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# UNANIMOUS IMPLEMENTATION: A CASE FOR APPROVAL MECHANISMS\*

MATÍAS NÚÑEZ<sup>a</sup> AND DIMITRIOS XEFTERIS<sup>b</sup>

**ABSTRACT.** We consider a class of indirect mechanisms, the Approval ones, in which the players' strategies coincide with the subsets of the outcome space. We focus on the single-peaked domain and we prove that: a) each of these rules is characterized by a unique equilibrium outcome and b) for every strategy-proof single-peaked rule there exists an Approval one that *unanimously* implements it. That is, Approval rules fix the problem of equilibrium-outcome multiplicity that is inherent to the ensuing games of strategy-proof single-peaked rules and, perhaps more importantly, promote social coherence: the implemented outcome is approved by every player.

**KEYWORDS.** Nash Implementation, Strategy-proof, Unanimity, Indirect Mechanisms.

**JEL CLASSIFICATION.** C9, D71, D78, H41.

## 1. INTRODUCTION

Incentives for revelation of true preferences and incentives to reach unanimous decisions are both desirable when decision-making is concerned but, at least at first inspection, mutually exclusive.

On the one hand, the desirability of true preference revelation stems from the fact that the social value of any outcome may be meaningfully measured only with reference to the society's true preferences. This is why the literature has focused so much on studying the existence and properties of strategy-proof decision rules (the Gibbard-Satterthwaite theorem is the most prominent example, see Barberà [2001] for an excellent review and Saporiti [2009] and Arribillaga and Massó [2015] for recent contributions). These rules promote the revelation and, subsequently, the aggregation of citizens' true preferences. In the context

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of single-peaked voting,<sup>1</sup> strategy-proof rules have a nice and well-defined structure: as proved by Moulin [1980], a rule is strategy-proof if and only if it is a (generalized) median.<sup>2</sup> Note that even though it seems intuitive that agents will behave sincerely using these rules, one can question their pertinence. Among the different objections,<sup>3</sup> the most salient one from a theoretical perspective is that the ensuing games of these rules might exhibit a large multiplicity of equilibrium outcomes. For example, in the ensuing game of the pure median rule, any alternative  $x \in A$  can be implemented in equilibrium for any preference profile! In order to address such issues, the literature has tried to find notions of implementation that deliver a reduced multiplicity of equilibria while keeping strategy-proofness as a requirement. However, this seems to be too ambitious for voting rules as shown by Saijo et al. [2007] among others: this inevitably leads to the usual dictatorship/negative result omnipresent in social choice theory.<sup>4</sup>

On the other hand, the appeal of reaching unanimity lies, mainly, on that it minimizes the probability of post-decision conflict. When a collective body has experienced great costs from conflict among its members (for example, the countries that compose the European Union),<sup>5</sup> it is straightforward that it values to a great extent consensual decision-making, even if it comes at a considerable deliberation cost. Moreover, elected officials that take decisions using advice from committees of experts are much more comfortable following unanimous recommendations than suggestions which are disputed by a number of experts in the committee. Unanimous recommendations minimize the responsibility of the decision maker and make her less accountable to groups of citizens that are negatively affected by her decisions. In addition, when experts agree on a policy recommendation, it is hard for elected officials to succumb to interest groups' pressures and neglect experts' advice, and this should maximize probability of

<sup>1</sup>The set of alternatives is  $A = [0, 1]$  and the set of possible preference relations consists of the single-peaked ones on  $A$ .

<sup>2</sup>Under such a rule each individual submits a message (a number in  $A$ ) and the rule implements the median of the set which consists of: a) the received messages and b) some exogenously fixed numbers (phantom voters).

<sup>3</sup>Experimental evidence also suggests that strategy-proof mechanisms might perform poorly in terms of sincere revelation. See for instance Cason et al. [2006] and Block et al. [2014].

<sup>4</sup>More precisely, Saijo et al. [2007] suggest to focus on secure implementation: a social choice function is securely implementable if there exists a game form that simultaneously implements it in dominant strategy equilibria and in Nash equilibria. While secure implementation paves the way to interesting mechanisms in some settings, such as quasi-linear environments, this is not the case in single-peaked voting. In this setting, every securely implementable social choice correspondence is either dictatorial or Pareto-inefficient.

<sup>5</sup>There is a large literature that rationalizes the use of the unanimity rule. While the classic argument is based on an ex-Post Pareto efficiency criterion (see Wicksell [1896] and Buchanan and Tullock [1962]), Maggi and Morelli [2006] provide an ex-ante argument in a dynamic model of international organizations.

informed decision making. These are only a few reasons why another branch of the literature has studied mechanisms that promote consensus building (see for example Bessette [1980], Gutmann and Thompson [1996], Gutmann and Thompson [2002] and Fishkin and Laslett [2003]).

Decision rules that promote aggregation of true preferences through sincere preference revelation, though, cannot result generically in consensual decisions and vice versa: if all society members report their true preferences, then the implemented policy should coincide with the preferred policy of only a fraction of the society (at best). In other words, the problem of having true preference aggregation and agreement on the implemented alternative seems to be a Gordian knot. This is surely the case as far as direct mechanisms are concerned. When the strategy space coincides with the players' type space then it is not generically possible to have in equilibrium both players' reporting their true types and unanimity. But is this equally obvious when we take decisions using an indirect mechanism?

Indirect mechanisms allow for more complex strategy spaces: this generates more possibilities as far as bridging true preference aggregation and consensus is concerned. Focusing on single-peaked voting, we consider that an indirect mechanism: a) promotes true preference aggregation if its equilibrium outcome coincides with the outcome of a strategy-proof rule and b) promotes consensus if the equilibrium outcome is declared at least as acceptable as any other alternative by all members of the society. Indeed, if an indirect mechanism implements a social choice rule then it has, outcome-wise, the same advantages as this rule: if strategy-proof rules are better than other rules in that they generate social outcomes based on true preferences and not on any other third factors, then the indirect mechanisms that implement them should be equally desirable in that respect. So the real question is: are there indirect mechanisms that implement strategy-proof rules in a way that a consensus regarding their outcome is guaranteed?

We propose a class of indirect mechanisms, the *Approval* ones, which allow every player to support, not just a single alternative, but as many alternatives as one wants (an arbitrary interval within the unit interval). After all individuals report their sets of approved alternatives, a publicly known aggregation rule is applied and an alternative is implemented. These aggregation rules might take very simple forms. The most intuitive examples are arguably the median and the mean aggregation rule. When players submit their sets of approved alternatives a distribution of approvals is generated: the density of this distribution at  $x \in [0, 1]$

is identical to the number of individuals that have approved of alternative  $x$ , normalized by the total measure of approvals. The median (resp. mean) aggregation rule simply implements the median (resp. mean) of this distribution.

Our main finding is that *for every strategy-proof rule in the single-peaked domain there exists an anonymous Approval mechanism that unanimously implements it.*<sup>6</sup> An Approval mechanism is understood to unanimously implement a social choice rule if: a) it implements it in *every* Nash equilibrium and b) there is at least one equilibrium in which each player includes in his strategy (set of approved outcomes) the implemented outcome. Players' equilibrium strategies are easy to describe: every player with a preferred alternative to the left (right) of the implemented one approves the implemented alternative and all the alternatives to its left (right). That is, in equilibrium at most one player may not include the implemented outcome and his own ideal outcome in his strategy, and this player's ideal outcome must coincide with the implemented one.

Hence, every equilibrium is essentially unanimous in the sense that, for each voter, the implemented outcome and his ideal one are either both contained in his strategy or they coincide with each other. Moreover, since these rules exhibit a unique equilibrium outcome which may be unanimously approved in equilibrium, it means again that these mechanisms trigger a decentralized unanimity over some strategy-proof mechanism.<sup>7</sup> In other words, these rules are outcome-equivalent to the unique class of direct mechanisms that lead to true preference aggregation. Finally, the fact that in equilibrium players approve, not only the implemented outcome, but their ideal one as well, shows that these rules promote sincere revelation of preferences to a certain extent.

The Approval mechanisms can be applied to a variety of decision-making problems. Consider for example a number of judges who disagree on the quality of an athletic performance (for example, in gymnastics or in figure skating) and that have to jointly assign a score to this performance, while each of them wants the joint score to be as close as possible to her individual performance evaluation.

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<sup>6</sup>From an implementation theory perspective, our result can be interpreted as follows. As previously discussed, every securely implementable social choice function must be a dictatorship in the single-peaked environment. The way we overcome this negative result is by weakening the strategy-proof requirement. Indeed, our indirect mechanisms implement a strategy-proof mechanism at any possible equilibrium, while this is not possible with direct mechanisms. Note that this does not clash with the Revelation Principle but it gives a nice application of it: indeed as argued by Myerson [2008], such a principle states that indirect mechanisms can be “simulated by an equivalent incentive-compatible direct-revelation mechanism”

<sup>7</sup>One may wonder whether our results could be extended to single-crossing preferences, a domain which generalizes single-peakedness while inhering many of its features. Indeed, Saporiti [2009] extends Moulin's characterization of strategy-proofness to single-crossing domains. However, it is far from clear how to define strategy-proof mechanisms over an infinite set of alternatives with single-crossing preferences, mainly due to the handling of individual indifferences.

Another potential application is the determination of LIBOR or the board members of the European Central Bank (ECB) deciding over the interest rate from a closed and convex set of interest rates (see Cai [2009], Rausser et al. [2015] and Rosar [2015] among others for recent analysis).<sup>8</sup> Our Approval mechanisms can be of interest in these settings since they can improve the quality of decision making by ensuring a unanimous final decision.

In what follows we describe the model (section 2) and present the formal results (section 3) and a discussion (section 4).

## 2. THE SETTING

Let  $A := [0, 1]$  denote the set of alternatives and  $N := \{1, \dots, n\}$  with  $n \geq 2$  stand for the finite set of players. Each player is endowed with preferences over  $A$ . The utility for player  $i$  when  $x \in A$  is the implemented policy equals  $u_i(x)$  with  $u_i : A \rightarrow \mathbb{R}$  where each  $u_i \in U$ , the set of single-peaked preferences. Note that each player  $i$  has a unique peak denoted  $t_i$  so that  $u_i(x') < u_i(x'')$  when  $x' < x'' \leq t_i$  and when  $t_i \leq x'' < x'$ .<sup>9</sup> We let  $(t_1, \dots, t_n)$  stand for a distribution of the players' peaks and  $u = (u_1, \dots, u_n) \in U := \prod_{j=1}^n U_j$ .

A social choice function is a function  $f : U \rightarrow A$  that associates to every  $u \in U$ , a unique alternative  $f(u)$  in  $A$ . A mechanism is a function  $\theta : S \rightarrow A$  that assigns to every  $s \in S$ , a unique element  $\theta(s)$  in  $A$ , where  $S := \prod_{i=1}^n S_i$  and  $S_i$  is the strategy space of agent  $i$ . The mechanism  $\theta$  is the direct revelation mechanism associated to a SCF  $f$  if  $S_i = U_i$  for all  $i \in N$  and  $\theta(u) = f(u)$  for every  $u \in U$ . A SCF  $f$  is strategy-proof if for all  $i \in N$ , all  $u_i, \tilde{u}_i \in U_i$ , and all  $u_{-i} \in U_{-i}$ ,  $u_i(f(u_i, u_{-i})) \geq u_i(f(\tilde{u}_i, u_{-i}))$ . As shown by Moulin [1980], these rules admit a simple characterization: they implement as an outcome the median of the peaks of the players plus  $(n - 1)$  exogenous parameters (phantoms). More formally,

<sup>8</sup>The London Interbank Offered Rate (Libor) is the interest rate at which banks can borrow from each other and plays a critical role in financial markets. Libor anchors contracts amount "to the equivalent of \$45000 for every human being on the planet" (see MacKenzie [2008]). The banks are asked to submit an interest rate at which their banks could borrow money. The lowest and highest quarter of the values are discarded and the Libor corresponds to the average of the remainder. In other words, the device used to determine this index is the trimmed mean rule. Theorists have mostly focused on the pure mean rule (without trimming) and their conclusion over its properties is qualified (see Renault and Trannoy [2005] and Yamamura and Kawasaki [2013] for theoretical works on this subject and Marchese and Montefiori [2011] and Block et al. [2014] for experimental ones). In environments with a large number of voters, this rule seems to be a natural candidate as it is the unique one satisfying a weakening of strategy-proofness (see Ehlers et al. [2004]). While the latter feature is quite desirable, the former violates the usual desiderata of voting theory: a voter drops his most preferred policy to announce an extremist policy that maximizes his impact on the final outcome. This extreme polarization of the voters' positions seems to posit a fundamental problem with the average method.

<sup>9</sup>For simplicity, we assume that  $t_i \neq t_j$  for any  $i, j \in N$ . Our results are not affected when relaxing this constraint.

for any finite collection of points  $x_1, \dots, x_m$  in  $[0, 1]$ , we let  $m(x_1, \dots, x_m)$  denote their median, that is the smallest number  $m(x_1, \dots, x_m) \in x_1, \dots, x_m$ , which satisfies:  $\frac{1}{m} \#\{x_i \mid x_i \leq m(x_1, \dots, x_m)\} \geq \frac{1}{2}$  and  $\frac{1}{m} \#\{x_i \mid x_i \geq m(x_1, \dots, x_m)\} \geq \frac{1}{2}$ . In the domain  $U$  and assuming that each agent's message is one element of  $A$ , a SCF  $f$  is anonymous, efficient and strategy-proof if and only if there exist  $(n-1)$  real numbers,  $\kappa_1, \dots, \kappa_{n-1}$  such that  $f(t_1, \dots, t_n) = m(t_1, \dots, t_n, \kappa_1, \dots, \kappa_{n-1})$ .

We let  $\mathcal{B}$  denote the collection of closed intervals of  $A$  and define an approval mechanism as a mechanism such that  $S_i = \mathcal{B}$  for every  $i \in N$ .<sup>10</sup> We write  $\underline{b}_i = \min b_i$  and  $\overline{b}_i = \max b_i$  for each  $b_i \in \mathcal{B}$ . Note that the strategy set  $\mathcal{B}$  allows elements of different dimensions: singletons and positive length intervals. To accommodate this fact, we let  $\lambda_d$  denote the Lebesgue measure on  $\mathbb{R}^d$  with  $d = 0, 1$ . Since each  $b_i$  is a convex set, its dimension is well-defined so that for each approval profile  $b = (b_i, b_{-i})$ , we let  $\dim(b) = \max_{i \in N} \dim(b_i)$ .

Given a mechanism  $\theta : S \rightarrow A$ , the strategy profile  $s \in S$  is a Nash equilibrium of  $\theta$  at  $u \in U$ , if  $u_i(\theta(s_i, s_{-i})) \geq u_i(\theta(s'_i, s_{-i}))$  for all  $i \in N$  and any  $s'_i \in S_i$ . Let  $N^\theta(u)$  be the set of Nash equilibria of  $\theta$  at  $u$ . The mechanism  $\theta$  implements the SCF  $f$  in Nash equilibria if for each  $u \in U$ , (i) there exists  $s \in N^\theta(u)$  such that  $\theta(s) = f(u)$  and (ii) for any  $s \in N^\theta(u)$ ,  $\theta(s) = f(u)$ . The SCF  $f$  is implementable if there exists a mechanism that implements  $f$  in Nash equilibria. An Approval Mechanism  $\theta$  unanimously implements the SCF  $f$  if (i)  $\theta$  implements  $f$  in Nash equilibria and (ii) there exists  $s \in N^\theta(u)$  such that  $\cap_{i=1}^n s_i \neq \emptyset$  with  $\theta(s) = \cap_{i=1}^n s_i$ . Our focus is on the unanimous implementation of strategy-proof rules.

### 3. AN EXAMPLE: THE MEDIAN APPROVAL MECHANISM

In order to clarify the main ideas behind unanimous implementation, this section presents an example that illustrates how an approval mechanism works. We are concerned here with the median approval mechanism that associates, to any distribution of approvals, its *median*. Therefore, we assume that the median approval mechanism associates to every strategy profile  $b$  (i.e. any announcement of intervals), the median  $\theta(b)$  of these intervals with

<sup>10</sup>This assumption can be relaxed by allowing as a pure strategy any finite union of closed and convex subsets of  $A$ . Relaxing it however would imply more cumbersome notation and proofs since then two strategies that differ by a zero-measure set can have equivalent consequences. Moreover, it will not affect much the result so that we prefer to stick to the simpler definition of strategy to keep the main message as simple as possible.

$$\theta(b) := \min\{x \in [0, 1] \mid \int_0^x f_b(t) dt = \frac{1}{2}\},$$

$$\text{with } f_b(t) = \frac{\#\{i \in N \mid t \in b_i\}}{\sum_{i \in N} \lambda_{\dim(b)}(b_i)} \text{ for any } t \in [0, 1].$$

In order to understand the definition of  $\theta$ , it suffices to understand that  $f_b(t)$  stands for the “score” of alternative  $t$  normalized by the size of the intervals in  $b$  and therefore  $\int_0^x f_b(t) dt$  counts the share of “approvals” located between 0 and  $x$ . It is hence a cumulative distribution in the usual sense<sup>11</sup> and therefore  $\theta$  implements the median  $\theta(b)$  as the value in which the share of approvals located below and above it is equal to  $1/2$ , the lowest value being chosen in case of ties.

Summarizing, the median approval mechanism works as follows:

- (1) Every player simultaneously and independently announces a closed interval  $b_i$  in  $A$  and
- (2) The mechanism implements  $\theta(b)$  with  $b = (b_1, \dots, b_n)$ .

As will be shown, with  $n$  players, the median approval mechanism unambiguously implements the generalized median rule  $m(t_1, t_2, \dots, t_n, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n})$ . In the particular situation in which  $N = \{1, 2, 3\}$  and  $(t_1, t_2, t_3)$  with  $t_1 < t_2 < \frac{1}{3} < \frac{2}{3} < t_3$ ,<sup>12</sup> it follows that the unique equilibrium outcome should equal  $\frac{1}{3}$ . This outcome can be thought as a compromise between the extreme types of the players. Moreover, such an outcome is supported by a unique equilibrium  $b^* = (b_1^*, b_2^*, b_3^*)$  such that

$$b_1^* = b_2^* = \left[0, \frac{1}{3}\right] \quad \text{and} \quad b_3^* = \left[\frac{1}{3}, 1\right].$$

It follows that  $\sum_{i \in N} \lambda_{\dim(b^*)}(b_i^*) = \frac{4}{3}$  and hence that  $f_{b^*}(t) = \frac{3}{2}$  whenever  $0 \leq t \leq \frac{1}{3}$  and  $f_{b^*}(t) = \frac{3}{4}$  otherwise. One can hence easily check that  $\theta(b^*) = \frac{1}{3}$ . Figure 1 depicts the distribution of approvals generated by  $b^*$ . The alternatives lower than  $\frac{1}{3}$  are selected by two players whereas the rest of them just by one. Hence it is graphically simple to understand that  $\theta(b^*) = \frac{1}{3}$  since it splits the area below the curve in two exact halves.

To explain why  $b^*$  is an equilibrium, we now describe the consequences of a possible deviation of player 1. Assume that 1 deviates to  $b'_1 = [0, x]$ . The size of  $b' = (b'_1, b_2^*, b_3^*)$  is equal to  $1 + x$  so that  $f_{b'}(t) = \frac{2}{1+x}$  whenever  $0 \leq t \leq x$  and  $f_{b'}(t) = \frac{1}{1+x}$  otherwise. When  $x$  is larger than  $\frac{1}{3}$ , the total size of  $b'$  is higher than the one of  $b^*$ ; hence the median must be located to the right of  $\frac{1}{3}$  since it is the

<sup>11</sup>A formal proof of this statement is provided by Lemma 1.

<sup>12</sup>A similar example is analyzed in Austen-Smith and Banks [2005], chapter 6, p.233. In their model, the three players also reach a consensus over an interior policy in the interval  $[0,1]$ . The reasons for consensus depend on the discount factors, which is to define the no-delay equilibrium. See also Banks and Duggan [2000] for a bargaining model of collective choice.



value that divides the area below  $f_b$  in two exact halves. On the contrary, when  $x$  is lower than  $\frac{1}{3}$ , the total size of  $b'$  is smaller than the one of  $b^*$ . However, the area located to the left of  $\frac{1}{3}$  now equals  $\frac{2x}{1+x} + \frac{\frac{1}{3}-x}{1+x}$  and hence represents less than half of the approvals. This leads again to a median larger than  $\frac{1}{3}$ .<sup>13</sup>

A similar reasoning proves the claim for the different deviation of this player and the different players.

If the distribution becomes less polarized and  $(t_1, t_2, t_3)$  with  $t_1 < \frac{1}{3} < t_2 < \frac{2}{3} < t_3$ , it follows that the unique equilibrium outcome should equal  $t_2$ . By similar reasonings as before, one can show that this outcome can be supported by an equilibrium  $b^+$  with, for some pair  $0 < \delta_1, \delta_2 < \min\{t_2, 1 - t_2\}$

$$b_1^+ = [0, t_2], \quad b_2^+ = [t_2 - \delta_1, t_2 + \delta_2] \quad \text{and} \quad b_3^+ = [t_2, 1]$$

Two features of the equilibria  $b^*$  and  $b^+$  deserve to be highlighted. The first one is that all players include the implemented alternative in their interval. The second one is that the voters' strategies are divided in three blocks: the ones with a peak to the left of the implemented alternative, the ones to the right and the one in which it coincides. Both characteristics are present in any equilibrium of any approval mechanism discussed in this work.

#### 4. CONDITIONS FOR UNANIMOUS IMPLEMENTATION

This section presents the main results of this work. After describing some axioms for the Approval mechanisms to satisfy, it proves that these axioms are enough to characterize unanimous implementation under Approval mechanisms.

**4.1. Axioms on Approval Mechanisms .** We restrict ourselves to anonymous Approval mechanisms<sup>14</sup> such that for each  $x \in A$ , there is some  $b \in \mathcal{B}^n$  with  $\theta(b) = x$ . We now introduce the axioms that will suffice to identify the Approval mechanism that induce unanimous implementation.

The first axiom deals with the two sort of strategies allowed in an Approval mechanism. Indeed, either a strategy contains finitely many alternatives (zero-dimensional strategy) or infinitely many (one-dimensional strategy). One might argue that zero-dimensional strategies are *stubborn* in the sense that the player is approving of a zero-measure set of the set of available alternatives. Similarly, a

<sup>13</sup>More generally, assume by contradiction that player 1 has a best response  $b'_1$  such that  $b'_1 \cap [\frac{1}{3}, 1]$  has positive Lebesgue measure. Then, take the strategy  $b''_1 := b'_1 \setminus \{b'_1 \cap [\frac{1}{3}, 1]\}$ . It is simple to see that  $\theta(b''_1, b_2^*, b_3^*) < \theta(b'_1, b_2^*, b_3^*)$  so that  $b''_1$  leads to a median closer to player 1's ideal policy than  $b'_1$ , a contradiction. Similarly, assume again by contradiction that there is some best response  $b'_1$  with  $b_1^* \setminus b'_1$  having positive Lebesgue measure. Then,  $\theta(b_1^*, b_2^*, b_3^*) < \theta(b'_1, b_2^*, b_3^*)$  since all these points are located to the left of  $\theta(b^*)$ .

<sup>14</sup>The mechanism  $\theta : \mathcal{B}^n \rightarrow A$  satisfies Anonymity if for any permutation  $\sigma : N \rightarrow N$ ,  $\theta(\sigma(b)) = \theta(b)$ .

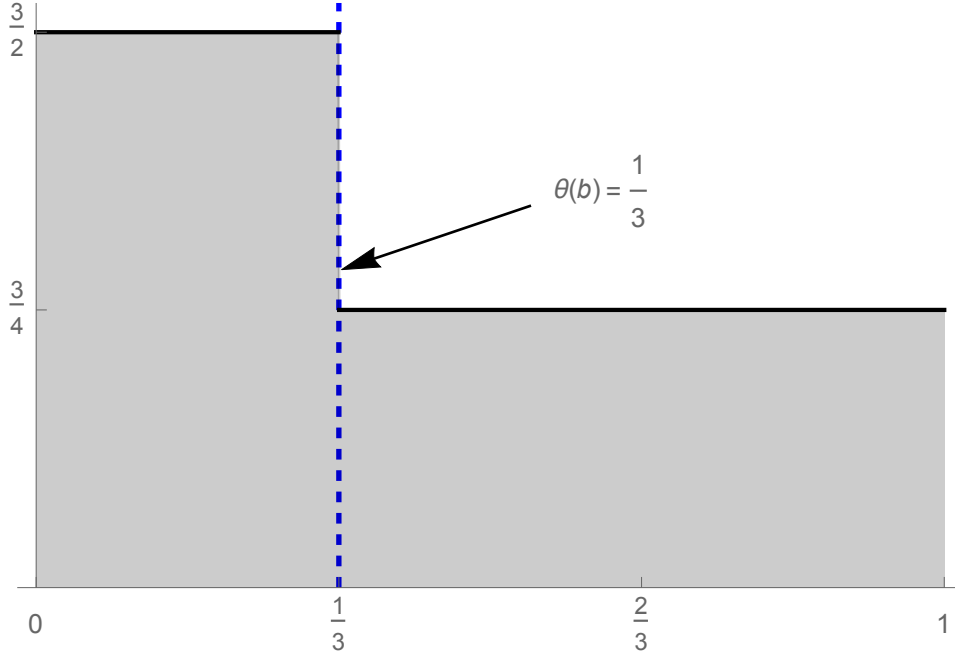


FIGURE 1. Distribution of approvals associated to  $b^*$ .

one-dimensional strategy is a *compromise* one by opposition to stubborn strategies. The set of stubborn and compromise strategies are respectively labeled by  $S(\mathcal{B})$  and  $C(\mathcal{B})$  with  $\mathcal{B} = S(\mathcal{B}) \cup C(\mathcal{B})$ . The Approval Mechanisms in which we will focus give incentives to players to select one-dimensional strategies, in the following sense:

*Incentives for Compromise (IC)*: The mechanism  $\theta : \mathcal{B}^n \rightarrow A$  satisfies Incentives for Compromise if, for any  $i \in N$  and for any  $b_i \in S(\mathcal{B})$ ,  $\exists c_i \in C(\mathcal{B})$  with  $u_i(\theta(c_i, b_{-i})) > u_i(\theta(b))$ , whenever  $\theta(b) \neq t_i$ .

This axiom ensures that each player has an incentive to submit a compromise strategy rather than a stubborn one, as long as the mechanism does not select his most preferred alternative. The main implication of *IC* is that there is no equilibrium in which each player announces a singleton as long as the axiom *IC* holds.

In order to define our second axiom, we introduce the following piece of notation. For each  $i \in N$  and any  $b_{-i} \in \mathcal{B}^{n-1}$ ,  $\theta(\mathcal{B}, b_{-i})$  denotes the attainable set of player  $i$  at  $b_{-i}$ ; it represents the set of available alternatives that player  $i$  can induce when the rest of the players select  $b_{-i}$ . Since  $\mathcal{B}$  is not finite, the set  $\theta(\mathcal{B}, b_{-i})$  need not have a maximum or a minimum. Monotonicity gives precise conditions to characterize the maximum and the minimum of the attainable set when they exist.

*Monotonicity (MON)*: The mechanism  $\theta : \mathcal{B}^n \rightarrow A$  satisfies Monotonicity if for any  $i \in N$  and any  $b_{-i} \in \mathcal{B}^{n-1}$ , then :

$$b_i^m \in \arg \min \theta(\mathcal{B}, b_{-i}) \text{ if and only if } \exists b_i^m = [0, x_i^m] \text{ with } x_i^m = \theta([0, x^m], b_{-i}), \text{ and} \quad (1)$$

$$b_i^M \in \arg \max \theta(\mathcal{B}, b_{-i}) \text{ if and only if } \exists b_i^M = [x_i^M, 1] \text{ with } x_i^M = \theta([x^M, 1], b_{-i}). \quad (2)$$

That is, when a player attempts to draw the implemented outcome as left as possible it should not be the case that he approves of outcomes to its right and it should not be the case that he does not approve of outcomes to its left, and vice versa.

To define our final two axioms, we consider the following class of strategy profiles. For any  $j = 0, 1, \dots, n$ , we define the strategy profile  $b^j(x) \in \mathcal{B}^n$  as the strategy profile in which  $n - j$  players use the strategy  $[0, x]$  and  $j$  players use the strategy  $[x, 1]$ . We let

$$\kappa_j := \{x \in A \mid \theta(b^j(x - \varepsilon)) > \theta(b^j(x)) = x > \theta(b^j(x + \varepsilon)) \text{ for any } \varepsilon > 0\}$$

for any  $j \in \{0, 1, \dots, n\}$ .

For simplicity, we say that  $\kappa_j$  denotes the fixed points of  $\theta(b^j(x))$  but, more accurately,  $\kappa_j$  is the set of points at which  $\theta(b^j(x))$  intersects with  $x$ .

For any  $b \in \mathcal{B}^n$ , we let  $\text{Supp}(b) = \bigcup b_i$  denote the support of profile  $b$ . The support denotes the set of alternatives that are selected by at least some player. When the support is convex, all alternatives located between the minimum and the maximum of the support are selected. Approval mechanisms are restricted to be continuous in the following sense, as long as they have a convex support.

*Continuity (C)*. The mechanism  $\theta : \mathcal{B}^n \rightarrow A$  satisfies Continuity if for any  $i \in N$ , any  $b, b^m \in \mathcal{B}^n$  with  $b^m = (b_i^m, b_{-i})$  such that  $\text{Supp}(b), \text{Supp}(b^m)$  are convex,

$$\lim_{m \rightarrow \infty} b_i^m = b_i \implies \lim_{m \rightarrow \infty} \theta(b^m) = \theta(b).$$

This technical axiom introduces a nice property of our mechanism, it should be continuous in each component. This is quite mild since it just applies to the strategy profiles such that  $\bigcup_{j \in N \setminus \{i\}} b_j \in C(\mathcal{B})$ .

The final axiom characterizes properties of the fixed points of the Approval mechanisms. We let  $h(n) := \frac{n}{2}$  when  $n$  is even and  $h(n) := \frac{n+1}{2}$  when  $n$  is odd, and  $G_{g,n} = \{g, \dots, n - g\}$  when  $g \leq \frac{n}{2}$  and  $G_{g,n} = \emptyset$  otherwise.

*Fixed-Point Monotonicity (FP)*. The mechanism  $\theta : \mathcal{B}^n \rightarrow A$  satisfies Fixed-Point Monotonicity if there exists  $g \in \{1, \dots, h(n)\}$  such that: a)  $\kappa_j$  is uniquely defined,

interior and strictly increasing in  $j \in G_{g,n}$ , and  $b$ ) for any  $j < \min\{g, \frac{n}{2}\}$  (resp.  $j > \max\{n - g, \frac{n}{2}\}$ ),  $\theta(b^j(x)) < x$  (resp.  $\theta(b^j(x)) > x$ ) when  $x \in (0, 1)$ .

This axiom clearly restricts the class of Approval mechanisms. It is essential to ensure the existence of pure strategy equilibrium and is also behind the uniqueness of the equilibrium outcome.

In order to illustrate which Approval mechanisms satisfy these axioms, we now present two leading examples that will be useful to understand the main intuitions behind our results. Given an approval profile  $b$  and an alternative  $x$ , we let  $s^x(b)$  denote the score of alternative  $x$  with  $s^x(b) = \#\{i \in N \mid x \in b_i\}$ . Thus, any approval profile  $b$  generates the function  $f_b$  with  $f_b(x) = \frac{s^x(b)}{\sum_{i \in N} \lambda_{\dim(b)}(b_i)}$  for any  $x \in [0, 1]$ . As shown by the next lemma,  $f_b$  is a well-defined density function for any approval profile  $b$ .

**Lemma 1.** *For any approval profile  $b = (b_i, b_{-i})$ ,  $f_b$  is a well-defined density function.*

**Proof.** : For each profile  $b$ , let  $V(b, j) \subseteq [0, 1]$  be the set such that  $V(b, j) = \{x \in [0, 1] \mid s^x(b) = j\}$ . Moreover,

$$\int_{[0,1]} f_b(x) dx = \frac{1}{\sum_{i \in N} \lambda_{\dim(b)}(b_i)} \int_{[0,1]} s^x(b) dx = \frac{1}{\sum_{i \in N} \lambda_{\dim(b)}(b_i)} \sum_{j=1}^n \int_{V(b,j)} j dx.$$

Since  $\int_{V(b,j)} j dx = j \lambda_d(V(b, j))$ , it follows that  $\int_{[0,1]} f_b(x) dx = 1$  as wanted. The previous equality combined with the function  $f_b(x)$  being non-negative for any  $x \in [0, 1]$  concludes the proof. **Q.E.D..**

The next two Approval mechanisms satisfy the axioms of Continuity, Monotonicity, Fixed Point Coherence and Incentives for Compromise.

*Average Approval Mechanism:* We let  $\mu_b$  stand for the mean of the approval profile  $b$  with  $\mu_b = \int_{[0,1]} x f_b(x) dx$ . Note that  $\mu_b \in [0, 1]$  and hence it always coincides with an alternative. The Average Approval Mechanism associates  $\mu_b$  to each approval profile  $b$  so that  $\theta(b) = \mu_b$ .

*Quantile Approval Mechanism:* The cumulative distribution of approvals,  $F(x)$ , is then given by  $F(x) = \int_0^x f_b(t) dt$ . The  $\alpha$ -Quantile Approval Mechanism associates to each approval profile  $b$  the lowest  $x^*$  such that  $F(x^*) = \alpha$  for some  $0 < \alpha < 1$ . The median approval mechanism employed in the previous example is a quantile mechanism with  $\alpha = \frac{1}{2}$ .

**4.2. Sufficiency.** Equipped with the previous results we are now ready to state the sufficient conditions for unanimous implementation.

**Theorem 1.** *If an Approval Mechanism  $\theta$  satisfies C, FP, MON and IC, then:*

- (1) *there is an equilibrium in pure strategies for every admissible preference profile*
- (2) *if  $g \leq \frac{n}{2}$ , then in every equilibrium  $b$  of  $\theta$  we have  $\theta(b) = m(t_1, t_2, \dots, t_n, \kappa_g, \dots, \kappa_{n-g})$  and if  $g = \frac{n+1}{2}$ , then in every equilibrium  $b$  of  $\theta$  we have  $\theta(b) = m(t_1, t_2, \dots, t_n)$ ,*
- (3) *there is an equilibrium  $b$  of  $\theta$  with  $\bigcap_{i=1}^n b_i = \theta(b)$ .*

**Proof.** Take some  $\theta : \mathcal{B}^n \rightarrow [0, 1]$  satisfying *C*, *FP*, *MON* and *IC*. We first notice that there should exist a unique  $g \in \{1, \dots, h(n)\}$  for which *FP* is satisfied. Throughout the proof we consider that  $g \leq \frac{n}{2}$  and in the end we comment why all developed arguments extend to the case in which  $g = \frac{n+1}{2}$ . For short, we write  $(t, \kappa)$  rather than  $(t_1, t_2, \dots, t_n, \kappa_g, \dots, \kappa_{n-g})$ . The proof first states the existence of equilibrium (**Step A.**), uniqueness of equilibrium outcome (**Step B.**) and finally the existence of a unanimous equilibrium (**Step C.**).

**Step A.: There is some equilibrium  $b$  of  $\theta$  with  $\theta(b) = m(t, \kappa)$ .**

Step A. is divided in two cases: either there is no  $t_h$  with  $t_h = m(t, \kappa)$  (Step A.I.) or there is such  $t_h$  to be developed in Step A.II.

**Step A.I.  $\nexists : t_h$  with  $t_h = m(t, \kappa)$ .** Since there is no  $t_h$  with  $t_h = m(t, \kappa)$ , there must exist  $j \in \{g, \dots, n-g\}$  such that  $\kappa_j = m(t, \kappa)$ . Therefore, the number of elements located below and above  $\kappa_j$  in  $(t, \kappa)$  is equal to  $n-g$ , which is equivalent to:

$$\underbrace{\#\{i \in N \mid t_i < \kappa_j\} + (j-g)}_{\text{elements lower than } \kappa_j} = \underbrace{\#\{i \in N \mid t_i > \kappa_j\} + (n-j-g)}_{\text{elements higher than } \kappa_j} = n-g.$$

The previous equalities jointly imply that  $\#\{i \in N \mid t_i < \kappa_j\} = n-j$  and  $\#\{i \in N \mid t_i > \kappa_j\} = j$ . Let  $b \in \mathcal{B}^n$  be an approval profile with:

$$b_i = \begin{cases} [0, \kappa_j] & \text{if } t_i < \kappa_j, \\ [\kappa_j, 1] & \text{if } t_i > \kappa_j. \end{cases}$$

Since  $\theta$  is anonymous by assumption, then  $\theta(b) = \theta(b^j(\kappa_j))$  so that  $\theta(b) = \kappa_j$  due to *FP*, and hence that  $\theta(b) = m(t, \kappa)$ . In order to prove that  $b$  is an equilibrium, assume that there is some  $i \in N$  with a profitable unilateral deviation  $b'_i$ , so that  $\theta(b'_i, b_{-i}) \neq \theta(b)$ . Assume first that  $\theta(b'_i, b_{-i}) < \theta(b)$ . If  $t_i > \kappa_j$  and given that preferences are single-peaked, it follows that  $u_i(\theta(b'_i, b_{-i})) < u_i(\theta(b_i, b_{-i}))$ . In other words,  $b'_i$  is not a profitable deviation, entailing a contradiction. If  $t_i < \kappa_j$ , then by definition  $b_i = [0, \kappa_j]$ . However, due to *MON*,  $b_i$  is player  $i$ 's unique best response, which proves that there is no profitable deviation. The same argument applies if  $\theta(b'_i, b_{-i}) > \theta(b)$ , which proves that  $b$  is an equilibrium of the game and concludes Step A.I.

**Step A.II.  $\exists : t_h$  with  $t_h = m(t, \kappa)$ .** If there exists  $j \in \{g, \dots, n-g\}$  such that  $\kappa_j = t_h$ , then  $j = n-h$  or  $j = n-h+1$ . Using the same line of reasoning as in A.I., one can

show that: a) when  $j = n - h + 1$ ,  $b^{n-h+1}(t_h)$  is an equilibrium with  $\theta(b^{n-h+1}(t_h)) = t_h$  and b) when  $j = n - h$ ,  $b^{n-h}(t_h)$  is an equilibrium with  $\theta(b^{n-h}(t_h)) = t_h$ .

If  $t_h = m(t, \kappa)$  and  $t_h \neq \kappa_j$ , there are  $n - g$  values smaller than  $t_h$  in  $(t, \kappa)$ . There are essentially two cases here: a)  $t_h \in (\kappa_g, \kappa_{n-g})$  and b)  $t_h < \kappa_g$  (the proof for the case  $t_h > \kappa_{n-g}$  is symmetric).

a) One can choose  $j$ , such that  $g < j < n - g$ , with  $\kappa_j < t_h = m(t, \kappa) < \kappa_{j+1}$ . Moreover  $\#\{\kappa_l \mid \kappa_l < t_h\} = j - g + 1$  and  $\#\{i \in N \mid t_i < t_h\} = h - 1$  so that:  $j - g + 1 + h - 1 = n - g \implies j = n - h$ . Therefore,  $\kappa_{n-h} < t_h < \kappa_{n-h+1}$ .

For each  $A \in \mathcal{B}$ , we define  $b^A$  as the approval profile with:

$$b_i^A = \begin{cases} [0, t_h] & \text{if } t_i < t_h, \\ A & \text{if } t_i = t_h, \\ [t_h, 1] & \text{if } t_i > t_h. \end{cases}$$

Our objective is to prove that there is at least one  $b^{A^*}$  with  $\theta(b^{A^*}) = t_h$ . Since  $\theta$  is continuous on a player's strategy, the result immediately follows from the Intermediate Value Theorem provided that there are some  $C$  and  $D$  with

$$\theta(C, b_{-h}) < t_h < \theta(D, b_{-h}).$$

Let  $\phi_{n-h}(x) = \theta(b^{n-h}(x))$  and  $\phi_{n-h+1}(x) = \theta(b^{n-h+1}(x))$  for any  $x \in [0, 1]$ . It follows that  $\phi_{n-h}(\kappa_{n-h}) = \kappa_{n-h} < t_h$  and  $\phi_{n-h+1}(\kappa_{n-h+1}) = \kappa_{n-h+1} > t_h$ . Moreover, since  $\theta$  satisfies *FP*, it follows that  $\kappa_t$  is a fixed point of  $\phi_t$  with  $\phi_t(\kappa_t) = \kappa_t$  whenever  $t = n - h, n - h + 1$ . We know that (i)  $\phi_t : [0, 1] \rightarrow [0, 1]$  and  $0 < \kappa_t < 1$ . Therefore, since  $\kappa_t$  is a fixed point of  $\phi_t$  and  $\phi_t$  is continuous on  $(0, 1)$ , it must be the case that for any  $x \in (0, \kappa_t)$ ,  $\phi_t(x) > x$  and for any  $x \in (\kappa_t, 1)$ ,  $\phi_t(x) < x$  whenever  $t = n - h, n - h + 1$ . Now, take  $C = [0, t_h]$ . Then  $b^C = b^{n-h}(t_h)$  and  $\theta(b^C) = \phi_{n-h}(t_h)$ . Similarly, take  $D = [t_h, 1]$  so that  $b^D = b^{n-h+1}(t_h)$  and  $\theta(b^D) = \phi_{n-h+1}(t_h)$ . Therefore, since  $\kappa_{n-h} < t_h$  and  $t_h < \kappa_{n-h+1}$  it must be respectively the case that  $t_h > \phi_{n-h}(t_h) = \theta(b^C)$  and  $t_h < \phi_{n-h+1}(t_h) = \theta(b^D)$ . We can hence conclude that there exists some  $A^*$  with  $\theta(b^{A^*}) = t_h$ .

In order to prove that  $b^{A^*}$  ( $b$  for short) is an equilibrium, suppose by contradiction that there exists some  $i \in N$  with a profitable deviation  $b'_i$ . Then, it cannot be the player with type  $t_h$  since  $\theta(b) = t_h$ . Suppose then that  $\theta(b'_i, b_{-i}) < \theta(b)$ . Then,  $t_i < t_h$ ; otherwise, if  $t_i > t_h$  then  $u_i(b'_i, b_{-i}) < u_i(b_i, b_{-i})$ , a contradiction with  $b'_i$  being a profitable deviation. However, any voter with  $t_i < t_h$  is playing his unique best response  $[0, t_h]$ , entailing again a contradiction. A symmetric argument applies when  $\theta(b'_i, b_{-i}) > \theta(b)$ . Therefore  $b$  must be an equilibrium concluding a) in Step A.

b) In this case  $t_h = m(t, \kappa) < \kappa_g$  and hence  $h = n - g + 1$ . According to *FP* we have that  $\theta(b^{n-h}(x)) < x$  for every  $x \in (0, 1)$  (because  $n-h = g-1 < g$ ) and  $\theta(b^{n-h+1}(x)) = x$  if and only if  $x = \kappa_g$  (because  $n-h+1 = g$ ). Therefore,  $\theta([0, t_h], b_{-h}) < t_h$  and  $\theta([t_h, 1], b_{-h}) > t_h$  and, hence, the continuity arguments used in case a) guarantee here the existence of an interval  $A^*$  such that  $\theta(b^{A^*}) = t_h$ , which ensures the existence of an equilibrium as the one described in a), which concludes the proof of step A. .

**Step B.: Any equilibrium  $b$  of  $\theta$  satisfies  $\theta(b) = m(t, \kappa)$ .** Suppose that, there is some  $\theta$  that admits an equilibrium  $b$  with  $\theta(b) > m(t, \kappa)$ . We let  $L_m := \{i \in N \mid t_i \leq m(t, \kappa)\}$  and  $F_m := \{j \in \{g, \dots, n-g\} \mid \kappa_j \leq m(t, \kappa)\}$  with  $\#L_m = i'$  and  $\#F_m = j'$ . However, by definition, it must be the case that  $i' + j' \geq n - (g - 1)$  so that  $n - (g - 1) - i' \leq j'$ . Thus,  $\kappa_{n-(g-1)-i'} \leq \kappa_{j'} = m(t, \kappa)$  (i). By Monotonicity, the unique best response for any player in  $S$  equals  $[0, \theta(b)]$  so that  $\theta(b) \leq \kappa_{n-(g-1)-i'}$  (ii). Combining both (i) and (ii), it follows that  $\theta(b) \leq m(t, \kappa)$  a contradiction with  $\theta(b) > m(t, \kappa)$ . A symmetric claim delivers also a contradiction whenever  $\theta(b) < m(t, \kappa)$ , proving that  $\theta(b) = m(t, \kappa)$  as wanted.

**Step C.: There exists some equilibrium  $b$  of  $\theta$  with  $\bigcap_{i=1}^n b_i = \theta(b)$ .** Note that by construction, the equilibrium built in Step A.I satisfies this claim. In Step A.II,  $n-1$  players announce  $t_h$  in their equilibrium strategy. Hence it suffices to show that there is some  $A^*$  with  $t_h \in A^*$ . If player  $h$  plays  $[0, t_h]$ , the outcome is lower than  $t_h$  whereas if he plays  $[t_h, 1]$  then the outcome is higher than  $t_h$  as proved in Step A.II. Observe that if  $h$  plays  $[\underline{c}, \bar{c}]$  and we start from  $\underline{c} = 0$  and  $\bar{c} = t_h$  and first we start increasing  $\bar{b}$  from  $t_h$  to 1 and then  $\underline{c}$  from 0 to  $t_h$ , we should have i) always  $t_h$  is included in the interval  $[\underline{c}, \bar{c}]$  and ii) at some point due to the continuity of outcome in  $\underline{c}$  and  $\bar{c}$  we should have the outcome being equal to  $t_h$ .

To see why all these steps hold for the case in which  $g = \frac{n+1}{2}$ , notice first that in such a case a)  $n$  must be odd and b)  $t_h = m(t_1, t_2, \dots, t_n)$  if and only if  $h = \frac{n+1}{2}$ . Then observe that from *FP* we know that  $\theta(b^{\frac{n-1}{2}}(t_h)) < t_h$  and  $\theta(b^{\frac{n+1}{2}}(t_h)) > t_h$ . That is, if players behave according to the profile  $b^A$  as presented in step A.II. for  $h = \frac{n+1}{2}$ , there must exist a strategy  $A \in C(\mathcal{B})$  such that  $\theta(b^A) = t_h$ . This establishes existence of an equilibrium  $b$  of  $\theta$  with  $\theta(b) = m(t_1, t_2, \dots, t_n)$ . The arguments that establish uniqueness of equilibrium outcome and existence of a unanimous equilibrium are trivial extensions of steps B and C respectively. **Q.E.D.**

As a by-product of the previous Theorem and the continuity axiom, we can establish the following interesting property of Approval mechanisms: these mechanisms are partially revealing in the sense that any player always has a best response in which he approves of his peak  $t_i$ .

**Lemma 2** (Partially Revealing). *Let  $\theta : \mathcal{B}^n \rightarrow A$  satisfy C, FP, MON and IC. For any approval profile  $b$  and any  $i \in N$ , there is some best response  $b_i \in \mathcal{B}$  with  $t_i \in b_i$ .*

The proof is an immediate consequence of Monotonicity whenever  $\theta(b) \neq t_i$ . If  $\theta(b) = t_i$ , the claim is a consequence of  $\theta$  being a deterministic mechanism and of the different axioms.

**4.3. Feasibility.** We now state feasible conditions for unanimous implementation. As we now show the axioms defined in the necessity part are not vacuous in the sense that for any strategy-proof direct mechanism, there exists some Approval mechanism that unanimously implements it.

Let  $Z : [0, 1] \rightarrow [0, 1]$  be a differentiable and strictly increasing function and  $q$  a non-negative real number with  $Z(0) = 0$  and  $Z(1) = 1$ . For any  $b \in \mathcal{B}^n$ , consider the approval mechanism  $\theta_{q,Z}$  such that:

a) if all voters submit singletons then the median report of the singletons is implemented so that

$$\theta_{q,Z}(b) = m(b_1, \dots, b_n) \quad \text{if } b_i \in S(\mathcal{B}) \quad \forall i \in N,$$

b) otherwise, there are  $m \geq 1$  voters who submit a positive length interval. In this case, we let the density function  $f_b^{q,Z}$  be such that:

$$f_b^{q,Z}(x) = \sum_{b_i \in C(b,x)} \frac{\left(\frac{q}{\bar{b}_i - \underline{b}_i} + Z'(x)\right)}{q \times m + \sum_{i \in N} (Z(\bar{b}_i) - Z(\underline{b}_i))}, \quad \text{for every } x \in [0, 1],$$

where  $C(b, x) := \{b_i \in b \mid b_i \in C(\mathcal{B}) \text{ and } x \in b_i\}$ . For each such  $b$ ,

$$\theta_{q,Z}(b) := \min\{x \in [0, 1] \mid \int_0^x f_b^{q,Z}(t) dt = \frac{1}{2}\},$$

so that  $\theta_{q,Z}$  selects as an outcome the median of the distribution function generated by  $f_b^{q,Z}$ . Each mechanism  $\theta_{q,Z}$  is characterized by the distribution function  $f_{q,Z}$  and is called a Generalized Median Approval Mechanism (GMAM).

To see which sort of aggregators are included within this family, we let, for instance,  $q = 0$  and  $Z(x) = x$ , so that

$$f_b^{q,Z}(x) = \frac{\#C(b,x)}{\sum_{i \in N} (\bar{b}_i - \underline{b}_i)} \quad \text{for every } x \in [0, 1].$$

This density function just differs from the one associated with the Median Approval mechanism, described in Section 3, in zero-measure sets, i.e. the singleton strategies used by the players in  $N \setminus C(b, x)$ . To see why, notice that for any approval profile  $b$  and any alternative  $x$ ,  $\#C(b, x) = s^x(b)$  whenever any  $b_i \in C(\mathcal{B})$ . Thus, it leads to the same outcome as the Median Approval mechanism.

As in Lemma 1, one can prove that any  $f_b^{q,Z}$  is a well-defined density function.



**Lemma 3.** For any approval profile  $b = (b_i, b_{-i})$ , any non-negative  $q$  and any  $Z : [0, 1] \rightarrow [0, 1]$ ,  $f_b^{q,Z}$  is a well-defined density function.

**Proof.** Take any  $f_b^{q,Z}$  and note first that  $f_b^{q,Z}(x) \geq 0$  for any  $x \in [0, 1]$ . It suffices to show that its integral over  $[0, 1]$  equals 1, which is equivalent to

$$\int_0^1 f_b^{q,Z}(x) dx = \int_0^1 \sum_{b_i \in C(b,x)} \frac{\left(\frac{q}{b_i - b_i} + Z'(x)\right)}{q \times m + \sum_{i \in N} (Z(\bar{b}_i) - Z(\underline{b}_i))} dx = 1.$$

Since  $f_b^{q,Z}(x) \geq 0$  for any  $x \in [0, 1]$ , we can express the integral of the sums as the sums of the integrals so that

$$\begin{aligned} \int_0^1 f_b^{q,Z}(x) dx &= \sum_{b_i \in C(b,x)} \int_{\underline{b}_i}^{\bar{b}_i} \frac{\left(\frac{q}{b_i - b_i} + Z'(x)\right)}{q \times m + \sum_{i \in N} (Z(\bar{b}_i) - Z(\underline{b}_i))} dx \\ &= \sum_{b_i \in C(b,x)} \frac{1}{q \times m + \sum_{i \in N} (Z(\bar{b}_i) - Z(\underline{b}_i))} \int_{\underline{b}_i}^{\bar{b}_i} \left(\frac{q}{b_i - b_i} + Z'(x)\right) dx = 1, \end{aligned}$$

which concludes the proof. Q.E.D..

The vector of fixed points of each mechanism  $\theta_{q,Z}$  is denoted by  $\kappa_j^{q,Z}$  and is defined as follows. Recall that for any  $j = 0, 1, \dots, n$  the strategy profile  $b^j(x) \in \mathcal{B}^n$  is the one in which  $n-j$  players use the strategy  $[0, x]$  and  $j$  players use the strategy  $[x, 1]$ . We let

$$\begin{aligned} \kappa_j^{q,Z} &:= \{x \in A \mid \theta_{q,Z}(b^j(x - \varepsilon)) > \theta_{q,Z}(b^j(x)) = x > \theta_{q,Z}(b^j(x + \varepsilon)) \text{ for any } \varepsilon > 0\} \\ &\text{for any } j \in \{0, 1, \dots, n\}. \end{aligned}$$

Therefore, we have for any  $j$ ,

$$\frac{(n-j)Z(\kappa_j^{q,Z}) + (n-j)q}{qn + (n-j)Z(\kappa_j^{q,Z}) + j[1 - Z(\kappa_j^{q,Z})]} = 1/2 \Leftrightarrow Z(\kappa_j^{q,Z}) = \frac{j+q(2j-n)}{n}.$$

**Proposition 1.** Any GMAM satisfies IC, MON, C and FP.

While the formal proof of this proposition is relegated to the appendix, we not briefly mention the intuition for these axioms to be satisfied by any GMAM. IC holds since a player submitting a singleton has no weight in the final decision. MON is satisfied since the maximal influence on the outcome of the mechanism is to include all the alternatives located to the left (to minimize it) or the right of the outcome (to maximize it). C holds since as long as one has convex support, the mechanism is implementing the median of a cumulative distribution. Finally,

*FP* holds almost by construction: indeed, the *GMAMs* are designed to exhibit trackable phantoms.

We are now ready to state the main result of this section.

**Theorem 2.** *If the number of voters is even (resp. odd), for any strategy proof mechanism  $m(t, \kappa)$  with  $p$  different phantoms for any odd  $p$  (resp. even), there exists some *GMAM* that unanimously implements it.*

**Proof.** By definition, each *GMAM*  $\theta_{q,Z}$  is characterized by a function  $Z$  and some non-negative number  $q$ . Moreover, since every  $\theta_{q,Z}$  satisfies our four axioms, each  $\theta_{q,Z}$  unanimously implements a strategy-proof rule  $f$  as stated by Theorem 1. Again due to Theorem 1, the game associated to  $\theta_{q,Z}$  has a unique equilibrium outcome characterized his vector of fixed points  $\kappa^{q,Z} = (\kappa_1^{q,Z}, \dots, \kappa_{n-1}^{q,Z})$ . The equilibrium outcome is hence equal to  $m(t, \kappa^{q,Z})$ . Note that each strategy-proof rule  $f = m(t, \kappa)$  is uniquely determined by  $\kappa = (\kappa_1, \dots, \kappa_{n-1})$ . Therefore, in order to establish validity of this theorem, it is sufficient to show that for each  $\kappa$ , there exists an admissible  $\theta_Z$  with a corresponding  $\kappa^Z = \kappa$ . As previously argued the fixed points of  $\theta_{q,Z}$  satisfy the following equation:

$$Z(\kappa_j^{q,Z}) = \frac{j+q(2j-n)}{n}.$$

Note that  $Z$  is invertible since, by definition,  $Z(x)$  is differentiable and strictly increasing on  $[0, 1]$  with  $Z(0) = 0$  and  $Z(1) = 1$ . Thus, for each  $\kappa$  with  $n-1$  distinct weights, there exists at least one *GMAM*  $\theta_{q,Z}$  with

$$\kappa^{q,Z} = (\kappa_1^{q,Z}, \dots, \kappa_{n-1}^{q,Z}) \text{ such that } \kappa_j^{q,Z} = Z^{-1}\left(\frac{j+q(2j-n)}{n}\right),$$

for each  $j \in \{1, \dots, n-1\}$ . In other words, for each  $\kappa$ , there exists an admissible  $\theta_{q,Z}$  with a corresponding  $\kappa^{q,Z} = \kappa$ .

Concerning the fixed points vector  $\kappa$  with less than  $n-1$  points, note that if  $j < n/2$  then  $\frac{j+q(2j-n)}{n} \leq 0$  if and only if  $q \geq \frac{j}{n-2j}$  and if  $j > n/2$  then  $\frac{j+q(2j-n)}{n} \geq 1$  if and only if  $q \geq \frac{n-j}{2j-n}$ . By increasing the  $q$ , one can trim any arbitrary number of extreme  $\kappa^{q,Z}$ s and then by appropriately choosing  $Z$ , one can give to each of the non-trimmed  $\kappa^{q,Z}$ s any value in  $[0, 1]$ , which concludes the proof. **Q.E.D.**

As an illustration of the previous result, consider the *GMAM* triggered by setting  $q = 1$  and  $Z(x) = x$ . In this case, the density function equals

$$f_b^{q,Z}(x) = \frac{\#C(b,x) + \sum_{b_i \in C(b,x)} \left(\frac{q}{b_i - b_i}\right)}{q \times m + \sum_{i \in N} (\bar{b}_i - b_i)} \text{ for every } x \in [0, 1].$$

In the particular situation in which  $N = \{1, 2, 3\}$  (so that  $n = 3$ ) and the voters' types satisfy  $t_1 < t_2 < t_3$ , this Approval mechanism implements the pure median rule since:

$$Z(\kappa_j^{q,Z}) = \frac{j+q(2j-n)}{n} \Leftrightarrow \kappa_j^{q,Z} = \frac{3j-3}{3} \Leftrightarrow \kappa_1^{q,Z} = 0 \text{ and } \kappa_2^{q,Z} = 1.$$

To see why this is true, consider the strategy profile  $b$  with  $b_1 = b_2 = [0, t_2]$  and  $b_3 = [t_2, 1]$ . Then:

$$\theta_{q,Z}(b) = \frac{2t_2+2}{3+2t_2+1-t_2} > \frac{1}{2} \text{ for every } t_2 \in (0, 1).$$

If player 2 deviates to  $b'_2 = [t_2, 1]$ , then the outcome of  $b' = (b_1, b'_2, b_3)$  equals:

$$\theta_{q,Z}(b') = \frac{t_2+1}{3+t_2+2(1-t_2)} < \frac{1}{2} \text{ for every } t_2 \in (0, 1).$$

That is, if the  $t_1$ -voter plays  $[0, t_2]$  and the  $t_3$ -voter plays  $[t_2, 1]$  then the  $t_2$ -voter by smoothly changing her strategy from  $[0, t_2]$  to  $[t_2, 1]$  can find a strategy that contains  $t_2$  and which leads to  $F_{q,Z}(t_2) = 1/2$ , that is, to the unanimous implementation of her ideal policy.

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#### APPENDIX A. THE CLASS OF GMAM

To describe the class of GMAMs, we start by describing the cumulative distribution of the median approval mechanism described in Section 3.

Suppose that there is just one player in the society so that  $n = 1$ . If he selects the interval  $b_i = [\min b_i, \max b_i]$ , the share of approvals lower than  $x$  equals  $\frac{F(b_i, x)}{\max b_i - \min b_i}$  with

$$F(b_i, x) = \begin{cases} 0, & \text{if } x < \min b_i. \\ x - \min b_i, & \text{if } \max b_i \leq x \leq \min b_i. \\ \max b_i - \min b_i, & \text{if } x > \max b_i. \end{cases}$$

For any  $b = (b_i, b_{-i})$ , the share of approvals until  $x$  equals

$$F(b, x) = \frac{\sum_{i \in N} F(b_i, x)}{\sum_{i \in N} (\max b_i - \min b_i)},$$

and since  $\theta$  is the Median Approval mechanism,

$$\theta(b) := \min\{x \in [0, 1] \mid F(b, x) = \frac{1}{2}\}.$$

The same logic applies to a Generalized Median Approval Mechanism. Let  $q$  be a non-negative real number and  $Z : [0, 1] \rightarrow [0, 1]$  a continuous and strictly increasing function with  $Z(0) = 0$  and  $Z(1) = 1$ . If there is just one player ( $n = 1$ ) and he selects the interval  $b_i = [\min b_i, \max b_i]$ , so that the share of approvals lower than  $x$  given  $q$  and  $Z$  equals  $\frac{F_{q,Z}(b_i, x)}{Z(\max b_i) - Z(\min b_i) + q}$  with

$$F_{q,Z}(b_i, x) = \begin{cases} 0, & \text{if } x < \min b_i. \\ Z(x) - Z(\min b_i) + \frac{(x - \min b_i)}{(\max b_i - \min b_i)} q, & \text{if } \max b_i \leq x \leq \min b_i. \\ Z(\max b_i) - Z(\min b_i) + q, & \text{if } x > \max b_i. \end{cases}$$

For any  $b = (b_i, b_{-i})$  with  $b_i \in C(\mathcal{B})$  for every  $i \in N$ , the share of approvals lower than  $x$  equals

$$F_{q,Z}(b, x) = \frac{\sum_{i \in N} F_{q,Z}(b_i, x)}{n \times q + \sum_{i \in N} (Z(\max b_i) - Z(\min b_i))}.$$

The outcome  $\theta_{q,Z}$  is the median of this cumulative distribution.

For ease of exposition, we introduce the following notation and focus on profiles with  $b_i \in C(\mathcal{B})$  for every  $i \in N$ . The similar argument applies if some player(s) select singletons.

For any  $b$  with convex support, we let  $\eta_{-i}(b, q, Z) := (n-1) \times q + \sum_{j \in N \setminus \{i\}} (Z(\max b_j) - Z(\min b_j))$ . Note that

$$F_{q,Z}(b, x) = \frac{Z(x) - Z(\min b_i) + \frac{(x - \min b_i)}{(\max b_i - \min b_i)} q + \sum_{j \neq i} F_{q,Z}(b_j, x)}{Z(\max b_i) - Z(\min b_i) + q + \eta_{-i}(b, q, Z)}.$$

Since  $F_{q,Z}(b_j, x)$  is a cumulative distribution, note that

$$\sum_{j \neq i} F_{q,Z}(b_j, x) < \eta_{-i}(b, q, Z)$$

for any  $b, q, Z$  and any  $i \in N$ . The next proposition show how the outcome varies when a player varies the lower and upper bound of his strategy.

**Lemma 4.** *Consider any profile  $b = (b_i, b_{-i})$  with convex support. Then,*

(1) *if  $\min b_i < \max b_i < \theta(b)$ , then*

$$\frac{\partial}{\partial \min b_i} \theta(b_i, b_{-i}) > 0 \text{ and } \frac{\partial}{\partial \max b_i} \theta(b_i, b_{-i}) < 0.$$

(2) *if  $\min b_i < \theta(b) < \max b_i$ , then*

$$\frac{\partial}{\partial \min b_i} \theta(b_i, b_{-i}) > 0 \text{ and } \frac{\partial}{\partial \max b_i} \theta(b_i, b_{-i}) > 0.$$

(3) if  $\theta(b) < \min b_i < \max b_i$ , then

$$\frac{\partial}{\partial \min b_i} \theta(b_i, b_{-i}) < 0 \text{ and } \frac{\partial}{\partial \max b_i} \theta(b_i, b_{-i}) > 0.$$

**Proof.** Consider first the case with  $\min b_i < \max b_i < \theta(b)$ . Consider  $x$  such that  $F_{q,Z}(b, x) = 1/2$ . Note that

$$\frac{\partial}{\partial \max b_i} F_{q,Z}(b_i, x) = \frac{\left( \eta_{-i}(b, q, Z) - \sum_{j \neq i} F_{q,Z}(b_j, x) \right) Z'(\max b_i)}{\left( Z(\max b_i) - Z(\min b_i) + \eta_{-i}(b, q, Z) + q \right)^2} > 0.$$

That is as  $\max b_i$  increases  $\theta(b)$  has to decrease for the median to be still equal to  $\frac{1}{2}$  so that  $\frac{\partial}{\partial \max b_i} \theta(b_i, b_{-i}) < 0$ , as wanted. As far as varying the lower bound of  $b_i$ , notice that

$$\frac{\partial}{\partial \min b_i} F_{q,Z}(b_i, x) = \frac{\left( \sum_{j \neq i} G_{q,Z}(b_j, x) - \eta_{-i}(b, q, Z) \right) Z'(\min b_i)}{\left( Z(\max b_i) - Z(\min b_i) + \eta_{-i}(b, q, Z) + q \right)^2} < 0.$$

Again, since  $\min b_i$  increases  $\theta(b)$  has to increase for the median to be still equal to  $\frac{1}{2}$  so that  $\frac{\partial}{\partial \min b_i} \theta(b_i, b_{-i}) > 0$ , as wanted. The case in which  $\theta(b) < \min b_i < \max b_i$  is symmetric and hence is omitted.

Consider now the case with  $\min b_i < \theta(b) < \max b_i$ . One can check that

$$\frac{\partial}{\partial \max b_i} \left( Z(x) - Z(\min b_i) + q \frac{x - \min b_i}{\max b_i - \min b_i} + \sum_{j \neq i} F_{q,Z}(b_j, x) \right) = \frac{q(\min b_i - x)}{(\min b_i - \max b_i)^2} < 0,$$

whereas

$$\frac{\partial}{\partial \max b_i} \left( Z(\max b_i) - Z(\min b_i) + q + \eta_{-i}(b, q, Z) \right) = Z'(\max b_i) > 0.$$

Thus,

$$\frac{\partial}{\partial \max b_i} F_{q,Z}(b_i, x) < 0,$$

so that as  $\max b_i$  increases  $x$  has to increase, showing that so that  $\frac{\partial}{\partial \max b_i} \theta(b_i, b_{-i}) > 0$ , as wanted. Symmetrically one can show that as  $\min b_i$  increases  $\theta(b)$  has to increase for  $F_{q,Z}(b, x)$  to be still equal to  $\frac{1}{2}$ . **Q.E.D.**

Once we have proved this key property of GMAM, we prove that each GMAM satisfies the different axioms used in the characterization.

**Lemma 5.** *Any GMAM satisfies IC.*

**Proof.** Take some  $b$  with  $\theta(b) \neq t_i$  and  $b_i \in C(\mathcal{B})$ . Let  $t_i < \theta(b)$  w.l.o.g. Applying Lemma 4, it is simple to see that  $\theta([t_i, t_i + \delta], b_{-i}) < \theta(b)$  so that  $\exists c_i \in C(\mathcal{B})$  with  $u_i(\theta(c_i, b_{-i})) > u_i(\theta(b))$ , as desired. **Q.E.D.**

**Lemma 6.** *Any GMAM satisfies MON.*

**Proof.** We now prove that for any GMAM  $\theta$ , the equivalence (1) holds. A similar proof applies to the characterization of the maximum of the attainable set.

**1. Sufficiency.** Take some  $i \in N$  and assume that there is some  $b_i^* \in \arg \min \theta(\mathcal{B}, b_{-i})$  with  $b_i^* \neq b_i^m$ . Let  $x^* = \theta(b_i^*, b_{-i})$ . Since  $b_i^* \neq b_i^m$ , this means that either  $b_i^* \cap [0, x^*] \neq \emptyset$  (1.a.) or  $b_i^* \cap [x^*, 1] \neq \emptyset$  (1.b) or both (1.c). In each of these cases, Lemma 4 directly implies that  $\theta([0, x^*], b_{-i}) < \theta(b_i^*, b_{-i})$ , a contradiction with  $b_i^* \in \arg \min \theta(\mathcal{B}, b_{-i})$ .

**2. Necessity.** Take some  $i \in N$  and assume that there is some  $b_i^m$  with  $b_i^m = [0, x_i^m]$  and  $x_i^m = \theta([0, x^m], b_{-i})$ . Assume that  $b_i^m \notin \arg \min \theta(\mathcal{B}, b_{-i})$ , so that there is some  $b_i^*$  with  $\theta(b_i^*, b_{-i}) < \theta([0, x^m], b_{-i})$ . By definition, it must be the case that this means that either  $b_i^* \cap [0, x_i^m] \neq \emptyset$  or  $b_i^* \cap [x_i^m, 1] \neq \emptyset$  or that both inequalities hold simultaneously. However, Lemma 4 again directly. proves that for any  $b_i^* \in \mathcal{B}$ ,  $\theta(b_i^*, b_{-i}) \geq \theta([0, x^m], b_{-i})$ , entailing a contradiction. **Q.E.D.**

**Lemma 7.** *Any GMAM satisfies C.*

**Proof. :** Take any GMAM  $\theta$  with density function  $f_b$ . Take some  $i \in N$  and any pair  $b, b^m \in B^n$  with  $b^m = (b_i^m, b_{-i})$  such that  $\text{Supp}(b), \text{Supp}(b^m) \in C(\mathcal{B})$  Assume moreover that  $\lim_{m \rightarrow \infty} b_i^m = b_i$ . It follows that

$$\lim_{m \rightarrow \infty} f_{b^m}(x) = f_b(x) \quad \text{for any } x \in \text{Supp}(b).$$

We let  $F_b(x)$  and  $F_{b^m}(x)$  respectively denote the cumulative distribution of  $f_b$  and  $f_{b^m}$ . Since  $\text{Supp}(b), \text{Supp}(b^m) \in C(\mathcal{B})$ ,  $F_b(x)$  and  $F_{b^m}(x)$  are strictly increasing and continuous (hence invertible) on  $\text{Supp}(b)$  and  $\text{Supp}(b^m)$ . The respective inverse functions are denoted by  $F_b^{-1} : [0, 1] \rightarrow \text{Supp}(b)$  and  $F_{b^m}^{-1} : [0, 1] \rightarrow \text{Supp}(b^m)$ . Therefore,

$$\lim_{m \rightarrow \infty} F_{b^m}^{-1}(x) = F_b^{-1}(x) \quad \text{for any } x \in \text{Supp}(b).$$

Since for any  $b$  with invertible  $F_b$ ,  $\theta(b) = F_b^{-1}(\frac{1}{2})$ , it follows that  $\lim_{m \rightarrow \infty} \theta(b^m) = \theta(b)$ , as wanted. **Q.E.D.**

**Lemma 8.** *Any GMAM satisfies FP.*

**Proof.** The vector of fixed points of each mechanism  $\theta_{q,Z}$  is denoted by  $\kappa^{q,Z}$  and is defined as follows. Recall that for any  $j = 0, 1, \dots, n$  the strategy profile  $b^j(x) \in \mathcal{B}^n$  is the one in which  $n-j$  players use the strategy  $[0, x]$  and  $j$  players use the strategy  $[x, 1]$ . We let

$$\kappa_j^{q,Z} := \{x \in A \mid \theta_{q,Z}(b^j(x - \varepsilon)) > \theta_{q,Z}(b^j(x)) = x > \theta_{q,Z}(b^j(x + \varepsilon))\}$$

for any  $\varepsilon > 0$  for any  $j \in \{0, 1, \dots, n\}$ .

Each such  $\kappa^{q,Z}$  must satisfy for any  $j \in \{1, \dots, n-1\}$ ,  $\kappa_j^{q,Z}$  the following equivalence:

$$\frac{Z(\kappa_j^{q,Z})^{\times(n-j)+(n-j)\times q}}{q \times n + Z(\kappa_j^{q,Z})^{\times(n-j)+[1-Z(\kappa_j^{q,Z})]^{\times j}} = 1/2 \Leftrightarrow Z(\kappa_j^{q,Z}) = \frac{j+q(2j-n)}{n}.$$

Note that by assumption  $Z$  is continuous and strictly increasing. It is hence invertible so that for any  $j$

$$\kappa_j^{q,Z} = Z^{-1}\left(\frac{j+q(2j-n)}{n}\right).$$

If  $j < n/2$  then  $\frac{j+q(2j-n)}{n} \leq 0$  if and only if  $q \geq \frac{j}{n-2j}$  and if  $j > n/2$  then  $\frac{j+q(2j-n)}{n} \geq 1$  if and only if  $q \geq \frac{n-j}{2j-n}$ . By increasing the  $q$  we can "trim" any arbitrary number of extreme  $\kappa^{q,Z}$ s. Moreover, by appropriately choosing  $Z$ , we can give to each of the non-trimmed  $\kappa^{q,Z}$ s any value between zero and one as required. **Q.E.D.**