r Hom(H, H')_{\mathbb{Q}} := \{f : Hom_{\mathbb{Q}}(H_{\mathbb{Q}}, H'_{\mathbb{Q}}) : \forall n, f(W_n H) \subset W_{n+r} H'\}$
iii) $F^r Hom(H, H')_{\mathbb{C}} := \{f : Hom_{\mathbb{C}}(H_{\mathbb{C}}, H'_{\mathbb{C}}) : \forall n, f(F^n H) \subset F^{n+r} H'\}$.
In particular the dual H^* of a mixed Hodge structure H is a MHS.

2.2.8. *Complex mixed Hodge structures.* Although the cohomology of algebraic varieties carries a MHS defined over $\mathbb{Z}$, we may need to work in analysis without the $\mathbb{Z}-$lattice.

Definition 2.28. *A complex mixed Hodge structure shifted by n on a complex vector space H is given by an increasing filtration W and two decreasing filtrations F and G such that $(Gr^W_k H, F, G)$, with the induced filtrations, is a complex HS of weight $n + k$.*

This shifted complex MHS by n is sometimes called of weight n. For $n = 0$, we recover the definition of a complex MHS.

2.2.9. *Variation of complex mixed Hodge structures.* The structure which appears in deformation theory on the cohomology of the fibers of a morphism of algebraic varieties leads one to introduce the concept of variation of MHS.

Definition 2.29. *i) A variation (VHS) of complex Hodge Structures on a complex manifold X of weight n is given by a data $(\mathcal{H}, F, \overline{F})$ where $\mathcal{H}$ is a complex local system, F (resp. $\overline{F}$) is a decreasing filtration varying holomorphically (resp. anti-holomorphically) by sub-bundles of the vector bundle $\mathcal{O}_X \otimes_{\mathbb{C}} \mathcal{H}$ (resp. $\mathcal{O}_{\overline{X}} \otimes_{\mathbb{C}} \mathcal{H}$ on the conjugate variety $\overline{X}$ with anti-holomorphic structural sheaf) such that for each point $x \in X$, data $(\mathcal{H}(x), F(x), \overline{F}(x))$ form a Hodge structure of weight n. Moreover, the connection ∇ defined by the local system satisfies Griffiths tranversality: for tangent vectors v holomorphic and $\overline{u}$ anti-holomorphic*

$$(\nabla_v F^p) \subset F^{p-1}, \quad (\nabla_{\overline{u}} \overline{F}^p) \subset \overline{F}^{p-1}$$

ii) A variation (VMHS) of complex mixed Hodge structures of weight n on X is given by the following data

$$(\mathcal{H}, W, F, \overline{F})$$

where $\mathcal{H}$ is a complex local system, W an increasing filtration by sub-local systems, F (resp. $\overline{F}$) is a decreasing filtration varying holomorphically (resp. anti-holomorphically) satisfying Griffiths tranversality

$$(\nabla_v F^p) \subset F^{p-1}, \quad (\nabla_{\overline{u}} \overline{F}^p) \subset \overline{F}^{p-1}$$

such that $(Gr^W_k \mathcal{H}, F, \overline{F})$, with the induced filtrations, is a complex VHS of weight $n + k$.

For $n = 0$ we just say a complex Variation of MHS. Let $\overline{\mathcal{H}}$ be the conjugate local system of $\mathcal{L}$. A linear morphism $S : \mathcal{H} \otimes_{\mathbb{C}} \overline{\mathcal{H}} \to \mathbb{C}_X$ defines a polarization of a VHS if it defines a polarization at each point $x \in X$. A complex MHS shifted by n is graded polarisable if $(Gr^W_k \mathcal{H}, F, \overline{F})$ is a polarized Variation Hodge structure. For a study of the degeneration of variations of MHS (see chapters 3 [7] and 11 [4] in this volume).

2.3. **Induced filtrations on spectral sequences.** To construct MHS we start from a bifiltered complex (K, F, W) satisfying conditions which will be introduced later under the terminology of mixed Hodge complex.
The fact that the two filtrations induce a MHS on the cohomology of the complex is based on a delicate study of the induced filtration by W on the spectral sequence defined by F. The result of Deligne known as the two filtrations lemma is presented here.
To explain the difficulty, imagine for a moment that we want to give a proof by induction on the length of W. Suppose that the weights of a mixed Hodge complex: (K, W, F) vary from $W_0 = 0$ to $W_l = K$ and suppose we did construct the mixed Hodge structure on cohomology for $l - 1$, then we consider the long exact sequence of cohomology:

$$H^{i-1}(Gr^W_l K) \to H^i(W_{l-1}K) \to H^i(W_l K) \to H^i(Gr^W_l K) \to H^{i+1}(W_{l-1}K)$$

the result would imply that the morphisms of the sequence are strict, hence the difficulty is a question of relative positions of the subspaces W_p and F^q on $H^i(W_l K)$ with respect to $\mathrm{Im} H^i(W_{l-1}K)$ and the projection on $H^i(Gr^W_l K)$.

2.3.1. *Deligne's two filtrations lemma.* This section relates results on various induced filtrations on terms of a spectral sequence, contained in [11] and [12]. Let (K, F, W) be a bi-filtered complex of objects of an abelian category. What we have in mind is to find axioms to define a MHS with induced filtrations W and F on the cohomology of K. The filtration W by subcomplexes define a spectral sequence $E_r(K, W)$. The second filtration F induces in various ways filtrations on $E_r(K, W)$. A detailed study will show that these filtrations coincide under adequate hypotheses and define a Hodge structure on the terms of the spectral sequence under suitable hypothesis on the bifiltered complex. Since the proof is technical but difficult, we emphasize here the main ideas as a guide to Deligne's proof.

2.3.2. Let (K, F, W) be a bi-filtered complex of objects of an abelian category, bounded below. The filtration F, assumed to be biregular, induces on the terms $E_r^{p,q}$ of the spectral sequence $E(K, W)$ various filtrations as follows:

Definition 2.30 (F_d, F_{d^*})**.** *Let $(E_r(K, W), d_r)$ denotes the graded complex consisting of the terms $E_r^{p,q}$. The first direct filtration on $E_r(K, W)$ is the filtration F_d defined for r finite or $r = \infty$, by the image:*

$$F_d^p(E_r(K, W)) = Im(E_r(F^pK, W) \to E_r(K, W)).$$

Dually, the second direct filtration F_{d^*} on $E_r(K, W)$ is defined by the kernel:

$$F_{d^*}^p(E_r(K, W)) = Ker(E_r(K, W) \to E_r(K/F^pK, W)).$$

The filtrations F_d, F_{d^*} are naturally induced by F, hence compatible with the differentials d_r.

They coincide on $E_r^{p,q}$ for $r = 0$, 1, since $B_r^{p,q} \subset Z_r^{p,q}$,

Lemma 2.31. *$F_d = F_{d^*}$ on $E_0^{p,q} = Gr_F^p(K^{p+q})$ and $E_1^{p,q} = H^{p+q}(Gr_p^W K)$.*

Definition 2.32 (F_{rec})**.** *The recurrent filtration F_{rec} on $E_r^{p,q}$ is defined by induction on r as follows*
i) On $E_0^{p,q}, F_{rec} = F_d = F_{d^*}$.
ii) The recurrent filtration F_{rec} on $E_r^{p,q}$ induces a filtration on ker d_r, which induces on its turn the recurrent filtration F_{rec} on $E_{r+1}^{p,q}$ as a quotient of ker d_r.

2.3.3. The precedent definitions of direct filtrations apply to $E_\infty^{p,q}$ as well and they are compatible with the isomorphism $E_r^{p,q} \simeq E_\infty^{p,q}$ for large r, from which we deduce also a recurrent filtration F_{rec} on $E_\infty^{p,q}$. The filtrations F and W induce each a filtration on $H^{p+q}(K)$. We want to prove that the isomorphism $E_\infty^{p,q} \simeq Gr_W^{-p} H^{p+q}(K)$ is compatible with F_{rec} on $E_\infty^{p,q}$ and F on the right term.

2.3.4. *Comparison of F_d, F_{rec}, F_{d^*}.* In general we have only the inclusions

Proposition 2.33. *i) On $E_r^{p,q}$, we have the inclusions*

$$F_d(E_r^{p,q}) \subset F_{rec}(E_r^{p,q}) \subset F_{d^*}(E_r^{p,q})$$

ii) On $E_\infty^{p,q}$, the filtration induced by the filtration F on $H^*(K)$ satisfy

$$F_d(E_\infty^{p,q}) \subset F(E_\infty^{p,q}) \subset F_{d^*}(E_\infty^{p,q}).$$

iii) The differential d_r is compatible with F_d and F_{d^*}.

We want to introduce conditions on the bifilterd complex in order that these three filtrations coincide, for this we need to know the compatibility of d_r with F_{rec}. Deligne proves an intermediary statement.

Theorem 2.34 (Deligne [11] 1.3.16, [12] 7.2)**.** *Let K be a complex with two filtrations W and F. We suppose W biregular and for a fixed integer $r_0 \geq 0$:*

($*r_0$) *For each non negative integer $r < r_0$, the differentials d_r of the graded complex $E_r(K,W)$ are strictly compatible with F_{rec}.*

then we have:

i) For $r \leq r_0$ the sequence

$$0 \to E_r(F^pK,W) \to E_r(K,W) \to E_r(K/F^pK,W) \to 0$$

is exact, and for $r = r_0 + 1$, the sequence

$$E_r(F^pK,W) \to E_r(K,W) \to E_r(K/F^pK,W)$$

is exact. In particular for $r \leq r_0 + 1$, the two direct and the recurrent filtration on $E_r(K,W)$ coincide $F_d = F_{rec} = F_{d^*}$.

ii) For $a < b$ and $r < r_0$, the differentials d_r of the graded complex $E_r(F^aK/F^bK,W)$ are strictly compatible with F_{rec}.

iii) If the above condition is satisfied for all r, the filtrations F_d, F_{rec}, F_{d^*} coincide into a unique filtration called F and the isomorphism $E_\infty^{p,q} \simeq Gr_W^{-p}H^{p+q}(K)$ is compatible with the filtrations F.

Moreover, we have an isomorphism of spectral sequences

$$Gr_F^pE_r(K,W) \simeq E_r(Gr_F^pK,W)$$

and the spectral sequence $E(K,F)$ (with respect to F) degenerates at rank $1 : (E_1 = E_\infty)$.

Proof. This surprising statement looks natural only if we have in mind the degeneration of $E(K,F)$ at rank 1 and the strictness in the category of mixed Hodge structures.

For fixed p, we consider the following property:

(P_r) $E_i(F^pK,W)$ injects into $E_i(K,W)$ for $i \leq r$ and its image is F^p_{rec} for $i \leq r+1$.

We already noted that (P_0) is satisfied. The proof by induction on r will apply as long as r remains $\leq r_0$. Suppose $r < r_0$ and (P_s) true for all $s \leq r$, we prove (P_{r+1}). The sequence:

$$E_r(F^pK,W) \xrightarrow{d_r} E_r(F^pK,W) \xrightarrow{d_r} E_r(F^pK,W)$$

injects into:

$$E_r(K,W) \xrightarrow{d_r} E_r(K,W) \xrightarrow{d_r} E_r(K,W)$$

with image $F_d = F_{rec}$, then, the image of F^p_{rec} in E_{r+1}:

$$F^p_{rec}E_{r+1} = Im[Ker(F^p_{rec}E_r(K,W) \xrightarrow{d_r} E_r(K,W)) \to E_{r+1}(K,W)]$$

coincides with the image of F^p_d which is by definition $Im[E_{r+1}(F^pK,W) \to E_{r+1}(K,W)]$.

Since d_r is strictly compatible with F_{rec}, we have:

$$d_rE_r(K,W) \cap E_r(F^pK,W) = d_rE_r(F^pK,W)$$

which means that $E_{r+1}(F^pK, W)$ injects into $E_{r+1}(K, W)$, hence we deduce the injectivity for $r+1$. Since ker d_r on F^p_{rec} is equal to ker d_r on $E_{r+1}(F^pK, W)$, we deduce $F^p_{rec} = F^p_d$ on $E_{r+2}(K, W)$, which proves (P_{r+1}).

Then, i) follows from a dual statement applied to F_{d^*} and
ii) follows, because we have an exact sequence:
$0 \to E_r(F^bK, W) \to E_r(F^aK, W) \to E_r(F^aK/F^bK, W) \to 0$.
iii) We deduce from i) and ii) the next exact sequence followed by its dual:

$$0 \to E_r(F^{p+1}K, W) \to E_r(F^pK, W) \to E_r(Gr^p_F K, W) \to 0$$
$$0 \leftarrow E_r(K/F^pK, W) \leftarrow E_r(F^{p+1}K, W) \leftarrow E_r(Gr^p_F K, W) \leftarrow 0.$$

In view of the injections in i) and the coincidence of $F_d = F_{rec} = F_{d^*}$ we have a unique filtration F, the quotient of the first two terms in the first exact sequence is isomorphic to $Gr^p_F E_r(K, W)$, hence we deduce an isomorphism

$$Gr^p_F E_r(K, W) \simeq E_r(Gr^p_F K, W)$$

compatible with d_r and autodual. If the hypothesis is now true for all r, we deduce an exact sequence:

$$0 \to E_\infty(F^pK, W) \to E_\infty(K, W) \to E_\infty(K/F^pK, W) \to 0$$

which is identical to:

$$0 \to Gr_W H^*(F^pK) \to Gr_W H^*(K) \to Gr_W H^*(K/F^pK) \to 0$$

from which we deduce, for all i:

$$0 \to H^i(F^pK) \to H^i(K) \to H^i(K/F^pK) \to 0$$

hence the spectral sequence with respect to F degenerates at rank 1 and the filtrations W induced on $H^i(F^pK)$ from (F^pK, W) and from $(H^i(K), W)$ coincide. $\square$

The condition $(*r_0)$ apply inductively for a category of complexes called mixed Hodge complexes which will be introduced later. It is applied in the next case for an example.

2.4. **MHS of a normal crossing divisor (NCD).** An algebraic subvariety Y of a complex smooth algebraic variety is called a Normal Crossing Divisor (NCD), if at each point $y \in Y$, there exists a neighborhoud for the transcendental topology U_y and coordinates $z = (z_1, \cdots, z_n) : U_y \to D^n$ to a product of the complex disc such that the image of $Y \cap U_y$ is defined by the equation $f(z) = z_1 \dots z_p = 0$ for some $p \leq n$. We consider a closed algebraic subvariety Y with NCD in a compact complex smooth algebraic variety X such that its irreducible components $(Y_i)_{i \in I}$ are smooth, and we put an order on the set of indices I of the components of Y.

2.4.1. *Mayer-Vietoris resolution.* The singular subset of Y is defined locally as the subset of points $\{z : df(z) = 0\}$. Let S_q denotes the set of strictly increasing sequences $\sigma = (\sigma_0, ..., \sigma_q)$ on the ordered set of indices I, $Y_\sigma = Y_{\sigma_0} \cap ... \cap Y_{\sigma_q}, Y_{\underline{q}} = \coprod_{\sigma \in S_q} Y_\sigma$ is the disjoint union, and for all $j \in [0, q], q \geq 1$ let $\lambda_{j,\underline{q}} : Y_{\underline{q}} \to Y_{\underline{q-1}}$ denotes the map inducing for each σ the embedding $\lambda_{j,\sigma} : Y_\sigma \to Y_{\sigma(\widehat{j})}$ where $\sigma(\widehat{j}) = (\sigma_0, ..., \widehat{\sigma_j}, ..., \sigma_q)$ is obtained by deleting σ_j. Let $\Pi_q : Y_{\underline{q}} \to Y$ denotes the canonical projection and $\lambda^*_{j,\underline{q}} : \Pi_* \mathbf{Z}_{Y_{\underline{q-1}}} \to \Pi_* \mathbf{Z}_{Y_{\underline{q}}}$ the restriction map defined by $\lambda_{j,\underline{q}}$ for $j \in [0,q]$. The various images of Π_q (or simply Π) define a natural stratification on X of dimension n

$$X \supset Y = \Pi(Y_{\underline{0}}) \supset \cdots \Pi(Y_{\underline{q}}) \supset \Pi(Y_{\underline{q+1}}) \supset \cdots \supset \Pi(Y_{\underline{n-1}}) \supset \emptyset$$

with smooth strata formed by the connected components of $\Pi_q(Y_{\underline{q}}) - \Pi_{q+1}(Y_{\underline{q+1}})$ of dimension $n-q-1$.

Lemma and Definition 2.35 (Mayer-Vietoris resolution of $\mathbb{Z}_Y$)**.** *The canonical morphism $\mathbb{Z}_Y \to \Pi_* \mathbb{Z}_{Y_{\underline{0}}}$ defines a quasi-isomorphism with the following complex of sheaves $\Pi_* \mathbb{Z}_{Y_{\underline{\bullet}}}$:*

$$0 \to \Pi_* \mathbb{Z}_{Y_{\underline{0}}} \to \Pi_* \mathbb{Z}_{Y_{\underline{1}}} \to \cdots \to \Pi_* \mathbb{Z}_{Y_{\underline{q-1}}} \stackrel{\delta_{q-1}}{\to} \Pi_* \mathbb{Z}_{Y_{\underline{q}}} \to \cdots$$

where $\delta_{q-1} = \sum_{j \in [0,q]} (-1)^j \lambda^*_{j,\underline{q}}$.

This resolution is associated to an hypercovering of Y by topological spaces in the following sense. Consider the diagram of spaces over Y:

$$Y_{\underline{\bullet}} = (Y_{\underline{0}} \; \substack{\leftarrow \\ \leftarrow} \; Y_{\underline{1}} \; \substack{\leftarrow \\ \leftarrow \\ \leftarrow} \; \cdots \; Y_{\underline{q-1}} \; \substack{\xleftarrow{\lambda_{j,\underline{q}}} \\ \vdots \\ \longleftarrow} \; Y_{\underline{q}} \cdots) \xrightarrow{\Pi} Y$$

This diagram is the strict simplicial scheme associated in [11] to the normal crossing divisor Y, called here after Mayer-Vietoris. The Mayer-Vietoris complex is canonically associated as direct image by Π of the sheaf $\mathbb{Z}_{Y_{\underline{\bullet}}}$ equal to $\mathbb{Z}_{Y_{\underline{i}}}$ on $Y_{\underline{i}}$. The generalization of such resolution is the basis of the later construction of mixed Hodge structure using simplicial covering of an algebraic variety.

2.4.2. *The cohomological mixed Hodge complex of a NCD.* The weight filtration W on $\Pi_* \mathbb{Q}_{Y_{\underline{\bullet}}}$ (it will define the weight of a MHS on the hypercohomology) is defined by:

$$W_{-q}(\Pi_* \mathbb{Q}_{Y_{\underline{\bullet}}}) = \sigma_{\bullet \geq q} \Pi_* \mathbb{Q}_{Y_{\underline{\bullet}}} = \Pi_* \sigma_{\bullet \geq q} \mathbb{Q}_{Y_{\underline{\bullet}}}, \quad Gr^W_{-q}(\Pi_* \mathbb{Q}_{Y_{\underline{\bullet}}}) \simeq \Pi_* \mathbb{Q}_{Y_{\underline{q}}}[-q]$$

To define the filtration F, we introduce the complexes $\Omega^*_{Y_{\underline{i}}}$ of differential forms on $Y_{\underline{i}}$. The simple complex $s(\Omega^*_{Y_{\underline{\bullet}}})$ (see 1.2.3) is associated to the double complex $\Pi_* \Omega^*_{Y_{\underline{\bullet}}}$ with the exterior differential d of forms and the differential δ_* defined by $\delta_{q-1} = \sum_{j \in [0,q]} (-1)^j \lambda^*_{j,\underline{q}}$ on $\Pi_* \Omega^*_{Y_{\underline{q-1}}}$. Then, the weight W and Hodge F filtrations are defined as:

$$W_{-q} = s(\sigma_{\bullet \geq q} \Omega^*_{Y_{\underline{\bullet}}}) = s(0 \to \cdots 0 \to \Pi_* \Omega^*_{Y_{\underline{q}}} \to \Pi_* \Omega^*_{Y_{\underline{q+1}}} \to \cdots)$$
$$F^p = s(\sigma_{* \geq p} \Omega^*_{Y_{\underline{\bullet}}}) = s(0 \to \cdots 0 \to \Pi_* \Omega^p_{Y_{\underline{\bullet}}} \to \Pi_* \Omega^{p+1}_{Y_{\underline{\bullet}}} \to \cdots)$$

We have an isomorphism of complexes of sheaves of abelian groups on Y compatible with the filtration:

$$(Gr^W_{-q} s(\Omega^*_{Y_{\underline{\bullet}}}), F) \simeq (\Pi_* \Omega^*_{Y_{\underline{q}}}[-q], F)$$

inducing quasi-isomorphisms:

$$(\Pi_* \mathbb{Q}_{Y_{\underline{\bullet}}}, W) \otimes \mathbb{C} = (\mathbb{C}_{Y_{\underline{\bullet}}}, W) \xrightarrow[\alpha]{\approx} (s(\Omega^*_{Y_{\underline{\bullet}}}), W)$$

$$Gr^W_{-q}(\Pi_* \mathbb{Q}_{Y_{\underline{\bullet}}}) \otimes \mathbb{C} = Gr^W_{-q}(\Pi_* \mathbb{C}_{Y_{\underline{\bullet}}}) \simeq \Pi_* \mathbb{C}_{Y_{\underline{q}}}[-q] \xrightarrow[\alpha_q]{\approx} \Pi_* \Omega^*_{Y_{\underline{q}}}[-q] \simeq Gr^W_{-q} s(\Omega^*_{Y_{\underline{\bullet}}})$$

The above situation is a model of the future constructions which lead to the MHS on cohomology of any algebraic variety. It is summarized by the construction of a system of filtered complexes with compatible isomorphisms:

$$\mathbf{K} = [\mathbf{K}_{\mathbb{Z}}; (\mathbf{K}_{\mathbb{Q}}, W_{\mathbb{Q}}), \mathbf{K}_{\mathbb{Z}} \otimes \mathbb{Q} \simeq \mathbf{K}_{\mathbb{Q}}; (\mathbf{K}_{\mathbb{C}}, W, F), (\mathbf{K}_{\mathbb{Q}}, W_{\mathbb{Q}}) \otimes \mathbb{C} \simeq (\mathbf{K}_{\mathbb{C}}, W)]$$

defined in our case by:

$$\mathbb{Z}_Y,\ (\Pi_* \mathbb{Q}_{Y_{\underline{\bullet}}}, W);\ \mathbb{Q}_Y \xrightarrow{\approx} \Pi_* \mathbb{Q}_{Y_{\underline{\bullet}}};\ (s(\Omega^*_{Y_{\underline{\bullet}}}), W, F);\ (\Pi_* \mathbb{Q}_{Y_{\underline{\bullet}}}, W) \otimes \mathbb{C} \xrightarrow{\approx} (s(\Omega^*_{Y_{\underline{\bullet}}}), W)$$

We extract the characteristic property of the system that we need, by considering $Gr^W(\mathbf{K})$ with the induced filtration by F, defined by:

$$Gr^W_{-q}(\Pi_* \mathbb{Q}_{Y_{\underline{\bullet}}}), \quad Gr^W_{-q}(\Pi_* \mathbb{Q}_{Y_{\underline{\bullet}}}) \otimes \mathbb{C} \simeq Gr^W_{-q} s(\Omega^*_{Y_{\underline{\bullet}}}), \quad (Gr^W_{-q} s(\Omega^*_{Y_{\underline{\bullet}}}), F)$$

This system will be treated like in the non-singular compact complex case defined by the various intersections $Y_{\underline{i}}$. In terms of Dolbeault resolutions : $(s(\mathcal{E}^{*,*}_{Y_{\underline{\bullet}}}), W, F)$, the above conditions induce on the complex of global sections $\Gamma(Y, s(\mathcal{E}^{*,*}_{Y_{\underline{\bullet}}}), W, F) := (\mathbb{R}\Gamma(Y, \mathbb{C}), W, F)$ a structure called a mixed Hodge complex in the following sense:

$$\begin{aligned}&(Gr^W_{-i}(\mathbb{R}\Gamma(Y, \mathbb{C}), F) := (\Gamma(Y, W_{-i} s(\mathcal{E}^{*,*}_{Y_{\underline{\bullet}}})/\Gamma(Y, W_{-i-1} s(\mathcal{E}^{*,*}_{Y_{\underline{\bullet}}}), F)\\ &\simeq (\Gamma(Y, Gr^W_{-i} s(\mathcal{E}^{*,*}_{Y_{\underline{\bullet}}}), F) \simeq (\mathbb{R}\Gamma(Y_{\underline{i}}, \Omega^*_{Y_{\underline{i}}}[-i]), F)\end{aligned}$$

The graded complex with respect to W has a structure called a Hodge complex of weight $-i$ in the sense that:

$$(H^n(Gr^W_{-i} \mathbb{R}\Gamma(Y, \mathbb{C})), F) \simeq (H^{n-i}(Y_{\underline{i}}, \mathbb{C}), F)$$

is a HS of weight $n - i$.
The terms of the spectral sequence $E_1(K, W)$ of (K, W) are written as:

$$_WE_1^{p,q} = \mathbb{H}^{p+q}(Y, Gr^W_{-p}(s\Omega^*_{Y_{\underline{\bullet}}})) \simeq \mathbb{H}^{p+q}(Y, \Pi_* \Omega^*_{Y_{\underline{p}}}[-p]) \simeq H^q(Y_{\underline{p}}, \mathbb{C})$$

They carry the HS of weight q on the cohomology of the space $Y_{\underline{p}}$. The differential is a combinatorial restriction map inducing a morphism of Hodge structures:

$$d_1 = \sum_{j \leq p+1} (-1)^j \lambda^*_{j,p+1} : H^q(Y_{\underline{p}}, \mathbb{C}) \to H^q(Y_{\underline{p+1}}, \mathbb{C}).$$

As morphisms of HS, the differentials are strict with respect to the filtration F, equal to F_{rec} (see 2.34). Hence, we can apply the theorem to deduce the condition $(*, 3)$ and obtain that the differential d_2 on the induced HS of weight q on the term $_WE_2^{p,q}$ is compatible with the HS, but since $d_2 : {_WE_2^{p,q}} \to {_WE_2^{p+2,q-1}}$ is a morphism of HS of different weight, it is strict and must vanish. Then, the argument applies inductively for $r \geq 2$ to show that the spectral sequence degenerates at rank 2: $(E_2 = E_\infty)$. Finally, we deduce

Proposition 2.36. *The system* $\mathbf{K}$ *associated to a normal crossing divisor* Y *with smooth proper irreducible components, defines a mixed Hodge structure on the cohomology* $H^i(Y, \mathbb{Q})$*, with weights varying between* 0 *and* i*.*

Corollary 2.37. *The Hodge structure on* $Gr^W_q H^{p+q}(Y, \mathbb{C})$ *is the cohomology of the complex of Hodge structure defined by* $(H^q(Y_{\underline{\bullet}}, \mathbb{C}), d_1)$ *equal to* $H^q(Y_{\underline{p}}, \mathbb{C})$ *in degree* $p \geq 0$*:*

$$(Gr^W_q H^{p+q}(Y, \mathbb{C}), F) \simeq ((H^p(H^q(Y_{\underline{\bullet}}, \mathbb{C}), d_1), F).$$

In particular, the weight of $H^i(Y, \mathbb{C})$ *varies in the interval* $[0, i]$ *(*$Gr^W_q H^i(Y, \mathbb{C}) = 0$ *for* $q \notin [0, i]$*).*

We will see that the last condition on the weight is true for all complete varieties.

3. Mixed Hodge Complex

The construction of mixed Hodge structures on the cohomology of algebraic varieties is similar to the case of normal crossing divisor. For each algebraic variety we need to construct a system of filtered complexes

$$\mathbf{K} = [\mathbf{K}_{\mathbb{Z}}; (\mathbf{K}_{\mathbb{Q}}, W_{\mathbb{Q}}), \mathbf{K}_{\mathbb{Z}} \otimes \mathbb{Q} \simeq \mathbf{K}_{\mathbb{Q}}; (\mathbf{K}_{\mathbb{C}}, W, F), (\mathbf{K}_{\mathbb{Q}}, W_{\mathbb{Q}}) \otimes \mathbb{C} \simeq (\mathbf{K}_{\mathbb{C}}, W)]$$

with a filtration W on the rational level and a filtration F on the complex level satisfying the following condition: the cohomology groups $\mathbb{H}^j(Gr_i^W(\mathbf{K}))$ with the induced filtration F, are HS of weight $j+i$.
The two filtrations lemma 2.3.1 on spectral sequences is used to prove that the jth-cohomology $(H^j(K), W[j], F)$ of the bifiltered complex (K, W, F) carries a mixed Hodge structure (the weight filtration of the MHS is deduced from W by adding j to the index). Such system is called a mixed Hodge complex (MHC).
The topological techniques used to construct W on the rational level are different from the geometrical techniques represented by de Rham complex used to construct the filtration F on the complex level. Comparison morphisms between the rational and complex levels must be added in order to obtain a satisfactory functorial theory of mixed Hodge structures with respect to algebraic morphisms.
However, the comparison between the rational and the complex filtrations W may not be defined by a direct morphism of complexes as in the previous NCD case but by a diagram of morphisms of one of the type:

$$(K_1, W_1) \underset{g_1}{\overset{\approx}{\longleftarrow}} (K_1', W_1') \xrightarrow{f_1} (K_2, W_2) \quad , \quad (K_1, W_1) \xrightarrow{f_2} (K_2',, W_2') \underset{g_2}{\overset{\approx}{\longleftarrow}} (K_2, W_2)$$

where g_1 and g_2 are filtered quasi-isomorphisms as, for example, in the case of the logarithmic complex in the next section.
This type of diagram of morphisms appears in the derived category of complexes of an abelian category constructed by Verdier [46, 47, 39, 11]. Defining the system $\mathbf{K}$ in a similar category called filtered derived category insures the correct identification of the cohomology with its filtrations defining a MHS independently of the choice of acyclic resolutions.
In such category, a diagram of morphisms of filtered complexes induces a morphism of the corresponding spectral sequences, but the reciprocal statement is not true: the existence of a diagram of quasi-isomorphisms is stronger than the existence of an isomorphism of spectral sequences.
The derived filtered categories described below have been used extensively in the more recent theory of perverse sheaves [2].
Finally, to put a mixed Hodge structure on the relative cohomology, we discuss the technique of the mixed cone which associates a new MHC to a morphism of MHC. The morphism must be defined at the level of the category of complexes and not up to homotopy: the mixed Hodge structure obtained depends on the homotopy between various resolutions [18].

3.1. **Derived category.** The hypercohomology of a functor of abelian categories $F : \mathbb{A} \to \mathbb{B}$ on a complex K of objects of an abelian category $\mathbb{A}$ with enough injectives is defined by considering an injective, or in general acyclic (see below), resolution $I(K)$ of K. Then, the hypercohomology of F at K is the cohomology $H^i(F(I(K)))$. If $I'(K)$ is a distinct resolution, there exists a unique isomorphism:

$$\phi_i(K) : H^i(F(I(K))) \to H^i(F(I'(K))).$$

Hence, we can choose an injective resolution and there is no ambiguity in the definition with respect to the choice.
Taking the hypercohomology, we loose information on the complex $F(I(K))$ which is not however necessarily isomorphic to $F(I'(K))$. This idea of Grothendieck is to construct a category where the various resolutions are isomorphic (not only their cohomology are isomorhic).
In [46] J.L. Verdier gives the construction of such category in two steps. In the first step he constructs the homotopy category where the morphisms are classes of morphisms of complexes defined up to homotopy ([46], see [39], [32]), and in the second step, a process of inverting all quasi-isomorphisms called localization is carried by a calculus of fractions similar to the process of inverting a multiplicative system in a ring, although in this case the system of quasi-isomorphisms is not commutative.
We give here the minimum needed to understand the mechanism in Deligne's definition of mixed Hodge structure. A full account may be found in ([46], [39],[32]). Recently, the formalism of derived category has been fundamental in the study of perverse sheaves [2]. To check various statements given here without proof, we recommend [39].

3.1.1. *The homotopy category* $K(\mathbb{A})$. Let $\mathbb{A}$ be an abelian category and let $C(\mathbb{A})$ (resp. $C^+(\mathbb{A})$, $C^-(\mathbb{A})$, $C^b(\mathbb{A})$) denotes the abelian category of complexes of objects in $\mathbb{A}$ (resp. complexes $X^\bullet$ satisfying $X^j = 0$ for $j << 0$, for $j >> 0$, and for both conditions, i.e. for j outside a finite interval).
A *homotopy* between two morphisms of complexes $f, g : X^\bullet \to Y^\bullet$ is a family of morphisms $h^j : X^j \to Y^{j-1}$ in $\mathbb{A}$ satisfying $f^j - g^j = d_Y^{j-1} \circ h^j + h^{j+1} \circ d_X^j$. Homotopy defines an equivalence relation on the additive group $Hom_{C(\mathbb{A})}(X^\bullet, Y^\bullet)$.

Definition 3.1. *The category* $K(\mathbb{A})$ *has the same object as the category of complexes* $C(\mathbb{A})$*, while the additive group of morphisms* $Hom_{K(\mathbb{A})}(X^\bullet, Y^\bullet)$ *is the group of morphisms of the two complexes of* $\mathbb{A}$ *modulo the homotopy equivalence relation.*

Similarly, we define $K^+(\mathbb{A})$, $K^-(\mathbb{A})$ and $K^b(\mathbb{A})$.

3.1.2. *Injective resolutions.* An abelian category $\mathbb{A}$ is said to have enough injectives if each object $A \in \mathbb{A}$ is embedded in an injective object of $\mathbb{A}$.

Any complex X of $\mathbb{A}$ bounded below is quasi-isomorphic to a complex of injective objects $I^\bullet(X)$ called its injective resolution [39] theorem 6.1.

Proposition 3.2. *Given a morphism* $f : A_1 \to A_2$ *in* $C^+(\mathbb{A})$ *and two injective resolutions* $A_i \xrightarrow{\approx} I^\bullet(A_i)$ *of* A_i*, there exists an extension of* f *as a morphism of resolutions* $I^\bullet(f) : I^\bullet(A_1) \to I^\bullet(A_2)$*; moreover two extensions of* f *are homotopic.*

See [39], sections $I.6$ and $I.7$, or [32], lemma 4.7. Hence, an injective resolution of an object in $\mathbb{A}$ becomes unique up to a unique isomorphism in the category $K^+(\mathbb{A})$. The category $K^+(\mathbb{A})$ is only additive but not abelian, even if $\mathbb{A}$ is abelian. Although we keep the same objects as in $C^+(\mathbb{A})$, the transformation on Hom is an important difference in the category since an homotopy equivalence between two complexes, i.e $f : X \to Y$ and $g : Y \to X$ such that $g \circ f$ (resp.$f \circ g$) is homotopic to the identity, becomes an isomorphism.

Example 3.3. In the category $\mathbb{A}$ of abelian sheaves on a topological space V, the ith-group of cohomology of a sheaf $\mathcal{F}$ is defined, up to an isomorphism, as the cohomology of the space of global sections of an injective resolution $H^i(I^\bullet(\mathcal{F})(V))$, while the complex of global sections $I^\bullet(\mathcal{F})(V)$ is defined, up to an homotopy, in the category of groups $C^+(\mathbb{Z})$. Hence, in the homotopy category of groups $K^+(\mathbb{Z})$, the complex $I^\bullet(\mathcal{F})(V)$ is defined, up to an isomorphism, and is called the higher direct image of $\mathcal{F}$ by the global section functor Γ and denoted $R\Gamma(V, \mathcal{F})$.

3.1.3. *The derived category* $D(\mathbb{A})$. For any two resolutions of a complex K, defined by quasi- isomorphisms $\phi_1 : K \xrightarrow{\approx} K_1$ and $\phi_2 : K \xrightarrow{\approx} K_2$, there exists a common injective resolution $\psi_1 : K_1 \xrightarrow{\approx} I^\bullet$ and $\psi_2 : K_2 \xrightarrow{\approx} I^\bullet$ inducing on K homotopic resolutions $\psi_i \circ \phi_i$, $i = 1, 2$. In this case, in classical Homological algebra, the i-th hypercohomology of a left exact functor F is defined by $H^i(F(I^\bullet))$.
The idea of Verdier is to construct in general, by inverting quasi-isomorphisms of complexes a new category $D(\mathbb{A})$ where all quasi-isomorphisms are isomorphic, without any reference to injective resolutions. The category $D(\mathbb{A})$ has the same objects as $K(\mathbb{A})$ but with a different additive group of morphisms. We describe now the additive group of two objects $Hom_{D(\mathbb{A})}(X, Y)$.
Let I_Y denotes the category whose objects are quasi-isomorphisms $s' : Y \xrightarrow{\approx} Y'$ in $K(\mathbb{A})$. Let $s'' : Y \xrightarrow{\approx} Y''$ be another object. A morphism $h : s' \to s''$ in I_Y is defined by a morphism $h : Y' \to Y''$ satisfying $h \circ s' = s''$. The key property is that we can take limits in $K(\mathbb{A})$, hence we define:

$$Hom_{D(\mathbb{A})}(X, Y) := \lim_{\longrightarrow I_Y} Hom_{K(\mathbb{A})}(X, Y')$$

A morphism $f : X \to Y$ in $D(\mathbb{A})$ is represented in the inductive limit by a diagram of morphisms: $X \xrightarrow{f'} Y' \underset{s'}{\overset{\approx}{\longleftarrow}} Y$ where s' is a quasi-isomorphism in $K(\mathbb{A})$. Two diagrams $X \xrightarrow{f'} Y' \underset{s'}{\overset{\approx}{\longleftarrow}} Y$ and $X \xrightarrow{f''} Y'' \underset{s''}{\overset{\approx}{\longleftarrow}} Y$ represent the same morphism f if and only if there exists a diagram

$$\begin{tikzcd} & X \arrow[dl, "f'"'] \arrow[d] \arrow[dr, "f''"] & \\ Y' \arrow[r, "\approx", "u"'] & Y''' & Y'' \arrow[l, "\approx"', "v"] \end{tikzcd}$$

such that $u \circ s' = v \circ s'' \in Hom(Y, Y''')$ and $u \circ f' = v \circ f'' \in Hom(X, Y''')$. In this case, the morphism f may be represented by a symbol ${s'}^{-1} \circ f'$ and this representation is not unique since in the above limit ${s'}^{-1} \circ f' = {s''}^{-1} \circ f''$. The construction of $D^+(\mathbb{A})$ is similar.
When there are enough injectives, the Hom of two objects A_1, A_2 in $D^+(\mathbb{A})$ is defined by their injective resolutions:

Corollary 3.4. *With the notations of the above proposition:*

$$Hom_{D^+(\mathbb{A})}(A_1, A_2) \simeq Hom_{D^+(\mathbb{A})}(I^\bullet(A_1), I^\bullet(A_2)) \simeq Hom_{K^+(\mathbb{A})}(I^\bullet(A_1), I^\bullet(A_2)).$$

See [46], chapter II §2, 2.2 Théorème p. 304. This is proposition 4.7 in [32], based on lemmas 4.4 and 4.5, see also [39], section I.6. In particular, all resolutions of a complex are isomorphic in the derived category.

Remark 3.5. We can equivalently consider the category J_X whose objects are quasi-isomorphisms $s' : X' \xrightarrow{\approx} X$ in $K(\mathbb{A})$ and define:

$$Hom_{D(\mathbb{A})}(X,Y) := \varinjlim_{J_X} Hom_{K(\mathbb{A})}(X',Y)$$

hence a morphism $f : X \to Y$ in $D(\mathbb{A})$ is represented by a diagram of morphisms in the inductive limit: $X \xleftarrow[s']{\approx} X' \xrightarrow{f'} Y$.

3.1.4. *The mapping cone construction.* We define the translate of a complex (K, d_K), denoted by TK or $K[1]$, by shifting the degrees:

$$(TK)^i = K^{i+1}, \quad d_{TK} = -d_K$$

Let $u : K \to L$ be a morphism of complexes in $C^+\mathbb{A}$, the mapping cone $C(u)$ is the complex $TK \oplus L$ with the differential

$$d : C(u)^k := K^{k+1} \oplus L^k \to C(u)^{k+1} := K^{k+2} \oplus L^{k+1} : (a,b) \mapsto (-d_K(a), u(a) + d_L(b)).$$

The exact sequence associated to $C(u)$ is:

$$0 \to L \xrightarrow{I} C(u) \to TK \to 0$$

Remark 3.6. Let h denotes an homotopy between two morphisms $u, u' : K \to L$, we define an isomorphism $I_h : C(u) \xrightarrow{\sim} C(u')$ by the matrix $\begin{pmatrix} Id & 0 \\ h & Id \end{pmatrix}$ acting on $TK \oplus L$, which commute with the injections of L in $C(u)$ and $C(u')$, and with the projections on TK.
Let h and h' be two homotopies of u to u'. A second homotopy of h to h', that is a family of morphisms $k^{j+2} : K^{j+2} \to L^j$ for $j \in \mathbb{Z}$, satisfying $h - h' = d_L \circ k - k \circ d_K$, defines an homotopy of I_h to $I_{h'}$.

3.1.5. *Distinguished Triangles.* We write the exact sequence of the mapping cone u as:

$$K \xrightarrow{u} L \xrightarrow{I} C(u) \xrightarrow{+1}$$

It is called a distiguished triangle since the last map may be continued to the same exact sequence but shifted by T. A distinguished triangle in $K(\mathbb{A})$ is a sequence of complexes isomorphic to the image in $K(\mathbb{A})$ of a distinguished triangle associated to a cone in $C(\mathbb{A})$.
Triangles are defined in $K(\mathbb{A})$, by short exact sequences of complexes which split in each degree [46] 2-4 p. 272, [39] I.4 definition 4.7, and [2] (1.1.2). We remark:
1) The cone over the identity morphism of a complex X is homotopic to zero.
2) Using the construction of the mapping cylinder ([39], I.4) over a morphism of complexes $u : X \to Y$, one can transform u, up to an homotopy equivalence into an injective morphism of complexes.
A distinguished triangle *in the derived category* $D(\mathbb{A})$ is a sequence of complexes isomorphic to the image in $D(\mathbb{A})$ of a distinguished triangle in $K(\mathbb{A})$. Long exact sequences of cohomologies are associated to triangles.

Remark 3.7. Each short exact sequence of complexes $0 \xrightarrow{u} X \xrightarrow{v} Y \to Z \to 0$ is isomorphic to the distinguished triangle in $D^+\mathbb{A}$ defined by the cone $C(u)$ over u. The morphism $C(u) \to Z$ is defined by v and we check it is a quasi-isomorphism using the connection morphism in the associated long exact sequence [46]chapter II 1.5 p. 295, [39] 6.8, 6.9, and [2] (1.1.3).

3.1.6. *Derived functor.* Let $F : \mathbb{A} \to \mathbb{B}$ be a functor of abelian categories. We denote also by $F : C^+\mathbb{A} \to C^+\mathbb{B}$ the corresponding functor on complexes, and by $Q_{\mathbb{A}} : C^+\mathbb{A} \to D^+\mathbb{A}$ (resp. $Q_{\mathbb{B}} : C^+\mathbb{B} \to D^+\mathbb{B}$) the canonical localizing functor. If *the category* $\mathbb{A}$ *has enough injective objects*, a derived functor:

$$RF : D^+\mathbb{A} \to D^+\mathbb{B}$$

satisfying $RF \circ Q_{\mathbb{A}} = Q_{\mathbb{B}} \circ F$ is defined as follows:
a) Given a complex K in $D^+(\mathbb{A})$, we start by choosing an injective resolution of K, that is a quasi-isomorphism $i : K \xrightarrow{\approx} I(K)$ where the components of I are injectives in each degree (see [39], 7.9 or [32] Lemma 4.6 p. 42).
b) We define $RF(K) = F(I(K))$.
A morphism $f : K \to K'$ gives rise to a morphism $RF(K) \to RF(K')$ functorially, since f can be extended to a morphism $I(f) : F(I(K)) \to F(I(K'))$, defined on the injective resolutions uniquely up to homotopy.
In particular, for a different choice of an injective resolution $J(K)$ of K, we have an isomorphism $F(I(K)) \simeq F(J(K))$ in $D^+(\mathbb{B})$.

Remark 3.8. The basic idea is that a functor F does not carry a quasi-isomorphim $\phi : K \xrightarrow{\approx} K'$ into a quasi-isomrphism $F(\phi) : F(K) \xrightarrow{\approx} F(K')$ in general, but $F(\phi)$ will be a quasi-isomorphism if the complexes are injective since then ϕ has an inverse up to homotopy.

Definition 3.9. *i) The cohomology $H^j(RF(K))$ is called the jth-hypercohomology $R^jF(K)$ of F at K.*
ii) An object $A \in \mathbb{A}$ is F-acyclic if $R^jF(A) = 0$ for $j > 0$.

Remark 3.10. i) We often add the condition that the functor $F : \mathbb{A} \to \mathbb{B}$ is left exact. In this case $R^0F \simeq F$ and we recover the theory of satellite functors in [6].
ii) If $K \xrightarrow{\approx} K'$ is a quasi-isomorphism of complexes, the morphism $FK \to FK'$ is not a quasi-isomorphism in general, while the morphism $RF(K) \to RF(K')$ must be a quasi-isomorphim since the image of an isomorphism in the derived category is an isomorphism.
iii) It is important to know that we can use acyclic objects to compute RF:
for any resolution $A(K)$ of a complex K by acyclic objects: $K \xrightarrow{\approx} A(K)$, $FA(K)$ is isomorphic to the complex $RF(K)$. For example, the hypercohomology of the global section functor Γ in the case of sheaves on a topological space, is equal to the cohomology defined via flasque resolutions or any "acyclic " resolution.
iv) The dual construction defines the left derived functor LF of a functor F if there exists enough projectives in the category $\mathbb{A}$.
v) Verdier defines a derived functor even if there is not enough injectives [46] chapter II §2 p. 301 and gives a construction of the derived functor in chapter II §2 p. 304 under suitable conditions.

3.1.7. *Extensions.* Fix a complex $B^\bullet$ of objects in $\mathbb{A}$. We consider the covariant functor $Hom^\bullet(B^\bullet, *)$ from the category of complexes of objects in $\mathbb{A}$ to the category of complexes of abelian groups defined for a complex $A^\bullet$ by

$$(Hom^\bullet(B^\bullet, A^\bullet))^n = \prod_{p\in\mathbb{Z}} Hom_{\mathbb{A}}(B^p, A^{n+p})$$

with the differential of f in degree n an element $d^n f$ defined by

$$[d^n f]^p = d^{n+p}_{A^\bullet} \circ f^p + (-1)^{n+1} f^{p+1} \circ d^p_{B^\bullet}$$

The associated derived functor is $RHom^\bullet(B^\bullet, *)$.
Suppose there is enough projectives and injectives in $\mathbb{A}$ and the complexes are bounded. If $P^\bullet \to B^\bullet$ is a projective resolution of the complex $B^\bullet$, and $A^\bullet \to I^\bullet$ is an injective resolution of $A^\bullet$, then, in $D^b(\mathbb{A})$, $RHom^\bullet(B^\bullet, A^\bullet)$ is isomorphic to $Hom^\bullet(P^\bullet, A^\bullet)$ and $Hom^\bullet(B^\bullet, I^\bullet)$.
One can check that the cycles (resp. the boundaries) of $Hom^\bullet(B^\bullet, I^\bullet)$ in degree n are the morphisms of complexes $Hom(B^\bullet, I^\bullet[n])$ (resp. consist of morphisms homotopic to zero). Hence, the cohomology group $H^0(RHom^\bullet(B^\bullet, A^\bullet)) \simeq H^0(RHom^\bullet(B^\bullet, I^\bullet))$ is naturally isomorphic as a group to the group of morphisms from $B^\bullet$ to $A^\bullet$ in the derived category, i.e.

$$H^0(RHom^\bullet(B^\bullet, A^\bullet)) \simeq Hom_{D^b(\mathbb{A})}(B^\bullet, A^\bullet).$$

Since for all k, $RHom^\bullet(B^\bullet, A^\bullet[k]) = RHom^\bullet(B^\bullet, A^\bullet)[k]$, we define:

$$Ext^k(B^\bullet, A^\bullet) := H^k(RHom^\bullet(B^\bullet, A^\bullet)) := H^0(RHom^\bullet(B^\bullet, A^\bullet[k])).$$

Lemma 3.11 (Extension groups)**.** *When the abelian category $\mathbb{A}$ has enough injectives, the group $Hom_{D(\mathbb{A})}(X^\bullet, Y^\bullet)$ of morphisms of two complexes in the derived category $D(\mathbb{A})$ has an interpretation as an extension group:*

$$Hom_{D(\mathbb{A})}(X^\bullet, Y^\bullet[n]) = Ext^n(X^\bullet, Y^\bullet)$$

where the groups on the right side are derived from the Hom functor. In general the group $Hom_{D(\mathbb{A})}(A, B[n])$ of two objects in $\mathbb{A}$ may be interpreted as the Yoneda $n-$extension group [39] $(XI.4)$.

3.1.8. *Filtered homotopy categories $K^+F(\mathbb{A}), K^+F_2(\mathbb{A})$.* For an abelian category $\mathbb{A}$, let $F\mathbb{A}$ (resp. $F_2\mathbb{A}$) denotes the category of filtered objects (resp. bi-filtered) of $\mathbb{A}$ with finite filtration(s), $C^+F\mathbb{A}$ (resp. $C^+F_2\mathbb{A}$) the category of complexes of $F\mathbb{A}$ (resp. $F_2\mathbb{A}$) bounded on the left (zero in degrees near $-\infty$) with morphisms of complexes respecting the filtration(s).
Two morphisms $u, u' : (K, F, W) \to (K', F, W)$, where W (resp. F) denotes uniformly the increasing (resp. decreasing) filtrations on K or K', are homotopic if there exists an homotopy from u to u' compatible with the filtrations, then it induces an homotopy on each term $k^{i+1} : F^jK^{i+1} \to F^jK'^i$ (resp. for W) and in particular $Gr_F(u - u')$ (resp. $Gr_F Gr_W(u - u')$) is homotopic to 0.
The homotopy category whose objects are bounded below complexes of filtered (resp. bi-filtered) objects of $\mathbb{A}$, and whose morphisms are equivalence classes modulo homotopy compatible with the filtration(s) is denoted by $K^+F\mathbb{A}$ (resp. $K^+F_2\mathbb{A}$).

Filtered resolutions. In presence of two filtrations by subcomplexes F and W on a complex K of objects of an abelian category $\mathbb{A}$, the filtration F induces by restriction a new filtration F on the terms W^iK, which also induces a quotient filtration F on $Gr^i_W K$. We define in this way the graded complexes $Gr_F K$, $Gr_W K$ and $Gr_F Gr_W K$.

Definition 3.12. *A morphism $f : (K, F, W) \xrightarrow{\approx} (K', F, W)$ of complexes with biregular filtrations F and W is a bi-filtered quasi-isomorphism if $Gr^*_F Gr^*_W(f)$ is a quasi-isomorphism.*

3.1.9. *Derived filtered categories* $D^+F(\mathbb{A}), D^+F_2(\mathbb{A})$. They are deduced from $K^+F\mathbb{A}$ (resp. $K^+F_2\mathbb{A}$) by inverting the filtered quasi-isomorphisms (resp. bi-filtered quasi-isomorphisms). The objects of $D^+F\mathbb{A}$ (resp.$D^+F_2\mathbb{A}$) are complexes of filtered objects of $\mathbb{A}$ as of $K^+F\mathbb{A}$ (resp. $K^+F_2\mathbb{A}$). Hence, the morphisms are represented by diagrams with filtered (resp. bi-filtered) quasi-isomorphisms.

3.1.10. *Triangles.* The complex $T(K,F,W)$ and the cone $C(u)$ of a morphism $u : (K,F,W) \to (K',F,W)$ are endowed naturally with filtrations F and W. The exact sequence associated to $C(u)$ is compatible with the filtrations. A filtered homotopy h of morphisms u and u' defines a filtered isomorphism of cones $I_h : C(u) \xrightarrow{\sim} C(u')$. Distinguished (or exact) triangles are defined similarly in $K^+F\mathbb{A}$ and $K^+F_2\mathbb{A}$ as well in $D^+F\mathbb{A}$ and $D^+F_2\mathbb{A}$. Long filtered (resp. bi- filtered) exact sequences of cohomologies are associated to triangles.

Remark 3.13. The morphisms of exact sequences are not necessarily strict for the induced filtrations on cohomology. However, this will be the case of the class of mixed Hodge complex giving rise to MHS that we want to define.

3.2. **Derived functor on a filtered complex.** Let $T : \mathbb{A} \to \mathbb{B}$ be a left exact functor of abelian categories with enough injectives in $\mathbb{A}$. We want to construct a derived functor $RT : D^+F\mathbb{A} \to D^+F\mathbb{B}$ (resp. $RT : D^+F_2\mathbb{A} \to D^+F_2\mathbb{B}$). For this, we need to introduce the concept of T-acyclic filtered resolutions. Given a filtered complex with biregular filtration(s) we define first the image of the filtrations via acyclic filtered resolutions. Then, we remark that a filtered quasi-isomorphism $\phi : (K,F) \xrightarrow{\approx} (K',F')$ of complexes filtered by complexes of T-acyclic sheaves has as image by T, a iltered quasi-isomorphism $T(\phi) : (TK,TF) \xrightarrow{\approx} (TK',TF')$, therefore the construction factors by RT through the derived filtered category.

3.2.1. *Image of a filtration by a left exact functor.* Let (A,F) be a filtered object in $\mathbb{A}$, with a finite filtration. Since T is left exact, a filtration TF of TA is defined by the sub-objects $TF^p(A)$.

If $Gr_F(A)$ is T-acyclic, then the objects $F^p(A)$ are T-acyclic as successive extensions of T-acyclic objects. Hence, the image by T of the sequence of acyclic objects:

$$0 \to F^{p+1}(A) \to F^p(A) \to Gr^p_F(A) \to 0$$

is exact. Then:

Lemma 3.14. *If $Gr_F(A)$ is a T-acyclic object, we have $Gr_{TF}(TA) \simeq TGr_F(A)$.*

3.2.2. Let A be an object with two finite filtrations F and W such that Gr_FGr_WA is T*-acyclic*, then the objects Gr_FA and Gr_WA are T-acyclic, as well $F^q(A)\cap W^p(A)$. As a consequence of acyclicity, the sequences:

$$0 \to T(F^q \cap W^{p+1}) \to T(F^q \cap W^p) \to T((F^q \cap W^p)/(F^q \cap W^{p+1})) \to 0$$

are exact, and $T(F^q(Gr^p_WA))$ is the image in $T(Gr^p_W(A))$ of $T(F^q\cap W^p)$. Moreover, the isomorphism $Gr_{TW}TA \simeq T(Gr_WA)$ transforms the filtration $Gr_{TW}(TF)$ on $Gr_{TW}(TA)$ into the filtration $T(Gr_W(F))$ on $T(Gr_WA)$.

3.2.3. $RT : D^+F(\mathbb{A}) \to D^+F(\mathbb{B})$.
Let F be a biregular filtration of K. A filtered T-acyclic resolution of K is given by a filtered quasi-isomorphism $i\colon (K,F) \to (K',F')$ to a complex with a biregular filtration such that for all p and n, $Gr^p_F(K^n)$ is acyclic for T.

Lemma 3.15 (Filtered derived functor of a left exact functor $T : \mathbb{A} \to \mathbb{B}$)**.** *Suppose we are given functorially for each filtered complex (K,F) a filtered T-acyclic resolution $i : (K,F) \to (K',F')$, we define $T' : C^+F(\mathbb{A}) \to D^+F(\mathbb{B})$ by the formula $T'(K,F) = (TK',TF')$. A filtered quasi-isomorphism $f : (K_1,F_1) \to (K_2,F_2)$ induces an isomorphism $T'(f) : T'(K_1,F_1) \simeq T'(K_2,F_2)$ in $D^+F(\mathbb{B})$.*
Hence T' factors through a derived functor $RT : D^+F(\mathbb{A}) \to D^+F(\mathbb{B})$ such that $RT(K,F) = (TK',TF')$, and we have $Gr_F RT(K) \simeq RT(Gr_F K)$ where $F RT(K) := T(F'(K'))$.

In particular for a different choice (K'',F'') of (K',F') we have an isomorphism $(TK'',TF'') \simeq (TK',TF')$ in $D^+F(\mathbb{B})$ and

$$RT(Gr_F K) \simeq Gr_{TF'} T(K') \simeq Gr_{TF''} T(K'').$$

Remark 3.16. Due to the above properties a bifiltered quasi-isomorphism of bifiltered comlexes induces a bifiltered isomorphism on their hypercohomology.

Example 3.17. In the particular case of interest, where $\mathbb{A}$ is the category of sheaves of A-modules on a topological space X, and where T is the global section functor Γ, an example of filtered T-acyclic resolution of K is the simple complex $\mathcal{G}^*(K)$, associated to the double complex defined by Godement resolution (see [20] chapter 7 of this volume, [39] Chapter II, §3.6 p.95 or [22] Chapter II, §4.3 p.167) $\mathcal{G}^*$ in each degree of K, filtered by $\mathcal{G}^*(F^pK)$.

This example will apply to the next result for bi-filtered complexes (K,W,F) with resolutions $(\mathcal{G}^*K, \mathcal{G}^*W, \mathcal{G}^*F)$ satisfying

$$Gr_{\mathcal{G}^*F} Gr_{\mathcal{G}^*W}(\mathcal{G}^*K) \simeq \mathcal{G}^*(Gr_F Gr_W K)$$

3.2.4. $RT : D^+F_2(\mathbb{A}) \to D^+F_2(\mathbb{B})$.
Let F and W be two biregular filtrations of K. A bi-filtered T-acyclic resolution of K is a bi-filtered quasi-isomorphism $f : (K,W,F) \to (K',W',F')$ such that W' and F' are biregular filtrations on K' and for all p, q, n: $Gr^p_F Gr^q_W({K'}^n)$ is acyclic for T.

Lemma 3.18. *Let (K,F,W) be a bi-filtered complex, $T : \mathbb{A} \to \mathbb{B}$ a left exact functor and $i : (K,F,W) \to (K',F',W')$ a functorial bi-filtered T-acyclic resolution. We define $T' : C^+F(\mathbb{A}) \to D^+F(\mathbb{B})$ by the formula $T'(K,F,W) = (TK',TF',TW')$.*
A bi-filtered quasi-isomorphism $f : (K_1,F_1,W_1) \to (K_2,F_2,W_2)$ induces an isomorphism $T'(f) : T'(K_1,F_1,W_1) \simeq T'(K_2,F_2,W_2)$ in $D^+F_2(\mathbb{B})$.
Hence T' factors through a derived functor $RT : D^+F_2(\mathbb{A}) \to D^+F_2(\mathbb{B})$ such that $RT(K,F,W) = (TK',TF',TW')$, and we have $Gr_F Gr_W RT(K) \simeq RT(Gr_F Gr_W K)$.

In particular for a different choice (K'',F'',W'') of (K',F',W') we have an isomorphism $(TK'',TF'',TW'') \simeq (TK',TF',TW'')$ in $D^+F_2(\mathbb{B})$ and

$$RT(Gr_F Gr_W K) \simeq Gr_{TF'} Gr_{TW'} T(K') \simeq Gr_{TF''} Gr_{TW''} T(K'')$$

3.2.5. *Hypercohomology spectral sequence.* An object of $D^+F(\mathbb{A})$ defines a spectral sequence functorial with respect to morphisms. Let $T : \mathbb{A} \to \mathbb{B}$ be a left exact functor of abelian categories, (K, F) an object of $D^+F\mathbb{A}$ and $RT(K, F) : D^+F\mathbb{A} \to D^+F\mathbb{B}$ its derived functor. Since $Gr_F RT(K) \simeq RT(Gr_F K)$, the spectral sequence defined by the filtered complex $RT(K, F)$ is written as:

$$_FE_1^{p,q} = R^{p+q}T(Gr_F^p K) \Rightarrow Gr_F^p R^{p+q}T(K)$$

Indeed, $H^{p+q}(Gr_F^p RT(K)) \simeq H^{p+q}(RT(Gr_F^p(K)))$. This is the hypercohomology spectral sequence of the filtered complex K with respect to the functor T. The spectral sequence depends functorially on K and a filtered quasi-isomorphism induces an isomorphism of spectral sequences. The differentials d_1 of this spectral sequence are the image by T of the connecting morphisms defined by the short exact sequences:

$$0 \to Gr_F^{p+1}K \to F^pK/F^{p+2}K \to Gr_F^p K \to 0.$$

For an increasing filtration W on K, we have:

$$_WE_1^{p,q} = R^{p+q}T(Gr_{-p}^W) \Rightarrow Gr_{-p}^W R^{p+q}T(K).$$

Example 3.19 (The τ and σ filtrations)**.**
1) Let K be a complex, the canonical filtration by truncation τ is the increasing filtration by sub-complexes:

$$\tau_p K^\bullet := (\cdots \to K^{p-1} \to Ker d_p \to 0 \cdots \to 0)$$

such that:

$$Gr_p^\tau K \xrightarrow{\approx} H^p(K)[-p], \; H^i(\tau_{\leq p}(K)) = H^i(K) \text{ if } i \leq p, \; H^i(\tau_{\leq p}(K)) = 0 \text{ if } i > p.$$

2) The sub-complexes of K:

$$\sigma_p K^\bullet := K^{\bullet \geq p} := (0 \to \cdots \to 0 \to K^p \to K^{p+1} \to \cdots)$$

define a decreasing biregular filtration, called the trivial filtration of K such that: $Gr_\sigma^p K = K^p[-p]$, i.e. it coincides with the filtration F on the de Rham complex.

A quasi-isomorphism $f : K \to K'$ is necessarily a filtered quasi-isomorphism for both filtrations: τ and σ. The hypercohomology spectral sequences of a left exact functor attached to both filtrations of K are the two natural hypercohomology spectral sequences of K.

3.2.6. *Construction of the hypercohomology spectral sequence and the filtration L.* Let $K^\bullet := ((K^i)_{i\in\mathbb{Z}}, d)$ be a complex of objects in an abelian category and F a left exact functor with values in the category of abelian groups (for example, $K^\bullet$ is a complex of sheaves on a topological space X and Γ the functor of global sections). To construct the hypercohomology spectral sequence we consider F-acyclic resolutions $(K^{i,*}, d'')$ of K^i forming a double complex $K^{i,j}$ with differentials $d' : K^{i,j} \to K^{i+1,j}$ and $d'' : K^{i,j} \to K^{i,j+1}$ such that the kernels $Z^{i,j}$ of d' (resp. the image $B^{i,j}$ of d', the cohomology $H^{i,j}$) form an acyclic resolution with varying index j, of the kernel Z_i of d on K^i (resp. the image B^i of d, the cohomology $H^i(K)$). The decreasing filtration L by sub-complexes of the simple complex $sFK := s(FK^{*,*})$ associated to the double complex, is defined by

$$L^r(sF(K))^n := \oplus_{i+j=n,\, j\geq r} F(K^{i,j})$$

The associated spectral sequence starts with the terms:

$$E_0^{p,q} := Gr_L^p(sF(K)^{p+q}) := F(K^{q,p}), d' : F(K^{q,p}) \to F(K^{p+1,p}),$$

$$E_1^{p,q} := H^{p+q}(Gr_L^p(sF(K)) = H^{p+q}(F(K^{*,p})[-p], d') = H^q(F(K^{*,p}), d')$$

where $F(K^{*,p})$ has degree $*+p$ in the complex $F(K^{*,p})[-p]$, and the terms $E_1^{p,q}$ form a complex for varying p with differential induced by $d'' : E_1^{p,q} \to E_1^{p+1,q}$.
It follows that: $E_2^{p,q} = H^p(H^q(F(K^{*,*}), d'), d'')$.
Since the cohomology groups $(H^q(K^{p,*}, d'), d'')$ for various p and induced differential d'' form an acyclic resolution of $H^q(K^\bullet)$, the cohomology for d'' is $E_2^{p,q} = R^pF(H^q(K))$, hence we have a spectral sequence

$$E_2^{p,q} := R^pF(H^qK) \Longrightarrow R^{p+q}F(K).$$

3.2.7. *Comparison lemma for L, τ and the filtration Dec(L).*
We have a comparison lemma for the above filtration L and τ on $s(FK^{*,*})$:

Lemma 3.20. *On the cohomology group $R^nF(K) := H^n(s(FK^{*,*}))$, the induced filtrations by L and τ coincide up to change in indices: $\tau_{-p} = L^{n+p}$.*

Since τ is increasing and $Gr^\tau_{-p}K = H^{-p}K[p]$, its associated spectral sequence is: $E_1^{p,q}(\tau) = R^{p+q}F(Gr^\tau_{-p}K) = R^{p+q}F(H^{-p}K[p]) = R^{2p+q}F(H^{-p}K)$.

The filtration τ is related to the filtration L by a process of decalage described in [11] (1.3.3).

3.2.8. *The filtration Dec(F).* Let F be a decreasing filtration on a complex K, then the filtration $Dec\,F$ is defined as:

$$(Dec\,F)^pK^n := Ker(F^{p+n}K^n \to K^{n+1}/(F^{p+n+1}K^{n+1})).$$

Then $E_0^{p,q}(Dec\,F) := Gr^p_{Dec\,F}K^{p+q}$ and we have a morphism in degree $p+q$: $(E_0^{p,q}(Dec\,F), d_0) \to (E_1^{2p+q,-p}(F), d_1)$ inducing an isomorphism in rank $r \geq 1$: $(E_r^{p,q}(Dec\,F), d_r) \simeq (E_{r+1}^{2p+q,-p}(F), d_{r+1})$ [11] proposition (1.3.4).
Hence, by definition:
$(Dec\,L)^p(s(FK))^n = (\oplus_{i+j=n, j>p+n}F(K^{i,j}))\oplus Ker[F(K^{-p,p+n}) \xrightarrow{d'} F(K^{-p+1,p+n}]$.
By construction, the sum of the double complex $\tau_pK^{*,j}$ for varying j define an acyclic resolution of τ_pK, then $RF(K,\tau)$ is the complex $s(FK^{*,*})$ filtered by: $(F\tau)_p := s(\tau_pK^{*,j})_{j\geq 0}$.
So that we have a morphism: $(s(FK^{*,*}), F\tau_{-p}) \to (s(FK^{*,*}), (Dec\,L)^p)$
inducing isomorphisms for $r > 0$: $E_r^{p,q}(\tau) \simeq E_r^{p,q}(Dec\,L) \simeq E_{r+1}^{2p+q,-p}(L)$ and
$E_1^{p,q}(\tau) \simeq E_2^{2p+q,-p}(L) = R^{2p+q}F(H^{-p}K)$.

3.2.9. *Leray's spectral sequence.* Let $f : X \to V$ be a continuous map of topological spaces and $\mathcal{F}$ be a sheaf of abelian groups. To construct $\mathcal{K} := Rf_*\mathcal{F}$, we use a flasque resolution of $\mathcal{F}$. Then we consider the filtrations τ and L on $Rf_*\mathcal{F}$ as above. Finally, we apply the functor Γ of global sections on V, then the above statement is on the cohomology of X since: $\mathbb{H}^p(V, Rf_*\mathcal{F}) \simeq \mathbb{H}^p(X, \mathcal{F})$

Lemma and Definition 3.21. *i) There exists a filtration L on the cohomology $H^n(X,\mathcal{F})$ and a convergent spectral sequence defined by f, starting at rank* 2

$$E_2^{p,q} = \mathbb{H}^p(V, R^qf_*\mathcal{F}) \Rightarrow E_\infty^{p,q} = Gr_L^pH^{p+q}(X,\mathcal{F})$$

ii) There exists a filtration τ on the cohomology $H^n(X,\mathcal{F})$ and a convergent spectral sequence defined by f, starting at rank 1

$$E_1^{p,q} = \mathbb{H}^{2p+q}(V, R^{-p}f_*\mathcal{F}) \Rightarrow E_\infty^{p,q} = Gr^\tau_{-p}H^{p+q}(X,\mathcal{F})$$

with isomorphisms for $r > 0$: $E_r^{p,q}(\tau) \simeq E_{r+1}^{2p+q,-p}(L)$.
On the cohomology group $H^n(X, \mathcal{F})$, the term of the induced filtration τ_p coincides with the term of the induced filtration L^{n-p}.

3.3. **Mixed Hodge complex (MHC).** In this section we introduce sufficient conditions on the filtrations of a bifiltered complex (K, W, F) in order to obtain a mixed Hodge structure with the induced filtrations by W and F on the cohomology of K. The structure defined on K by these conditions is called a mixed Hodge complex (MHC).
Let Γ denotes the global sections functor on On an algebraic variety V. We want to construct on V a bifiltered complex of sheaves $(K_{\mathbb{C}}, F, W)$ where the filtration W is rationally defined. It is called a cohomological MHC if its image by the derived functor $R\Gamma(V, K, W, F)$ is a MHC. Then, by the two filtrations lemma (2.3.1), the filtrations induced by W and F on the hypercohomology $\mathbb{H}^i(V, K_{\mathbb{C}})$ define a MHS. This result is so powerful that the rest of the theory will consist in the construction of a cohomological MHC for all algebraic varieties. The theoretical path to construct a MHS on a variety follows the pattern:

$$\text{cohomological MHC} \Rightarrow \text{ MHC} \Rightarrow \text{MHS}$$

It is true that a direct study of the logarithmic complex by Griffiths and Schmid [26] is very attractive, but the above pattern in the initial work of Deligne is easy to apply, flexible and helps to go beyond this case towards a general theory.

The de Rham complex of a smooth compact complex variety is a special case of a mixed Hodge complex, called a Hodge complex (HC) with the characteristic property that it induces a Hodge structure on its hypercohomology. We start by rewriting the Hodge theory of the first section with this terminology since it is fitted to generalization to MHC.

3.3.1. *Definitions.* Let A denote $\mathbb{Z}$, $\mathbb{Q}$ or $\mathbb{R}$ as in section (2.2), $D^+(\mathbb{Z}))$ (resp. $D^+(\mathbb{C}))$ denotes the derived category of $\mathbb{Z}-$modules (resp. $\mathbb{C}-$vector spaces), and the corresponding derived category of sheaves on V: $D^+(V, \mathbb{Z}))$, $D^+(V, \mathbb{C}))$.

Definition 3.22 (Hodge complex (HC))**.** *A Hodge A-complex K of weight n consists of:*
i) a complex K_A of A-modules, such that $H^k(K_A)$ is an A-module of finite type for all k;
ii) a filtered complex $(K_{\mathbb{C}}, F)$ of $\mathbb{C}-$vector spaces;
iii) an isomorphism $\alpha : K_A \otimes \mathbb{C} \simeq K_{\mathbb{C}}$ in $D^+(\mathbb{C})$.
The following axioms must be satisfied:
(HC 1) the differential d of $K_{\mathbb{C}}$ is strictly compatible with the filtration F, i.e., $d^i : (K_A^i, F) \to (K_A^{i+1}, F)$ is strict, for all i;
(HC 2) for all k, the filtration F on $H^k(K_{\mathbb{C}}) \simeq H^k(K_A) \otimes \mathbb{C}$ defines an A-Hodge structure of weight $n + k$ on $H^k(K_A)$.

The condition (HC1) is equivalent to the degeneration of the spectral sequence defined by $(K_{\mathbb{C}}, F)$ at rank 1, i.e. $E_1 = E_\infty$ (see 2.5). By (HC2) the filtration F is $(n + k)$-opposed to its complex conjugate (conjugation makes sense since $A \subset \mathbb{R}$).

Definition 3.23. *Let X be a topological space. An A-cohomological Hodge complex K of weight n on X, consists of:*
i) a complex of sheaves K_A of $A-$modules on X;

ii) a filtered complex of sheaves $(K_{\mathbb{C}}, F)$ *of* $\mathbb{C}$*-vector spaces on* X*;*
iii) an isomorphism $\alpha : K_A \otimes \mathbb{C} \xrightarrow{\approx} K_{\mathbb{C}}$ *in* $D^+(X, \mathbb{C})$ *of* $\mathbb{C}-$*sheaves on* X*.*
Moreover, the following axiom must be satisfied:
(CHC) The triple $(R\Gamma(X, K_A), R\Gamma(X, K_{\mathbb{C}}, F), R\Gamma(\alpha))$ *is a Hodge complex of weight* n*.*

If (K, F) is a HC (resp. cohomological HC) of weight n, then $(K[m], F[p])$ is a Hodge complex (resp. cohomological HC) of weight $n + m - 2p$.

Example 3.24. The following statement is a new version of Hodge decomposition theorem. Let X be a compact complex algebraic manifold and consider:
i) $K_{\mathbb{Z}}$ the complex reduced to a constant sheaf $\mathbb{Z}$ on X in degree zero;
ii) $K_{\mathbb{C}}$ the analytic de Rham complex Ω_X^* with its trivial filtration $F^p = \Omega_X^{*\geq p}$ by subcomplexes:

$$F^p\Omega_X^* := 0 \to \cdots 0 \to \Omega_X^p \to \Omega_X^{p+1} \to \cdots \to \Omega_X^n \to 0;$$

iii) The quasi-isomorphism $\alpha \colon K_{\mathbb{Z}} \otimes \mathbb{C} \xrightarrow{\approx} \Omega_X^*$ (Poincaré lemma).
Then $(K_{\mathbb{Z}}, (K_{\mathbb{C}}, F), \alpha)$ is a cohomological HC of weight 0.
which means that its hypercohomology on X is isomorphic to the cohomology of X and carries a Hodge structure.

Now, we define the structure including two filtrations by weight W and F needed on a complex, in order to define a MHS on its cohomology.

Definition 3.25 (MHC)**.** *An A-mixed Hodge complex (MHC) K consists of:*
i) a complex K_A *of* $A-$*modules such that* $H^k(K_A)$ *is an A-module of finite type for all k;*
ii) a filtered complex $(K_{A\otimes\mathbb{Q}}, W)$ *of* $(A \otimes \mathbb{Q})-$*vector spaces with an increasing filtration* W*;*
iii) an isomorphism $K_A \otimes \mathbb{Q} \xrightarrow{\sim} K_{A\otimes\mathbb{Q}}$ *in* $D^+(A \otimes \mathbb{Q})$*;*
iv) a bi-filtered complex $(K_{\mathbb{C}}, W, F)$ *of* $\mathbb{C}-$*vector spaces with an increasing (resp. decreasing) filtration* W *(resp.* F*) and an isomorphism:*

$$\alpha : (K_{A\otimes\mathbb{Q}}, W) \otimes \mathbb{C} \xrightarrow{\sim} (K_{\mathbb{C}}, W)$$

in $D^+F(\mathbb{C})$*.*
Moreover, the following axiom is satisfied:
(MHC) For all n, the system consisting of
- the complex $Gr_n^W(K_{A\otimes\mathbb{Q}})$ *of* $(A \otimes \mathbb{Q})-$*vector spaces,*
- the complex $Gr_n^W(K_{\mathbb{C}}, F)$ *of* $\mathbb{C}-$*vector spaces with induced* F *and*
- the isomorphism $Gr_n^W(\alpha) : Gr_n^W(K_{A\otimes\mathbb{Q}}) \otimes \mathbb{C} \xrightarrow{\sim} Gr_n^W(K_{\mathbb{C}})$*,*
is an $A \otimes \mathbb{Q}$*-Hodge complex of weight n.*

The above structure has a corresponding structure on a complex of sheaves on X called a cohomological MHC:

Definition 3.26 (cohomological MHC)**.** *An A-cohomological mixed Hodge complex K on a topological space X consists of:*
i) a complex of sheaves K_A *of sheaves of A-modules on X such that* $H^k(X, K_A)$ *are A-modules of finite type;*
ii) a filtered complex $(K_{A\otimes\mathbb{Q}}, W)$ *of sheaves of* $(A \otimes \mathbb{Q})-$*vector spaces on X with an increasing filtration W and an isomorphism* $K_A \otimes \mathbb{Q} \simeq K_{A\otimes\mathbb{Q}}$ *in* $D^+(X, A \otimes \mathbb{Q})$*;*

iii) a bi-filitered complex of sheaves $(K_{\mathbb{C}}, W, F)$ of $\mathbb{C}-$vector spaces on X with an increasing (resp. decreasing) filtration W (resp. F) and an isomorphism:

$$\alpha : (K_{A\otimes\mathbb{Q}}, W) \otimes \mathbb{C} \xrightarrow{\sim} (K_{\mathbb{C}}, W)$$

in $D^+F(X,\mathbb{C})$.
Moreover, the following axiom is satisfied:
For all n, the system consisting of:
- the complex $Gr_n^W(K_{A\otimes\mathbb{Q}})$ of sheaves of $(A\otimes\mathbb{Q})$-vector spaces on X;
- the complex $Gr_n^W(K_{\mathbb{C}}, F)$ of sheaves of $\mathbb{C}$-vector spaces on X with induced F;
- the isomorphism $Gr_n^W(\alpha) : Gr_n^W(K_{A\otimes\mathbb{Q}}) \otimes \mathbb{C} \xrightarrow{\sim} Gr_n^W(K_{\mathbb{C}})$,
is an $A\otimes\mathbb{Q}$-cohomological HC of weight n.

If (K, W, F) is a MHC (resp. cohomological MHC), then for all m and $n \in \mathbb{Z}, (K[m], W[m-2n], F[n])$ is a MHC (resp. cohomological MHC).

In fact, any HC or MHC described here is obtained from de Rham complexes with modifications (at infinity) as the logarithmic complex described in the next section. A new construction of HC has been later introduced with the theory of differential modules and perverse sheaves [2] and [43] following the theory of the intersection complex but will not be covered in these lectures.

Now we describe how we first obtain the MHC from a cohomological MHC, then a MHS from a MHC.

Proposition 3.27. *If $K = (K_A, (K_{A\otimes\mathbb{Q}}, W), (K_{\mathbb{C}}, W, F))$ and the isomorphism α is an A-cohomological MHC then:*

$$R\Gamma K = (R\Gamma K_A, R\Gamma(K_{A\otimes\mathbb{Q}}, W), R\Gamma(K_{\mathbb{C}}, W, F))$$

together with the isomorphism $R\Gamma(\alpha)$ is an A-MHC.

The main result of Deligne in [11] and [12] states in short:

Theorem 3.28 (Deligne)**.** *The cohomology of a mixed Hodge complex carries a mixed Hodge structure.*

The proof of this result requires a detailed study of spectral sequences based on the two filtrations lemma 2.3.1. We give first the properties of the various spectral sequences which may be of interest as independent results. Precisely, the weight spectral sequence of a MHC is in the category of HS. So, the MHS on cohomology is approached step by step by HS on the terms of the weight spectral sequence ${}_WE_r^{p,q}$ of $(K_{\mathbb{C}}, W)$. However, the big surprise is that the spectral sequence degenerates quickly, at rank two for W and at rank one for F: these terms are all that is needed in computation.

We show first that the first terms $E_1^{p,q}$ of the spectral sequence with respect to W carry a HS of weight q defined by the induced filtration by F. Moreover, the differentials d_1 are morphisms of HS, hence the terms $E_2^{p,q}$ carry a HS of weight q. Then the proof based on 2.3.1 consists to show that d_r is compatible with the induced HS; but for $r > 1$ it is a morphism between two HS of different weight, hence it must vanish.

Proposition 3.29 (MHS on the cohomology of a MHC)**.** *Let K be an A-MHC.*
i)The filtration $W[n]$ of $H^n(K_A)\otimes\mathbb{Q} \simeq H^n(K_{A\otimes\mathbb{Q}})$:

$$(W[n])_q(H^n(K_{A\otimes\mathbb{Q}}) := Im\,(H^n(W_{q-n}K_{A\otimes\mathbb{Q}}) \to H^n(K_{A\otimes\mathbb{Q}}))$$

and the filtration F on $H^n(K_{\mathbb{C}}) \simeq H^n(K_A) \otimes_A \mathbb{C}$:

$$F^p(H^n(K_{\mathbb{C}}) := Im\,(H^n(F^pK_{\mathbb{C}}) \to H^n(K_{\mathbb{C}}))$$

define on $H^n(K)$ an A-mixed Hodge structure:

$$(H^n(K_A), (H^n(K_{A\otimes\mathbb{Q}}), W), (H^n(K_{\mathbb{C}}), W, F)).$$

ii) On the terms ${}_WE_r^{p,q}(K_{\mathbb{C}}, W)$, the recurrent filtration and the two direct filtrations coincide $F_d = F_{rec} = F_{d^*}$ and define the Hodge filtration F of a Hodge sructure of weight q and d_r is compatible with F.
iii) The morphisms $d_1 : {}_WE_1^{p,q} \to {}_WE_1^{p+1,q}$ are strictly compatible with F.
iv) The spectral sequence of $(K_{A\otimes\mathbb{Q}},\ W)$ degenerates at rank $2\,({}_WE_2 = {}_WE_\infty)$.
v) The spectral sequence of $(K_{\mathbb{C}}, F)$ degenerates at rank $1\,({}_FE_1 = {}_FE_\infty)$.
vi) The spectral sequence of the complex $Gr_F^p(K_{\mathbb{C}})$, with the induced filtration W, degenerates at rank 2.

Remark that the indices of the weight filtration are not given by the indices of the induced filtration W on cohomology, but are shifted by n. One should recall that the weight of the HS on the terms ${}_WE_r^{p,q}$ is always q, hence the weight of $Gr_{-p}^W H^{p+q}(K)$ is q i.e the induced term W_{-p} is considered with index q: $W_{-p} = (W[p+q])_q$.

3.3.2. *Proof of the existence of a MHS on the cohomology of a MHC.* We need to check that the hypothesis $(*r_0)$ in the theorem 2.34 applies to MHC, which is done by induction on r_0. If we assume that the filtrations $F_d = F_{rec} = F_{d^*}$ coincide for $r < r_0$ and moreover define the same Hodge filtration F of a Hodge structure of weight q on $E_r^{p,q}(K, W)$ and $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$ is compatible with such Hodge structure, then in particular d_r is strictly compatible with F, hence the induction apply.

Lemma 3.30. *For $r \geq 1$, the differentials d_r of the spectral sequence ${}_WE_r$ are strictly compatible with the recurrent filtration $F = F_{rec}$. For $r \geq 2$, they vanish.*

The initial statement applies for $r = 1$ by definition of a MHC since the complex $Gr_{-p}^W K$ is a HC of weight $-p$. Hence, the two direct filtrations and the recurrent filtration F_{rec} coincide with the Hodge filtration F on ${}_WE_1^{p,q} = H^{p+q}(Gr_{-p}^W K)$. The differential d_1 is compatible with the direct filtrations, hence with F_{rec}, and commutes with complex conjugation since it is defined on $A \otimes \mathbb{Q}$, hence it is compatible with $\overline{F}_{rec}$. Then it is strictly compatible with the Hodge filtration $F = F_{rec}$.

The filtration F_{rec} defined in this way is q-opposed to its complex conjugate and defines a HS of weight q on ${}_WE_2^{pq}$.

We suppose by induction that the two direct filtrations and the recurrent filtration coincide on ${}_WE_s (s \leq r+1) : F_d = F_{rec} = F_{d^*}$ and ${}_WE_r = {}_WE_2$. On ${}_WE_2^{p,q} = {}_WE_r^{p,q}$, the filtration $F_{rec} := F$ is compatible with d_r and q-opposed to its complex conjugate. Hence the morphism $d_r : {}_WE_r^{p,q} \to {}_WE_r^{p+r,q-r+1}$ is a morphism of a HS of weight q to a Hodge Structure of weight $q-r+1$ and must vanish for $r > 1$. In particular, we deduce that the weight spectral sequence degenerates at rank 2.

The filtration on ${}_WE_\infty^{p,q}$ induced by the filtration F on $H^{p+q}(K)$ coincides with the filtration F_{rec} on ${}_WE_2^{p,q}$.

3.4. **Relative cohomology and the mixed cone.** The notion of morphism of MHC involves compatibility between the rational level and complex level. They are stated in the derived category to give some freedom in the choice of resolutions while keeping track of this compatibility. This is particularly interesting in the proof of functoriality of MHS.
However, to put a MHS on the relative cohomology it is natural to use the cone construction over a morphism u. If u is given as a class $[u]$ up to homotopy, the cone construction depends on the choice of the representative u as we have seen.
To define a mixed Hodge structure on the relative cohomology, we must define the notion of mixed cone with respect to a representative of the morphism on the level of complexes.
The isomorphism between two structures obtained for two representatives depends on the choice of a homotopy, hence it is not naturally defined.
Nevertheless this notion is interesting in applications since in general the MHC used to define a MHS on a variety X is in fact defined in $C^+F(X,\mathbb{Q})$ and $C^+F^2(X,\mathbb{C})$.

3.4.1. Let (K,W) be a complex of objects of an abelian category $\mathbb{A}$ with an increasing filtration W. We denote by (T_MK,W) or $(K[1],W[1])$ the complex shifted by a translation on degrees of K and of W such that $(W[1]_nK[1]) := (W_{n-1}K)[1]$ or $W_n(T_MK) = W_{n-1}TK$, i.e $(W[1]_nK[1])^i := (W_{n-1}K)^{i+1}$. Then:
$Gr_n^W(K[1],W[1]) = (Gr_{n-1}^WK)[1]$ and if (K,W,F) is a MHC,
$H^i(Gr_n^W(K[1],W[1]),F) = H^{i+1}(Gr_{n-1}^WK,F)$ is a HS of weight $n+i$, in other terms $(K[1],W[1],F)$ is a MHC.

Definition 3.31 (Mixed cone). *Let $u : (K,W,F) \to (K',W',F')$ be a morphism of complexes in $C^+F(\mathbb{A})$ (resp. $C^+F_2(\mathbb{A})$) with increasing filtrations W,W' (resp. decreasing filtrations F,F'). The structure of mixed cone $C_M(u)$ is defined on the cone complex $C(u) := K[1]\oplus K'$ with the filtrations $W[1]\oplus W'$ (resp. $F\oplus F'$).*

The definition is not in the derived category but on the level of filtered complexes. In particular, the mixed cone of a MHC is a MHC, since: $Gr_n^W(C_M(u),W,F) = ((Gr_{n-1}^WK)[1],F)\oplus(Gr_n^{W'}K'),F')$ is a HC of weight n.

3.4.2. *Morphisms of MHC.* A morphism $u : K \to K'$ of MHC (resp. cohomological MHC) consists of morphisms:

$$u_A : K_A \to K'_A \text{ in } D^+A(\text{resp.}D^+(X,A)),$$
$$u_{A\otimes\mathbb{Q}} : (K_{A\otimes\mathbb{Q}},W) \to ({K'}_{A\otimes\mathbb{Q}},W) \text{ in } D^+F(A\otimes\mathbb{Q})(\text{ resp. } D^+F(X,A\otimes\mathbb{Q})),$$
$$u_{\mathbb{C}} : (K_{\mathbb{C}},W,F) \to (K'_{\mathbb{C}},W,F) \text{ in } D^+F_2\mathbb{C} \text{ (resp. } D^+F_2(X,\mathbb{C})).$$

and commutative diagrams:

$$\begin{array}{ccc} K_{A\otimes\mathbb{Q}} & \xrightarrow{u_{A\otimes\mathbb{Q}}} & K'_{A\otimes\mathbb{Q}} \\ \wr\downarrow\alpha & & \wr\downarrow\alpha' \\ K_A\otimes\mathbb{Q} & \xrightarrow{u_A\otimes\mathbb{Q}} & K'_A\otimes\mathbb{Q} \end{array} \qquad \begin{array}{ccc} (K_{A\otimes\mathbb{Q}},W)\otimes\mathbb{C} & \xrightarrow{u_{A\otimes\mathbb{Q}}\otimes\mathbb{C}} & (K'_{A\otimes\mathbb{Q}},W)\otimes\mathbb{C} \\ \wr\downarrow\beta & & \wr\downarrow\beta' \\ (K_{\mathbb{C}},W) & \xrightarrow{u_{\mathbb{C}}} & (K'_{\mathbb{C}},W) \end{array}$$

in $D^+(A\otimes\mathbb{Q})$ (resp. $D^+(X,A\otimes\mathbb{Q})$ at left and compatible with W in $D^+F(\mathbb{C})$ (resp. $D^+F(X,\mathbb{C})$) at right.

3.4.3. Let $u : K \to K'$ be a morphism of a MHC. There exists a quasi-isomorphism $v = (v_A, v_{A\otimes\mathbb{Q}}, v_{\mathbb{C}}) : \tilde{K} \xrightarrow{\approx} K$ and a morphism $\tilde{u} = (\tilde{u}_A, \tilde{u}_{A\otimes\mathbb{Q}}, \tilde{u}_{\mathbb{C}}) : \tilde{K} \to K'$ of MHC such that v and $\tilde{u}$ are defined successively in C^+A, $C^+F(A\otimes\mathbb{Q})$ and $C^+F_2\mathbb{C}$, i.e. we can find, by definition, diagrams:

$$K_A \xleftarrow{\approx} \tilde{K}_A \to K'_A, \quad K_{A\otimes\mathbb{Q}} \xleftarrow{\approx} \tilde{K}_{A\otimes\mathbb{Q}} \to K'_{A\otimes\mathbb{Q}}, \quad K_{\mathbb{C}} \xleftarrow{\approx} \tilde{K}_{\mathbb{C}} \to K'_{\mathbb{C}},$$

or in short $K \underset{v}{\xleftarrow{\approx}} \tilde{K} \xrightarrow{\tilde{u}} K'$ (or equivalently $K \xrightarrow{\tilde{u}} \tilde{K}' \underset{v}{\xleftarrow{\approx}} K'$) representing u.

3.4.4. *Dependence on homotopy.* Consider a morphism $u\colon K \to K'$ of MHC, represented by a morphism of complexes $\tilde{u} : \tilde{K} \to K'$.
To define the mixed cone $C_M(\tilde{u})$ out of:
(i) the cones $C(\tilde{u}_A) \in C^+(A)$, $C_M(\tilde{u}_{A\otimes\mathbb{Q}}) \in C^+F(A\otimes\mathbb{Q})$, $C_M(\tilde{u}_{\mathbb{C}}) \in C^+F_2(\mathbb{C})$, we still need to define compatibility isomorphisms:

$$\gamma_1 : C_M(\tilde{u}_{A\otimes\mathbb{Q}}) \simeq C(\tilde{u}_A) \otimes \mathbb{Q}, \quad \gamma_2 : (C_M(\tilde{u}_{\mathbb{C}}, W) \simeq (C_M(\tilde{u}_{A\otimes\mathbb{Q}}), W) \otimes \mathbb{C}$$

successively in $C^+(A\otimes\mathbb{Q})$ and $C^+F(\mathbb{C})$. With the notations of 3.4.2 the choice of isomorphisms $C_M(\tilde{\alpha}, \tilde{\alpha}')$ and $C_M(\tilde{\beta}, \tilde{\beta}')$ representing the compatibility isomorphisms in K and K' does not define compatibility isomorphisms for the cone since the diagrams of compatibility are commutative only up to homotopy, that is there exists homotopies h_1 and h_2 such that:

$$\tilde{\alpha}' \circ (\tilde{u}_{A\otimes\mathbb{Q}}) - (\tilde{u}_A \otimes \mathbb{Q}) \circ \tilde{\alpha} = h_1 \circ d + d \circ h_1,$$

and:

$$\tilde{\beta}' \circ \tilde{u}_{\mathbb{C}} - (\tilde{u}_{A\otimes\mathbb{Q}} \otimes \mathbb{C}) \circ \tilde{\beta} = h_2 \circ d + d \circ h_2.$$

ii) Then we can define the compatibility isomorphism as:

$$C_M(\tilde{\alpha}, \tilde{\alpha}', h_1) := \begin{pmatrix} \tilde{\alpha} & 0 \\ h_1 & \tilde{\alpha}' \end{pmatrix} : C_M(\tilde{u}_{A\otimes\mathbb{Q}}) \xrightarrow{\sim} C(\tilde{u}_A) \otimes \mathbb{Q}$$

and a similar formula for $C_M(\tilde{\beta}, \tilde{\beta}', h_2)$.

Definition 3.32. *Let $u\colon K \to K'$ be a morphism of MHC. The mixed cone $C_M(\tilde{u}, h_1, h_2)$ constructed above depends on the choices of the homotopies (h_1, h_2) and the choice of a representative $\tilde{u}$ of u, such that:*

$$Gr_n^W(C_M(\tilde{u}), F) \simeq (Gr_{n-1}^W(T\tilde{K}), F) \oplus (Gr_n^W K', F)$$

is a HC of weight n; hence $C_M(\tilde{u}, h_1, h_2)$ is a MHC.

Remark 3.33. The MHC used in the case of a projective NCD case and its complement are naturally defined in $C^+F(X,\mathbb{Q})$ and $C^+F_2(X,\mathbb{C})$.

4. MHS on the cohomology of a complex algebraic variety

The aim of this section is to prove:

Theorem 4.1 (Deligne)**.** *The cohomology of complex algebraic varieties is endowed with a functorial mixed Hodge structure with respect to algebraic morphisms.*

The uniqueness follows easily, once we have fixed the case of compact normal crossing divisors which in particular includes the non singular compact case. All what we need, is to construct explicitly on each algebraic variety X a cohomological MHC, to which we apply the previous abstract algebraic study to define the MHS on the cohomology groups of X.

- First, on a smooth complex variety X containing a normal crossing divisor (NCD) Y with smooth irreducible components, we shall construct the complex of sheaves of differential forms with logarithmic singularities along Y denoted $\Omega^*_X(Log\, Y)$, or $\Omega^*_X < Y >$ whose hypercohomology on X is isomorphic to the cohomology of $X - Y$ with coefficients in $\mathbb{C}$. We shall endow this complex with two filtrations W and F.

When X is also compact algebraic, the bi-filtered complex $(\Omega^*_X(LogY), W, F)$ underlies the structure of a cohomological MHC which defines a mixed Hodge structure on the cohomology of $X - Y$. In other terms, to construct the MHS of the smooth variety V we have to consider a compactification of V by a compact algebraic variety X, which always exists by a result of Nagata [41]. Moreover, by Hironaka's desingularization theorem [35], we can suppose X smooth and the complement $Y = X - V$ to be a NCD with smooth irreducible components. Then, the MHS on the cohomology of $V = X - Y$ will be deduced from the logarithmic complex $(\Omega^*_X(LogY), W, F)$. It is not difficult to show that such a MHS does not depend on the compactification X and will be referred to as the MHS on V. In some sense it depends on the asymptotic properties at infinity of V. The weights of the MHS on the cohomology $H^i(V)$ of a smooth variety V, i.e the weights j of the HS on Gr^W_j are $\geq i$ and to be precise $W_{i-1} = 0, W_{2i} = H^i(V)$.

The Verdier - Poincaré dual of the logarithmic complex which hypercohomology on X is equal to the cohomology of compact support $H^*_c(X - Y, \mathbb{C})$ of $X - Y$ is more natural to construct as it can be deduced from the mixed cone construction (shifted by [-1]) over the restriction map $\Omega^*_X \to \Omega^*_{Y_\bullet}$ from the de Rham complex on X to the MHC on the normal crossing divisor Y described in section 2 [18]. It is associated to the natural morphism $\mathbb{Z}_X \to \mathbb{Z}_{Y_\bullet}$.
Let Z be a sub-NCD of Y, union of some components of Y, then the complement $Y - Z$ is an open NCD and its cohomology may be described by a double complex combination of the open case and the NCD case [18]. It is a model for the simplicial general case in this section. In this case, the weights of the MHS on the cohomology $H^j(Y - Z)$ vary from 0 to $2j$ which is the general case.

- For any algebraic variety, the construction is based on a diagram of algebraic varieties:

$$X_\bullet = \left(X_0 \;\begin{array}{c}\leftarrow\\ \leftarrow\end{array}\; X_1 \;\begin{array}{c}\leftarrow\\ \leftarrow\\ \leftarrow\end{array}\; \cdots \quad X_{q-1} \;\begin{array}{c}\xleftarrow{X_*(\delta_i)}\\ \vdots\\ \longleftarrow\end{array}\; X_q \cdots \right)$$

similar to the model in the case of a NCD. Here the $X_*(\delta_i)$ are called the face maps (see below 4.12 for the definition), one for each $i \in [0, q]$ and satisfy commutativity relations under composition.
When we consider diagrams of complexes of sheaves, the resolutions of such sheaves are defined with compatibility relations with respect to the maps $X_*(\delta_i)$. We are interested in such diagrams when they form a simplicial hypercovering of an algebraic variety X by non singular varieties; in other terms, when the diagram defines a resolution of the constant sheaf $\mathbb{Z}_X$ on X by the direct images of constant sheaves on the various non singular X_i.
Using a general simplicial technique combined with Hironaka's desingularization at the various steps, Deligne shows the existence of such simplicial resolutions [12]. This construction is admitted here.

Starting with the various logarithmic complexes on the terms of the simplicial hypercovering, we describe the construction of an associated cohomological MHC on the variety X. The construction is based on a diagonal process which generalizes the construction of the MHC in the case of a NCD and it is similar to a repeated mixed cone construction without the ambiguity of the choice of homotopy, since resolutions of simplicial complexes of sheaves are functorial in the simplicial derived category.

In particular, we should view the simplicial category as a set of diagrams and the construction is carried out with respect to such diagrams. In fact, there exists another construction based on a set of diagrams defined by cubical schemes [31, 42]. At the end we give an alternative construction for embedded varieties with diagrams of four edges only [18], which shows that the ambiguity in the mixed mapping cone construction may be overcome.

In all cases, the mixed Hodge structure is constructed first for smooth varieties and normal crossing divisors, then it is deduced for general varieties. The uniqueness follows from the compatibility of the MHS with Poincaré duality and classical exact sequences on cohomology as we will see at the end.

4.1. **MHS on the cohomology of smooth algebraic varieties.** As we already said above, to construct the mixed Hodge structure on the cohomology of a smooth complex algebraic variety V, we use a result of Nagata [41] to embed V as an open Zariski subset of a complete variety Z (here we need the algebraic structure on V). Then the singularities of Z are included in $D := Z - V$. Since Hironaka's desingularization process in characteristic zero (see [35]) is carried out by blowing up smooth centers above D, there exists a variety $X \to Z$ above Z such that the inverse image of D is a normal crossing divisor Y with smooth components in X such that $X - Y \simeq Z - D$.

Hence, we may start with the hypothesis that $V = X^* := X - Y$ is the complement of a normal crossing divisor Y in a smooth compact algebraic variety X, so the construction of the mixed Hodge structure is reduced to this situation. Still we need to prove that it does not depend on the choice of X.

We introduce now the logarithmic complex underlying the structure of cohomological mixed Hodge complex on X which computes the cohomology of V.

4.1.1. *The Logarithmic complex.* Let X be a complex manifold and Y be a NCD in X. We denote by $j : X^* \to X$ the embedding of $X^* := X - Y$ into X. We say that a meromorphic form ω has a pole of order at most 1 along Y if at each point $y \in Y$, $f\omega$ is holomorphic for some local equation f of Y at y. Let $\Omega^*_X(*Y)$ denote the sub-complex of $j_*\Omega^*_{X^*}$ defined by meromorphic forms along Y, holomorphic on X^*.

Definition 4.2. *The logarithmic de Rham complex of X along a normal crossing divisor Y is the subcomplex $\Omega^*_X(Log\, Y)$ of the complex $\Omega^*_X(*Y)$ defined by the sections ω such that ω and $d\omega$ both have a pole of order at most 1 along Y.*

By definition, at each point $y \in Y$, there exist local coordinates $(z_i)_{i\in[1,n]}$ on X and a subset $I_y \subset [1, n]$ depending on $y \in Y$ such that Y is defined at y by the equation $\Pi_{i\in I_y} z_i = 0$. Then ω and $d\omega$ have logarithmic poles along Y if and only if ω can be written locally as:

$$\omega = \sum_{i_1,\cdots,i_r \in I_y} \varphi_{i_1,\cdots,i_r} \frac{dz_{i_1}}{z_{i_1}} \wedge \cdots \wedge \frac{dz_{i_r}}{z_{i_r}} \quad \text{where} \quad \varphi_{i_1,\cdots,i_r} \text{ is holomorphic.}$$

Indeed, $d(1/z_i) = -dz_i/(z_i)^2$ has a pole of order 2, and $d\omega$ will have a pole along along $z_i = 0$ of order 2, unless ω is divisible by dz_i/z_i i.e $\omega = \omega' \wedge (dz_i/z_i)$.
This formula may be used as a definition, then we prove the independence of the choice of coordinates, that is ω may be written in this form with respect to any set of local coordinates at y.

The $\mathcal{O}_X$-module $\Omega^1_X(Log\,Y)$ is locally free at $y \in Y$ with basis $(dz_i/z_i)_{i\in I_y}$ and $(dz_j)_{j\in[1,n]-I_y}$ and $\Omega^p_X(Log\,Y) \simeq \wedge^p\Omega^1_X(Log\,Y)$.

Let $f : X_1 \to X_2$ be a morphism of complex manifolds, with normal crossing divisors Y_i in X_i for $i = 1, 2$, such that $f^{-1}(Y_2) = Y_1$. Then, the reciprocal morphism $f^* : f^*(j_{2*}\Omega^*_{X_2^*}) \to j_{1*}\Omega^*_{X_1^*}$ induces a morphism on logarithmic complexes:

$$f^* : f^*\Omega^*_{X_2}(Log\,Y_2) \to \Omega^*_{X_1}(Log\,Y_1).$$

4.1.2. *The weight filtration W.* Let $Y = \cup_{i\in I} Y_i$ be the union of smooth irreducible divisors. We put an order on I. Let S^q denotes the set of strictly increasing sequences $\sigma = (\sigma_1, ..., \sigma_q)$ in the set of indices I, such that $Y_\sigma \neq \emptyset$, where $Y_\sigma = Y_{\sigma_1...\sigma_q} = Y_{\sigma_1} \cap ... \cap Y_{\sigma_q}$. Denote $Y^q = \coprod_{\sigma\in S^q} Y_\sigma$ the disjoint union of Y_σ. Set $Y^0 = X$ and let $\Pi : Y^q \to Y$ be the canonical projection. An increasing filtration W, called the weight filtration, is defined as follows:

$$W_m(\Omega^p_X(Log\,Y)) = \sum_{\sigma\in S^m} \Omega^{p-m}_X \wedge dz_{\sigma_1}/z_{\sigma_1} \wedge ... \wedge dz_{\sigma_m}/z_{\sigma_m}$$

The sub-$\mathcal{O}_X$-module $W_m(\Omega^p_X(Log\,Y)) \subset \Omega^p_X(Log\,Y)$ is the smallest sub-module stable by exterior multiplication with local sections of Ω^*_X and containing the products $dz_{i_1}/z_{i_1} \wedge ... \wedge dz_{i_k}/z_{i_k}$ for $k \leq m$ for local equations z_j of the components of Y.

4.1.3. *The Residue isomorphism.* We define now Poincaré residue isomorphisms:

$$Res : Gr^W_m(\Omega^p_X(Log\,Y)) \to \Pi_*\Omega^{p-m}_{Y^m}, \quad Res : Gr^W_m(\Omega^*_X(Log\,Y)) \xrightarrow{\sim} \Pi_*\Omega^*_{Y^m}[-m].$$

Locally, at a point y on the intersection of an ordered set of m components $Y_{i_1}, \ldots, Y_{i_m}$ of Y, we choose a set of local equations z_i for $i \in I_y$ of the comonents of Y at y and an order of the indices $i \in I_y$, then Poincaré residue, defined on W_m by

$$Res_{Y_{i_1},...,Y_{i_m}}([\alpha \wedge (dz_{i_1}/z_{i_1} \wedge ... \wedge (dz_{i_m}/z_{i_m})]) = \alpha_{|Y_{i_1,...,i_m}}$$

vanishes on W_{m-1}, hence it induces on Gr^W_m a morphism independent of the choice of the equations and compatible with the differentials. To prove the isomorphism, we construct its inverse. For each sequence of indices $\sigma = (i_1, \ldots, i_m)$, we consider the morphism $\rho_\sigma : \Omega^p_X \to Gr^W_m(\Omega^{p+m}_X(Log\,Y))$, defined locally as:

$$\rho_\sigma(\alpha) = \alpha \wedge dz_{\sigma_1}/z_{\sigma_1} \wedge ... \wedge dz_{i_m}/z_{i_m}$$

It does not depend on the choice of z_i, since for another choice of coordinates z'_i, z_i/z'_i are holomorphic and the difference $(dz_i/z_i) - (dz'_i/z'_i) = d(z_i/z'_i)/(z_i/z'_i)$ is holomorphic; then $\rho_\sigma(\alpha) - \alpha \wedge dz'_{i_1}/z'_{i_1} \wedge ... \wedge dz'_{i_m}/z'_{i_m} \in W_{m-1}\Omega^{p+m}_X(LogY)$, and successively $\rho_\sigma(\alpha) - \rho'_\sigma(\alpha) \in W_{m-1}\Omega^{p+m}_X(Log\,Y)$. We have $\rho_\sigma(z_{i_j} \cdot \beta) = 0$ and

$\rho_\sigma(dz_{i_j} \wedge \beta') = 0$ for sections β of Ω^p_X and β' of Ω^{p-1}_X; hence ρ_σ factors by $\overline{\rho}_\sigma$ on $\Pi_*\Omega^p_{Y_\sigma}$ defined locally and glue globally into a morphism of complexes on X:

$$\overline{\rho}_\sigma : \Pi_*\Omega^p_{Y_\sigma} \to Gr^W_m(\Omega^{p+m}_X(Log\,Y)), \quad \overline{\rho} : \Pi_*\Omega^*_{Y^m}[-m] \to Gr^W_m\Omega^*_X(Log\,Y).$$

Lemma 4.3. *We have the following isomorphisms of sheaves:*
i) $H^i(Gr^W_m\Omega^*_X(Log\,Y)) \simeq \Pi_*\mathbf{C}_{Y^m}$ *for* $i = m$ *and* 0 *for* $i \neq m$,
ii) $H^i(W_r\Omega^*_X(Log\,Y)) \simeq \Pi_*\mathbf{C}_{Y^i}$ *for* $i \leq r$ *and* $H^i(W_r\Omega^*_X(Log\,Y)) = 0$ *for* $i > r$,
in particular $H^i(\Omega^*_X(Log\,Y)) \simeq \Pi_*\mathbb{C}_{Y^i}$.

Proof. The statement in i) follows from the residue isomorphism.

The statement in ii) follows easily by induction on r, from i) and the long exact sequence associated to the short exact sequence $0 \to W_r \to W_{r+1} \to Gr^W_{r+1} \to 0$, written as

$$H^i(W_r) \to H^i(W_{r+1}) \to H^i(Gr^W_{r+1}) \to H^{i+1}(W_r)$$

□

Proposition 4.4 (Weight filtration W). *The morphisms of filtered complexes:*

$$(\Omega^*_X(Log\,Y), W) \xleftarrow{\alpha} (\Omega^*_X(Log\,Y), \tau) \xrightarrow{\beta} (j_*\Omega^*_{X^*}, \tau)$$

where τ *is the truncation filtration, are filtered quasi-isomorphisms.*

Proof. The quasi-isomorphism α follows from the lemma.

The morphism j is Stein, since for each polydisc $U(y)$ in X centered at a point $y \in Y$, the inverse image $X^* \cap U(y)$ is Stein as the complement of an hypersurface, hence j is acyclic for coherent sheaves, that is $Rj_*\Omega^*_{X^*} \simeq j_*\Omega^*_{X^*}$. By Poincaré lemma $\underline{\mathbb{C}}_{X^*} \simeq \Omega^*_{X^*}$, so that $Rj_*\underline{\mathbb{C}}_{X^*} \simeq j_*\Omega^*_{X^*}$, hence it is enough to prove $Gr^\tau_i Rj_*\mathbb{C}_{X^*} \simeq \Pi_*\mathbb{C}_{Y^i}$, which is a local statement.

For each polydisc $U(y) = U$ with $U^* = U - U \cap Y \simeq (D^*)^m \times D^{n-m}$, hence $(D^*)^m \times D^{n-m}$ is homotopic to an $i-$dimensional torus $(S^1)^m$. The cohomology $\mathbb{H}^i(U, Rj_*\underline{\mathbb{C}}_{X^*}) = H^i(U^*, \mathbb{C})$ can be computed by Künneth formula and is equal to $\wedge^i H^1(U^*, \mathbb{C}) \simeq \wedge^i\Gamma(U, \Omega^1_U(Log\,Y)))$ where $dz_i/z_i, i \in [1, m]$ form a basis dual to the homology basis defined by products of S^1. □

Corollary 4.5. *The weight filtration is rationally defined.*

The main point here is that the τ filtration is defined with rational coefficients as $(Rj_*\underline{\mathbb{Q}}_{X^*}, \tau) \otimes \mathbb{C}$, which gives the rational definition for W.

4.1.4. *Hodge filtration* F. It is defined by the formula $F^p = \Omega^{*\geq p}_X(Log\,Y)$, which includes all forms of type (p', q') with $p' \geq p$. We have:

$$Res : F^p(Gr^W_m\Omega^*_X(Log\,Y)) \simeq \Pi_*F^{p-m}\Omega^*_{Y^m}[-m]$$

hence a filtered isomorphism:

$$Res : (Gr^W_m\Omega^*_X(Log\,Y), F) \simeq (\Pi_*\Omega^*_{Y^m}[-m], F[-m]).$$

Corollary 4.6. *The following system* $\mathbf{K}$*:*

(1) $(\mathbf{K}^\mathbb{Q}, W) = (Rj_*\underline{\mathbb{Q}}_{X^*}, \tau) \in ObD^+F(X, \mathbb{Q})$
(2) $(\mathbf{K}^\mathbb{C}, W, F) = (\Omega^*_X(Log\,Y), W, F) \in ObD^+F_2(X, \mathbb{C})$
(3) *The isomorphism* $(\mathbf{K}^\mathbb{Q}, W) \otimes \mathbb{C} \simeq (\mathbf{K}^\mathbb{C}, W)$ in $D^+F(X, \mathbb{C})$

is a cohomological MHC on X.

Remark 4.7. In the above system $\mathbf{K}$ we can choose adequate resolutions, for example Godement resolutions denoted below by $\mathcal{G}^\bullet$ to define the morphisms of filtered complexes in $C^+F(X,\mathbb{C})$:

$$\mathcal{G}^\bullet(\Omega^*_X(Log\,Y),W) \xleftarrow{\alpha} \mathcal{G}^\bullet(\Omega^*_X(Log\,Y),\tau) \xrightarrow{\beta} j_*\mathcal{G}^\bullet(\Omega^*_{X^*},\tau) \xleftarrow{I} j_*\mathcal{G}^\bullet(\mathbb{C}_{X^*},\tau)$$

then we can apply the mixed cone construction to the system. For example the mixed cone over the mapping $(\Omega^*_X, F) \to (\Omega^*_X(Log\,Y), W, F)$ from the HC on X of weight 0 is quasi isomorphic to $R\Gamma_Y\mathbb{C}[1]$. It puts a MHS on the cohomology with support in Y. As well the quotient complex $(\Omega^*_X(Log\,Y)/\Omega^*_X, W, F)$ define the same MHS.

Theorem 4.8 (Deligne)**.** *The system $K = R\Gamma(X,\mathbf{K})$ is a mixed Hodge complex. It endows the cohomology of $X^* = X - Y$ with a canonical mixed Hodge structure.*

Proof. The result follows directly from the general theory of cohomological MHC. Nevertheless, it is interesting to understand what is needed for a direct proof and to compute the weight spectral sequence at rank 1:

$$\begin{aligned}&{}_WE_1^{pq}(R\Gamma(X,\Omega^*_X(Log\,Y)) = \mathbb{H}^{p+q}(X, Gr^W_{-p}\Omega^*_X(LogY)) \simeq \mathbb{H}^{p+q}(X, \Pi_*\Omega^*_{Y^{-p}}[p])\\ &\simeq H^{2p+q}(Y^{-p},\mathbb{C}) \Rightarrow Gr^W_q H^{p+q}(X^*,\mathbb{C}).\end{aligned}$$

where the double arrow means that the spectral sequence degenerates to the cohomology graded with respect to the filtration W induced by the weight on the complex level. In fact, we recall the proof up to rank 2. The differential d_1:

$$d_1 = \sum_{j=1}^{-p}(-1)^{j+1}G(\lambda_{j,-p}) = G : H^{2p+q}(Y^{-p},\mathbb{C}) \longrightarrow H^{2p+q+2}(Y^{-p-1},\mathbb{C})$$

where $\lambda_{j,-p}$ is defined as in 2.4.1 and d_1 is equal to an alternate Gysin morphism, Poincaré dual to the alternate restriction morphism:

$$\rho = \sum_{j=1}^{-p}(-1)^{j+1}\lambda^*_{j,-p} : H^{2n-q}(Y^{-p-1},\mathbb{C}) \to H^{2n-q}(Y^{-p},\mathbb{C})$$

therefore, the first term:

$$({}_WE_1^{pq}, d_1)_{p\in\mathbb{Z}} = (H^{2p+q}(Y^{-p},\mathbb{C}), d_1)_{p\in\mathbb{Z}}$$

is viewed as a complex in the category of HS of weight q. It follows that the terms:

$${}_WE_2^{pq} = H^p({}_WE_1^{*,q}, d_1)$$

are endowed with a HS of weight q. We need to prove that the differential d_2 is compatible with the induced Hodge filtration. For this we introduced the direct filtrations compatible with d_2 and proved that they coincide with the induced Hodge filtration. The differential d_2 is necessarily zero since it is a morphism of HS of different weights: the HS of weight q on E_2^{pq} and the HS of weight $q-1$ on $E_2^{p+2,q-1}$. The proof is the same for any MHC and consists of a recurrent argument to show in this way that the differentials d_i for $i \geq 2$ are zero (see 2.34). ☐

4.1.5. *Independence of the compactification and functoriality.* Let U be a smooth complex algebraic variety, X (resp. X') a compactification of U by a normal crossing divisor Y (resp. Y') at infinity, $j : U \to X$ (resp. $j' : U \to X'$) the open embedding; then $j \times j' : U \to X \times X'$ is a locally closed embedding, with closure V. By desingularizing V outside the image of U, we are reduced to the case where we have a smooth variety $X'' \xrightarrow{f} X$ such that $Y'' := f^{-1}(Y)$ is a NCD and $U \simeq X'' - Y''$, then we have an induced morphism f^* on the corresponding logarithmic complexes, compatible with the structure of MHC. It follows that the induced morphism f^* on hypercohomology is compatible with the MHS and is an isomorphism on the hypercohomology groups, hence it is an isomorphism of MHS. *Functoriality.* Let $f : U \to V$ be a morphism of smooth varieties, and let X (resp. Z) be smooth compactifications of U (resp. V) by NCD at infinity, then taking the closure of the graph of f in $X \times Z$ and desingularizing, we are reduced to the case where there exists a compactification X with an extension $\overline{f} : X \to Z$ inducing f on U. The induced morphism $\overline{f}^*$ on the corresponding logarithmic complexes is compatible with the filtrations W and F and with the structure of MHC, hence it is compatible with the MHS on hypercohomology.

Proposition 4.9. *Let U be a smooth complex algebraic variety.*
i) The Hodge numbers $h^{p,q} := dim.H^{p,q}(Gr^W_{p+q}H^i(U,\mathbb{C}))$ vanish for $p,q \notin [0,i]$. In particular, the weight of the cohomology $H^i(U,\mathbb{C})$ vary from i to $2i$.
ii) let X be a smooth compactification of U, then:

$$W_iH^i(U,\mathbb{Q}) = Im\ (H^i(X,\mathbb{Q}) \to H^i(U,\mathbb{Q})).$$

Proof. i) The space $Gr^W_r H^i(U,\mathbb{Q})$ is isomorphic to the term $E_2^{i-r,r}$ of the spectral sequence with a HS of weight r, hence it is a sub-quotient of $E_1^{i-r,r} = H^{2i-r}(Y^{r-i},\mathbb{Q})$ twisted by $\mathbb{Q}(i-r)$, which gives the following relation with the Hodge numbers $h^{p',q'}$ on Y^{r-i}: $h^{p,q} = h^{p',q'}$ for $(p,q) = (p'+r-i, q'+r-i)$. Since $h^{p',q'}(H^{2i-r}(Y^{r-i},\mathbb{Q})) = 0$ unless $r-i \geq 0$ ($Y^{r-i} \neq \emptyset$) and $2i-r \geq 0$ (the degree of cohomology), we deduce: $i \leq r \leq 2i$. Moreover, $p',q' \in [0,2i-r]$, hence $p,q \in [r-i,i]$.

ii) Suppose U is the complement of a NCD and denote by $j : U \to X$ the inclusion. By definition $W_iH^i(U,\mathbb{Q}) = \mathrm{Im}\ (\mathbb{H}^i(X,\tau_{\leq 0}Rj_*\mathbb{Q}_U) \to H^i(U,\mathbb{Q}))$, hence it is equal to the image of $H^i(X,\mathbb{Q})$ since $\mathbb{Q}$ is quasi-isomorphic to $\tau_{\leq 0}Rj_*\mathbb{Q}_U$. If $X-U$ is not a NCD, there exists a desingularization $\pi : X' \to X$ with an embedding of U in X' as the complement of a NCD, then we use the trace map $Tr\,\pi : H^i(X',\mathbb{Q}) \to H^i(X,\mathbb{Q})$ satisfying $(Tr\pi) \circ \pi^* = Id$ and compatible with Hodge structures. In fact the trace map is defined as a morphism of sheaves $R\pi_*\mathbb{Q}_{X'} \to \mathbb{Q}_X$ [47] (2.3.4) and [32] (III.10), (VI,4), hence commutes with the restriction to U. In particular, the images of both cohomology groups coincide in $H^i(U,\mathbb{Q})$. □

Exercise 4.10 (Riemann Surface)**.** Let $\overline{C}$ be a connected compact Riemann surface of genus g, $Y = \{x_1, \dots, x_m\}$ a subset of m points, and $C = \overline{C} - Y$ the open surface with $m > 0$ points in $\overline{C}$ deleted. The long exact sequence:

$$0 \to H^1(\overline{C},\mathbb{Z}) \to H^1(C,\mathbb{Z}) \to H^2_Y(\overline{C},\mathbb{Z}) = \oplus_{i=1}^{i=m}\mathbb{Z} \to H^2(\overline{C},\mathbb{Z}) \simeq \mathbb{Z} \to H^2(C,\mathbb{Z}) = 0$$

reduces to the following short exact sequence of mixed Hodge structures:

$$0 \to H^1(\overline{C},\mathbb{Z}) \to H^1(C,\mathbb{Z}) \to \mathbb{Z}^{m-1} \simeq Ker(\oplus_{i=1}^{i=m}\mathbb{Z} \to \mathbb{Z}) \to 0$$

where $H^1(C,\mathbb{Z}) = W_2H^1(C,\mathbb{Z})$ is of rank $2g+m-1$ contains:
$W_1H^1(C,\mathbb{Z}) = H^1(\overline{C},\mathbb{Z})$ of rank $2g$ and $Gr_2^W H^1(C,\mathbb{Z}) \simeq \mathbb{Z}^{m-1}$.
The Hodge filtration is given by: $F^0H^1(C,\mathbb{C}) = H^1(C,\mathbb{C})$ of rank $2g+m-1$, while

$$F^1H^1(C,\mathbb{C}) \simeq \mathbb{H}^1(\overline{C},(0 \to \Omega^1_{\overline{C}}(Log\{x_1,\dots,x_m\}) \simeq H^0(\overline{C},\Omega^1_{\overline{C}}(Log\{x_1,\dots,x_m\})$$

has dimension: $g+m-1$, since $Gr^0_F H^1(C,\mathbb{C}) \simeq H^1(C, Gr^0_F\Omega^*_{\overline{C}}(Log\{x_1,\dots,x_m\}) = \mathcal{O}_{\overline{C}})$ is of rank g, finally $F^2H^1(C,\mathbb{C}) = 0$.

The exact sequence : $0 \to \Omega^1_{\overline{C}} \to \Omega^1_{\overline{C}}(Log\{x_1,\dots,x_m\}) \to \mathcal{O}_{\{x_1,\dots,x_m\}} = 0 \to 0$ is defined by the residue morphism and has the associated long exact sequence:
$0 \to H^0(\overline{C},\Omega^1_{\overline{C}}) \simeq \mathbb{C}^g \to H^0(\overline{C},\Omega^1_{\overline{C}}(Log\{x_1,\dots,x_m\})) \to H^0(\overline{C},\mathcal{O}_{\{x_1,\dots,x_m\}}) \simeq \mathbb{C}^m \to H^1(\overline{C},\Omega^1_{\overline{C}}) \simeq \mathbb{C} \to H^1(\overline{C},\Omega^1_{\overline{C}}(Log\{x_1,\dots,x_m\})) = 0 \to 0$.

Exercise 4.11 (Hypersurfaces)**.** Let $i : Y \hookrightarrow P$ be a smooth hypersurface in a projective variety P. To describe the cohomology of the affine open set $U = P - Y$ we may use,
i) Rational algebraic forms on P regular on U denoted $\Omega^*(U) = \Omega^*_P(*Y)$, by Grothendieck's result on algebraic de Rham cohomology, or
ii) Forms on the analytic projective space meromorphic along Y, holomorphic on U denoted by $\Omega^*_{P_{an}}(*Y)$, where the Hodge filtration is described by the order of the pole ([11] Prop. 3.1.11) (the trivial filtration F described above on $\Omega^*(U)$ does not induce the correct Hodge filtration if U is not compact), or
iii) Forms with logarithmic singularities denoted by the logarithmic complex $\Omega^*_{P_{an}}(LogY)$ with its natural filtration F.
1) For example, in the case of a curve Y in a plane $\mathbb{P}^2$, the global holomorphic forms are all rational by Serre's result on cohomology of coherent sheaves. The residue along Y fits into an exact sequence of sheaves:

$$0 \to \Omega^2_P \to \Omega^2_P(LogY) \to i_*\Omega^1_Y \to 0.$$

We deduce from the associated long exact sequence, the isomorphism:

$$Res : H^0(P,\Omega^2_P(LogY)) \xrightarrow{\sim} H^0(Y,\Omega^1_Y),$$

since $h^{2,0} = h^{2,1} = 0$, $H^0(P,\Omega^2_P) = H^1(P,\Omega^2_P) = 0$. Hence, the 1-forms on Y are residues of rational $2-$forms on P with simple pole along the curve.
In homogeneous coordinates, let $F = 0$ be the homogeneous equation of Y. The 1-forms on Y are residues along Y of the rational forms:

$$\frac{A(z_0dz_1 \wedge dz_2 - z_1dz_0 \wedge dz_2 + z_2dz_0 \wedge dz_1)}{F}$$

where A is homogeneous of degree $d-3$ if F has degree d ([5] example 3.2.8).
2) The exact sequence defined by the relative cohomology (or cohomology with support in Y):

$$H^{k-1}(U) \xrightarrow{\partial} H^k_Y(P) \to H^k(P) \xrightarrow{j^*} H^k(U)$$

reduces via Thom's isomorphism, to:

$$H^{k-1}(U) \xrightarrow{r} H^{k-2}(Y) \xrightarrow{i_*} H^k(P) \xrightarrow{j^*} H^k(U)$$

where r is the topological Leray's residue map, dual to the tube over a cycle map $\tau : H_{k-2}(Y) \to H_{k-1}(U)$ associating to a cycle c the boundary in U of a tube over

c, and i_* is Gysin map, Poincaré dual to the map i^* in cohomology.
For $P = \mathbb{P}^{n+1}$ and n odd, the map r below is an isomorphism:

$$H^{n-1}(Y) \simeq H^{n+1}(P) \to H^{n+1}(U) \xrightarrow{r} H^n(Y) \xrightarrow{i_*} H^{n+2}(P) = 0 \xrightarrow{j^*} H^{n+2}(U)$$

and for n even the map r is injective and surjective onto the primitive cohomology $H^n_{\rm prim}(X)$, defined as Kernel of i_*:

$$H^{n+1}(P) = 0 \to H^{n+1}(U) \xrightarrow{r} H^n(Y) \xrightarrow{i_*} H^{n+2}(P) = \mathbb{Q} \xrightarrow{j^*} H^{n+2}(U).$$

4.2. **MHS on cohomology of simplicial varieties.** To construct a natural mixed Hodge structure on the cohomology of an algebraic variety S, not necessarily smooth or compact, Deligne considers a *simplicial smooth variety* $\pi : U_\bullet \to S$ which is a *cohomological resolution* of the original variety in the sense that the direct image $\pi_*\mathbb{Z}_{U_\bullet}$ is a resolution of $\mathbb{Z}_S$ (descent theorem, see Theorem 4.17).

On each term of the simplicial resolution, which consists of the complement of a normal crossing divisor in a smooth compact complex variety, the various logarithmic complexes are connected by functorial relations and form a *simplicial cohomological MHC* giving rise to the cohomological MHC defining the mixed Hodge structure we are looking for on the cohomology of S. Although such a construction is technically elaborate, the above abstract development of MHC leads easily to the result without further difficulty.

4.2.1. *Simplicial category.* The simplicial category Δ is defined as follows:
i) The objects of Δ are the subsets of integers $\Delta_n := \{0, 1, \dots, n\}$ for $n \in \mathbf{N}$,
ii) The set of morphisms of Δ are the sets $H_{p,q}$ of increasing mappings from Δ_p to Δ_q for integers $p, q \geq 0$, with the natural composition of mappings : $H_{pq} \times H_{qr} \to H_{pr}$. Notice that $f : \Delta_p \to \Delta_q$ is increasing in the non-strict sense $\forall i < j$, $f(i) \leq f(j)$.

Definition 4.12. *We define for $0 \leq i \leq n+1$ the $i-$th face map as the unique strictly increasing mapping such that $i \notin \delta_i(\Delta_n)$: $\delta_i : \Delta_n \to \Delta_{n+1}$ is increasing and such that: $i \notin Im\, \delta_i$.*

The semi-simplicial category $\Delta_>$ is obtained when we consider only the strictly increasing morphisms in Δ. In what follows we could restrict the constructions to semi-simplicial spaces which underly the simplicial spaces and work only with such spaces, since we use only the face maps.

Definition 4.13. *A simplicial (resp. co-simplicial) object $X_\bullet := (X_n)_{n \in \mathbb{N}}$ of a category $\mathcal{C}$ is a contravariant (resp. covariant) functor F from Δ to $\mathcal{C}$.*
A morphism $a : X_\bullet \to Y_\bullet$ of simplicial (resp. co-simplicial) objects is defined by its components $a_n : X_n \to Y_n$ compatible with the various maps $F(f)$, image by the functor F of simplicial morphisms $f \in H_{pq}$ for all $p, q \in \mathbb{N}$.

The functor $T : \Delta \to \mathcal{C}$ is defined by $T(\Delta_n) := X_n$ and for each $f : \Delta_p \to \Delta_q$, by $T(f) : X_q \to X_p$ (resp. $T(f) : X_p \to X_q$); $T(f)$ will be denoted by $X_\bullet(f)$.

4.2.2. *Sheaves on a simplicial space.* If $\mathcal{C}$ is the category of topological spaces, a simplicial functor define a simplicial topological space. A sheaf F^* on a simplicial topological space $X_\bullet$ is defined by:
1) A family of sheaves F^n on X_n,
2) For each $f : \Delta_n \to \Delta_m$ with $X_\bullet(f) : X_m \to X_n$, an $X_\bullet(f)-$morphism $F(f)_*$ from F^n to F^m, that is maps: $X_\bullet(f)^* F^n \to F^m$ on X_m satisfying for all $g : \Delta_r \to \Delta_n$,

$F(f\circ g)_* = F(f)_* \circ F(g)_*$.
A morphism $u : F^* \to G^*$ is a family of morphisms $u^n : F^n \to G^n$ such that for all $f : \Delta_n \to \Delta_m$, $u^m F(f)_* = G(f)_* u^n$:

$$X_*(f)^* F^n \xrightarrow{F(f)_*} F^m \xrightarrow{u^m} G^m, \quad X_*(f)^* F^n \xrightarrow{X_*(f)^*(u_n)} X_*(f)^* G^n \xrightarrow{G(f)_*} G^m.$$

The image of the i-th face map by a functor is also denoted abusively by the same symbol $\delta_i \colon X_{n+1} \to X_n$.
For a ring A, we will consider the derived category of co-simplicial sheaves of A-modules.

4.2.3. *Derived filtered category on a simplicial space.* The definition of a complex of sheaves K on a simplicial topological space X_* follows from the definition of sheaves. Such complex has two degrees $K := K^{p,q}$ where p is the degree of the complex and q is the simplicial degree, hence for each p, $K^{p,*}$ is a simplicial sheaf and for each q, $K^{*,q}$ is a complex on X_q.

A quasi-isomorphism (resp. filtered, bi-filtered) $\gamma : K \to K'$ (resp. with filtrations) of simplicial complexes on $X_\bullet$, is a morphism of simplicial complexes inducing a quasi-isomorphism $\gamma^{*,q} : K^{*,q} \to K'^{*,q}$ (resp. filtered, bi-filtered) for each space X_q.

The definition of the derived category (resp. filtered, bi-filtered) of the abelian category of abelian sheaves of groups (resp. vector spaces) on a simplicial space is obtained by inverting the quasi-isomorphisms (resp. filtered, bi-filtered).

4.2.4. A topological space S defines a simplicial constant space $S_\bullet$ such that $S_n = S$ for all n and $S_\bullet(f) = Id$ for all $f \in H^{p,q}$.

An augmented simplicial space $\pi : X_\bullet \to S$ is defined by a family of maps $\pi_n : X_n \to S_n = S$ defining a morphism of simplicial spaces.

4.2.5. The structural sheaves $\mathcal{O}_{X_n}$ of a simplicial complex analytic space form a simplicial sheaf of rings. Let $\pi : X_\bullet \to S$ be an augmentation to a complex analytic space S. The various de Rham complexes of sheaves $\Omega^*_{X_n/S}$ for $n \in \mathbb{N}$ form a complex of sheaves on $X_\bullet$ denoted $\Omega^*_{X_\bullet/S}$.

A simplicial sheaf F^* on the constant simplicial space $S_\bullet$ defined by S corresponds to a co-simplicial sheaf on S; hence if F^* is abelian, it defines a complex via the face maps, with:

$$d = \sum_i (-1)^i \delta_i : F^n \to F^{n+1}.$$

A complex of abelian sheaves K on $S_\bullet$, denoted by $K^{n,m}$ with m the co-simplicial degree, defines a simple complex sK (see 1.2.3):

$$(sK)^n := \oplus_{p+q=n} K^{pq} : \; d(x^{pq}) = d_K(x^{pq}) + \sum_i (-1)^i \delta_i x^{pq}.$$

The following filtration L with respect to the second degree will be useful:

$$L^r(sK) = s(K^{pq})_{q \geq r}.$$

4.2.6. *Direct image in the derived category of abelian sheaves (resp. filtered, bi-filtered).* For an augmented simplicial space $a : X_\bullet \to S$, we define a functor denoted Ra_* on complexes K (resp. filtered (K, F), bi-filtered (K, F, W))) of abelian sheaves on $X_\bullet$. We may view S as a constant simplicial scheme $S_\bullet$ and a as a morphism $a_\bullet : X_\bullet \to S_\bullet$. In the first step we construct a complex I (resp. (I, F), (I, F, W)) of acyclic (for example flabby) sheaves, quasi-isomorphic (resp. filtered, bi-filtered) to K (resp. (K, F), (K, F, W))); we can always take Godement resolutions ([39] Chapter II, §3.6 p. 95 or [22] Chapter II, §4.3 p.167) for example, then in each degree p, $(a_q)_* I^p$ on $S_q = S$ defines for varying q a co-simplicial sheaf on S denoted $(a_\bullet)_* I^p$, and a differential graded complex for varying p, which is a double complex whose associated simple complex is denoted $s(a_\bullet)_* I := Ra_* K$:

$$(Ra_*K)^n := \oplus_{p+q=n}(a_q)_* I^{p,q}; \quad dx^{pq} = d_I(x^{pq}) + (-1)^p \sum_{i=0}^{q+1} (-1)^i \delta_i x^{pq} \in (Ra_*K)^{n+1}$$

where q is the simplicial index: $\delta_i(x^{pq}) \in I^{p,q+1}$ and p is the degree. In particular for S a point we define the hypercohomology of K:

$$R\Gamma(X_\bullet, K) := sR\Gamma_\bullet(X_\bullet, K); \quad \mathbb{H}^i(X_\bullet, K) := H^i(R\Gamma(X_\bullet, K)).$$

Respectively, the definition of $Ra_*(K, F)$ and $Ra_*(K, F, W)$ is similar.

The filtration L on $s(a_\bullet)_* I := Ra_* K$ defines a spectral sequence:

$$E_1^{p,q} = R^q(a_p)_*(K_{|X_p}) := H^q(R(a_p)_*(K_{|X_p})) \Rightarrow H^{p+q}(Ra_*K) := R^{p+q}a_*K$$

Remark 4.14 (Topological realization). Recall that a morphism of simplices $f : \Delta_n \to \Delta_m$ has a geometric realization $|f| : |\Delta_n| \to |\Delta_m|$ as the affine map defined when we identify a simplex Δ_n with the vertices of its affine realization in $\mathbb{R}^{\Delta_n}$. We construct the topological realization of a topological semi-simplicial space $X_\bullet$ as the quotient of the topological space $Y = \coprod_{n \geq 0} X_n \times |\Delta_n|$ by the equivalence relation $\mathcal{R}$ generated by the identifications:

$$\forall f : \Delta_n \to \Delta_m, x \in X_m, a \in |\Delta_n|, \quad (x, |f|(a)) \equiv (X_\bullet(f)(x), a)$$

The topological realization $|X_\bullet|$ is the quotient space of Y, modulo the relation $\mathcal{R}$, with its quotient topology. The construction above of the cohomology amounts to the computation of the cohomology of the topological space $|X_\bullet|$ with coefficient in an abelian group A:

$$H^i(X_\bullet, A) \simeq H^i(|X_\bullet|, A).$$

4.2.7. *Cohomological descent.* Let $a : X_\bullet \to S$ be an augmented simplicial scheme; any abelian sheaf F on S, lifts to a sheaf a^*F on $X_\bullet$ and we have a natural morphism:

$$\varphi(a) : F \to Ra_* a^* F \text{ in } D^+(S).$$

Definition 4.15 (cohomological descent). *The morphism $a : X_\bullet \to S$ is of cohomological descent if the natural morphism $\varphi(a)$ is an isomorphism in $D^+(S)$ for all abelian sheaves F on S.*

The definition amounts to the following conditions:

$$F \xrightarrow{\sim} Ker(a_{0*}a_0^*F \xrightarrow{\delta_1 - \delta_0} a_1^*F); \quad R^i a_* a^* F = 0 \text{ for } i > 0.$$

In this case for all complexes K in $D^+(S)$:

$$R\Gamma(S, K) \simeq R\Gamma(X_\bullet, a^*K)$$

and we have a spectral sequence:

$$E_1^{p,q} = \mathbb{H}^q(X_p, a_p^* K) \Rightarrow \mathbb{H}^{p+q}(S, K), \quad d_1 = \sum_i (-1)^i \delta_i : E_1^{p,q} \to E_1^{p+1,q}.$$

4.2.8. *MHS on cohomology of algebraic varieties.* A simplicial complex variety $X_\bullet$ is smooth (resp. compact) if every X_n is smooth (resp. compact).

Definition 4.16 (NCD). *A simplicial normal crossing divisor is a family $Y_n \subset X_n$ of NCD such that the family of open subsets $U_n := X_n - Y_n$ form a simplicial subvariety $U_\bullet$ of $X_\bullet$, hence the family of filtered logarithmic complexes $(\Omega^*_{X_n}(LogY_n))_{n\geq 0}, W)$ form a filtered complex on $X_\bullet$.*

The following theorem is admitted here:

Theorem 4.17 (Deligne [12] 6.2.8). *For each separated complex variety S,*
i) There exist a simplicial variety compact and smooth $X_\bullet$ over $\mathbb{C}$ containing a simplicial normal crossing divisor $Y_\bullet$ in $X_\bullet$ and an augmentation $a : U_\bullet = (X_\bullet - Y_\bullet) \to S$ satisfying the cohomological descent property.
Hence for all abelian sheaves F on S, we have an isomorphism $F \xrightarrow{\sim} Ra_* a^* F$.
ii) Moreover, for each morphism $f : S \to S'$, there exists a morphism $f_\bullet : X_\bullet \to X'_\bullet$ of simplicial varieties compact and smooth with simplicial normal crossing divisors $Y_\bullet$ and $Y'_\bullet$ and augmented complements $a : U_\bullet \to S$ and $a' : U'_\bullet \to S'$ satisfying the cohomological descent property, with $f_\bullet(U_\bullet) \subset U'_\bullet$ and $a' \circ f = a$.

The proof is based on Hironaka's desingularisation theorem and on a general contruction of hypercoverings described briefly by Deligne in [12] after preliminaries on the general theory of hypercoverings. The desingularisation is carried at each step of the construction by induction.

Remark 4.18. We can and shall assume that the normal crossing divisors have smooth irreducible components.

4.2.9. An A-cohomological mixed Hodge complex K on a topological simplicial space $X_\bullet$ consists of:
i) A complex K_A of sheaves of A-modules on $X_\bullet$ such that $\mathbb{H}^k(X_\bullet, K_A)$ are A-modules of finite type,
ii) A filtered complex $(K_{A\otimes\mathbb{Q}}, W)$ of filtered sheaves of $A \otimes \mathbb{Q}$ modules on $X_\bullet$ with an increasing filtration W and an isomorphism $K_{A\otimes\mathbb{Q}} \simeq K_A \otimes \mathbb{Q}$ in the derived category on $X_\bullet$,
iii) A bi-filitered complex $(K_\mathbb{C}, W, F)$ of sheaves of complex vector spaces on $X_\bullet$ with an increasing (resp. decreasing) filtration W (resp. F) and an isomorphism $\alpha : (K_{A\otimes\mathbb{Q}}, W) \otimes \mathbb{C} \xrightarrow{\sim} (K_\mathbb{C}, W)$ in the derived category on $X_\bullet$.
Moreover, the following axiom is satisfied:
(CMHC) The restriction of K to each X_n is an A-cohomological MHC.

4.2.10. *Simplicial NCD.* Let $X_\bullet$ be a simplicial complex compact smooth algebraic variety with $Y_\bullet$ a simplicial NCD in $X_\bullet$ such that $j_\bullet : U_\bullet = (X_\bullet - Y_\bullet) \to X_\bullet$ is an open simplicial embedding, then

$$(Rj_{\bullet *}\mathbb{Z}, (Rj_{\bullet *}\mathbb{Q}, \tau_\leq), (\Omega^*_{X_\bullet}(LogY_\bullet), W, F))$$

is a cohomological MHC on $X_\bullet$.

4.2.11. If we apply the global section functor to an A-cohomological mixed Hodge complex K on $X_\bullet$, we get an A-co-simplicial MHC defined as follows:
1) A co-simplicial complex $R\Gamma_\bullet K_A$ in the derived category of co-simplicial A-modules,
2) A filtered co-simplicial complex $R\Gamma_\bullet(K_{A\otimes\mathbb{Q}}, W)$ in the derived category of filtered co-simplicial vector spaces, and an isomorphism $(R\Gamma_\bullet K_A)\otimes\mathbb{Q} \simeq R\Gamma_\bullet(K_{A\otimes\mathbb{Q}})$.
3) A bi-filtered co-simplicial complex $R\Gamma_\bullet(K_\mathbb{C}, W, F)$ in the derived category of bi-filtered co-simplicial vector spaces,
4) An isomorphism $R\Gamma_\bullet(K_{A\otimes\mathbb{Q}}, W)\otimes\mathbb{C} \simeq R\Gamma_\bullet(K_\mathbb{C}, W)$ in the derived category of filtered co-simplicial vector spaces (see 4.2.3).

4.2.12. *Diagonal filtration.* To a co-simplicial mixed Hodge complex K, we associate here a differential graded complex which is viewed as a double complex whose associated simple complex is denoted sK. We put on sK a weight filtration by a diagonal process.

Definition 4.19 (Differential graded A-MHC)**.** *A differential graded DG^+-complex (or a complex of graded objects) is a bounded below complex with two degrees, the first is defined by the degree of the complex and the second by the degree of the grading. It can be viewed as a double complex.*
A differential graded A-MHC is defined by a system of DG^+-complex (resp. filtered, bi-filtered):

$$K_A, (K_{A\otimes\mathbb{Q}}, W), K_A\otimes\mathbb{Q} \simeq K_{A\otimes\mathbb{Q}}, (K_\mathbb{C}, W, F), (K_{A\otimes\mathbb{Q}}, W)\otimes\mathbb{C} \simeq (K_\mathbb{C}, W)$$

such that for each degree n of the grading, the component at the complex level $(K_\mathbb{C}^{*,n}, W, F)$ form with the components at the A and $A\otimes\mathbb{Q}$ levels an A-MHC .

A co-simplicial MHC: (K, W, F) defines a DG^+-A-MHC

$$sK_A, (sK_{A\otimes\mathbb{Q}}, W), sK_A\otimes\mathbb{Q} \simeq sK_{A\otimes\mathbb{Q}}, (sK_\mathbb{C}, W, F), (sK_{A\otimes\mathbb{Q}}, W)\otimes\mathbb{C} \simeq (sK_\mathbb{C}, W)$$

where the degree of the grading is the co-simplicial degree. The hypercohomology of an A-cohomological mixed Hodge complex on $X_\bullet$ is such a complex.

Definition 4.20 (Diagonal filtration)**.** *The diagonal $\delta(W, L)$ of the two filtrations W and L on sK is defined by:*

$$\delta(W, L)_n(sK)^i := \oplus_{p+q=i} W_{n+q}K^{p,q}.$$

where $L^r(sK) = s(K^{p,q})_{q\geq r}$. For a bi-filtered complex (K, W, F) with a decreasing filtration F, the sum over F is natural (not diagonal).

4.2.13. *Properties.* We have:

$$Gr_n^{\delta(W,L)}(sK) \simeq \oplus_p Gr_{n+p}^W K^{*,p}[-p]$$

In the case of a DG^+-complex defined as the hypercohomology of a complex (K, W) on a simplicial space $X_\bullet$, we have:

$$Gr_n^{\delta(W,L)} R\Gamma K \simeq \oplus_p R\Gamma(X_p, Gr_{n+p}^W K^{*,p})[-p].$$

and for a bi-filtered complex with a decreasing F:

$$Gr_n^{\delta(W,L)} R\Gamma(K, F) \simeq \oplus_p R\Gamma(X_p, (Gr_{n+p}^W K, F))[-p].$$

Next we remark:

Lemma 4.21. *If $H = (H_A, W, F)$ is an A-mixed Hodge structure, a filtration L of H_A is a filtration of mixed Hodge structure, if and only if, for all n,*

$$(Gr^n_L H_A, Gr^n_L(W), Gr^n_L(F))$$

is an A-mixed Hodge structure.

Theorem 4.22 (Deligne ([12] theorem 8.1.15)). *Let K be a graded differential A-mixed Hodge complex (for example, defined by a co-simplicial A-mixed Hodge complex).*
i) Then, $(sK, \delta(W,L), F)$ is an A-mixed Hodge complex.
The first terms of the weight spectral sequence:

$${}_{\delta(W,L)}E_1^{pq}(sK \otimes \mathbb{Q}) = \oplus_n H^{q-n}(Gr^W_n K^{*,p+n})$$

form the simple complex $({}_{\delta(W,L)}E_1^{pq}, d_1)$ of $A \otimes \mathbb{Q}$-Hodge structures of weight q associated to the double complex where $m = n+p$ and E_1^{pq} is represented by the sum of the terms on the diagonal :

$$\begin{array}{ccccc}
H^{q-(n+1)}(Gr^W_{n+1}K^{*,m+1}) & \xrightarrow{\partial} & H^{q-n}(Gr^W_n K^{*,m+1}) & \xrightarrow{\partial} & H^{q-(n-1)}(Gr^W_{n-1}K^{*,m+1}) \\
d'' \uparrow & & d'' \uparrow & & d'' \uparrow \\
H^{q-(n+1)}(Gr^W_{n+1}K^{*,m}) & \xrightarrow{\partial} & H^{q-n}(Gr^W_n K^{*,m}) & \xrightarrow{\partial} & H^{q-(n-1)}(Gr^W_{n-1}K^{*,m})
\end{array}$$

where ∂ is a connecting morphism and d'' is simplicial.
ii) The terms ${}_LE_r$ for $r > 0$, of the spectral sequence defined by $(sK_{A\otimes\mathbb{Q}}, L)$ are endowed with a natural A-mixed Hodge structure, with differentials d_r compatible with such structures.
iii) The filtration L on $H^*(sK)$ is a filtration in the category of mixed Hodge structures and:

$$Gr^p_L(H^{p+q}((sK), \delta(W,L)[p+q], F) = ({}_LE^{pq}_\infty, W, F).$$

4.2.14. In the case of a smooth simplicial variety $U_\bullet$ complement of a normal crossing divisor at infinity, the cohomology groups $H^n(U_\bullet, \mathbb{Z})$ are endowed with the mixed Hodge structure defined by the following mixed Hodge complex:

$$R\Gamma(U_\bullet, \mathbb{Z}),\ R\Gamma(U_\bullet, \mathbb{Q}), \delta(W,L)), R\Gamma(U_\bullet, \Omega^*_{X_\bullet}(Log Y_\bullet)), \delta(W,L)), F)$$

with natural compatibility isomorphisms, satisfying:

$$Gr_n^{\delta(W,L)} R\Gamma(U_\bullet, \mathbb{Q}) \simeq \oplus_m Gr^W_{n+m} R\Gamma(U_m, , \mathbb{Q}))[-m] \simeq \oplus_m R\Gamma(Y_m^{n+m}, \mathbb{Q})[-n-2m]$$

where the first isomorphism corresponds to the diagonal filtration and the second to the logarithmic complex for the open set U_m; recall that Y_m^{n+m} denotes the disjoint union of intersections of $n+m$ components of the normal crossing divisor Y_m of simplicial degree m. Moreover:

$${}_{\delta(W,L)}E_1^{p,q} = \oplus_n H^{q-2n}(Y^n_{n+p}, \mathbb{Q}) \Rightarrow H^{p+q}(U_\bullet, \mathbb{Q})$$

The filtration F induces on ${}_{\delta(W,L)}E_1^{p,q}$ a Hodge Structure of weight b and the differentials d_1 are compatible with the Hodge Structures. The term E_1 is the simple complex associated to the double complex of Hodge Structure of weight q where G is an alternating Gysin map:

$$\begin{array}{ccccc}
H^{q-(2n+2)}(Y^{n+1}_{p+n+1}, \mathbb{Q}) & \xrightarrow{G} & H^{q-2n}(Y^n_{p+n+1}, \mathbb{Q}) & \xrightarrow{G} & H^{q-2(n-2)}(Y^{n-1}_{p+n+1}, \mathbb{Q}) \\
\sum_i (-1)^i \delta_i \uparrow & & \sum_i (-1)^i \delta_i \uparrow & & \sum_i (-1)^i \delta_i \uparrow \\
H^{q-(2n+2)}(Y^{n+1}_{p+n}, \mathbb{Q}) & \xrightarrow{G} & H^{q-2n}(Y^n_{p+n}, \mathbb{Q}) & \xrightarrow{G} & H^{q-2(n-2)}(Y^{n-1}_{p+n}, \mathbb{Q})
\end{array}$$

where the Hodge Structure on the columns are twisted respectively by $(-n-1),(-n),(-n+1)$, the lines are defined by the logarithmic complex, while the vertical differentials are simplicial. We deduce from the general theory:

Proposition 4.23. *i) Let $(U_\bullet, X_\bullet)$ be a simplicial NCD (see 4.2.10).The mixed Hodge structure on $H^n(U_\bullet,\mathbb{Z})$ is defined by the graded differential mixed Hodge complex associated to the simplicial MHC defined by the logarithmic complex on each term of $X_\bullet$. It is functorial in the couple $(U_\bullet, X_\bullet)$.*
ii) The rational weight spectral sequence degenerates at rank 2 and the Hodge Structure on E_2 induced by E_1 is isomorphic to the Hodge Structure on $Gr_W H^n(U_\bullet,\mathbb{Q})$.
iii)The Hodge numbers h^{pq} of $H^n(U_\bullet,\mathbb{Q})$ vanish for $p\notin[0,n]$ or $q\notin[0,n]$.
iv) For $Y_\bullet=\emptyset$, the Hodge numbers h^{pq} of $H^n(X_\bullet,\mathbb{Q})$ vanish for $p\notin[0,n]$ or $q\notin[0,n]$ or $p+q>n$.

Definition 4.24. *The mixed Hodge structure on the cohomology $H^n(X,\mathbb{Z})$ of a complex algebraic variety X is defined by any logarithmic simplicial resolution of X via the isomorphism with $H^n(U_\bullet,\mathbb{Z})$ defined by the augmentation $a: U_\bullet\to X$.*

The mixed Hodge structure just defined does not depend on the resolution and is functorial in X since we can reduce to the case where a morphism $f: X\to Z$ is covered by a morphism of hypercoverings.

4.2.15. *Problems.* 1) Let $i: Y\to X$ be a closed subvariety of X and $j: U := (X-Y)\to X$ the embedding of the complement. Then the two long exact sequences of cohomology:

$$\cdots\to H^i(X,X-Y,\mathbb{Z})\to H^i(X,\mathbb{Z})\to H^i(Y,\mathbb{Z})\to H^{i+1}_Y(X,\mathbb{Z})\to\cdots$$
$$\cdots\to H^i_Y(X,\mathbb{Z})\to H^i(X,\mathbb{Z})\to H^i(X-Y,\mathbb{Z})\to H^{i+1}_Y(X,\mathbb{Z})\to\cdots$$

underlie exact sequences of mixed Hodge structure.

The idea is to use a simplicial hypercovering of the morphism i in order to define two mixed Hodge complexes: $K(Y)$ on Y and $K(X)$ on X with a well defined morphism on the level of complexes of sheaves $i^*: K(X)\to K(Y)$ (resp. $j^*: K(X)\to K(X-Y)$), then the long exact sequence is associated to the mixed cone $C_M(i^*)$(resp. $C_M(j^*)$).

In particular, one deduces associated long exact sequences by taking the graded spaces with respect to the filtrations F and W.
2) *Künneth formula* [42]. Let X and Y be two algebraic varieties, then the isomorphisms of cohomology vector spaces:

$$H^i(X\times Y,\mathbb{C})\simeq\oplus_{r+s=i}H^r(X,\mathbb{C})\otimes H^s(Y,\mathbb{C})$$

underlie isomorphisms of $\mathbb{Q}$-mixed Hodge structure. The answer is in two steps:
i) Consider the tensor product of two mixed Hodge complex defining the mixed Hodge structure of X and Y from which we deduce the right term, direct sum of tensor product of mixed Hodge structures.
ii) Construct a quasi-isomorphism of the tensor product with a mixed Hodge complex defining the mixed Hodge structure of $X\times Y$.
iii) Deduce that the cup product on the cohomology of an algebraic variety is compatible with mixed Hodge structures.

4.3. **MHS on the cohomology of a complete embedded algebraic variety.** For embedded varieties into smooth varieties, the mixed Hodge structure on cohomology can be obtained by a simple method using exact sequences, once the mixed Hodge structure for normal crossing divisor has been constructed, which should easily convince of the natural aspect of this theory. The technical ingredients consist of Poincaré duality and its dual the trace (or Gysin) morphism.

Let $p : X' \to X$ be a proper morphism of complex smooth varieties of same dimension, Y a closed subvariety of X and $Y' = p^{-1}(Y)$. We suppose that Y' is a NCD in X' and that the restriction of p induces an isomorphism $p_{X'-Y'} : X' - Y' \xrightarrow{\sim} X - Y$:

$$\begin{array}{ccccc} Y' & \xrightarrow{i'} & X' & \xleftarrow{j'} & X' - Y' \\ \downarrow p_Y & & \downarrow p & & \downarrow p_{X'-Y'} \\ Y & \xrightarrow{i} & X & \xleftarrow{j} & X - Y \end{array}$$

The trace morphism $Tr\,p$ is defined as Poincaré dual to the inverse image p^* on cohomology, hence $Tr\,p$ is compatible with HS. It can be defined at the level of sheaf resolutions of $\mathbb{Z}_{X'}$ and $\mathbb{Z}_X$ as constructed by Verdier [47] in the derived category $Tr\,p$: $Rp_*\mathbb{Z}_{X'} \to \mathbb{Z}_X$ hence we deduce morphisms: $Trp : H^i_c(X'-Y',\mathbb{Z}) \to H^i_c(X-Y,\mathbb{Z})$ and by restriction morphisms depending on the embeddings of Y and Y' into X and X':

$$(Tr\,p)_{|Y} : Rp_*\mathbb{Z}_{Y'} \to \mathbb{Z}_Y,\ (Tr\,p)_{|Y} : H^i(Y',\mathbb{Z}) \to H^i(Y,\mathbb{Z}).$$

Remark 4.25. Let U be a neighbourhood of Y in X, retract by deformation onto Y such that $U' = p^{-1}(U)$ is a retract by deformation onto Y'. Then the morphism $(Tr\,p)_{|Y}$ is deduced from $Tr\,(p_{|U})$ in the diagram:

$$\begin{array}{ccc} H^i(Y',\mathbb{Z}) & \xleftarrow{\sim} & H^i(U',\mathbb{Z}) \\ \downarrow (Tr\,p)_{|Y} & & \downarrow Tr(p_{|U}) \\ H^i(Y,\mathbb{Z}) & \xleftarrow{\sim} & H^i(U,\mathbb{Z}) \end{array}$$

Consider now the diagram:

$$\begin{array}{ccccc} R\Gamma_c(X'-Y',\mathbb{Z}) & \xrightarrow{j'_*} & R\Gamma(X',\mathbb{Z}) & \xrightarrow{i'^*} & R\Gamma(Y',\mathbb{Z}) \\ Trp \downarrow & & \downarrow Trp & & \downarrow (Trp)_{|Y} \\ R\Gamma_c(X-Y,\mathbb{Z}) & \xrightarrow{j_*} & R\Gamma(X,\mathbb{Z}) & \xrightarrow{i^*} & R\Gamma(Y,\mathbb{Z}) \end{array}$$

Proposition 4.26. [18] *i) The morphism $p^*_Y\colon H^i(Y,\mathbb{Z}) \to H^i(Y',\mathbb{Z})$ is injective with retraction $(Tr\,p)_{|Y}$.*
ii) We have a quasi-isomorphism of $i_*\mathbb{Z}_Y$ with the cone $C(i'^* - Tr\,p)$ of the morphism $i'^* - Tr\,p$. The long exact sequence associated to the cone splits into short exact sequences:

$$0 \to H^i(X',\mathbb{Z}) \xrightarrow{i'^*-Tr\,p} H^i(Y',\mathbb{Z}) \oplus H^i(X,\mathbb{Z}) \xrightarrow{(Tr\,p)_{|Y}+i^*} H^i(Y,\mathbb{Z}) \to 0.$$

Moreover $i'^* - Tr\,p$ is a morphism of mixed Hodge structures. In particular, the weight of $H^i(Y,\mathbb{C})$ varies in the interval $[0,i]$ since this is true for Y' and X.

Lemma and Definition 4.27. *The mixed Hodge structure of Y is defined as cokernel of $i'^* - Tr\,p$ via its isomorphism with $H^i(Y,\mathbb{Z})$, induced by $(Tr\,p)_{|Y} + i^*$. It coincides with Deligne's mixed Hodge structure.*

This result shows the uniqueness of the theory of mixed Hodge structure, once the MHS of the normal crossing divisor Y' has been constructed. The above technique consists in the realization of the MHS on the cohomology of Y as relative cohomology with MHS on X, X' and Y' all smooth compact or NCD. Notice that the MHS on $H^i(Y,\mathbb{Z})$ is realized as a quotient and not as an extension.

Proposition 4.28. *Let X, X' be compact algebraic varieties with X' non singular and let $p : X' \to X$ be a surjective proper morphism. Then for all integers i, we have*

$$W_{i-1}H^i(X,\mathbb{Q}) = Ker\,(H^i(X,\mathbb{Q}) \xrightarrow{p^*} H^i(X',\mathbb{Q}))$$

In particular, this result applies to a desingularization of X.

We have trivially $W_{i-1}H^i(X,\mathbb{Q}) \subset Ker\, p^*$ since $H^i(X',\mathbb{Q})$ is of pure weight i. Let $i : Y \to X$ be the subvariety of singular points in X and let $Y' := p^{-1}(Y)$, denote $p_{Y'} : Y' \to Y$ the morphism induced by p and $i' : Y' \to X'$ the injection into X', then we have a long exact sequence:

$$H^{i-1}(Y',\mathbb{Q}) \to H^i(X,\mathbb{Q}) \xrightarrow{(p^*,-i^*)} H^i(X',\mathbb{Q}) \oplus H^i(Y,\mathbb{Q}) \xrightarrow{i'^*+p^*_{Y'}} H^i(Y',\mathbb{Q}) \to \cdots$$

It is enough to prove $Ker\, p^* \subset Ker\, i^*$, since then

$$Ker\, p^* = Ker\, p^* \cap Ker\, i^* = Im\,(H^{i-1}(Y',\mathbb{Q}) \to H^i(X,\mathbb{Q}))$$

where the weight of $H^{i-1}(Y',\mathbb{Q})$ is $\leq i-1$. By induction on the dimension of X, we may suppose that the proposition is true for Y. Let $\alpha : Y'' \to Y'$ be a desingularization of Y', $q := p_{Y'} \circ \alpha$ and $i'' := i' \circ \alpha$, then we have a commutative diagram

$$\begin{array}{ccc} Y'' & \xrightarrow{i''} & X' \\ q\downarrow & & \downarrow p \\ Y & \xrightarrow{i} & X \end{array}$$

where Y'' is compact and non singular.
Let $a \in Gr_i^W H^i(X,\mathbb{Q})$ such that $Gr_i^W p^*(a) = 0$, then $Gr_i^W(p\circ i'')^*(a) = 0$. Hence $Gr_i^W(i\circ q)^*(a) = 0$ since $Gr_i^W(i\circ q)^* = Gr_i^W(p\circ i'')^*$. By induction $Gr_i^W(q)^*$ is injective, then we deduce $Gr_i^W(i^*)(a) = 0$.

Remark 4.29 (mixed Hodge structure on the cohomology of an embedded algebraic variety). The construction still applies for non proper varieties if we construct the MHS of an open NCD.
Hypothesis. Let $i_Z : Z \to X$ a closed embedding and $i_X : X \to P$ a closed embedding in a projective space (or any proper smooth complex algebraic variety). By Hironaka desingularization we construct a diagram:

$$\begin{array}{ccccc} Z'' & \to & X'' & \to & P'' \\ \downarrow & & \downarrow & & \downarrow \\ Z' & \to & X' & \to & P' \\ \downarrow & & \downarrow & & \downarrow \\ Z & \to & X & \to & P \end{array}$$

first by blowing up centers over Z so to obtain a smooth space $p : P' \to P$ such that $Z' := p^{-1}(Z)$ is a normal crossing divisor; set $X' := p^{-1}(X)$, then:

$$p_| : X' - Z' \xrightarrow{\sim} X - Z, \quad p_| : P' - Z' \xrightarrow{\sim} P - Z$$

are isomorphisms since the modifications are all over Z. Next, by blowing up centers over X' we obtain a smooth space $q : P'' \to P'$ such that $X'' := q^{-1}(X')$ and $Z'' := q^{-1}(Z')$ are NCD, and $q_| : P'' - X'' \xrightarrow{\sim} P' - X'$. Then, we deduce the diagram:

$$\begin{array}{ccccc} X'' - Z'' & \xrightarrow{i''_X} & P'' - Z'' & \xleftarrow{j''} & P'' - X'' \\ q_X \downarrow & & q_| \downarrow & & q_| \downarrow \wr \\ X' - Z' & \xrightarrow{i'_X} & P' - Z' & \xleftarrow{j'} & P' - X' \end{array}$$

Since the desingularization is a sequence of blowing-ups above X', we still have an isomorphism induced by q on the right side of the preceding diagram. For $\dim P = d$ and all integers i, the morphism $q_|^* : H_c^{2d-i}(P'' - Z'', \mathbb{Q}) \to H_c^{2d-i}(P' - Z', \mathbb{Q})$ is well defined on cohomology with compact support since q is proper. Its Poincaré dual is called the trace morphism $Tr\, q_| : H^i(P'' - Z'', \mathbb{Q}) \to H^i(P' - Z', \mathbb{Q})$ and satisfies the relation $Tr\, q_| \circ q_|^* = Id$. Moreover, the trace morphism is defined as a morphism of sheaves $q_{|*}\mathbb{Z}_{P''-Z''} \to \mathbb{Z}_{P'-Z'}$ [47], hence an induced trace morphism $(Trq)_| : H^i(X'' - Z'', \mathbb{Q}) \to H^i(X' - Z', \mathbb{Q})$ is well defined.

Proposition 4.30. *With the notations of the above diagram, we have short exact sequences:*

$$0 \to H^i(P'' - Z'', \mathbb{Q}) \xrightarrow{(i''_X)^* - Trq_|} H^i(X'' - Z'', \mathbb{Q}) \oplus H^i(P' - Z', \mathbb{Q})$$
$$\xrightarrow{(i'_X)^* - (Trq)_|} H^i(X' - Z', \mathbb{Q}) \to 0$$

Since we have a vertical isomorphism q on the right side of the above diagram, we deduce a long exact sequence of cohomology spaces containing the sequences of the proposition; the injectivity of $(i''_X)^* - Trq_|$ and the surjectivity of $(i'_X)^* - (Trq)_|$ are deduced from $Trq_| \circ q_|^* = Id$ and $(Trq)_| \circ q_X^* = Id$, hence the long exact sequence splits into short exact sequences.

Corollary 4.31. *The cohomology $H^i(X - Z, \mathbb{Z})$ is isomorphic to $H^i(X' - Z', \mathbb{Z})$ since $X - Z \simeq X' - Z'$ and then carries the MHS isomorphic to the cokernel of $(i''_X)^* - Trq_|$ which is a morphism of MHS.*

The left term carry a MHS as the special case of the complementary of the normal crossing divisor: Z'' into the smooth proper variety P'' , while the middle term is the complementary of the normal crossing divisor Z'' into the normal crossing divisor X''. Both cases can be treated by the above special cases without the general theory of simplicial varieties. This shows that the MHS is uniquely defined by the construction on an open NCD and the logarithmic case for a smooth variety.

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