An extension of the Tusnady inequality for general distributions

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Under mild assumptions an analog of Tusnady's inequality for conditional distributions of partial sums of independent nonidentically distributed r.v.'s is obtained. These type of inequalities provide very sharp bounds for couplings of random variables via quantile transformations, which are main tools for proving various strong invariance principles for sums of independent random variables. We reffer to Komlós, Major and Tusnády (1975, 1976), Sakhanenko (1984), Zaitsev (1996), Grama, I. and Nussbaum, M. (2002), Carter and Pollard (2004) and to references therein for details. Recently quantile transformations have been applied for establishing a constructive asymptotic equivalence of statistical experiments, see Brown, Carter, Low and Zhang (2004).

Let $(\Omega, \mathcal{F}, P)$ be a probability space and λ be a real number satisfying $0 < \lambda < \infty$. Denote by $\mathfrak{D}(\lambda)$ the subset of all real valued random variables S on the probability space $(\Omega, \mathcal{F}, P)$, which admits for some $n \geq 1$, a representation of the form $S = X_1 + ... + X_n$, where $X_1, ..., X_n$ are independent r.v.'s, satisfying assumptions

A1 The r.v.'s $X_1, ..., X_n$ are of means zero and finite variances:

$$\mathrm{E}X_i = 0, \quad 0 < \mathrm{E}X_i^2 < \infty, \quad i = 1, ..., n.$$

A2 The following Sakhanenko's condition holds true:

$$\lambda \mathrm{E} \left| X_i \right|^3 \exp \left\{ \lambda \left| X_i \right| \right\} \leq \mathrm{E}X_i^2, \quad i = 1, ..., n.$$

Let $\mathbf{i} = \sqrt{-1}$ and ε, ν be real numbers, such that $0 < \varepsilon \leq 1/96$ and $0 < \nu < \infty$. Denote by $\mathfrak{D}_1(\lambda)$ the subset of all r.v.'s $S \in \mathfrak{D}(\lambda)$ which additionally satisfy the smoothness assumption

S1 The r.v. S is such that,

$$\sup_{|h| \leq \lambda} \int_{|t| > \varepsilon\lambda} \left| \frac{\mathrm{E} \exp \left\{ (\mathbf{i}t + h) S \right\}}{\mathrm{E} \exp \left\{ hS \right\}} \right| dt \leq \frac{\nu}{\lambda \mathrm{E} S^2}.$$

Denote by $\mathfrak{D}_2(\lambda)$ the subset of all r.v.'s $S \in \mathfrak{D}(\lambda)$ which take discrete values $a + \Delta k$, $k \in \{0, \pm 1, \pm 2, ...\}$ with maximal span Δ, where $0 \leq a < \Delta$ and $\lambda \Delta \leq \pi / \varepsilon$, and which additionally satisfy the assumption (S2)

S2 The r.v. S is such that,

$$\sup_{|h| \leq \lambda} \int_{\varepsilon\lambda < |t| \leq \pi/\Delta} \left| \frac{\mathrm{E} \exp \left\{ (\mathbf{i}t + h) S \right\}}{\mathrm{E} \exp \left\{ hS \right\}} \right| dt \leq \frac{\nu}{\lambda \mathrm{E} S^2}.$$

Let us agree that $\mathcal{X}$ is the range of a r.v. S in the class $\mathfrak{D}_1(\lambda)$ or $\mathfrak{D}_2(\lambda)$, i.e. $\mathcal{X} = R$, if $S \in \mathfrak{D}_1(\lambda)$ and $\mathcal{X} = \{a + \Delta k : k = 0, \pm 1, \pm 2, ...\}$, if $S \in \mathfrak{D}_2(\lambda)$.

Assume that we are given three r.v.'s S_0, S_1 and S_2 with mean 0 and finite variance, so that S_1 and S_2 are independent and $S_0 = S_1 + S_2$, $S_i \in \mathfrak{D}(\lambda)$. Set $B_k^2 = \mathrm{E}S_k^2$, $k = 0, 1, 2$ and $B = B_1 B_2 / B_0$. Let $\Psi_k(h)$ be the cumulant generating function of the r.v. S_k, i.e.

$$\Psi_k(h) = \log \mathrm{E} \exp\{hS_k\}, \quad |h| \leq \lambda, \quad k = 0, 1, 2.$$

It is easy to see that $\Psi_0'(h)$ is strictly increasing, for $|h| \leq \lambda$. Let $H_0(x) = h$ be the solution of the equation $\Psi_0'(h) = x$, for x in the range of $\Psi_0'(h)$. Set $B = B_1 B_2 / B_0$,

$$\mu(y) = \Psi_1'(H_0(y)) = y - \Psi_2'(H_0(y)),$$

and

$$\sigma^2(y) = \Psi_1''(H_0(y)) \Psi_2''(H_0(y)) / \Psi_0''(H_0(y)).$$

Define the conditional distrubution function

$$F(x|y) = P\left(S_1 \leq x | S_0 = y\right).$$

In the sequel c_i, $i = 0, 1, ...$ denote positive absolute constants and θ_i, $i = 1, 2, ...$ real numbers satisfying $|\theta_i| \leq 1$.

Theorem 1. *Assume that $S_0, S_1, S_2 \in \mathfrak{D}_1(\lambda)$ or $S_0, S_1, S_2 \in \mathfrak{D}_2(\lambda)$, for some $0 < \lambda < \infty$. Then, for any $x, y \in \mathcal{X}$ satisfying $|x| \leq c_0 \lambda B^2$ and $|y| \leq c_0 \lambda B^2$, we have*

$$F\left(x + \mu(y) | y\right) = \Phi\left(\frac{x}{\sigma(y)} + \theta_1 \frac{c_1}{\lambda \sigma(y)} \left(1 + \frac{x^2}{\sigma^2(y)}\right)\right),$$

provided $\lambda B \geq c_2$. Moreover, for $y \in \mathcal{X}$ satisfying $|y| \leq c_0 \lambda B^2$, we have

$$\mu(y) = \beta_1 y + \theta_2 c_3 \frac{y^2}{\lambda B^2}, \quad \frac{1}{\sigma^2(y)} = \frac{1}{B^2} + \theta_3 c_4 \frac{|y|}{\lambda B^4},$$

where $\beta_1 = B_1^2 / B_0^2$.

In the case case of sums of i.i.d. r.v.'s we get the following result. Let $\xi_1, ..., \xi_n$, $n \geq 1$ be a sequence of i.i.d. r.v.'s of means 0 and finite variances $\sigma^2 = \mathrm{E}\xi_i^2 < \infty$, obeying one of the two conditions (G1) or (G2) below:

G1 For some $0 < \lambda < \infty$ and $q \geq 1$,

$$\sup_{|h| \leq \lambda} \int_{-\infty}^{\infty} \left|\mathrm{E} \exp((it + h)\xi_i)\right|^q dt < \infty.$$

G2 The r.v. ξ_i takes values on the lattice $a + \Delta k$, $k \in \mathbb{Z}$ with maximal span Δ and for some $0 < \lambda < \infty$,

$$\mathrm{E} \exp(\lambda |\xi_i|) < \infty.$$

The common cumulant generating function of ξ_i-th is denoted by $\psi(t) = \log \mathrm{E} \exp(t\xi_i)$, for $|t| \leq \lambda$. Let $h(x)$, $|x| \leq \lambda/8$ be the inverse of the monotone function $\psi'(t)$, $|t| \leq \lambda/4$. Assume that n_1, n_2 are positive integers such that $n_1 + n_2 = n$. Set $S_1 = \xi_1 + ... + \xi_{n_1}$, $S_0 = \xi_1 + ... + \xi_n$ and $F(x|y) = P(S_1 \leq x | S_0 = y)$.

Theorem 2. *For any $|x| \leq c_0 \lambda n \sigma^2$ and $|y| \leq c_0 \lambda n \sigma^2$, we have*

$$F(x + \mu(y)|y) = \Phi\left(\frac{x}{\sigma(y)} + \theta_1 \frac{c_1}{\lambda \sigma(y)}\left(1 + \frac{x^2}{\sigma^2(y)}\right)\right).$$

where

$$\mu(y) = \frac{n_1}{n} y, \quad \sigma^2(y) = \frac{n_1 n_2}{n} \psi''\left(h\left(\frac{y}{n}\right)\right).$$

Proofs of Theorems 1 and 2 rely on Lemmas 1 and 2 below, which are of independent interest.

Assume that we are given a real valued r.v. X, which has a density $p(x)$ w.r.t. Lebesque measure on the real line. We shall impose conditions (C1-C2) below, where $\gamma_0, ..., \gamma_4 > 0$ and $\sigma > 1$ denote real quantities not depending on x.

C1 For any $|x| \leq \gamma_0 \sigma$,

$$p(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2} + \theta_1 \frac{\gamma_1}{2} \frac{x^3}{\sigma}\right)\left(1 + \theta_2 \gamma_2 \frac{1 + |x|}{\sigma}\right),$$

where $|\theta_i| \leq 1$, $i = 1, 2$.

C2 For any $|x| \leq \gamma_0 \sigma$ and any real z satisfying $zx \geq 0$,

$$p(x + z) \leq \gamma_3 \exp\left(-\frac{x^2}{2} + \frac{\gamma_1}{2} \frac{x^3}{\sigma} - \gamma_4 z x\right).$$

Lemma 1. *Assume that the density $p(x)$ satisfies conditions (C1-C2). Then, for any x satisfying $|x| \leq c_0 \sigma$, we have*

$$\Phi\left(x - c_1 \frac{1 + x^2}{\sigma}\right) \leq P(X \leq x) \leq \Phi\left(x + c_1 \frac{1 + x^2}{\sigma}\right),$$

provided $\sigma \geq c_2$, where c_0, c_1 and c_2 are constants depending only on $\gamma_0, ..., \gamma_4$.

Let $\Delta_0 > 0$ and $\Delta \in (0, \Delta_0]$. Assume that we are given a r.v. X, which takes discrete values $a_k = a + k\Delta$, $k \in \{0, \pm 1, \pm 2, ...\}$, with maximal span Δ and with $a \in [0, \Delta)$. In the sequel we shall impose conditions (D1-D2) below, where we assume that $\gamma_0, ..., \gamma_4 > 0$ and $\sigma > 1$ denote real quantities not depending on values a_k, $k \in \{0, \pm 1, \pm 2, ...\}$. For the sake of brevity we set $x_k = a_k/\sigma$.

D1 For any integer k satisfying $|x_k| \leq \gamma_0 \sigma$,

$$P(X = a_k) = \frac{\Delta}{\sqrt{2\pi}\sigma} \exp\left(-\frac{x_k^2}{2} + \theta_1 \frac{\gamma_1}{2} \frac{x_k^3}{\sigma}\right)\left(1 + \theta_2 \gamma_2 \frac{1 + |x_k|}{\sigma}\right).$$

D2 For any integers k, j satisfying $|x_k| \leq \gamma_0 \sigma$ and $(x_{k+j} - x_k)x_k \geq 0$,

$$P(X = a_{k+j}) \leq \gamma_3 \frac{\Delta}{\sigma} \exp\left(-\frac{x_k^2}{2} + \frac{\gamma_1}{2} \frac{x_k^3}{\sigma} - \gamma_4 (x_{k+j} - x_k) x_k\right).$$

Lemma 2. *Assume that conditions (D1-D2) are satisfied. Then, for any* x_k *satisfying* $|x_k| \le c_0\sigma$,

$$\Phi\left(x_k - c_1\frac{1+x_k^2}{\sigma}\right) \le P\left(X \le a_k\right) \le \Phi\left(x_k + c_1\frac{1+x_k^2}{\sigma}\right),$$

provided $\sigma \ge c_2$, *where* c_0, c_1 *and* c_2 *are constants depending only on* $\gamma_0, ..., \gamma_4$ *and* Δ_0.

Note that condition $\Delta \in (0, \Delta_0]$ is used to assure the uniformity of the assertion of the lemma in Δ.

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