Spectral rigidity of group actions on homogeneous spaces

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Abstract

Actions of a locally compact group G on a measure space X give rise to unitary representations of G on Hilbert spaces. We review results on the rigidity of these actions from the spectral point of view, that is, results about the existence of a spectral gap for associated averaging operators and their consequences. We will deal both with spaces X with an infinite measure as well as with spaces with an invariant probability measure. The spectral gap property has several striking applications to group theory, geometry, ergodic theory, operator algebras, graph theory, theoretical computer science, etc.

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1 Introduction

Let G be a separable locally compact group. The study of actions of G on various spaces is of course of fundamental importance, both for the understanding of properties of the groups and the spaces under consideration. When G acts on such a space X, which might be a manifold or a graph, there is often a positive measure m on the measurable subsets of X which is quasi-invariant under G and sometimes even invariant. We will consider two kinds of actions, which require different approaches:

(A) the case where $m(X) = \infty$, that is, m is an infinite measure;

(B) the case where m is a finite measure which is G-invariant.

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In case (B), we may of course assume that m is a probability measure. Attached to these data, there is a natural unitary representation π_X of G on the Hilbert space $L^2(X,m)$; see Section 2. In case m is a probability measure and is G-invariant, the space $\mathbf{C1}_X$ of the constant functions on X is contained in $L^2(X,m)$ and is G-invariant as well as its orthogonal complement

$$L^2_0(X,m) = \left\{ f \in L^2(X,m) \,:\, \int_X f(x)dm(x) = 0 \right\}.$$

In this survey, we will be concerned with the study of a spectral rigidity property for actions $G \curvearrowright X$, in each of the two situations (A) and (B) above. We say that the action of G on (X,m) has the Spectral Gap Property if the representation π_X on $L^2_0(X,m)$ does not have almost invariant vectors, that is, if there is no sequence of unit vectors ξ_n in $L^2_0(X,m)$ such that $\lim_n \|\pi_X(g)\xi_n - \xi_n\| = 0$ for all g in G. Here, we have denoted the space $L^2(X,m)$ by $L^2_0(X,m)$ in the case where m is an infinite measure.

Two classes of groups are distinguished with respect to this Spectral Gap Property: amenable groups and Kazhdan groups. Indeed, an amenable group G never has the Spectral Gap Property (in case (B), we have to assume that G is countable and X non-atomic) and Kazhdan groups always have the Spectral Gap Property for ergodic actions (see Corollary 5.16 and Theorem 7.1 as well as Corollary 5.17). We will first recall some basic facts on these classes of groups. Then, we turn to the negation of the Spectral Gap Property in case (A) above. Actions $G \curvearrowright X$ without the Spectral Gap Property are called *co-amenable,* as they are characterized by the existence of an invariant mean on $L^\infty(X,m)$. Such actions have been studied, with various degrees of generality, by several authors (see [Eyma72], [Gree69], [Guiv80], among others). We will give a comprehensive account about the main characterizations of such actions in Section 5. In the case where G is a non compact simple Lie group, the co-amenable proper subgroups H (that is, such that $G \curvearrowright G/H$ is co-amenable) are Zariski dense in G and are characterized as the discrete subgroups with the maximal critical exponent (Theorems 5.22 and 5.27).

Next, we will deal with the Spectral Gap Property in case (B) above, that is, for actions with an invariant probability measure. We will review some specific examples for actions with the Spectral Gap Property on a homogeneous $X = H/\Lambda$, where Λ is a locally compact group and Λ is a lattice in H; here the action of G on H/Λ is given by left translations or by automorphisms, that is, by a homomorphism $G \to H$ or a homomorphism $G \to \mathrm{A}ut(X)$, where $\mathrm{A}ut(X)$ is the subgroup of continuous automorphisms φ of H such that $\varphi(\Lambda) = \Lambda$. Apart from a few exceptions (such as Bernoulli actions), showing the Spectral Gap Property for these actions is usually a difficult problem. As a novelty, we establish by completely elementary means (using an idea from [BeLu11]) the Spectral Gap Property for the action of $G = PGL_2(\mathbf{F}_q((t^{-1})))$ on $X = PGL_2(\mathbf{F}_q((t^{-1})))/PGL_2(\mathbf{F}_q[t])$, where $\mathbf{F}_q$ is the finite field with q elements and $\mathbf{F}_q((t^{-1}))$ the local field of Laurent series (see Section 8.3). As a crucial tool in our approach, we prove and use a Cheeger type inequality for Markov chains established in [LaSo88] and [SiJe89].

We approach the Spectral Gap Property mainly in terms of averaging opera-

tors, also known as Markov operators. Let μ be a probability measure on G and $\pi_X(\mu)$ the convolution operator defined on $L^2_0(X,m)$ by

$$\pi_X(\mu)f = \int_G \pi_X(g)f d\mu(g) \qquad \text{for all} \quad f \in L^2_0(X,m).$$

We have $r_{spec}(\pi_X(\mu)) \leq 1$ for the spectral radius $r_{spec}(\pi_X(\mu))$ of $\pi_X(\mu)$. Assume that μ is adapted, that is, the support $\mathrm{supp}(\mu)$ of μ generates a dense subgroup of G. Then the action of G on X has the Spectral Gap Property if and only if $r_{spec}(\pi_X(\mu)) < 1$.

The point of view of Markov operators is relevant for applications in probability theory. Given an action $G \curvearrowright (X,m)$ and a probability measure μ on G, consider a sequence of independent μ-distributed random variables X_n with values in G and the corresponding random products $S_n = X_n \dots X_1$ for $n \in \mathbf{N}$. This defines a random walk on X, given by the transitions probabilities

$$A \mapsto p(x,A) = \int_G \mathbf{1}_A(g^{-1}x)d\mu(g) = (\pi_X(\mu)\mathbf{1}_A)(x),$$

for x in X and A a measurable subset of X.

The Spectral Gap Property has several interesting applications in ergodic theory; for instance, if m is a probability measure, then, for every $f \in L^2(X,m)$, the sequence of functions $x \mapsto \mathbf{E}(f(S_n(x)))$ converges to $\int_X f dm$ in the L^2-norm, with an exponentially fast rate of order λ^n with $\lambda = \|\pi_X(\mu)\|$. Other ergodic theoretic applications to random walks (see [CoGu13], [CoLe11], [FuSh99], [GoNe10] and [Guiv15]) include the rate of convergence in the random ergodic theorem, pointwise ergodic theorems, analogues of the law of large numbers and of the central limit theorem, etc. Another application of the Spectral Gap Property is the uniqueness of m as G-invariant mean on $L^\infty(X,m)$; for this as well as for further applications, see [BHV], [BoGa10],[Lubo94], [Popa08], [Sarn90]. To illustrate the use of the Spectral Gap Property in both situations (A) and (B), we present in the last section of this survey (Section 10) two such applications: one to expanders graphs and one to the escape rate of random matrix products. We also discuss (Section 6) the question of quantifying the Spectral Gap Property, that is, giving upper bounds for the norm or the spectral radius of the Markov operator $\pi_X(\mu)$.

2 Group actions on measure spaces and associated representations

Let G be locally compact group, which we always assume to be second countable. We will then say that G is a *separable locally compact group*. The group G has an (essentially unique) Haar measure λ, that is, a σ-finite measure λ on the Borel subsets of G which is invariant under left translations.

Let (X,m) be a measure space, where m is a positive σ-finite measure on a fixed σ-algebra of subsets of X. We will only consider actions of G of X which are measurable, that is, actions $G \curvearrowright X$ for which

$$G \times X \to X \quad (g,x) \mapsto gx$$

is measurable.

We will always assume that the measure m is *quasi-invariant:* m and its image gm under g are equivalent measures (that is, m and gm have the same sets of measure 0) for every $g \in G$.

Our actions will usually be ergodic group actions.

Definition 2.1. The action $G \curvearrowright X$ on the measure space (X, m) is *ergodic* if there are no nontrivial invariant subsets of X in the following sense: if A is a measurable subset of X such that $gA = A$ for all $g \in G$, then $m(A) = 0$ or $m(X \setminus A) = 0$.

Ergodicity can be expressed in terms of functions on X as follows. We say that a measurable function $f : X \to \mathbf{R}$ is G-invariant if, for every $g \in G$, we have $f(gx) = f(x)$ for m-almost every $x \in X$. An action $G \curvearrowright X$ is ergodic if and only if every G-invariant function is constant m-almost everywhere (see Theorem 1.3 in [BeMa00]).

2.1 Examples of group actions on measure spaces

We list some examples of group actions, which will appear throughout this survey.

1. Let Γ be a countable group, $X = \{0,1\}^\Gamma$, and m the probability measure $m = \bigotimes_{\gamma\in\Gamma} \nu$ for the measure ν on $\{0,1\}$ given by $\nu(\{0\}) = \nu(\{1\}) = 1/2$. The Bernoulli action of Γ is the measure preserving action $\Gamma \curvearrowright X$ defined by shifting coordinates:

$$\gamma(x_\delta)_{\delta\in\Gamma} = (x_{\gamma^{-1}\delta})_{\delta\in\Gamma}.$$

2. Let H be a separable locally compact group and L a closed subgroup of H. The homogeneous space $X = H/L$ has a unique (up to equivalence) non-zero σ-finite measure m on its Borel subsets, which is quasi-invariant under the action of H by left translations (see [Foll95, (2.59)]). Every subgroup G of H acts on X by left translations.

3. An important special case in the previous example arises when L is a lattice in the locally compact group H, that is, L is a discrete subgroup of H and there exists a H-invariant probability measure m on the Borel subsets of H/L. Every subgroup G of H acts in measure preserving way on $X = H/L$ by left translations. Examples are given by $H = \mathbf{R}^n$ and $L = \mathbf{Z}^n$, in which case $X = \mathbf{T}^n$ is the n-torus or by $H = SL_n(\mathbf{R})$ and $L = SL_n(\mathbf{Z})$, in which case X is the space of unimodular lattices in $\mathbf{R}^n$.

4. Let H, L and m be as in Example 2. Let $\mathrm{A}ut(H)$ be the group of continuous automorphisms of H. The subgroup $\mathrm{A}ut(H/L)$ of $\mathrm{A}ut(H)$, defined by

$$\mathrm{A}ut(H/L) = \{\varphi \in \mathrm{A}ut(H) \,:\, \varphi(L) = L\},$$

acts on H/L, leaving m quasi-invariant. If m is H-invariant and finite, then m is also invariant under $\mathrm{A}ut(H/L)$.

In the example given by $H = \mathbf{R}^n$, $L = \mathbf{Z}^n$ and $X = \mathbf{T}^n$, the group $\mathrm{A}ut(H/L)$ can be identified with $GL_n(\mathbf{Z})$. Other examples arise when H is a nilpotent Lie group and L is a lattice in H (see Section 9).

2.2 Unitary representations associated to actions

Let G a separable locally compact group and $G \curvearrowright X$ an action on a σ-finite measure space (X, m), with m quasi-invariant.

For $g \in G$, denote by $c(g, x) = \dfrac{dgm}{dm}(x)$ the Radon-Nikodym derivative of gm with respect to m. The mapping $\pi_X : G \to \mathcal{B}(L^2(X, m))$, defined by

$$\pi_X(g)f(x) = c(g^{-1}, x)^{1/2} f(g^{-1}x) \qquad \text{for all} \quad f \in L^2(X, m), x \in X,$$

is a continuous unitary representation of G on $L^2(X, m)$, often called the *Koopman representation* associated to $G \curvearrowright X$ (for more details, see [BHV, A.6]).

Assume now that m is a G-invariant probability measure. Then

$$L^2_0(X) = \{f \in L^2(X) \ : \ \int_X f dm = 0\} = (\mathbf{C}\mathbf{1}_X)^\perp$$

is G-invariant. Denote again by π_X the restriction of Koopman representation to $L^2_0(X)$. Then $G \curvearrowright X$ is ergodic if and only if π_X has no non-zero invariant vectors: $L^2_0(X)^G = \{0\}$.

Most of the actions we consider will be mixing in the following sense. The probability measure preserving action $G \curvearrowright X$ is called *mixing* if π_X is a C_0 representation: for all $f_1, f_2 \in L^2_0(X)$, the matrix coefficient

$$C_{f_1,f_2} : G \to \mathbf{C}, \qquad g \mapsto \langle \pi_X(g)f_1, f_2 \rangle$$

belongs to $C_0(G)$, that is, $\lim_{g\to\infty} \langle \pi_X(g)f_1, f_2 \rangle = 0$. Of course, mixing actions are ergodic.

Example 2.2. (i) The Bernoulli action of an infinite countable group Γ on $X = \{0, 1\}^\Gamma$ is mixing.
(ii) Let Γ be a subgroup of $GL_n(\mathbf{Z})$. The action of Γ on the n-torus $\mathbf{T}^n = \mathbf{R}^n/\mathbf{Z}^n$ is ergodic if and only if every Γ^t-orbit in $\mathbf{Z}^n$ is infinite; this action is mixing if and only if every point stabilizer for the action of Γ^t on $\mathbf{Z}^n$ is finite.

The claims in the previous examples follow from the next proposition, which illustrates the power of the use of the Koopman representation. Concerning the first example, observe that we can view $\{0, 1\}^\Gamma$ as the compact abelian group $A = \prod_{\gamma\in\Gamma} \mathbf{Z}/2\mathbf{Z}$ and Γ as a subgroup of $\mathrm{A}ut(A)$.

Proposition 2.3. *Let A be a compact abelian group with normalized Haar measure m and Γ a subgroup of $\mathrm{A}ut(A)$. Let $\widehat{A}$ be the (discrete) dual group of A (on which $\mathrm{A}ut(A)$ acts naturally).*

- *The action $\Gamma \curvearrowright A$ is ergodic if and only if every Γ-orbit in $\widehat{A}\setminus\{1\}$ is infinite;*

- *the action $\Gamma \curvearrowright A$ is mixing if and only if Γ acts properly on $\widehat{A} \setminus \{1\}$ (that is, point stabilizers are finite).*

Proof By Fourier transform, we have isometric isomorphisms $L^2(A) \cong \ell^2(\widehat{A})$ and $L^2_0(A) \cong \ell^2(\widehat{A} \setminus \{1\})$ as Γ-representations.

Now, $\ell^2(\widehat{A} \setminus \{1\})^\Gamma$ consists exactly of the Γ-invariant ℓ^2-functions on $\widehat{A} \setminus \{1\}$. Moreover, the Γ-representation on $\ell^2(\widehat{A}\setminus\{1\})$ is C_0 if and only if all point stabilizers are finite.■

The following result is the celebrated Howe-Moore theorem from [HoMo79], which shows that, as a rule, ergodic actions of simple Lie groups are mixing.

Theorem 2.4. ***(Howe-Moore Theorem)*** *Let G be a connected simple Lie group with finite center and $(\pi, \mathcal{H})$ a unitary representation of G on a Hilbert space $\mathcal{H}$. Assume that $\mathcal{H}^G = \{0\}$. Then π is a C_0-representation: the matrix coefficients*

$$C_{\xi,\eta} : G \to \mathbf{C}, \quad g \mapsto \langle \pi(g)\xi, \eta \rangle$$

belong to $C_0(G)$ for all $\xi, \eta \in \mathcal{H}$.

Here is one striking consequence of the Howe-Moore theorem.

Corollary 2.5. *Let G be as in Theorem 2.4 and Λ a lattice in G. Let H be a subgroup with a non-compact closure in G. Then $H \curvearrowright G/\Lambda$ is ergodic. Morevover, $H \curvearrowright G/\Lambda$ is even mixing if H is closed.*

Proof Set $X = G/\Lambda$. Obviously, we have $L^2_0(X)^G = \{0\}$. Let $f \in L^2_0(X)^H$. Then $f \in L^2_0(X)^{\overline{H}}$. Now, by Theorem 2.4, the matrix coefficient $C_{f,f}$ belongs to $C_0(G)$. Since $\overline{H}$ is not compact, it follows that $f = 0$. ■

3 The Spectral Gap Property

Let G a separable locally compact group and $G \curvearrowright X$ an action on a measure space (X, m), with m quasi-invariant.

We set $L^2_0(X) = (\mathbf{C}\mathbf{1}_X)^\perp$ in case m is a G-invariant probability measure, and $L^2_0(X) = L^2(X, m)$ otherwise. The corresponding unitary representation of G on $L^2_0(X)$ will always be denoted by π_X.

Definition 3.1. (Actions with the Spectral Gap Property) The action of G on X has the *Spectral Gap Property*, if there exists a compact set Q of G and $\varepsilon > 0$ such that

$$\sup_{s \in Q} \|\pi_X(s)f - f\| \geq \varepsilon \|f\| \qquad \text{for all} \quad f \in L^2_0(X).$$

We can extend this definition to arbitrary unitary representations.

Definition 3.2. (Representations with the Spectral Gap Property) A unitary representation $(\pi, \mathcal{H})$ of G has the Spectral Gap Property, if there exists a compact subset Q of G and $\varepsilon > 0$ such that

$$\sup_{s \in Q} \|\pi(s)\xi - \xi\| \geq \varepsilon \|\xi\| \qquad \text{for all} \quad \xi \in \mathcal{H}.$$

Remark 3.3. (i) **(Negation of the Spectral Gap Property)** The unitary representation $(\pi, \mathcal{H})$ of G does not have the Spectral Gap Property if, for every pair (Q, ε), where Q is a compact subset of G and $\varepsilon > 0$, there exists a unit vector unit $\xi \in \mathcal{H}$ which is (Q, ε)-invariant:

$$\sup_{s \in Q} \|\pi(s)\xi - \xi\| < \varepsilon.$$

Since G is σ-compact, observe that $(\pi, \mathcal{H})$ does not have the Spectral Gap Property if and only if there exists a sequence of unit vectors $(\xi_n)_n$ in $\mathcal{H}$ such that

$$\lim_n \|\pi(g)\xi_n - \xi_n\| = 0 \qquad \text{for all} \quad g \in G.$$

(The "if" part of the previous statement follows from a standard Baire category argument.)
(ii) The negation of the Spectral Gap Property may be formulated in terms of Fell's notion of weak containment: π does not have the Spectral Gap Property if and only if the trivial representation 1_G is weakly contained in π. Recall that a unitary representation ρ is said to be *weakly contained* in another unitary representation σ, if every diagonal matrix coefficient $C^{\rho}_{\xi,\xi}$ of ρ can be approximated, uniformly on compact subsets of G, by convex combinations of diagonal matrix coefficients of σ (see Appendix F in [BHV]).

3.1 Spectral Gap Property in terms of averaging operators

Let $(\pi, \mathcal{H})$ be a unitary representation of a locally compact group G and μ a probability measure on the Borel subsets of G. Define the averaging operator $\pi(\mu) \in \mathcal{B}(\mathcal{H})$ by

$$\pi(\mu)\xi = \int_G \pi(g)\xi d\mu(g) \qquad \text{for all} \quad \xi \in \mathcal{H}.$$

Then clearly $\|\pi(\mu)\| \leq 1$ and hence $r_{spec}(\pi(\mu)) \leq 1$ for the spectral radius $r_{spec}(\pi(\mu))$ of $\pi(\mu)$.

We say that μ is *adapted* if the subgroup generated by the support $\mathrm{supp}(\mu)$ of μ is dense in G. We will also consider the stronger condition that $\mathrm{supp}(\mu)$ is not contained in the coset of a proper closed subgroup of G. In this case, we say that μ is *strongly adapted*.

It is easy to see that μ is strongly adapted if and only if the convolution product $\check{\mu} * \mu$ is adapted, where $\check{\mu}$ is defined by $d\check{\mu}(g) = d\mu(g^{-1})$. We say that μ is absolutely continuous if it is absolutely continuous with respect to a Haar measure on G.

Proposition 3.4. *Let $(\pi, \mathcal{H})$ be a unitary representation of the separable locally compact group G. The following statements are equivalent:*

(i) π has the Spectral Gap Property;

(ii) $\|\pi(\mu)\| < 1$ for any (or for some) probability measure μ on G which is strongly adapted and absolutely continuous;

(iii) 1 is not a spectral value of $\pi(\mu)$ for any (or for some) probability measure μ on G which is adapted and absolutely continuous.

Proof We just give the proof of the equivalence of (i) and (ii) in the case where $G = \Gamma$ discrete, and refer to Proposition G.4.2 in [BHV] for the complete proof.

Assume that π does not have the Spectral Gap Property; so, there exists a sequence ξ_n of unit vectors in $\mathcal{H}$ such that

$$\lim_n \|\pi(\gamma)\xi_n - \xi_n\| = 0 \qquad \text{for all} \quad \gamma \in \Gamma.$$

Summing against μ gives

$$\lim_n \|\pi(\mu)\xi_n - \xi_n\| \leq \lim_n \sum_{\gamma\in\Gamma} \mu(\gamma)\|\pi(\gamma)\xi_n - \xi_n\| = 0.$$

So, 1 is in the spectrum of $\pi(\mu)$. Since $\|\pi(\mu)\| \leq 1$, we have $\|\pi(\mu)\| = 1$.

Conversely, assume that $\|\pi(\mu)\| = 1$. Since μ is strongly adapted, $\mathrm{supp}(\check{\mu} * \mu)$ generates Γ. Now,

$$\|\pi(\check{\mu} * \mu)\| = \|\pi(\mu)^*\pi(\mu)\| = \|\pi(\mu)\|^2 = 1$$

and $\pi(\check{\mu} * \mu)$ is a positive self-adjoint operator. Hence, 1 is a spectral value of $\pi(\check{\mu} * \mu)$ and there exists a sequence η_n in $\mathcal{H}$ of approximate eigenvectors, that is, a sequence of unit vectors η_n with $\lim_n \|\pi(\check{\mu} * \mu)\eta_n - \eta_n\| = 0$. Then

$$\lim_n \|\pi(\gamma)\eta_n - \eta_n\| = 0$$

for all $\gamma \in \mathrm{supp}(\check{\mu} * \mu)$ and therefore for all $\gamma \in \Gamma$.■

3.2 Kazhdan's Property (T)

We review a few basic facts on Kazhdan groups, with an emphasis on the existence of a spectral gap for averaging operators. We refer to the monograph [BHV] for missing details.

Definition 3.5. ([Kazh67]) A locally compact group G has *Property (T)* or is a *Kazhdan group*, if every unitary representation $(\pi, \mathcal{H})$ of G with $\mathcal{H}^G = \{0\}$ has the Spectral Gap Property.

Recall that a local field $\mathbf{k}$ is a locally compact non discrete field. As is well-known, a local field is isomorphic either to $\mathbf{R}$, to $\mathbf{C}$, to a finite extension of the field of p-adic numbers $\mathbf{Q}_p$, or to the field $k((X))$ of Laurent series over a finite field k.

Theorem 3.6. ***([Kazh67])*** *Let* $\mathbf{k}$ *be a local field and* G *the group of* $\mathbf{k}$*-rational points of a simple algebraic group over* $\mathbf{k}$ *with* $\mathbf{k}$*-rank at least two. Then* G *has Property (T).*

Example 3.7. The groups $SL_n(\mathbf{R})$ for $n \geq 3$ and $Sp_{2n}(\mathbf{R})$ for $n \geq 2$ have Property (T).

When G has Property (T), we can find a uniform pair (Q, ε) as in Definition 3.2 for all representations without invariant vectors.

Proposition 3.8. *Let* G *be a Kazhdan group. Then there exists a pair* (Q, ε) *with* $\varepsilon > 0$ *and* Q *a compact subset of* G *so that, for every unitary representation* $(\pi, \mathcal{H})$ *of* G *with* $\mathcal{H}^G = \{0\}$*, we have*

$$\sup_{s \in Q} \|\pi(s)\xi - \xi\| \geq \varepsilon \|\xi\| \qquad \textit{for all} \quad \xi \in \mathcal{H}.$$

The pair (Q, ε) *is called a* Kazhdan pair

Proof Assume, by contradiction, that no such a pair exists. Denote by I the set of pairs (Q, ε) with $\varepsilon > 0$ and $Q \subset G$ compact. So, for every $i = (Q, \varepsilon) \in I$, there exists a unitary representation $(\pi_i, \mathcal{H}_i)$ of G with $\mathcal{H}_i^G = \{0\}$ and a unit vector $\xi_i \in \mathcal{H}_i$ such that

$$\sup_{s \in Q} \|\pi_i(s)\xi_i - \xi_i\| < \varepsilon.$$

Then the direct sum $\bigoplus_{i \in I} \pi_i$ is a unitary representation of G in $\bigoplus_{i \in I} \mathcal{H}_i$ which has almost invariant vectors and has no non-trivial invariant vectors. This is a contradiction.■

Here is one important application of Kazhdan's Property (T).

Theorem 3.9. ***([Kazh67])*** *If* G *has Property (T), then* G *is compactly generated. In particular, a discrete group with Property (T) is finitely generated.*

Proof Let $\mathcal{C}$ be the family of open and compactly generated subgroups of G. Consider the representation $\pi = \bigoplus_{H \in \mathcal{C}} \pi_{G/H}$, where $\pi_{G/H}$ is the regular representation on $\ell^2(G/H)$. Then π does not have the Spectral Gap Property. So, π has a non-zero invariant vector. This implies that G/H is compact for some $H \in \mathcal{C}$ and hence that G is compactly generated. ■

An important feature of Property (T) is that it is inherited by lattices. The proof involves induced representations.

Let G a separable locally compact group and Γ a lattice in G. Let $(\pi, \mathcal{H})$ be a unitary representation of Γ. The induced representation $\widetilde{\pi} = Ind_{\Gamma}^{G} \pi$ is the unitary representation of G which can be defined as follows. Let $X \subset G$ be a Borel set which is a fundamental domain for the action of Γ, so that

$$G = \coprod_{\gamma \in \Gamma} \gamma X.$$

Given $g \in G$ and $x \in X$, there are uniquely determined elements $c(x,g) \in \Gamma$ and $x \cdot g \in X$ such that

$$xg = c(x,g)(x \cdot g).$$

Then $\widetilde{\pi} = \mathrm{I}nd_\Gamma^G \pi$ is defined on the space $\widetilde{\mathcal{H}}$ of measurable maps $F \in L^2(X, \mathcal{H})$ by

$$(\widetilde{\pi}(g)F)(x) = \pi(c(x,g))(F(x \cdot g)).$$

For more details on induced representations, see [Mack76].

Proposition 3.10. *Let G a separable locally compact group and Γ a lattice in G. Let $(\pi, \mathcal{H})$ be a unitary representation of Γ. Assume that π does not have the Spectral Gap Property. Then $\mathrm{I}nd_\Gamma^G \pi$ does not have the Spectral Gap Property.*

Proof Since π does not have the Spectral Gap Property, there exist a sequence of unit vectors $\xi_n \in \mathcal{H}$ with $\lim_n \|\pi(\gamma)\xi_n - \xi_n\| = 0$ for all $\gamma \in \Gamma$. Define $F_n \in \widetilde{\mathcal{H}}$ by $F_n(x) = \xi_n$ for all $x \in X$. Then $\|F_n\| = 1$ and

$$\|\widetilde{\pi}(g)F_n - F_n\|^2 = \int_X \|\pi(c(x,g))\xi_n - \xi_n\|^2 dm(x),$$

where $dm(x)$ denotes the Haar measure restricted to X, normalized by $m(X) = 1$. Hence, $\lim_n \|\widetilde{\pi}(g)F_n - F_n\| = 0$. So, $\widetilde{\pi} = \mathrm{I}nd_\Gamma^G \pi$ does not have the Spectral Gap Property.■

We deduce from the previous proposition the following important theorem, a key result for producing examples of *discrete* Kazhdan groups.

Theorem 3.11. ***([Kazh67])*** *Let G a locally compact group and Γ a lattice in G. If G has Property (T), then Γ has property (T).*

Proof Let $(\pi, \mathcal{H})$ be a unitary representation of Γ without the Spectral Gap Property. Then $\widetilde{\pi} = \mathrm{I}nd_\Gamma^G \pi$ does not have the Spectral Gap Property, by the previous proposition. So, $\widetilde{\mathcal{H}}^G \neq \{0\}$. On the other hand, it is immediate from the definition of $\mathrm{I}nd_\Gamma^G \pi$ that $\widetilde{\mathcal{H}}^G$ consists exactly of the constant mappings F in $L^2(X, \mathcal{H})$ with $F(x) \in \mathcal{H}^\Gamma$ for m-almost every $x \in X$. Hence, $\mathcal{H}^\Gamma \neq \{0\}$.■

Example 3.12. The discrete groups $SL_n(\mathbf{Z})$ for $n \geq 3$ and $Sp_{2n}(\mathbf{Z})$ for $n \geq 2$ have Property (T), as they are lattices in the Kazhdan groups $SL_n(\mathbf{R})$ and $Sp_{2n}(\mathbf{R})$, respectively.

3.3 Uniform Spectral Gap Property for Kazhdan groups

Let G be a locally compact group with Property (T) and π a unitary representation of G without non-zero invariant vectors. Then, by definition, π has the Spectral Gap Property. Hence, given a strongly adapted and absolutely continuous probability measure μ on G, we have $\|\pi(\mu)\| < 1$, by Proposition 3.4. In fact, we can find a uniform bound for $\|\pi(\mu)\|$, independent of π.

Theorem 3.13. ***(Uniform Spectral Gap Property)*** *Let G be a separable locally compact group with Kazhdan's Property (T) and μ a strongly adapted and absolutely continuous probability measure on G. Then there exists $C < 1$ such that $\|\pi(\mu)\| < C$ for every unitary representation $(\pi, \mathcal{H})$ of G without non-zero invariant vectors.*

Proof Assume, by contradiction, that this is not the case. Then there exists a sequence $(\pi_n, \mathcal{H}_n)$ of unitary representations of G without non-zero invariant vectors such that $\lim_n \|\pi_n(\mu)\| = 1$. Then $\pi = \bigoplus_n \pi_n$ is a unitary representation of G on $\mathcal{H} = \bigoplus_n \mathcal{H}_n$ which has no non-trivial invariant vectors and for which $\|\pi(\mu)\| = 1$. This is a contradiction. ■

Remark 3.14. Let G be a locally compact group with the following property: every ergodic measure preserving action $G \curvearrowright X$ on a probability space (X, m) has the Spectral Gap Property. Of course, Kazhdan groups have this property. In fact, it was shown in [CoWe80] that this property characterizes the class of Kazhdan groups.

3.4 Amenable groups

Amenability of locally compact groups may be expressed in several equivalent ways. We give a brief review of a few number of these equivalent reformulations.

Definition 3.15. Let G be a locally compact group and denote by m a left Haar measure on G. The group G is *amenable* if there exists a G-invariant mean on $L^\infty(G, m)$, that is, a positive linear functional M on $L^\infty(G, m)$ such that $M(\mathbf{1}_G) = 1$ and $M({}_g\varphi) = M(\varphi)$ for all g in G and φ in $L^\infty(G, m)$, where ${}_g\varphi(x) = \varphi(g^{-1}x)$.

We mention the following useful characterizations of amenable groups and refer to Appendix G in [BHV] for proofs.

Proposition 3.16. *Let G be a locally compact group. The following properties are equivalent:*

(i) G is amenable;

(ii) every continuous action $G \curvearrowright \mathcal{C}$ by affine mappings on a non-empty compact convex subset $\mathcal{C}$ of a locally convex topological vector space has a fixed point;

(iii) for every continuous action $G \curvearrowright X$ on a compact non-empty set X, there exists a G-invariant probability measure on the Borel subsets of X.

Examples of amenable groups include abelian groups and more generally solvable groups. On the other hand, free non-abelian groups are not amenable.

We now rephrase amenability of a group in terms of its regular representation.

Let G be a locally compact group with left Haar measure m. The left regular representation of G is the unitary representation π_G which is defined by left translations on $L^2(G, m)$. The following result is due to Hulanicki ([Hula66]) and Reiter ([Reit65]).

Theorem 3.17. ***(Hulanicki-Reiter's Theorem)*** *The following statements are equivalent for a locally compact group G.*

(i) The group G is amenable;

(ii) the regular representation π_G does not have the Spectral Gap Property.

We will give the proof of Hulanicki-Reiter's theorem in the much wider context of co-amenable actions of groups on measure spaces in Section 5 (see Remark 5.9).

Følner's theorem [Foel55] is a refinement of the Hulanicki-Reiter Theorem. When G is amenable, we find a sequence of functions $f_n \in L^2(G)$ with $\|f_n\|_2 = 1$ and $\lim_n \|\pi_G(g)f_n - f\|_2 = 0$ for all $g \in G$. One may ask whether the f_n's can be chosen as normalized indicator functions of Borel sets in G. This is indeed the case.

Theorem 3.18. ***(Følner's theorem)*** *Let G be an amenable locally compact group with left Haar measure m. Then there exist so-called Følner sets: for every compact subset Q of G and every $\varepsilon > 0$, there exists a Borel subset U of G with $0 < m(U) < \infty$ such that*

$$\frac{m(xU \triangle U)}{m(U)} \leq \varepsilon \qquad \text{for all} \quad x \in Q,$$

where $\triangle$ denotes the symmetric difference.

We will generalize Følner's theorem to the context of co-amenable actions (Theorem 5.11).

The following proposition on the relation between amenability and Property (T) is an obvious consequence of Hulanicki-Reiter's Theorem.

Proposition 3.19. *An amenable locally compact group G has Property (T) if and only if G is compact.*

Amenability may be expressed in terms of averaging operators. Kesten proved the following result in case G is a discrete group and μ a symmetric probability measure on G ([Kest59a]). The general case is due to [DeGu73].

Theorem 3.20. ***([Kest59a], [DeGu73])*** *Let G be a locally compact group. and μ a strongly adapted probability measure on G. The following statements are equivalent.*

1. *the group G is amenable;*
2. $\|\pi_G(\mu)\| = 1$.

When μ is absolutely continuous, this is a consequence of the Hulanicki-Reiter theorem and the characterization of the Spectral Gap Property for a unitary representation π in terms of the averaging operator $\pi(\mu)$ from Proposition 3.4. We will prove a more general result in the context of co-amenable actions in Section 5.

Remark 3.21. (Spectral radius versus operator norm) It may happen that $\|\pi_G(\mu)\| = 1$ for an adapted probability measure μ even when the group G is non-amenable (of course, we then have $r_{spec}(\pi_G(\mu)) < 1$). Indeed, let $G = \mathbf{F}_2$ be the free group on 2 generators a and b and μ the uniform distribution on $\{a, b\}$. Then $\|\pi_G(\mu)\| = 1$. We have

$$\|\pi_G(\mu)\| = \|\pi_G(\delta_{a^{-1}} * \mu)\| = \|\pi_G(\nu)\|,$$

where $\nu = \delta_{a^{-1}} * \mu$ is the uniform distribution on $\{e, a^{-1}b\}$. Since the subgroup generated by $a^{-1}b$ is cyclic and hence amenable, we have $\|\pi_G(\nu)\| = 1$. So, $\|\pi_G(\mu)\| = 1$, although $\mathrm{supp}(\mu)$ generates $\mathbf{F}_2$ and $\mathbf{F}_2$ is not amenable.

4 Random walks and spectral radius of averaging operators

We introduce the spectral radius of a finitely generated group (that is, the spectral radius of the simple random associated to a fixed generating set), determine its exact value for free groups after Kesten and relate norms of averaging operators of Bernoulli actions of groups to their spectral radius.

4.1 Random walks on groups

Let Γ be a finitely generated group. Let S be a finite generating set of Γ with $S^{-1} = S$. Let $\mathcal{G}(\Gamma, S)$ be the associated Cayley graph, which is the graph defined as follows: the vertex set is Γ and $(x, y) \in \Gamma \times \Gamma$ is an edge if and only if $y = xs$ for some $s \in S$.

The simple random walk on $\mathcal{G}(\Gamma, S)$ is the random walk X_n defined as follows: every step consists in moving from a vertex x to a neighbour xs with probability $1/|S|$. The associated Markov operator

$$M = \pi_\Gamma(\mu_S) = \frac{1}{|S|} \sum_{s \in S} \pi_\Gamma(\delta_s)$$

acts on $\ell^2(\Gamma)$, where μ_S is the uniform distribution on S.

Then

$$\langle M^n \delta_e, \delta_e \rangle = \mu_S^n(e)$$

is the probability $\mathbb{P}(X_n = e | X_0 = e)$ of return to the group unit at time n, where μ_S^n denotes n-fold convolution. Now, one can show (see Proposition 5.14) that

$$\lim_n \mu_S^{2n}(e)^{1/2n} = \|\pi_\Gamma(\mu_S)\|.$$

The number

$$\rho = \lim_n \mu_S^{2n}(e)^{1/2n}$$

is usually called the *spectral radius* of the random walk (ρ is indeed the spectral radius of the self-adjoint operator M). So

$$\rho = \lim_n \mathbb{P}(X_{2n} = e | X_0 = e)^{1/2n}.$$

In the case where Γ is non-amenable, $\rho < 1$ and hence $\mathbb{P}(X_{2n} = e | X_0 = e)$ decreases exponentially fast as $n \to +\infty$.

4.2 An example of the computation of the spectral radius

Kesten determined in [Kest59b] the exact value for the spectral radius of simple random walk on a free group.

Let $\Gamma = \mathbf{F}_N$ be the free group on N generators $a_1, \ldots, a_N$. Let μ be the uniform distribution on $\{a_1^{\pm 1}, \ldots, a_N^{\pm 1}\}$:

$$\mu(a_i) = \mu(a_i^{-1}) = \frac{1}{2N} \qquad \text{for all} \quad 1 \leq i \leq N.$$

We claim that the spectral radius of the associated random walk is

$$\rho = \|\pi_\Gamma(\mu)\| = \frac{\sqrt{2N-1}}{N}.$$

More generally, for $d \geq 2$, let T be the d-regular tree. We consider the random walk on the vertices of T with transition probability equal to $1/d$ to go from one vertex to one of its neighbours. Let $M : \ell^2(T) \to \ell^2(T)$ be the associated Markov operator, defined by

$$Mf(v) = \frac{1}{d} \sum_{w \sim v} f(w),$$

where the sum is over the vertices w which are neighbours of v. (The Cayley graph of $(\mathbf{F}_N, \{a_1^{\pm 1}, \ldots, a_N^{\pm 1}\})$ is the $2N$-regular tree, and M can be identified with $\pi_{\mathbf{F}_N}(\mu)$). We are going to show that

$$\|M\| = \frac{2\sqrt{d-1}}{d}.$$

Apart from Kesten's original one, there are several proofs of this formula (see for instance Proposition 4.5.2 in [Lubo94]). We will follow a short argument from [Frie91].

Since T is normal and since $|\langle Mf, f\rangle| \leq \langle M|f|, |f|\rangle$ and $\|f\|_2 = \||f|\|_2$ for $f \in \ell^2(T)$, we have

$$\|M\| = \sup\left\{\langle Mf, f\rangle \,:\, f \in \ell^2(T), f \geq 0, \|f\|_2 = 1\right\}.$$

Fix an origin $o \in T$ and denote by $\delta(v)$ the graph distance of a vertex v to o. Observe that every vertex $v \neq 0$ has exactly $d-1$ neighbours w with $\delta(w) = \delta(v)+1$ and one neighbour $w = w(v)$ with $\delta(w) = \delta(v) - 1$ and that o has d neighbours w all with $\delta(w) = 1$.

Let $f \in \ell^2(T)$ with $f \geq 0$ and $\|f\|_2 = 1$. Then

$$\begin{aligned} d\langle Mf, f\rangle &= \sum_{v\in T}\left(\sum_{w\sim v} f(v)f(w)\right) \\ &= \sum_{w:\delta(w)=1} f(o)f(w) + \sum_{v\neq o}\left(\sum_{w:\delta(w)=\delta(v)+1} f(v)f(w)\right) + \sum_{v\neq o} f(v)f(w(v)) \\ &= 2\sum_{v\in T}\left(\sum_{w:\delta(w)=\delta(v)+1} f(v)f(w)\right) \end{aligned}$$

Using the estimate

$$f(v)f(w) \leq \frac{1}{2}\left(\frac{1}{\sqrt{d-1}} f(v)^2 + \sqrt{d-1} f(w)^2\right),$$

we obtain

$$\begin{aligned} d\langle Mf, f\rangle &\leq \sum_{v\neq o}\left(\frac{d-1}{\sqrt{d-1}} f(v)^2 + \sqrt{d-1} f(v)^2\right) + \frac{d}{\sqrt{d-1}} f(o)^2 \\ &\leq \sqrt{d-1}\sum_{v\in T} 2f(v)^2 = 2\sqrt{d-1}. \end{aligned}$$

This proves that $\|M\| \leq \dfrac{2\sqrt{d-1}}{d}$.

To establish the lower bound, take an increasing sequence of real numbers $\lambda_n < 1$ with $\lim_n \lambda_n = 1$. Let f_n be the radial function on T defined by $f_n(v) = \left(\dfrac{\lambda}{\sqrt{d-1}}\right)^m$ if $\delta(v) = m$. Then

$$\|f_n\|_2^2 = 1 + \sum_{m\geq 1}\left(d(d-1)^{m-1}\left(\frac{\lambda_n}{\sqrt{d-1}}\right)^{2m}\right) = 1 + \frac{d}{d-1}\left(\frac{\lambda_n^2}{1-\lambda_n^2}\right)$$

and

$$\begin{aligned} \langle Mf_n, f_n\rangle &= \frac{2}{d}\sum_{m\geq 0}\left(d(d-1)^{m-1}(d-1)\left(\frac{\lambda_n}{\sqrt{d-1}}\right)^{m+1}\left(\frac{\lambda_n}{\sqrt{d-1}}\right)^{m}\right) \\ &= \frac{2}{\sqrt{d-1}}\left(\frac{\lambda_n}{1-\lambda_n^2}\right). \end{aligned}$$

So, $\lim_n \dfrac{\langle Mf_n, f_n\rangle}{\|f_n\|_2^2} = \dfrac{2\sqrt{d-1}}{d}$ and this proves the claim.

Remark 4.1. (i) With a little more effort, one can show that the spectrum of the operator M as above is the whole interval $\left[-\dfrac{2\sqrt{d-1}}{d}, \dfrac{2\sqrt{d-1}}{d}\right]$ (see [Frie91] or [Kest59b]).

(ii) For $N \geq 2$, let Γ be a group generated by N elements $a_1, \ldots, a_N$. For the spectral radius ρ of the random walk on Γ defined by the uniform distribution on $\{a_1^{\pm 1}, \ldots, a_N^{\pm 1}\}$, one has $\rho \geq \frac{\sqrt{2N-1}}{N}$. Indeed, Γ is a quotient of $\mathbf{F}_N$ and the claim follows from Proposition 5.15 below. Kesten (see [Kest59b]) proved that, if one has equality $\rho = \frac{\sqrt{2N-1}}{N}$, then Γ is the free group on $a_1, \ldots, a_N$.
(iii) Given a specific group Γ generated by a finite set S, it is usually difficult to compute or even to find bounds for the spectral radius of the corresponding random walk. For a recent result in the case where Γ is a surface group, see [Goue15]. The monography [Woes00] provides a comprehensive overview on results about random walks on infinite groups as well as on infinite graphs.

4.3 The norm of averaging operators for Bernoulli actions

Let Γ be an infinite countable group and $\Gamma \curvearrowright X$ its Bernoulli action on $X = \{0,1\}^\Gamma$ (see Section 2.1). Let μ be a symmetric and adapted probability measure on Γ. We claim that

$$\|\pi_X(\mu)\| = \|\pi_\Gamma(\mu)\|.$$

In particular, it will follow from the Hulanicki-Reiter theorem that $\Gamma \curvearrowright X$ has the Spectral Gap Property if and only if Γ is not amenable.

To prove the claim, view X as the compact abelian group $X = \prod_{\gamma\in\Gamma} \mathbf{Z}/2\mathbf{Z}$. The dual group is the discrete group $\widehat{X} = \bigoplus_{\gamma\in\Gamma} \mathbf{Z}/2\mathbf{Z}$. By Fourier transform, we have $L^2_0(X) \cong \ell^2(\widehat{X} \setminus \{0\})$ as Γ-representations. Let $\Omega \subset \widehat{X} \setminus \{0\}$ be a set of representatives for the Γ-orbits in $\widehat{X} \setminus \{0\}$. Then $\ell^2(\widehat{X} \setminus \{0\})$ decomposes as a direct sum of Γ-invariant subspaces:

$$\ell^2(\widehat{X} \setminus \{0\}) = \bigoplus_{x\in\Omega} \ell^2(\Gamma x) \cong \bigoplus_{x\in\Omega} \ell^2(\Gamma/\Gamma_x),$$

where Γ_x is the stabilizer of x. It is clear that Γ_x is finite for every x. Hence, we have $\ell^2(\Gamma/\Gamma_x) \subset \ell^2(\Gamma)$ in an obvious sense. So,

$$\ell^2(\widehat{X} \setminus \{0\}) \subset \bigoplus_{x\in\Omega} \ell^2(\Gamma).$$

Therefore, $\|\pi_X(\mu)\| \leq \|\pi_\Gamma(\mu)\|$. On the other hand, we have, for every $x \in \Omega$

$$\|\pi_\Gamma(\mu)\| \leq \|\pi_X|_{\ell^2(\Gamma/\Gamma_x)}(\mu)\|,$$

as follows from Proposition 5.15 below.

5 Actions on homogeneous spaces with infinite measure

In this section, we introduce and study a class of actions which we call co-amenable. As for amenability of groups, these actions admit several characterizations, the most notable one in our context being the absence of the Spectral Gap Property.

5.1 Co-amenable actions

A useful generalization of amenable groups is the notion of amenable homogeneous spaces in the sense of Eymard ([Eyma72]). One can extend this notion to actions on measure spaces as follows. Let $G \curvearrowright X$ be an action of the separable locally compact group G on the measure space (X, m), where as always m is a σ-finite quasi-invariant measure.

Definition 5.1. We say that the action of G on X is *co-amenable* if there exists a G-invariant mean on $L^\infty(X, m)$, that is, a positive linear functional M on $L^\infty(X, m)$ such that that $M(\mathbf{1}_X) = 1$ and $M({}_g\varphi) = M(\varphi)$ for all g in G and φ in $L^\infty(X, m)$, where ${}_g\varphi(x) = \varphi(g^{-1}x)$.

Remark 5.2. (i) Consider the action of the locally compact group G by left translation on (G, m), where m is a left Haar measure. This action is co-amenale if and only if G is amenable.
(ii) Co-amenable actions $G \curvearrowright X$ were first considered by Greenleaf ([Gree69]). There were intensively studied by Eymard in the case of the action of G on a homogeneous space $X = G/H$ in [Eyma72], where such an action (or space) is called amenable. We prefer to call them co-amenable in order to avoid confusion with the well-established notion of an amenable action $G \curvearrowright X$ due to Zimmer ([Zimm84]), which in the case $X = G/H$ corresponds to the amenability of H; a unification of both notions by means of an appropriate definition of amenable actions on pairs of measure spaces is given in [Zimm78]. For a further extension of these notions to the context on non-commutative measure spaces (that is, to the context of von Neumann algebras), see [Anan08].
(iii) If X is a locally compact space, one may define, as in [Gree69] or [Guiv80], co-amenability of the action $G \curvearrowright X$ through the existence of a G-invariant mean on the space $C^b(X)$ of continuous bounded functions on X; as Example 5.10 below shows, this is in general a weaker condition than co-amenability of the action of G on the measure space (X, m), even in the case of a homogeneous space $X = \widetilde{G}/H$ (for a group $\widetilde{G}$ containing G), where a natural quasi-invariant measure (class) m is given.

Let $G \curvearrowright (X, m)$ be an action of G on the measure space m. Given a measure space (X, m), a mean M on $L^\infty(X, m)$ defines a *finitely additive* probability measure μ_M on the measurable subsets A of X, given by $\mu_M(A) = M(\mathbf{1}_A)$. Such a finitely additive probability measure is absolutely continuous with respect to m, in the sense that if $m(A) = 0$ then $\mu_M(A) = 0$. Conversely, a finitely additive probability measure μ on X which is absolutely continuous with respect to m defines a unique mean M on $L^\infty(X, m)$, given by $M(\varphi) = \sum_{i=1}^m \alpha_i \mu(A_i)$, if $\varphi = \sum_{i=1}^m \alpha_i \mathbf{1}_{A_i}$ is a linear combination of indicator functions of measurable subsets A_i of X.

Remark 5.3. (i) Let X be a *compact* space. Then every mean M on $L^\infty(X, m)$ defines in a natural way a genuine (that is, σ-additive) probability measure on X. Indeed, let $\Phi : C(X) \to L^\infty(X, m)$ be the obvious mapping. Then $M \circ \Phi$ is a positive linear functional on $C(X)$ with $M \circ \Phi(\mathbf{1}_X) = 1$. By Riesz representation

theorem, there exists a probability measure μ on X such that

$$M \circ \Phi(f) = \int_X f(x) d\mu(x) \qquad \text{for all} \quad f \in C(X).$$

Observe that, if a locally compact group G acts on (X, m) and if M is G-invariant, then the associated probability measure μ is G-invariant.

(ii) Recall that the space of probability measures $\mathcal{P}(X)$ on a compact space X is compact for the weak topology $\sigma(\mathcal{P}(X)), C(X))$; this is no longer true when X is, say, a non compact locally compact space. However, for any measure space (X, m), the space $\mathcal{M}(X)$ of means on $L^\infty(X, m)$ has an equally useful compactness property: $\mathcal{M}(X)$ is compact for the weak*-topology $\sigma(L^\infty(X, m), L^1(X, m))$.

Means share with probability measures the property that they may be pushed forward through measurable mappings.

Let (X, m) and (Y, m') be σ-finite measure spaces and $\theta : X \to Y$ a measurable mapping such that m' is absolutely continuous with respect to the push-forward measure $\theta_*(m)$. Then

$$\Phi : L^\infty(Y, m') \to L^\infty(X, m), \quad \varphi \mapsto \varphi \circ \theta$$

is a well-defined linear mapping which preserves positivity and maps $\mathbf{1}_Y$ to $\mathbf{1}_X$. So, if M is a mean on $L^\infty(X, m)$, then $\theta_*(M) := M \circ \Phi$ is a mean on $L^\infty(Y, m')$, which we call the *push-forward mean* or the image of M through θ.

An immediate consequence of the consideration of push-forward means is the following useful fact.

Corollary 5.4. *Let $G \curvearrowright X$ and $G \curvearrowright Y$ be actions of the locally compact group G on measure spaces (X, m) and (Y, m'). Assume that there exists a measurable mapping $\theta : X \to Y$ which intertwines the respective G-actions and such that m' is absolutely continuous with respect to $\theta_*(m)$. If $G \curvearrowright X$ is co-amenable, then $G \curvearrowright Y$ is co-amenable.*■

Proof If M is a G-invariant mean on $L^\infty(X, m)$, then its push-forward $\theta_*(M)$ is a G-invariant mean on $L^\infty(Y, m')$. ■

Given an action $G \curvearrowright (X, m)$, it will be convenient in the sequel to consider G-invariant means on the subspace $L^\infty(X)_{G,u}$ of G-continuous functions in $L^\infty(X)$, that is, the space $L^\infty(X)_{G,u}$ of all $\varphi \in L^\infty(X)$ for which the mapping

$$G \to L^\infty(X), \qquad g \mapsto {}_g\varphi$$

is norm-continuous.

The space $L^\infty(X)_{G,u}$ is a large subspace, obtained by averaging functions from $L^\infty(X, m)$. Indeed, it is easy to see that $f * \varphi \in L^\infty(X, m)_{G,u}$ for every $f \in L^1(G, \lambda)$ and $\varphi \in L^\infty(X)$, where $f * \varphi(x) = \int_G f(g)\, {}_g\varphi(x) d\lambda(g)$ and λ is Haar left measure on G. (In fact, using Cohen's factorization theorem [HeRo63], one can show that $L^\infty(X)_{G,u} = \{f * \varphi \ : \ f \in L^1(G, \lambda), \varphi \in L^\infty(X)\}$.) It is obvious that $L^\infty(X)_{G,u}$ is a G-invariant closed subspace of $L^\infty(X)$ containing the constant functions.

Let $L^1(G)_{1,+}$ denote the convex set of all $f \in L^1(G, \lambda)$ with $f \geq 0$ and $\|f\|_1 = 1$. Observe that $L^1(G)_{1,+}$ is closed under convolution.

Lemma 5.5. *The following properties are equivalent.*

(i) There exists a G-invariant mean on $L^\infty(X)$.

(ii) There exists a G-invariant mean on $L^\infty(X)_{G,u}$.

(iii) There exists a topologically invariant *mean on $L^\infty(X)$, that is, a mean M such that $M(f*\varphi) = M(\varphi)$ for all $\varphi \in L^\infty(X)$ and $f \in L^1(G)_{1,+}$.*

Proof The implication $(i) \Rightarrow (ii)$ is trivial. To show that $(ii) \Rightarrow (iii)$, let M be a G-invariant mean on $L^\infty(X)_{G,u}$. Since the mapping $G \to L^\infty(X)_{G,u},\ g \mapsto {}_g\varphi$ is norm continuous for $\varphi \in L^\infty(X)_{G,u}$, one has

$$M(f*\varphi) = M(\varphi) \qquad \text{for all} \quad f \in L^1(G)_{1,+},\ \varphi \in L^\infty(X)_{G,u}.$$

Let $(f_n)_n$ be an approximate identity in $L^1(G)_{1,+}$ for $L^1(G)$. Then, for each $\varphi \in L^\infty(X)$ and $f \in L^1(G)_{1,+}$, we have $\lim_n \|f*f_n*\varphi - f*\varphi\|_\infty = 0$ and, hence,

$$M(f*\varphi) = \lim_n M(f*f_n*\varphi) = \lim_n M(f_n*\varphi).$$

This shows that $M(f*\varphi) = M(f'*\varphi)$ for all $f, f' \in L^1(G)_{1,+}$ and all $\varphi \in L^\infty(X)$.

Fix any $f_0 \in L^1(G)_{1,+}$, and define a mean $\widetilde{M}$ on $L^\infty(X)$ by

$$\widetilde{M}(\varphi) = M(f_0*\varphi) \qquad \text{for all} \quad \varphi \in L^\infty(X).$$

Then $\widetilde{M}$ is topologically invariant, since, for $f \in L^1(G)_{1,+}$ and $\varphi \in L^\infty(X)$, we have

$$\widetilde{M}(f*\varphi) = M(f_0*f*\varphi) = M(f_0*\varphi) = \widetilde{M}(\varphi).$$

To show that $(iii) \Rightarrow (i)$, let M be a topologically invariant mean on $L^\infty(X)$. Then M is G-invariant. Indeed, fix $f \in L^1(G)_{1,+}$. Then, for $\varphi \in L^\infty(X)$ and $g \in G$, we have $f *{}_g\varphi = f_g * \varphi$ and hence

$$M({}_g\varphi) = M(f *{}_g\varphi) = M(f_g*\varphi) = M(\varphi),$$

for $f_g \in L^1(G)_{1,+}$ defined by $f_g(h) = \Delta(g^{-1})f(hg^{-1})$, where Δ is the modular function of G.■

Recall that the unitary representation π_X of G associated to the G-action on X is defined on $L^2(X,m)$ by

$$\pi_X(g)\xi(x) = \sqrt{\frac{dm(g^{-1}x)}{dm(x)}}\,\xi(g^{-1}x), \qquad g \in G,\ x \in X,\ \xi \in L^2(X).$$

(Observe that $G \curvearrowright X$ is trivially co-amenable if m is a G-invariant probability measure; so, our interest is in the case where m is either infinite or not invariant.)

We will need the following general fact concerning the spectral radius of a convolution operator acting on $L^2(X,m)$.

Lemma 5.6. *Let μ be a probability measure on G with $r_{spec}(\pi_X(\mu)) = 1$. Then 1 belongs to the spectrum $\sigma(\pi_X(\mu))$ of $\pi_X(\mu)$; more precisely, 1 is an approximate eigenvalue of $\pi_X(\mu)$.*

Proof Set $T := \pi_X(\mu)$. Since $r_{spec}(T) = 1$, there exists $\lambda \in \sigma(T)$ with $|\lambda| = 1$.

We claim that λ is an approximate eigenvalue of T. Indeed, otherwise, $\mathrm{I}m(T - \lambda I)$ would be a proper closed subspace of $L^2(X)$. So, we would have $\ker(T^* - \overline{\lambda} I) \neq \{0\}$. However, since T is a contraction, the equality case of Cauchy-Schwarz inequality shows that $\ker(T^* - \overline{\lambda} I) = \ker(T - \lambda I)$. Hence, λ would be an eigenvalue of T and this would be a contradiction.

So, there exists a sequence $(\xi_n)_n$ in $L^2(X)$ with $\|\xi_n\| = 1$ such that $\lim_n \|T\xi_n - \lambda\xi_n\| = 0$ or, equivalently,

$$\lim_n \int_G \langle \pi_X(g)\xi_n, \xi_n\rangle d\mu(g) = \lim_n \langle T\xi_n, \xi_n\rangle = \lambda.$$

In particular, we have

$$\lim_n \left| \int_G \langle \pi_X(g)\xi_n, \xi_n\rangle d\mu(g) \right| = 1.$$

Since

$$\begin{aligned} 1 &= \int_G \|\pi_X(g)\xi_n\|\|\xi_n\| d\mu(g) \geq \int_G \langle \pi_X(g)|\xi_n|, |\xi_n|\rangle d\mu(g) \\ &\geq \int_G |\langle \pi_X(g)\xi_n, \xi_n\rangle| d\mu(g) \geq \left| \int_G \langle \pi_X(g)\xi_n, \xi_n\rangle d\mu(g) \right|, \end{aligned}$$

it follows that

$$\lim_n \langle T|\xi_n|, |\xi_n|\rangle = \lim_n \int_G \langle \pi_X(g)|\xi_n|, |\xi_n|\rangle d\mu(g) = 1,$$

that is, $\lim_n \| T|\xi_n| - |\xi_n| \| = 0$. Hence, 1 is an approximate eigenvalue of T. ■

The following result, which is the main result of this section, was obtained by Guivarc'h in [Guiv80, Proposition 1]. The equivalence $(i) \Leftrightarrow (ii)$ is due to Eymard ([Eyma72]) in the case of an action $G \curvearrowright G/H$ for a closed subgroup H of G. The equivalence $(ii) \Leftrightarrow (iv)$ (or $(ii) \Leftrightarrow (v)$) has been extended to a class of more general unitary representations in [BeGu06].

Theorem 5.7. ***([Guiv80])*** *Let $G \curvearrowright X$ be an action of the separable locally compact group G on the measure space (X, m), where m is a σ-finite quasi-invariant measure on X. The following properties are equivalent:*
(i) The action $G \curvearrowright X$ is co-amenable;
(ii) the representation π_X of G does not have the Spectral Gap Property;
(iii) we have $r_{spec}(\pi_X(\mu)) = 1$ for every probability measure μ on G;
(iv) we have $\|\pi_X(\mu)\| = 1$ for some strongly adapted probability measure μ on G;
(v) we have $r_{spec}(\pi_X(\mu)) = 1$ for some adapted probability measure μ on G.

Proof $(i) \Rightarrow (ii)$: The set $\mathcal{M}$ of all means on $L^\infty(X)$ is a weak* closed (and hence compact) convex subset of the unit ball of $L^\infty(X)^*$. We can view the set $L^1(X)_{1,+}$ of densities as a subset of $\mathcal{M}$, since every ξ in $L^1(X)_{1,+}$ defines an element in $\mathcal{M}$, via integration against ξ. Hahn-Banach's theorem shows that $L^1(X)_{1,+}$ is weak* dense in $\mathcal{M}$.

The group G acts by isometries on $L^1(X,m)$, through the formula

$$\pi_X(g)\xi(x) = \frac{dm(g^{-1}x)}{dm(x)}\xi(g^{-1}x) \qquad \text{for all} \quad g \in G,\ \xi \in L^1(X,m),\ x \in X,$$

and so $L^1(X)$ is a continuous $L^1(G)$-module, via

$$f * \xi(x) := \int_G f(g)(\pi_X(g)\xi)(x)d\lambda(g) \qquad \text{for all} \quad f \in L^1(G),\ \xi \in L^1(X).$$

One checks that, for $f \in L^1(G), \xi \in L^1(X)$ and $\varphi \in L^\infty(X)$, one has

$$\int_X (f * \xi)(x)\varphi(x)dm(x) = \int_X \xi(x)(\check{f} * \varphi)(x)dm(x),$$

where $\check{f} \in L^1(G)$ is defined by $\check{f}(g) = \Delta(g^{-1})f(g^{-1})$.

Let $\mathcal{L}$ be a countable dense subset of $L^1(G)_{1,+}$ for the L^1-norm. For each $f \in \mathcal{L}$, take a copy of $L^1(X)$ and consider the product space $E = \prod_{f\in\mathcal{L}} L^1(X)$, with the product of the norm topologies. Then E is a locally convex space, and the weak topology on E is the product of the weak topologies. Consider the convex set

$$\Sigma = \{(f * \xi - \xi)_{f\in\mathcal{L}} :\ \xi \in L^1(X)_{1,+}\} \subset E.$$

Since $G \curvearrowright X$ is co-amenable, there exists a G-invariant mean M on $L^\infty(X)$, which is even topologically invariant (see Lemma 5.5). Take a net $(\xi_i)_i$ in $L^1(X)_{1,+}$ with

$$M(\varphi) = \lim_i \int_X \xi_i(x)\varphi(x)dm(x) \qquad \text{for all} \quad \varphi \in L^\infty(X).$$

Then, since M is topologically invariant, we have $\lim_i f * \xi_i - \xi_i = 0$ in the weak topology of $L^1(X)$, for every $f \in L^1(G)_{1,+}$. So, 0 belongs to the closure of Σ for the product of the weak topologies and hence for the product of the norm-topologies, since Σ is convex. Therefore, we can find a sequence $(\xi_n)_n$ in $L^1(X)_{1,+}$ such that, for every $f \in \mathcal{L}$, we have

$$\lim_n \|f * \xi_n - \xi_n\|_1 = 0.$$

It follows that $\lim_n \|f * \xi_n - \xi_n\|_1 = 0$ for every $f \in L^1(G)_{1,+}$, by density of $\mathcal{L}$. Since

$$\|\pi_X(g)(f * \xi_n) - f * \xi_n\|_1 \leq \|({}_g f) * \xi_n - \xi_n\|_1 + \|f * \xi_n - \xi_n\|_1,$$

we have therefore

$$\lim_n \|\pi_X(g)(f * \xi_n) - f * \xi_n\|_1 = 0 \qquad \text{for all} \quad g \in G.$$

Choose $f \in \mathcal{L}$ and set $\eta_n = \sqrt{f * \xi_n}$. Then $\|\eta_n\|_2^2 = \|f * \xi_n\|_1 = 1$; moreover, for every $g \in G$, we have

$$\lim_n \|\pi_X(g)\eta_n - \eta_n\|_2 = 0,$$

since

$$\|\pi_X(g)\eta_n - \eta_n\|_2^2 \leq \|\pi_X(g)(f * \xi_n) - (f * \xi_n)\|_1,$$

using the elementary inequality $|a-b|^2 \leq |a^2-b^2|$ for all non-negative real numbers a and b. It follows then that $\lim_n \|\pi_X(g)\eta_n - \eta_n\|_2 = 0$ uniformly on compact subsets of G (see Remark 3.3.i). This proves (ii).
$(ii) \Rightarrow (iii)$: Assume that π_X has does not have Spectral Gap Property. So, there exists a sequence $\eta_n \in L^2(X)$ with $\|\eta_n\|_2 = 1$ and $\lim_n \|\pi_X(g)\eta_n - \eta_n\|_2 = 0$ for all $g \in G$. Set $\xi_n = |\eta_n|^2$. Then $\xi_n \in L^1(X)_{1,+}$ and $\lim_n \|\pi_X(g)\xi_n - \xi_n\|_1 = 0$ for all $g \in G$, since

$$\|\pi_X(g)\xi_n - \xi_n\|_1 \leq 2\|\pi_X(g)\eta_n - \eta_n\|_2,$$

by Cauchy-Schwarz inequality.

Let μ be any probability measure on G. Since

$$\|\pi_X(\mu)\eta_n - \eta_n\|_2 \leq \int_G \|\pi_X(g)\eta_n - \eta_n\|_2 d\mu(g),$$

it follows from Lebesgue convergence theorem that $\lim_n \|\pi_X(\mu)\eta_n - \eta_n\|_2 = 0$. Hence, $r_{spec}(\pi_X(\mu)) = 1$.
$(iii) \Rightarrow (iv)$ is obvious.
$(iv) \Rightarrow (v)$ Let μ be a strongly adapted probability measure on G with $\|\pi_X(\mu)\| = 1$. Then $\check{\mu} * \mu$ is adapted and $\pi_X(\check{\mu} * \mu)$ is a self-adjoint operator on $L^2(X)$ with $\|\pi_X(\check{\mu} * \mu)\| = \|\pi_X(\mu)\|^2 = 1$. Hence, $r_{spec}(\pi_X(\check{\mu} * \mu)) = 1$.
$(v) \Rightarrow (i)$: Let μ be an adapted probability measure on G with $r_{spec}(\pi_X(\mu)) = 1$.

By Lemma 5.6, 1 belongs to $\sigma(\pi_X(\mu))$ and is an approximate eigenvalue. Hence, there exists a sequence of unit vectors η_n in $L^2(X)$ with

$$\lim_n \|\pi_X(\mu)\eta_n - \eta_n\|_2 = 0.$$

So, $\lim_n \langle \pi_X(\mu)\eta_n, \eta_n\rangle = 1$, that is,

$$\lim_n \int_G \langle \pi_X(g)\eta_n, \eta_n\rangle d\mu(g) = 1.$$

It follows that there exists a subsequence, again denoted by η_n, such that

$$\lim_n \langle \pi_X(g)\eta_n, \eta_n\rangle = 1.$$

for μ-almost every $g \in G$. Since

$$\|\pi_X(g)|\eta_n| - |\eta_n|\|_2 \leq \|\pi_X(g)\eta_n - \eta_n\|_2,$$

we can assume that $\eta_n \geq 0$. Set

$$H = \{g \in G \ : \ \lim_n \langle \pi_X(g)\eta_n, \eta_n\rangle = 1\} = \{g \in G \ : \ \lim_n \|\pi_X(g)\eta_n - \eta_n\|_2 = 0\}.$$

Then H is a measurable subgroup of G with $\mu(H) = 1$. Hence, H contains $\mathrm{supp}(\mu)$ and it follows that H is dense in G.

Set $\xi_n = \sqrt{\eta_n} \in L^1(X)_{1,+}$. Let now M be a mean on $L^\infty(X)_{G,u}$ which is a limit of $(\xi_n)_n$ in the weak-*-topology. Then M is H-invariant. Since, $g \mapsto {}_g\varphi$ is norm-continuous for $\varphi \in L^\infty(X)_{G,u}$, it follows that M is G-invariant. Hence, $G \curvearrowright X$ is co-amenable by Lemma 5.5. ■

The following consequence of the equivalence between (i), (iii) and (v) in the previous theorem is worth mentioning.

Corollary 5.8. *Let $G \curvearrowright X$ be an action of the separable locally compact group G on the measure space (X, m). Let H be a separable locally compact group, $j : H \to G$ a continuous homomorphism and $H \curvearrowright X$ the corresponding action.*
(i) If $G \curvearrowright X$ is co-amenable, then $H \curvearrowright X$ is co-amenable. In particular, $H \curvearrowright X$ is co-amenable if H is a closed subgroup or a countable dense subgroup of G.
(ii) Assume that $G \curvearrowright X$ is not co-amenable and that $j(H)$ is dense in G. Then $H \curvearrowright X$ is not co-amenable.

Remark 5.9. (i) Considering the action $G \curvearrowright G$ given by left translation, we see that Theorem 3.20 is a special case of Theorem 5.7.
(ii) For an extension of Theorem 5.7 to amenable pair of actions (including actions on von Neumann algebras), see [Anan03] and [Anan08]).

Example 5.10. Identify the group $G = SL_2(\mathbf{R})$ with the subgroup $\begin{pmatrix} 1 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix}$ of $H = SL_3(\mathbf{R})$; the standard action of G on $X = \mathbf{R}^3 \backslash \{0\}$ (which is a homogeneous space of the form H/L), preserves the Lebesgue measure m on X, and fixes the first unit vector $e_1 \in \mathbf{R}^3$. The Dirac measure δ_{e_1} is a G-invariant mean on the space $C^b(X)$; however, there exists no G-invariant mean on $L^\infty(X, m)$. Indeed, otherwise, the corresponding unitary representation π_X of G on $L^\infty(X) = L^2(\mathbf{R}^3)$ would have almost invariant vectors (by Theorem 5.7). On the other hand, upon neglecting a set of m-measure 0, the G-orbits in X are the sets

$$X_x = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid y, z \in \mathbf{R} \right\}$$

for $x \in \mathbf{R}$, which all are isomorphic to G/N where N is the subgroup of unipotent upper-triangular matrices. So, π_X is weakly equivalent to the quasi-regular representation $\pi_{G/N}$ in G/N. Since, N is amenable, $\pi_{G/N}$ is weakly contained in the regular representation π_G (see Theorem F.3.5 in [BHV]). Therefore, π_G would have almost invariant vectors and this impossible, since $G = SL_2(\mathbf{R})$ is not amenable.

5.2 Følner sequences

We now prove the existence of Følner sequences in X for co-amenable actions $G \curvearrowright (X, m)$, provided the measure m is *G-invariant*. More precisely, the following extension of Theorem 3.18 holds (see also [Gree69] and [IoNe96]).

Theorem 5.11. ***(Existence of Følner sequences)*** *Let G be a separable locally compact group and let $G \curvearrowright X$ be a co-amenable action of G on the measure space (X, m). Assume that m is G-invariant. Then, for every compact subset Q of G and every $\varepsilon > 0$, there exists a measurable subset U of X with $0 < m(U) < \infty$ such that*

$$\frac{m(gU \triangle U)}{m(U)} \leq \varepsilon \qquad \text{for all} \quad g \in Q.$$

Proof We follow the proof given in [BHV, Theorem G.5.1] for the group case (Theorem 3.18), which carries over *mutatis mutandis* in our situation.

Let Q be a compact subset of G containing e, and let $\varepsilon > 0$. Set $K := Q^2$. Since π_X does not have the Spectral Gap Property, we can find $\xi \in L^2(X)$ with $\|\xi\|_2 = 1$ such that

$$\sup_{g \in K} \|\pi_X(g)\xi - \xi\|_2 \leq \frac{\varepsilon |Q|}{2|K|},$$

where we denote by $|A|$ the Haar measure of a subset A of G.

Set $f = |\xi|^2$. Then $f \in L^1(X)_{1,+}$ and

$$\sup_{g \in K} \|_{g^{-1}} f - f\|_1 \leq \frac{\varepsilon |Q|}{2|K|}.$$

For $t \geq 0$, let $E_t := \{y \in X \ : \ f(y) \geq t\}$. Then

$$gE_t = \{y \in X \ : \ _{g^{-1}} f(y) \geq t\}.$$

By the lemma below, we have

$$\|_{g^{-1}} f - f\|_1 = \int_0^\infty m(gE_t \triangle E_t) dt.$$

for every $g \in G$. Hence, for every $g \in K$, we have

$$\int_0^\infty m(E_t) \left(\int_K \frac{m(gE_t \triangle E_t)}{m(E_t)} dg \right) dt = \int_K \|_{g^{-1}} f - f\|_1 dg \leq \frac{\varepsilon |Q|}{2}.$$

Since $\int_0^\infty m(E_t) dt = \|f\|_1 = 1$, it follows that there exists t with $0 < m(E_t) < \infty$ and such that

$$\int_K \frac{m(gE_t \triangle E_t)}{m(E_t)} dg \leq \frac{\varepsilon |Q|}{2}.$$

Set

$$A = \{g \in K : \ \frac{m(gE_t \triangle E_t)}{m(E_t)} \leq \varepsilon\}.$$

Then $|K \setminus A| < |Q|/2$ and we claim that $Q \subset AA^{-1}$.

To show this, let $g \in Q$. Then $|gK \cap K| \geq |gQ| = |Q|$ and, hence,

$$\begin{aligned} |Q| \leq |gK \cap K| &\leq |gA \cap A| + |K \setminus A| + |g(K \setminus A)| = |gA \cap A| + 2|K \setminus A| \\ &< |gA \cap A| + |Q|. \end{aligned}$$

Therefore, $|gA \cap A| > 0$, and this implies that $g \in AA^{-1}$, as claimed.

Now, for $g_1, g_2 \in A$, we have

$$g_1 g_2^{-1} E_t \triangle E_t \subset (g_1 g_2^{-1} E_t \triangle g_1 E_t) \cup (g_1 E_t \triangle E_t),$$

and hence

$$\begin{aligned} m(g_1 g_2^{-1} E_t \triangle E_t) &\leq m(g_2^{-1} E_t \triangle E_t) + m(g_1 E_t \triangle E_t) \\ &= m(g_2 E_t \triangle E_t) + m(g_1 E_t \triangle E_t) \leq 2\varepsilon m(E_t). \blacksquare \end{aligned}$$

The following formulas, which are versions of the area and co-area formulas from Lemma 8.6, have a similar elementary proof (see Lemma G.5.2 in [BHV]).

Lemma 5.12. *Let (X, m) be a measure space. Let f, f' be non-negative functions in $L^1(X)$. For every $t \geq 0$, let $E_t = \{x \in X : \ f(x) \geq t\}$ and $E'_t = \{x \in X : f'(x) \geq t\}$. Then*

$$\|f - f'\|_1 = \int_0^\infty m(E_t \triangle E'_t) dt.$$

In particular, $\|f\|_1 = \int_0^\infty m(E_t) dt$.

Remark 5.13. As noticed in [Gree69], the previous theorem may fail if one drops the assumption that the measure m on X is G-invariant. Indeed, consider for example the action of $G = \mathbf{Z}$ on $X = \mathbf{Z}$ by translations, where X is equipped with the measure defined by $m(\{n\}) = 2^{|n|}$ for all $n \in \mathbf{Z}$. It is easy to see that, for $Q = \{\pm 1\}$ and for every finite set U of $\mathbf{Z}$, we have $\sup_{g \in Q} \frac{m(gU \triangle U)}{m(U)} \geq 1$.

5.3 Norm of averaging operators under the regular representation

Let G be a separable locally compact group, and denote by m a left Haar measure on G. We consider the action of G on itself by left translation.

The associated unitary representation π_G is the left regular representation of G on on $L^2(G, m)$. It is a remarkable fact that, given a probability measure μ on G, the norm of $\pi_G(\mu)$ under the regular representation gives a lower bound for $\pi_X(\mu)$ for *every* action $G \curvearrowright X$ on a measure space X. This fact, which may be viewed as a version of Herz's majoration principle from [Herz70], was proved by Shalom (see Lemma 2.3 in [Shal00]).

We first give a formula, due to Kesten in the discrete case and to Berg and Christensen in general, for the norm of the convolution operators defined by probability measures on G.

Proposition 5.14. ***([Kest59b], [BeCh74])*** *Let μ be a probability measure on the separable locally compact group G. Then, for every compact neighbourhood U of the group unit e, we have*

$$\|\pi_G(\mu)\| = \lim_{n \to +\infty} \langle \pi_G(\check{\mu} * \mu)^n \mathbf{1}_U, \mathbf{1}_U \rangle^{1/2n} = \lim_{n \to +\infty} \left((\check{\mu} * \mu)^n (U) \right)^{1/2n}.$$

In particular, if G is discrete, we have

$$\|\pi_G(\mu)\| = \lim_{n \to +\infty} (\check{\mu} * \mu)^n (e)^{1/2n} = \lim_{n \to +\infty} \langle \pi_G(\check{\mu} * \mu)^n) \delta_e, \delta_e \rangle^{1/2n}.$$

Proof We give the proof of the formula in the case where G is discrete.

First, we observe that the sequence $(\check{\mu} * \mu)^n(e)^{1/2n}$ converges. Indeed, set $a_n = (\check{\mu} * \mu)^n(e)$. Let T be the square root of the positive selfadjoint operator $\pi_G(\check{\mu} * \mu)$. Then

$$a_n = \|T^n \delta_e\|^2 = \langle T^n \delta_e, T^n \delta_e \rangle$$

and, using Cauchy-Schwarz inequality, we see that $a_n^2 \leq a_{n+1} a_{n-1}$. Hence $(a_{n+1}/a_n)_n$ is increasing and therefore convergent. It follows that $(a_n^{1/2n})_n$ converges (to the same limit).

Next, since $\pi_G(\check{\mu} * \mu)$ is selfadjoint, we have

$$\|\pi_G(\check{\mu} * \mu)\| = \lim_n \|\pi_G(\check{\mu} * \mu)^n\|^{1/n},$$

so that

$$\|\pi_G(\mu)\|^2 = \sup_{f \in \mathbf{C}[G]} \left(\limsup_n \langle \pi_G(\check{\mu} * \mu)^n f, f \rangle^{1/n} \right),$$

where $\mathbf{C}[G]$ is the group algebra of G, that is, the linear span of $\{\delta_x : x \in G\}$. Now

$$\limsup_n \langle \pi_G(\check{\mu} * \mu)^n \sum_{i=1}^k c_i \delta_{x_i}, \sum_{i=1}^k c_i \delta_{x_i} \rangle^{1/n} \leq \max_{i=1}^k \left(\limsup_n \langle \pi_G(\check{\mu} * \mu)^n \delta_{x_i}, \delta_{x_i} \rangle^{1/n} \right),$$

for all $x_1, \ldots, x_k \in G$ and $c_1, \ldots, c_k$ in $\mathbf{C}$. Since

$$\langle \pi_G(\check{\mu} * \mu)^n \delta_{x_i}, \delta_{x_i} \rangle = (\check{\mu} * \mu)^n(e),$$

this proves the claim. ■

Proposition 5.15. ***([Shal00])*** *Let $G \curvearrowright X$ be an action of the separable locally compact group G on the measure space (X, m), where m is a σ-finite quasi-invariant measure on X. For every probability measure μ on G, we have*

$$\|\pi_X(\mu)\| \geq \|\pi_G(\mu)\|.$$

Proof Set $\nu = \check{\mu} * \mu$ and let ξ be a unit vector in $L^2(X)$ with $\xi \geq 0$. The non-negative function

$$G \mapsto \mathbf{R}, \quad g \mapsto \langle \pi_X(g)\xi, \xi \rangle$$

is continuous and takes the value 1 at e. Hence, there exists a compact neighbourhood U of e such that

$$\langle \pi_X(g)\xi, \xi \rangle \geq \frac{1}{2} \qquad \text{for all} \quad g \in U.$$

For every $n \geq 1$, we have

$$\begin{aligned} \langle \pi_X(\nu)^n \xi, \xi \rangle = \langle \pi_X(\nu^n)\xi, \xi \rangle &= \int_G \langle \pi_X(g)\xi, \xi \rangle d\nu^n(g) \\ &\geq \frac{1}{2} \nu^n(U). \end{aligned}$$

Since

$$\|\pi_X(\mu)\| \geq \langle \pi_X(\mu)^n \xi, \pi_X(\mu)^n \xi \rangle^{1/2n} = \langle \pi_X(\nu)^n \xi, \xi \rangle^{1/2n},$$

it follows from Proposition 5.14 that

$$\|\pi_X(\mu)\| \geq \|\pi_G(\mu)\|. \blacksquare$$

As a consequence of the previous proposition and the spectral characterization of amenability (Theorem 3.20), we see that an action of an amenable group on a measure space *never* has the Spectral Gap Property.

Corollary 5.16. *Let G be an amenable separable locally compact group. Then every action $G \curvearrowright X$ on a measure space (X, m) is co-amenable.* ■

Alternatively, the previous corollary follows also from the fixed point property of amenable groups (Proposition 3.16.ii), applied to the convex set of means on (X, m).

In contrast to this, the co-amenable actions of a Kazhdan group are the actions which are co-amenable for trivial reasons. More precisely, the following result holds.

Corollary 5.17. *Let G be locally compact group with Kazhdan's Property (T) and $G \curvearrowright X$ a co-amenable ergodic action on a measure space (X, m). Then m is equivalent to a G-invariant probability measure on X.*

Proof Indeed, the assumptions imply that there exists a function $f \in L^1(X, m)_{1,+}$ which is G-invariant, that is,

$$\frac{dgm}{dm}(x) f(gx) = f(x) \qquad \text{for all} \quad g \in G,\ x \in X.$$

One checks that the density $f(x)dm(x)$ is a G-invariant measure on X. Moreover, since the action the action is ergodic, $f > 0$ almost everywhere.■

5.4 Linear actions with the Spectral Gap Property

Given a separable locally compact group G and an action $G \curvearrowright X$ on a measure space (X, m), there are only few general results ensuring the Spectral Gap Property for this action.

We will consider linear actions on a vector spaces $V = \mathbf{k}^d$ over an arbitrary local field $\mathbf{k}$. Let G be a subgroup of $GL(V) = GL_d(\mathbf{k})$. We have the following sufficient condition for the Spectral Gap Property for the linear action $G \curvearrowright V \setminus \{0\}$, where V is equipped with a translation invariant measure m. This condition involves the induced action of G on the projective space $\mathbf{P}(V)$ of V. Recall that $\mathbf{P}(V)$ is compact for the quotient topology induced from that of V.

Lemma 5.18. *Let G is a locally compact group which embeds continuously in $PGL(V) = PGL_d(\mathbf{k})$. Assume that there exists no G-invariant probability measure on $\mathbf{P}(V)$. Then the action $G \curvearrowright V \setminus \{0\}$ has the Spectral Gap Property.*

Proof Indeed, assume, by contraposition, that the action $G \curvearrowright V \setminus \{0\}$ is co-amenable. The canonical mapping $p : V \setminus \{0\} \to \mathbf{P}(V)$ is G-equivariant and the Lebesgue measure m' on $\mathbf{P}(V)$ is absolutely continuous with respect to $p_*(m)$. Hence, $G \curvearrowright \mathbf{P}(V)$ is co-amenable (see Corollary 5.4). So, there exists a G-invariant mean M on $L^\infty(\mathbf{P}(V), m')$. Since $\mathbf{P}(V)$ is a compact space, M defines a a probability measure on $\mathbf{P}(V)$, which is G-invariant (see Remark 5.3.i). ■

The following result is a consequence of Furstenberg's celebrated lemma (see [Furs76] or [Zimm84, Corollary 3.2.2]) on stabilizers of probability measures on projective spaces. A subgroup G of $GL(V)$ is said to be *totally irreducible* if every finite index subgroup of G acts irreducibly on V.

Theorem 5.19. *Let G be a totally irreducible subgroup in $SL(V)$. Assume that G is not relatively compact in $SL(V)$. Then $G \curvearrowright V \setminus \{0\}$ has the Spectral Gap Property.*

Proof By the previous lemma, it suffices to show that there exists no G-invariant probability measure on $\mathbf{P}(V)$.

Since G is not relatively compact, there exists a sequence $(g_n)_n$ in G with $\lim_n \|g_n\| = \infty$ (for any norm on $End(V)$). Set $u_n = \dfrac{g_n}{\|g_n\|}$. Then $\|u_n\| = 1$ and $\lim_n \det(u_n) = \lim_n 1/\|g_n\|^d = 0$. Upon passing to a subsequence, we can therefore assume that $u = \lim_n u_n$ exists in $End(V)$ with $\|u\| = 1$ and $\det(u) = 0$. So, $\ker(u)$ and $u(V)$ are proper non-zero subspaces of V. Denote by X_1 and X_2 their images in $\mathbf{P}(V)$.

Assume now, by contradiction, that there exists a G-invariant probability measure ν on $\mathbf{P}(V)$. Write $\nu = \nu_1 + \nu_2$, where ν_1 and ν_2 are positive measures on $\mathbf{P}(V)$ with

$$\nu_1(\mathbf{P}(V) \setminus X_1) = 0 \qquad \text{and} \qquad \nu_2(X_1) = 0.$$

By compactness of the set of bounded positive measures on $\mathbf{P}(V)$, we can assume that $\lim_n g_n\nu_1 = m_1$ and $\lim_n g_n\nu_2 = m_2$ exist. Then

$$\nu = \lim_n g_n\nu = m_1 + m_2.$$

Since ν_2 is supported on $\mathbf{P}(V) \setminus X_1$ and since $\lim_n g_n x = ux \in X_2$ for every $x \in \mathbf{P}(V) \setminus X_1$, the measure m_2 is supported on X_2. We can also assume that $X' = \lim_n g_n X_1$ exists, where X' is the image in $\mathbf{P}(V)$ of a subspace of V of the same dimension as $\ker(u)$. So, ν is supported on the union $X' \cup X_2$ of two proper projective subspaces. This contradicts the fact that G is totally irreducible. Indeed, let F be a projective subspace of minimal dimension with $\nu(F) > 0$. If $\nu(gF \cap F) > 0$, then $gF = F$ by the minimality of dimension of F. Since ν is a probability measure, it follows that $\{gF \ : \ g \in G\}$ is a finite set of projective subspaces. Therefore, the proper projective subspace F is fixed by a subgroup of G of finite index. ■

Remark 5.20. The proof of Theorem 5.19 shows that actually the following stronger statement holds (see Lemma 3 in [Furs76]). Let G be locally compact

group which is minimally almost periodic (that is, every continuous homomorphism of G into a compact group is trivial) and let $\pi : G \mapsto GL(V)$ be a representation of G on a finite dimensional vector space V over a local field. Consider the action of G on $\mathbf{P}(V)$ associated to π. If G preserves a probability measure ν on $\mathbf{P}(V)$, then ν is concentrated on the G-fixed points in $\mathbf{P}(V)$.

Remark 5.21. Shalom obtained in [Shal99] the following nice characterization of co-amenable actions of an algebraic group G over a local field $\mathbf{k}$ on an arbitrary algebraic $\mathbf{k}$-variety: there exists an amenable quotient of G through which every such action factorizes.

5.5 Co-amenable subgroups of semisimple groups

Given a locally compact group G, one may ask which closed subgroups H are co-amenable in G, that is, for which closed subgroups H is $G \curvearrowright G/H$ co-amenable. Observe that, by Corollary 5.17, if G has Kazhdan property, then $G \curvearrowright G/H$ is co-amenable if and only H has finite covolume in G (that is, G/H has a non trivial finite G-invariant Borel measure). So, the only simple real Lie groups interesting for this question are the isometry groups of real or complex hyperbolic spaces. As we will see (Theorem 5.27), in this case, the co-amenability of a subgroup can be read off from its critical exponent.

It is a remarkable fact that co-amenable subgroups of semi-simple algebraic groups are Zariski-dense, a result established in [Guiv80, Corollaire, p. 192] for the case of real algebraic groups and in [Shal99, Corollary 1.7] in general (this generalizes a result from [Stuc92]; see also [IoNe96]).

Theorem 5.22. ***([Guiv80],[Shal99])*** *Let $G = \mathbf{G}(\mathbf{k})$ be the group of $\mathbf{k}$-rational points of a semi-simple connected algebraic group over the local field $\mathbf{k}$ without compact factors. Let H be a closed co-amenable subgroup of G. Then H is Zariski dense in G.*

Proof Let $L \subset G$ be the Zariski-closure of H in G. By a well-known result of Chevalley (see Theorem 5.1 in [Bore91]), there exist a $\mathbf{k}$-rational representation $G \to GL(V)$ over a finite dimensional $\mathbf{k}$-vector space V and a line $\ell \subset V$ such that L is the stabilizer of ℓ in G. The next proposition shows that ℓ is G-fixed and so $L = G$.■

Proposition 5.23. *Let G be as in Theorem 5.22 and let $\pi : G \to GL(V)$ be a continuous representation of G on a finite dimensional vector space V over $\mathbf{k}$. Let H be a closed co-amenable subgroup of G and let $\ell \subset V$ be a line which is H-invariant. Then ℓ is G-invariant (and hence G fixes every point in ℓ).*

Proof We proceed by induction on $\dim V$. The case $\dim V = 1$ being trivial, assume that $\dim V \geq 2$.

Let $x \in \mathbf{P}(V)$ be the image of ℓ in $\mathbf{P}(V)$. The orbital mapping

$$G \to \mathbf{P}(V), \quad g \mapsto gx$$

induces a G-equivariant continuous mapping $f : G/H \to \mathbf{P}(V)$. If M be an invariant mean on $L^\infty(G/H)$, then $f_*(M)$ defines a G-invariant probability measure ν on $\mathbf{P}(V)$. Since G is minimally almost periodic, ν is supported by the G-fixed points in $\mathbf{P}(V)$; see Remark 5.20 above. So, G fixes a line ℓ' in V.

Consider now the representation $\overline{\pi} : G \to GL(V/\ell')$ associated to π. Since $\dim(V/\ell') < \dim V$, the induction hypothesis shows that the image of ℓ in V/ℓ' is $\overline{\pi}(G)$-invariant.

Let $W = \ell + \ell'$. Then W is a G-invariant subspace of V. Moreover, the corresponding homomorphism $G \to GL(W)$ has its image contained in a solvable group. Hence, G acts trivially on W and the claim is proved.■

Corollary 5.24. *Let G be a connected simple non compact real Lie group with trivial center and let H be a proper closed co-amenable subgroup of G. Then H is a discrete subgroup of G.*

Proof Let $\mathfrak{g}$ be the Lie algebra of G and $\mathrm{A}d : G \to GL(\mathfrak{g})$ the adjoint representation. Then G can be identified with the connected component (in the Hausdorff topology) of the real points of the simple algebraic group $\mathrm{A}ut(\mathfrak{g})$ over $\mathbf{R}$.

The Lie subalgebra $\mathfrak{h}$ corresponding to H is $\mathrm{A}d(H)$-invariant and hence $\mathrm{A}d(G)$-invariant, since H is Zariski dense in G. So, $\mathfrak{h}$ is an ideal in $\mathfrak{g}$. Since $\mathfrak{g}$ is a simple Lie algebra, it follows that either $\mathfrak{h} = \mathfrak{g}$ or $\mathfrak{h} = \{0\}$. The first case does not occur as H would be an open proper subgroup of the connected group G. So, $\mathfrak{h} = \{0\}$ and this means that H is discrete.■

We give below examples of co-amenable subgroups H in $SL_2(\mathbf{R})$ which are not lattices. These examples are based on the following proposition about "transitivity" of co-amenability.

Proposition 5.25. *Let G be a locally compact group and $H \subset L$ be closed subgroups of G. If L is co-amenable in G and H is co-amenable in L, then H is co-amenable in G.*

Proof Since H is co-amenable in L, the regular representation $\pi_{L/H}$ of L in $L^2(L/H)$ weakly contains the trivial representation 1_L. By continuity of induction (see Theorem F.3.5 in [BHV]), the induced representation $\mathrm{I}nd_L^G(\pi_{L/H})$ weakly contains $\mathrm{I}nd_L^G 1_L$. Now, $\mathrm{I}nd_L^G(\pi_{L/H})$ is equivalent to the regular representation $\pi_{G/H}$ in $L^2(G/H)$ and $\mathrm{I}nd_L^G 1_L$ to the regular representation $\pi_{G/L}$ in $L^2(G/L)$. It follows that 1_G is weakly contained in $\pi_{G/H}$, since 1_G is weakly contained in $\pi_{G/L}$ by the co-amenability of L in G. So, H is co-amenable in G.■

Example 5.26. Let G be a locally compact group and Γ a lattice in G. Let $\varphi : \Gamma \to S$ be a surjective homomorphism into an amenable discrete group S. The previous proposition shows that $H = \ker(\varphi)$ is a co-amenable subgroup in G.

For instance, if $G = PSL_2(\mathbf{R})$ and $\Gamma = F_2$ is the free group on two generators realized as lattice in G, then $H = [\Gamma, \Gamma]$ is co-amenable in G and is not a lattice. This is also true for every co-compact lattice Γ in $G = PSL_2(\mathbf{R})$. For other examples, see [IoNe96] and [Stuc92].

As a consequence of results from [Corl90] and [Sull87], we now characterize the co-amenability of a (discrete) subgroup of $SO(n,1)$ or $SU(n,1)$ in terms of the value of its critical exponent. Fix a G-invariant Riemannian metric d on the hyperbolic space $\mathbf{H}^n(\mathbf{R})$ in case $G = SO(n,1)$ or $\mathbf{H}^n(\mathbf{C})$ in case $G = SU(n,1)$, normalized to have constant (respectively, maximal) sectional curvature equal to -1 in the real case (respectively, in the complex case).

Recall that the *critical exponent* $\delta(\Gamma)$ of a discrete subgroup Γ of $G = SO(n,1)$ or $G = SU(n,1)$, is defined as

$$\delta(\Gamma) := \inf \left\{ s \in \mathbf{R} \ : \ \sum_{\gamma \in \Gamma} e^{-sd(\gamma x, x)} < \infty \right\},$$

where x is an arbitrary point in $\mathbf{H}^n(\mathbf{R})$ or $\mathbf{H}^n(\mathbf{C})$. If Γ is a lattice in G, then $\delta(\Gamma) = n-1$ in case $G = SO(n,1)$ and $\delta(\Gamma) = 2n$ in case $G = SU(n,1)$; these are the maximal possible values of $\delta(\Gamma)$ for a discrete subgroup Γ.

Theorem 5.27. *Let $G = SO(n,1)$ for $n \geq 2$ or $G = SU(n,1)$ for $n \geq 1$. The closed proper subgroups of G which are co-amenable are exactly the discrete subgroups Γ with maximal critical exponent, that is, such that $\delta(\Gamma) = n-1$ in case $G = SO(n,1)$ and $\delta(\Gamma) = 2n$ in case $G = SU(n,1)$.*

Proof Let Γ be a closed co-amenable proper subgroup of G; in view of Corollary 5.24, we can assume that Γ is discrete,

Let $\lambda_0 \geq 0$ be the bottom of the spectrum of the Laplace-Beltrami operator on the locally symmetric space $\Gamma\backslash X$, where $X = \mathbf{H}^n(\mathbf{R})$ or $\mathbf{H}^n(\mathbf{C})$. It is well-known and easy to prove that 1_G is not weakly contained in the regular representation $\pi_{\Gamma\backslash G}$ on $L^2(\Gamma\backslash G)$ if and only if $\lambda_0 > 0$. Now, it is shown in [Corl90, §4] (see also [Sull87, 2.17]) that $\lambda_0 = d^2/4$ if $\delta(\Gamma) \leq 1/2d$ and $\lambda_0 = \delta(\Gamma)(d - \delta(\Gamma))$ if $\delta(\Gamma) \geq 1/2d$, where $d = n-1$ in case $G = SO(n,1)$ and $d = 2n$ in case $G = SU(n,1)$. So, $\lambda_0 = 0$ if and only if $\delta(\Gamma) = d$.■

6 Quantifying the Spectral Gap Property

Let G be a separable locally compact group and $G \curvearrowright X$ an action on a measure space (X, m). Assume that the corresponding representation π_X has the Spectral Gap Property. (Recall that in case m is G-invariant probability measure, π_X denotes the corresponding representation in $L^2_0(X)$.) So, given a strongly adapted probability measure μ on G, we have $\|\pi_X(\mu)\| < 1$. For various applications (see Section 10), it is important to have an upper bound for $\|\pi_X(\mu)\|$. Such a bound may involve the norm of μ under some "known" representations of the group G, such as the regular representation π_G. As we now see, estimates of this kind are available when G is a subgroup of a simple Lie group, as a consequence of the strong decay of the matrix coefficients of their unitary representations.

Let $(\pi, \mathcal{H})$ be a unitary representation of the locally compact group G. For a real number p with $1 \leq p < +\infty$, the representation π is said to be *strongly L^p*, if

there is dense subspace $D \subset \mathcal{H}$. such that, for every $\xi, \eta \in D$, the matrix coefficient

$$C^{\pi}_{\xi,\eta} : G \to \mathbf{C}, \quad g \mapsto \langle \pi(g)\xi, \eta \rangle$$

belongs to $L^p(G)$. Observe that then π is strongly L^q for any $q > p$, since $C^{\pi}_{\xi,\eta}$ is bounded.

Strongly L^p-representations for $p = 2$ or $p \geq 2+\varepsilon$ are closely tied to the regular representation π_G.

Proposition 6.1. *Let $(\pi, \mathcal{H})$ be a unitary representation of the locally compact group G.*

(i) If π is strongly L^2, then π is contained in an infinite multiple of the regular representation π_G.

(ii) If π is strongly L^p for every $p > 2$, then π is weakly contained in the regular representation π_G.

Concerning the proofs, see Proposition 1.2.3 in Chapter V of [HoTa92] for (i) and Theorem 1 in [CHH88] for (ii). Representations which are strongly L^p for every $p > 2$ also often called tempered representations.

A crucial fact for the sequel is the following theorem; part (i) is due to Cowling (Theorem 2.4.2 in [Cowl79]) and part (ii) to Moore (Proposition 3.6 in [Moor87]).

Theorem 6.2. *Let G be a simple real Lie group with finite center.*

*(i) **([Cowl79])** If the real rank of G is at least two, then there exists $p(G)$ in $(2, +\infty)$ such that every unitary representation of G with no no-zero invariant vectors is strongly L^p for every $p > p(G)$.*

*(ii) **([Moor87])** If the real rank of G is one, then every unitary representation of G with the Spectral Gap Property is strongly L^p for some $p \in [2, +\infty)$*

Remark 6.3. Part (i) of the previous theorem holds more generally when G is the group of $\mathbf{k}$-rational points of a simple algebraic group over a local field $\mathbf{k}$, with $\mathbf{k}$-rank at least two (see Theorem 5.6 in [HoMo79]). Estimates for the optimal bounds $p(G)$ in (i) have been given in [Howe82], [Li95], [LiZh96] and [Oh02]. For instance, one has $p(SL_n(\mathbf{R})) = 2n - 2$ for $n \geq 3$. Cowling's result also covers the case where G has $\mathbf{R}$-rank one and is a Kazhdan group (that is, when G is locally isomorphic to $Sp(1, n)$ or F_4^{-20})).

To our knowledge, it is not known whether (ii) is true for all local fields $\mathbf{k}$ and all simple algebraic groups with $\mathbf{k}$-rank 1. We suspect that it is indeed the case.

The results above are derived using pointwise estimates for matrix coefficients of the involved unitary representation, and this provides a more precise and often useful information about them (see [Cowl79], [HoTa92]).

Combining Theorem 6.2 with Proposition 6.1, we obtain the following remarkable result. Recall that, for an integer $k \geq 1$, the k-fold tensor product $\pi^{\otimes k}$ of a unitary representation $(\pi, \mathcal{H})$ is the unitary representation of G defined on the tensor product Hilbert space $\mathcal{H}^{\otimes k}$ by

$$\pi^{\otimes k}(g)(\xi_1 \otimes \cdots \otimes \xi_k) = \pi(g)\xi_1 \otimes \cdots \otimes \pi(g)\xi_k \qquad \text{for all} \quad g \in G,\, \xi_1, \ldots, \xi_k \in \mathcal{H}.$$

Corollary 6.4. *Let G be a simple real Lie group with finite center and π a unitary representation of G with the Spectral Gap Property. Then there exists $N \geq 1$ such that $\pi^{\otimes N}$ is contained in an infinite multiple of the regular representation π_G. Moreover, in case G has Property (T), the integer N can be chosen independently of π.*

Proof Let $1 \leq p < \infty$ be such that π is strongly L^p. Let N be an integer with $N \geq p/2$. Then $\pi^{\otimes N}$ is strongly L^2 and the claim follows. ■

Given a probability measure, we now deduce from the previous corollary estimates for the norm $\pi(\mu)$ in terms of $\pi_G(\mu)$, For this, we will use in a crucial way the following estimate which appears in the proof of Theorem 1 in [Nevo98]. Recall that, for a unitary representation $(\pi, \mathcal{H})$ of G, the contragredient (or conjugate) representation $\overline{\pi}$ acts on the conjugate Hilbert space $\overline{\mathcal{H}}$.

Proposition 6.5. ***([Nevo98])*** *Let μ be a probability measure on the locally compact group G. Let $(\pi, \mathcal{H})$ be a unitary representation of G. For every integer $k \geq 1$, we have*

$$\|\pi(\mu)\| \leq \| \left(\pi \otimes \overline{\pi}\right)^{\otimes k} (\mu)\|^{1/2k},$$

Proof Using Jensen's inequality, we have for every vector $\xi \in \mathcal{H}$,

$$\begin{aligned}
\|\pi(\mu)\xi\|^{4k} = |\langle \pi(\check{\mu} * \mu)\xi, \xi\rangle|^{2k} &= \left| \int_G \langle \pi(g)\xi, \xi\rangle d(\check{\mu} * \mu)(g) \right|^{2k} \\
&\leq \int_G \langle |\pi(g)\xi, \xi\rangle|^{2k} d(\check{\mu} * \mu)(g) \\
&= \int_G |\langle (\pi \otimes \overline{\pi})(g)(\xi \otimes \xi), \xi \otimes \xi\rangle|^k d(\check{\mu} * \mu)(g) \\
&= \int_G \langle (\pi \otimes \overline{\pi})^{\otimes k}(g)(\xi \otimes \xi)^{\otimes k}, (\xi \otimes \xi)^{\otimes k}\rangle d(\check{\mu} * \mu)(g) \\
&= |\langle (\pi \otimes \overline{\pi})^{\otimes k}(\check{\mu} * \mu)(\xi \otimes \xi)^{\otimes k}, (\xi \otimes \xi)^{\otimes k}\rangle| \\
&= \|(\pi \otimes \overline{\pi})^{\otimes k}(\mu)(\xi \otimes \xi)^{\otimes k}\|^2.
\end{aligned}$$

and the claim follows. ■

Corollary 6.6. *Let G be a simple real Lie group with finite center and π a unitary representation of G with the Spectral Gap Property. Then there exists $N \geq 1$ such that, for every probability measure μ on G, we have*

$$\|\pi(\mu)\| \leq \|\pi_G(\mu)\|^{1/N}.$$

Moreover, in case G has Property (T), the integer N can be chosen to be independent of π.

Proof Let $1 \leq p < \infty$ be such that π is strongly L^p. Let k be an integer with $k \geq p/4$. Then $(\pi \otimes \overline{\pi})^{\otimes k}$ is strongly L^2 and is hence is contained in a multiple of π_G. The claim follows now from Proposition 6.5. ■

Remark 6.7. Results about decay of matrix coefficients of semisimple algebraic groups G as described in Theorem 6.2 and Corollary 6.6 have been used in the monograph [GoNe10] in order to prove impressive quantitative ergodic theorems for families of averaging operators on G or on a lattice in G as well as results on counting lattice points (with explicit error term), with various applications to counting problems from number theory.

7 Probability measure preserving actions

From now on, we will deal only with group actions which preserve a probability measure. So, let G be a locally compact group acting on a measure space (X, m), where m is a G-invariant probability measure. Recall that $G \curvearrowright X$ has the Spectral Gap Property if the Koopman representation π_X on $L^2_0(X) = (\mathbf{C}\mathbf{1}_X)^\perp$ does not have almost invariant vectors.

7.1 Amenability as obstruction to the Spectral Gap Property

The following result shows that amenability is an obstruction to the Spectral Gap Property also in the probability measure preserving case, at least for discrete groups.

Theorem 7.1. ***([JuRo79])*** *Let Γ be a countable amenable group with a measure preserving action on a* non atomic *probability space (X, m). Then $\Gamma \curvearrowright X$ does not have the Spectral Gap Property.*

Proof Let S be a finite symmetric subset of Γ and $\varepsilon > 0$. We want to construct a function $f \in L^2_0(X)$ which is (S, ε)-invariant. Since Γ is amenable, there exists a finite subset F of Γ with $|Fs \triangle F| \leq \varepsilon |F|$ for all $s \in S$, by Følner's Theorem 3.18.

Since m is not atomic, there exists a measurable subset A of X with $m(A) = 1/2|F|$. Consider the function $\xi : X \to \mathbf{N}$ defined by $\xi = \sum_{\gamma \in F} \pi_X(\gamma^{-1})(\mathbf{1}_A)$:

$$\xi(x) = \sum_{\gamma \in F} \mathbf{1}_{\gamma^{-1}A}(x) \qquad \text{for all} \quad x \in X.$$

We have

$$\|\xi\|_1 = \sum_{\gamma \in F} \int_X \pi_X(\gamma^{-1})(\mathbf{1}_A) dm(x) = \sum_{\gamma \in F} m(A) = |F| m(A) = \frac{1}{2}.$$

Let $s \in S$ be fixed. Then

$$\begin{aligned}\|\pi_X(s^{-1})\xi - \xi\|_1 &= \left\|\sum_{\gamma\in F} \mathbf{1}_{(\gamma s)^{-1}A} - \mathbf{1}_{\gamma^{-1}A}\right\|_1 \\ &= \left\|\sum_{\gamma\in Fs} \mathbf{1}_{\gamma^{-1}A} - \sum_{\gamma\in F} \mathbf{1}_{\gamma^{-1}A}\right\|_1 \\ &= \left\|\sum_{\gamma\in Fs\setminus F} \mathbf{1}_{\gamma^{-1}A} - \sum_{\gamma\in F\setminus Fs} \mathbf{1}_{\gamma^{-1}A}\right\|_1 \\ &\leq |Fs\triangle F| m(A) \leq \varepsilon |F| m(A) = \varepsilon/2.\end{aligned}$$

Let $g = \sqrt{\xi}$. Then $\|g\|_2^2 = \|\xi\|_1 = 1/2$ and

$$\|\pi_X(s^{-1})g - g\|_2^2 \leq \|\pi_X(s^{-1})\xi - \xi\|_1 \leq \varepsilon/2.$$

for all $s \in S$. Since ξ takes its values in $\mathbf{N}$, we have $g = \sqrt{\xi} \leq \xi$. Hence,

$$(\int_X g dm)^2 \leq (\int_X \xi dm)^2 = \|\xi\|_1^2 = 1/4$$

and so $\|g - \int_X g dm \mathbf{1}_X\|_2^2 \geq \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$. Set now

$$f := \frac{g - \int_X g dm \mathbf{1}_X}{\|g - \int_X g dm \mathbf{1}_X\|_2}.$$

Then $\|f\|_2 = 1$ and

$$\|\pi_X(s^{-1})f - f\|_2^2 \leq \frac{4\varepsilon}{2} = 2\varepsilon.\blacksquare$$

Remark 7.2. Theorem 7.1 may fail when G is amenable and not discrete. Indeed, when G is a compact infinite group, the action G on itself by left translation has the Spectral Gap Property (as probability measure preserving action).

Let $\Gamma \curvearrowright X$ be a measure preserving action on a probability space (X, m). A *factor* of the triple (X, m, Γ) is a triple (Y, m', Γ') where Γ' is a group acting on a probability space (Y, m') such that there exists a homomorphism $\Gamma \to \Gamma'$ and a measurable map $\Phi : X \to Y$ with $\Phi_*(m) = m'$ and intertwining the two actions. Here is a corollary of Theorem 7.1.

Corollary 7.3. *Let (Y, m', Γ') be a factor of (X, m, Γ). Assume that m' is non atomic and that Γ' is amenable. Then $\Gamma \curvearrowright X$ does not have the Spectral Gap Property.*

Proof Observe that $\{f \circ \Phi : f \in L_0^2(Y, m')\}$ is a Γ-invariant closed subspace of $L_0^2(X, m)$.$\blacksquare$

Remark 7.4. The previous corollary shows that, given a discrete group Γ and an action $\Gamma \curvearrowright X$ on a probability space (X, m), the existence of a factor (Y, m', Γ') of (X, m, Γ), with Γ' amenable and m' non atomic, is an obstruction to the Spectral Gap Property for $\Gamma \curvearrowright X$. In some cases, such as $\Gamma \subset SL_n(\mathbf{Z})$ acting by automorphisms on $X = \mathbf{T}^n$, this is the only obstruction (see Theorem 9.1).

7.2 Spectral Gap Property and Orbit Equivalence

Orbit Equivalence is a subject where one aims to study groups G and group actions $G \curvearrowright X$ on measure spaces through the equivalence relation on X defined by the partition into G-orbits (for a recent survey on this theme, see [Gabo10]).

Definition 7.5. Two measure preserving ergodic actions of two countable groups Γ_1 and Γ_2 on probability spaces (X_1, m_1) and (X_2, m_2) are orbit equivalent if there exist measurable subsets X_1' and X_2' with measure 1 in X_1 and X_2 and a Borel isomorphism $f : X_1' \to X_2'$ with $f_*(m_1) = m_2$ such that, for m_1-almost every $x \in X_1'$, we have

$$f(\Gamma_1 x) = \Gamma_2 f(x).$$

A natural question is whether the Spectral Gap Property is an invariant of orbit equivalence. This is the not case, as is shown by the following result of Hjorth and Kechris (Theorem A. 3.2 in [HjKe05]). Recall that, if Γ is a non amenable group, its Bernoulli action $\Gamma \curvearrowright \{0,1\}^\Gamma$ has the Spectral Gap Property (see Example 4.3).

Theorem 7.6. ***([HjKe05])*** *The Bernoulli action of the free group $\Gamma = F_2$ on 2 generators on $X = \{0,1\}^\Gamma$ is orbit equivalent to another action of Γ on X which does not have the Spectral Gap Property.*

There is however a property related to the Spectral Gap Property which is an invariant of orbit equivalence, namely strong ergodicity in the sense of Schmidt (see [Schm80], [Schm81]),

Let $\Gamma \curvearrowright X$ be a measure preserving action of a countable group Γ on the probability space (X, m). If this action is ergodic, there is no non-trivial invariant measurable subset of X. Nevertheless, there might exist non-trivial asymptotically invariant subsets in the following sense.

Definition 7.7. A sequence of measurable subsets $(A_n)_n$ of X is said to be asymptotically invariant if

$$\lim_n m(\gamma A_n \triangle A_n) = 0 \qquad \text{for all} \quad \gamma \in \Gamma.$$

An asymptotically invariant sequence $(A_n)_n$ is said to be non-trivial if

$$\inf_n m(A_n)(1 - m(A_n)) > 0.$$

The action of Γ on X is *strongly ergodic* if there exists no non-trivial asymptotically invariant sequence in X.

Strong ergodicity is an invariant of orbit equivalence. Indeed, let $[\Gamma]$ denote the *full group* of Γ, that is, the group of all measure preserving measurable bijections $\varphi : X \to X$ with $\varphi(x) \in \Gamma x$ for m-almost every $x \in X$. The following lemma shows that the collection of asymptotically invariant sequences for Γ and hence strong ergodicity only depends on the equivalence relation on X defined by the action of Γ.

Lemma 7.8. *Let $(A_n)_n$ be an asymptotically invariant sequence for Γ. Then $\lim_n m(\varphi(A_n) \triangle A_n) = 0$, for every $\varphi \in [\Gamma]$.*

Proof Let $\varepsilon > 0$. There exist Borel subsets $X_1, \ldots, X_k$ of X with

$$m(X \setminus \bigcup_{i=1}^{k} X_i) \leq \varepsilon/4$$

and elements $\gamma_1, \ldots, \gamma_k$ in Γ such that $\varphi(x) = \gamma_i x$ for all $x \in X_i$ and $i = 1, \ldots, k$. Let $N \in \mathbf{N}$ be such that

$$m(\gamma_i A_n \triangle A_n) \leq \varepsilon/4k \qquad \text{for all} \quad n \geq N,\, i = 1, \ldots, k.$$

Since

$$\varphi(A_n) \setminus A_n \subset \bigcup_{i=1}^{k} (\gamma_i A_n \setminus A_n) \cup \varphi(X \setminus \bigcup_{i=1}^{k} X_i),$$

it follows that

$$m(\varphi(A_n) \setminus A_n) \leq \varepsilon/2 \qquad \text{for all} \quad n \geq N,\, i = 1, \ldots, k.$$

As $m(A_n \setminus \varphi(A_n)) = m(\varphi^{-1}(A_n) \setminus A_n)$ and, similarly,

$$\varphi^{-1}(A_n) \setminus A_n \subset \bigcup_{i=1}^{k} (\gamma_i^{-1} A_n \setminus A_n) \cup \varphi^{-1}(X \setminus \bigcup_{i=1}^{k} \gamma_i X_i),$$

we obtain $m(\varphi(A_n) \triangle A_n) \leq \varepsilon$ for all $n \geq N$.■

The following elementary proposition shows that the Spectral Gap Property implies strong ergodicity and this is often the way strong ergodicity is established in specific examples.

Proposition 7.9. *Let $\Gamma \curvearrowright X$ be a measure preserving action of a countable group Γ on the probability space (X, m). If $\Gamma \curvearrowright X$ has the Spectral Gap Property, then $\Gamma \curvearrowright X$ is strongly ergodic.*

Proof Assume, by contradiction, that there exists a non-trivial asymptotically invariant sequence $(A_n)_n$ of measurable subsets of X. Set

$$f_n = \mathbf{1}_{A_n} - m(A_n)\mathbf{1}_X.$$

Then $f_n \in L^2_0(X, m)$ and, for every $\gamma \in \Gamma$, we have

$$\|f_n\|^2 = m(A_n)(1 - m(A_n)) \qquad \text{and} \qquad \|\pi_X(\gamma) f_n - f_n\|^2 = m(\gamma A_n \triangle A_n).$$

So, π_X^0 has almost invariant vectors and this is a contradiction. ■

The following example, due to K. Schmidt, shows that the converse does not hold in the previous proposition (that is, strong ergodicity and the Spectral Gap Property do not coincide). Of course, since strong ergodicity is an invariant of orbit equivalence, this also follows from Theorem 7.6 above.

Example 7.10. ([Schm81]) Let $\Gamma = F_3$ be the free group on the three generators a, b, c. Let Γ_2 be the free subgroup of Γ generated by a, b and let $Y = \mathbf{T}^2$ be the 2-torus endowed with Lebesgue measure λ. Then Γ_2 acts as a group of automorphisms on Y through a surjective homomorphism $\Gamma_2 \to SL_2(\mathbf{Z})$. The action $\Gamma_2 \curvearrowright Y$ has the Spectral Gap Property (see Example 9.4) and is hence strongly ergodic by the previous proposition.

Set $X = Y \times \mathbf{N}$ and define a finite positive measure m on X by

$$m(A) = \sum_{n=1}^{\infty} \frac{1}{n^2} \lambda\left(\{y \in Y \ : \ (y,n) \in A\}\right)$$

for every Borel subset A of X. We define a measure preserving action of Γ_2 on (X, m) by

$$(\gamma, (y,n)) \mapsto (\gamma y, n) \qquad \text{for all} \quad \gamma \in \Gamma_2,\ (y,n) \in X.$$

Let $T : X \to X$ be a measure preserving bijection with $T(X_n) \subset X_{n-1}$ for all $n \geq 2$, where $X_n = Y \times \{n\}$. Such a mapping T can be constructed as follows. For every $n \geq 2$, choose a Borel subset Z_{n-1} of X_{n-1} with $m(Z_{n-1}) = m(X_n) = 1/n^2$. There exist a measure preserving bijection $T_n : X_n \to Z_{n-1}$ for every $n \geq 2$ as well as a measure preserving bijection $T_1 : X_1 \to X \setminus \cup_{n \geq 2} Z_{n-1}$. This gives rise to a measure preserving bijection $T : X \to X$ defined by $T|_{X_n} = T_n$ for all $n \geq 1$.

We extend the action of Γ_2 on X to a measure preserving action of Γ by letting c act as T. As we are going to show, the action $\Gamma \curvearrowright X$ is strongly ergodic and does not have the Spectral Gap Property.

Let $(A_k)_k$ be an asymptotically invariant sequence for Γ. For every $n \geq 1$, let $A_{k,n} = A_k \cap X_n$. Since Γ_2 leaves X_n invariant, $(A_{k,n})_k$ is an asymptotically invariant sequence for Γ_2. Hence,

$$\lim_k m(A_{k,n})(m(X_n) - m(A_{k,n})) = 0$$

and therefore, for every $n \geq 1$, we have either $\lim_k m(A_{k,n}) = 0$ or $\lim_k m(A_{k,n}) = m(X_n)$. We claim that we have either $\lim_k m(A_{k,n}) = 0$ for all n or $\lim_k m(A_{k,n}) = m(X_n)$ for all n. Once proved, this will imply that $\lim_k m(A_k) = 0$ or $\lim_k m(A_k) = m(X)$

To prove the claim, assume that $\lim_k m(A_{k,n}) = m(X_n)$ for some n. Then $\lim_k m(A_{k,n-1}) = m(X_{n-1})$ and $\lim_k m(A_{k,n+1}) = m(X_{n+1})$ in case $n \geq 2$ and $\lim_k m(A_{k,l}) = m(X_l)$ for all $l \geq 1$, in case $n = 1$.

Indeed, let $n \geq 2$. Since $T(A_{k,n}) \subset X_{n-1}$ and $T^{-1}(A_{k,n}) \subset X_{n+1}$ and since $\lim_k m(T^{\pm 1} A_k \triangle A_k) = 0$, it follows that $\lim_k m(A_{k,n-1}) \neq 0$ and $\lim_k m(A_{k,n+1}) \neq 0$ and so $\lim_k m(A_{k,n-1}) = m(X_{n-1})$ and $\lim_k m(A_{k,n+1}) = m(X_{n+1})$. The case $n = 1$ is treated similarly. This proves that the action $\Gamma \curvearrowright X$ is strongly ergodic.

For every $n \geq 2$, let $C_n = \cup_{k \geq n} X_k$. Then C_n is Γ_2-invariant,

$$m(T(C_n) \triangle C_n) = \frac{1}{(n-1)^2},$$

and

$$m(C_n) = \sum_{k \geq n} \frac{1}{k^2} = \mathrm{O}\left(\frac{1}{n}\right).$$

Hence, $a(C_n) \triangle C_n = \emptyset, b(C_n) \triangle C_n = \emptyset$, and

$$\lim_n \frac{m(T(C_n) \triangle C_n)}{m(C_n)} = 0.$$

This implies that

$$\lim_n \frac{m(\gamma C_n \triangle C_n)}{m(C_n)} = 0 \qquad \text{for all} \quad \gamma \in \Gamma.$$

With $f_n := \mathbf{1}_{C_n} - m(C_n)\mathbf{1}_X \in L^2_0(X, m)$, we have, for every $\gamma \in \Gamma$,

$$\lim_n \frac{\|\pi_X(\gamma) f_n - f_n\|^2}{\|f_n\|^2} = \lim_n \frac{m(\gamma C_n \triangle C_n)}{m(C_n)(1 - m(C_n))} = 0,$$

and this shows that $\Gamma \curvearrowright X$ does not have the Spectral Gap Property.

Schmidt proved in [Schm81, (2.4) Theorem] the following stronger version of Theorem 7.1.

Theorem 7.11. ***([Schm81])*** *Let Γ be a countable amenable group. Then no measure preserving action $\Gamma \curvearrowright X$ on a non-atomic probability space (X, m) is strongly ergodic.*

8 Spectral gap property for homogeneous spaces

In this section, we will deal with the class of probability measure preserving actions arising from homogeneous spaces associated to lattices. The setting we consider is as follows.

Let G be a locally compact group (for instance, a connected Lie group) and Γ a lattice in G. Let G/Γ be equipped with its unique G-invariant probability measure m. We consider the action of $G \curvearrowright G/\Gamma$ and the corresponding Koopman representation $\pi_{G/\Gamma}$ of G on $L^2_0(G/\Gamma)$. We then ask: does the action $G \curvearrowright G/\Gamma$ have the Spectral Gap Property?

8.1 The case of a cocompact lattice

As we now see, the question above has a positive answer for *uniform* lattices.

Proposition 8.1. *Let G be a locally compact group and Γ a cocompact lattice in G. Then $G \curvearrowright G/\Gamma$ has the Spectral Gap Property.*

Proof We denote by π the Koopman representation on $L^2(G/\Gamma)$ (and *not* on $L^2_0(G/\Gamma)$).

We first check the crucial fact that, for every $f \in C_c(G)$, the convolution operator $\pi(f) : L^2(G/\Gamma) \to L^2(G/\Gamma)$ is a compact operator. Indeed, let $X \subset G$ be a compact fundamental domain for the action of Γ on G. Thus, X is a compact subset of G such that

$$G = \coprod_{\gamma\in\Gamma} X\gamma.$$

We denote by m a Haar measure on G. (Observe that G is unimodular, since it contains a lattice and so m is right and left invariant; see Proposition B.2.2 in [BHV].)

View $\xi \in L^2(G/\Gamma)$ as a function on G which is Γ-invariant on the right; then, for every $x \in X \cong G/\Gamma$,

$$\begin{aligned}\pi(f)\xi(x) &= \int_G f(g)\xi(g^{-1}x)dm(g)\\ &= \int_G f(xg^{-1})\xi(g)dm(g)\\ &= \int_X \sum_{\gamma\in\Gamma} f(x\gamma^{-1}y^{-1})\xi(y\gamma)dm(y)\\ &= \int_X K(x,y)\xi(y)dm(y),\end{aligned}$$

where $K(x,y) = \sum_{\gamma\in\Gamma} f(x\gamma^{-1}y^{-1})$. Observe that there are only finitely many γ in Γ for which $x\gamma^{-1}y^{-1}$ is in the compact support of f for some $(x,y) \in X \times X$. So, K is continuous on $X \times X$ and hence $\pi(f)$ is an integral operator with continuous kernel. Since, X is compact, $\pi(f)$ is therefore a Hilbert-Schmidt operator on $L^2(X) = L^2(G/\Gamma)$.

Now, let $f \in C_c(G)$ with $f \geq 0, \int_G f(g)dm(g) = 1$ and $\check{f} = f$ and such that $\mathrm{supp}(f)$ generates G. Then $\pi(f)$ is a compact self-adjoint operator. Hence, 1 is an isolated spectral value of $\pi(f)$ and so $G \curvearrowright G/\Gamma$ has the Spectral Gap Property, by Proposition 3.4.■

Remark 8.2. In fact, pushing the analysis a bit further, one can show that $L^2(G/\Gamma)$ decomposes as a Hilbert space direct sum $L^2(G/\Gamma) = \bigoplus_i m_i\mathcal{H}_i$ of irreducible G-invariant subspaces $\mathcal{H}_i$, each of which occuring with finite multiplicity m_i (see Theorem in Chap. I, Section 2.3 in [GGPS90]).

8.2 The case of a non cocompact lattice

. The problem of establishing the Spectral Gap Property for $G \curvearrowright G/\Gamma$ is much harder in the case of a non cocompact lattice Γ. Only partial results are known.

Part (i) of the following theorem has been conjectured in [Marg91, Chapter III. Remark 1.12] and proved in [BeCo08]; part (ii) is from [BeLu11].

Theorem 8.3. ***([BeCo08], [BeLu11])*** *Let G be a locally compact group and Γ a lattice in G. Then $G \curvearrowright G/\Gamma$ has the Spectral Gap Property in the following cases:*

(i) G is a real Lie group;

(i) $G = \mathbf{G}(\mathbf{k})$ is the group of $\mathbf{k}$-rational points of a simple algebraic group $\mathbf{G}$ over a local field $\mathbf{k}$.

Concerning part (ii) of the theorem, observe that when $\mathbf{k}$ is non-archimedean with characteristic 0, every lattice Γ in $\mathbf{G}(\mathbf{k})$ is cocompact (see [Serr, p.84]) and the result follows from Proposition 8.1. By way of contrast, G has many non uniform lattices when the characteristic of $\mathbf{k}$ is non zero (see [Serr] and [Lubo91]). So, for the proof, it suffices to consider the case where the characteristic of $\mathbf{k}$ is non-zero and where $\mathbf{k}-\mathrm{rank}(\mathbf{G}) = 1$. (Recall that $\mathbf{G}(\mathbf{k})$ has Kazhdan's Property (T) when $\mathbf{k}-\mathrm{rank}(\mathbf{G}) \geq 2$; see Theorem 3.6.) In this situation, it is known that $G = \mathbf{G}(\mathbf{k})$ acts by automorphisms on the associated Bruhat-Tits tree T (see [Serr]), which is a regular or a bi-partite bi-regular tree. The proof then is reduced to showing that the projection on the quotient graph $X = \Gamma\backslash T$ of the standard random walk on T has a spectral gap.

8.3 An example: the case of $PGL_2(\mathbf{F}_q((t^{-1})))/PGL_2(\mathbf{F}_q[t])$

We will give in this section a complete proof of part (ii) of Theorem 8.3 for the special case $G = PGL_2(\mathbf{F}_q((t^{-1})))$ and $\Gamma = PGL_2(\mathbf{F}_q[t])$, where $\mathbf{F}_q((t^{-1}))$ is the local field of formal Laurent series with coefficients in the finite field with q elements. Following an idea from [BeLu11], the proof uses a version of Cheeger's inequality for Markov chains on a countable state space for which we give a full proof.

8.3.1 A Cheeger inequality for Markov chains on a countable state space

Let X be a countable set and let μ be a Markov kernel on X, that is, a mapping $\mu : X \times X \to \mathbf{R}^+$ such that $\sum_{y\in X} \mu(x, y) = 1$ for all $x \in X$. Such a kernel defines a Markov chain $(Z_n)_{n\geq 0}$ on X, with transition probabilities

$$\mathbf{P}(Z_{n+1} = y | Z_n = x) = \mu(x, y).$$

We assume that μ is irreducible, that is, given any pair (x, y) of distinct points in X, there exist an integer $n \geq 1$ and a sequence $x = x_0, x_1, \ldots, x_n = y$ in X such that $\mu(x_{j-1}, x_j) > 0$ for any $j \in \{1, \ldots, n\}$. We also assume that μ is reversible: there exists a stationary measure m for μ, that is a function $m : X \to \mathbf{R}^*_+$ such that

$$m(x)\mu(x, y) = m(y)\mu(y, x) \quad \text{for all} \quad x, y \in X.$$

The corresponding Markov operator M_μ on $\ell^2(X, m)$ is defined by

$$M_\mu f(x) = \sum_{(x,y)\in X^2} \mu(x, y)f(y) \qquad \text{for all} \quad f \in \ell^2(X, m).$$

Since m is stationary measure for μ, one checks that the operator M_μ is self-adjoint with $\|M_\mu\| \leq 1$.

From now on, we assume that the stationary measure is a *finite measure* and, hence without loss of generality, that m is a *probability measure.* Then 1 is an eigenvalue of M_μ, with $\mathbf{1}_X$ as eigenfunction. We will be concerned with finding upper bounds for the spectrum $\sigma(M_\mu)$ of M_μ restricted to $\ell^2_0(X,m) = (\mathbf{C}\mathbf{1}_X)^\perp$.

It is convenient to consider the Laplacian $\Delta_\mu = I - M_\mu$, which is a nonnegative operator on $\ell^2(X,m)$ with $\|\Delta_\mu\| \leq 2$. So, we seek a lower bound for

$$\lambda_1 = \inf \sigma\left(\Delta_\mu|_{\ell^2_0(X,m))}\right).$$

Such a bound will be given in terms of the *Cheeger constant* $h(X)$ of the random walk μ which is defined as follows.

Let $\widetilde{\mu}$ be the (symmetric) measure on $X \times X$ defined by

$$\widetilde{\mu}(x,y) = m(x)\mu(x,y) \quad \text{for all} \quad (x,y) \in X \times X.$$

Set

$$h(X) := \inf \frac{\widetilde{\mu}(S \times S^c)}{m(S)m(S^c)},$$

where the infimum is taken over all non empty subsets S of X and where $S^c = X \backslash S$.

The Cheeger inequality is an isoperimetric inequality originally proved for the Laplacian acting on the L^2-space of a compact Riemannian manifold (see [Chee70], [Chav93]) and carried over to the setting of Markov chains in [LaSo88],[SiJe89]. Versions of Cheeger's inequality for weighted graphs were considered by several authors (see, for instance, [Alon86], [DiSt91], [Dodz84], [Mokh03], [Morg94], [BKW15]).

Theorem 8.4. ***([LaSo88],[SiJe89])*** *We have*

$$\frac{h(X)^2}{8} \leq \lambda_1.$$

Consequently, if $h(X) > 0$ then $\lambda_1 > 0$.

We now proceed with the proof of Cheeger's inequality. The first ingredient is the following lemma, which is straightforward to check.

Lemma 8.5. *(i) λ_1 is the infimum of the Rayleigh quotients $\frac{\langle \Delta_\mu f, f\rangle}{\|f\|^2}$ over all $f \in \ell^2_0(X)$, $f \neq 0$ and f real valued.*
(ii) For every $f \in \ell^2(X,m)$, we have

$$\langle \Delta_\mu f, f\rangle = \frac{1}{2} \sum_{(x,y)\in X^2} |f(y)-f(x)|^2 m(x)\mu(x,y) = \frac{1}{2} \sum_{(x,y)\in X^2} |f(y)-f(x)|^2 \widetilde{\mu}(x,y). \blacksquare$$

The next ingredient in the proof are the so-called area and co-area formulas.

Lemma 8.6. *Let $u : X \to \mathbf{R}^+$ be in $\ell^1(X, m)$ and set $S_t = \{x \in X \ : \ u(x) > t\}$ for $t \geq 0$. Then the following formulas holds:*
(i) ***(Area formula)***

$$\sum_{x\in X} u(x)m(x) = \int_0^\infty m(S_t)dt.$$

(ii) ***(Co-area formula)***

$$\frac{1}{2}\sum_{(x,y)\in X^2} |u(y) - u(x)|\widetilde{\mu}(x, y) = \int_0^\infty \widetilde{\mu}(S_t \times S_t^c)dt.$$

Proof (i) For $x \in X$, we have $x \in S_t$ if and only if $\mathbf{1}_{(t,\infty)}(u(x)) = 1$ and hence

$$\begin{aligned}\int_0^\infty m(S_t)dt &= \int_0^\infty \left(\sum_{x\in X} m(x)\mathbf{1}_{(t,\infty)}(u(x))\right) dt = \sum_{x\in X} m(x)\int_0^\infty \mathbf{1}_{(t,\infty)}(u(x))dt \\ &= \sum_{x\in X} m(x)\int_0^\infty \mathbf{1}_{[0,u(x))}(t)dt = \sum_{x\in X} m(x)u(x).\end{aligned}$$

(ii) For $(x, y) \in X \times X$, denote by $I_{x,y} \subset \mathbf{R}^+$ the interval between $u(x)$ and $u(y)$. So, $|I_{x,y}| = |u(y) - u(x)|$. We have $(x, y) \in (S_t \times S_t^c) \cup (S_t^c \times S_t)$ if and only if $t \in I_{x,y}$, that is if and only if $\mathbf{1}_{I_{x,y}}(t) = 1$. Since $\widetilde{\mu}$ is symmetric, we have

$$\widetilde{\mu}\left((S_t \times S_t^c) \cup (S_t^c \times S_t)\right) = 2\widetilde{\mu}(S_t \times S_t^c)$$

and hence

$$\begin{aligned}2\int_0^\infty \widetilde{\mu}(S_t \times S_t^c)dt &= \int_0^\infty \left(\sum_{x,y} \widetilde{\mu}(x, y)\mathbf{1}_{I_{x,y}}(t)\right) dt \\ &= \sum_{x,y} \widetilde{\mu}(x, y)\int_0^\infty \mathbf{1}_{I_{x,y}}(t)dt \\ &= \sum_{x,y} \widetilde{\mu}(x, y)|u(y) - u(x)|. \ \blacksquare\end{aligned}$$

Proof of Cheeger's inequality (Theorem 8.4) Let $f \in \ell_0^2(X, m)$ with real values. In view of Lemma 8.5, we have to prove the following inequality:

$$\frac{h(X)^2}{4}\|f\|^2 \leq \sum_{(x,y)\in X^2} |f(y) - f(x)|^2\widetilde{\mu}(x, y). \qquad (*)$$

Let $c \in \mathbf{R}$ be such that $m(\{x \in X \ : \ (f+c)(x) > 0\}) \leq 1/2$. Since $\sum_{x\in X} f(x)m(x) = 0$, one checks that $\|f + c\|^2 \geq \|f\|^2$. So, upon replacing f by $f + c$, we can assume that $m(\{x \in X \ : \ f(x) > 0\}) \leq 1/2$ (observe that the right hand side of $(*)$ does not change when f is replaced by $f + c$).

Let f_+ and f_- be the positive and negative parts of f, so that $f = f_+ - f_-$ and $\|f\|^2 = \|f_+\|^2 + \|f_-\|^2$. Set $u = f_+^2$ or $u = f_-^2$. Writing $S_t = \{x \in X \ : \ u(x) > t\}$,

observe that $m(S_t) \leq 1/2$ for all $t > 0$ (and hence $m(S_t^c) \geq 1/2$). Using first the area formula, then the definition of $h = h(X)$ and finally the co-area formula, we have

$$\begin{aligned} h \sum_{x \in X} u(x)m(x) &= h \int_0^\infty m(S_t)dt \\ &\leq 2 \int_0^\infty \widetilde{\mu}(S_t \times S_t^c)dt = \sum_{x,y} |u(y) - u(x)|\widetilde{\mu}(x,y). \end{aligned}$$

So,

$$h\|f\|^2 \leq \sum_{x,y} |f_+^2(y) - f_+^2(x)|\widetilde{\mu}(x,y) + \sum_{x,y} |f_-^2(y) - f_-^2(x)|\widetilde{\mu}(x,y)$$

By Cauchy-Schwarz inequality, we have

$$\begin{aligned} &\sum_{x,y} |f_\pm^2(y) - f_\pm^2(x)|\widetilde{\mu}(x,y) \\ &\leq \left(\sum_{x,y} |f_\pm(y) - f_\pm(x)|^2 \widetilde{\mu}(x,y) \right)^{1/2} \left(\sum_{x,y} |f_\pm(y) + f_\pm(x)|^2 \widetilde{\mu}(x,y) \right)^{1/2}. \end{aligned}$$

Now, using the symmetry of $\widetilde{\mu}$, the fact that $\sum_y \widetilde{\mu}(x,y) = m(x)$ and again Cauchy-Schwarz inequality, we have

$$\begin{aligned} \sum_{x,y} |f_\pm(y) + f_\pm(x)|^2 \widetilde{\mu}(x,y) &= \sum_{x,y} (f_\pm^2(y) + 2f_\pm(x)f_\pm(y) + f_\pm^2(x))\widetilde{\mu}(x,y) \\ &= 2 \sum_{x,y} f_\pm^2(x)\widetilde{\mu}(x,y) + 2 \sum_{x,y} f_\pm(x)f_\pm(y)\widetilde{\mu}(x,y) \\ &\leq 4\|f_\pm\|^2 \end{aligned}$$

Hence,

$$\begin{aligned} h\|f\|^2 \leq 2\|f_+\| \left(\sum_{x,y} |f_+(y) - f_+(x)|^2 \widetilde{\mu}(x,y) \right)^{1/2} + \\ + 2\|f_-\| \left(\sum_{x,y} |f_-(y) - f_-(x)|^2 \widetilde{\mu}(x,y) \right)^{1/2} \end{aligned}$$

Now, it is straightforward to check that

$$\sum_{x,y} |f_\pm(y) - f_\pm(x)|^2 \widetilde{\mu}(x,y) \leq \sum_{x,y} |f(y) - f(x)|^2 \widetilde{\mu}(x,y).$$

Hence,

$$h^2\|f\|^4 \leq 4\|f\|^2 \sum_{x,y} |f(y) - f(x)|^2 \widetilde{\mu}(x,y)$$

and inequality $(*)$ follows.■

Remark 8.7. There is also an upper bound which is almost trivial, namely

$$\lambda_1 \leq 2h(X).$$

Indeed, it is straightforward to check that, for a non empty subset S of X, we have

$$\frac{\langle \Delta_\mu f, f\rangle}{\|f\|^2} = \frac{\widetilde{\mu}(S \times S^c)}{m(S)m(S^c)}$$

for the function $f = \mathbf{1}_S - m(S)\mathbf{1}_X$ in $\ell^2_0(X)$ and the inequality follows from Lemma 8.5.

8.3.2 Spectral Gap Property for $PGL_2(\mathbf{F}_q((t^{-1})))/PGL_2(\mathbf{F}_q[t])$

Let $\mathbf{F}_q$ be the finite field with q elements and $\mathbf{F}_q((t^{-1}))$ the local field of Laurent series, which is the the completion of $\mathbf{F}_q(t)$ with respect to the valuation at infinity defined by $v(a/b) = deg(b) - deg(a)$ for $a, b \in \mathbf{F}_q[t], b \neq 0$. The corresponding compact subring is the ring of formal series $\mathbf{F}_q[[t^{-1}]]$.

Let $G = PGL_2(\mathbf{F}_q((t^{-1})))$. The subgroup $K = PGL_2(\mathbf{F}_q[[t^{-1}]])$ is compact and open in G. As described in II.1.1 and II.1.6 of [Serr] (see also [Efra91]), $T = G/K$ can be endowed with the structure of a $q+1$-regular tree: one can take as set of representatives for the cosets in G/K the set of matrices

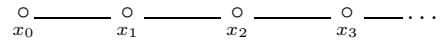
$$\begin{pmatrix} t^n & \alpha \\ 0 & 1 \end{pmatrix},$$

with $n \in \mathbf{Z}$ and α from a set of representatives of $\mathbf{F}_q[[t^{-1}]]/(t^n)$; the neighbours of the vertex $\begin{pmatrix} t^n & \alpha \\ 0 & 1 \end{pmatrix}$ are the $q+1$ vertices

$$\begin{pmatrix} t^{n+1} & \alpha \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} t^{n-1} & \alpha + \beta t^n \\ 0 & 1 \end{pmatrix}, \qquad \beta \in \mathbf{F}_q.$$

The group G acts on T by isometries on the left.

Let $\Gamma = PGL_2(\mathbf{F}_q[t])$, which is a discrete subgroup of G. The quotient graph $X := \Gamma\backslash T \cong \Gamma\backslash G/K$ is a half-line tree



given by (the cosets of) the elements

$$x_n = \begin{pmatrix} t^n & 0 \\ 0 & 1 \end{pmatrix} \qquad n \geq 0.$$

The $q+1$ edges $(x_0, y) \in E(T)$ are mapped to the edge $(x_0, x_1) \in E(X)$; for $n \geq 1$, the q-edges (x_n, y) with $y = \begin{pmatrix} t^{n-1} & \beta t^n \\ 0 & 1 \end{pmatrix}$ for $\beta \in \mathbf{F}_q$ are mapped to the edge $(x_n, x_{n-1}) \in E(X)$ and the edge (x_n, y) with $y = \begin{pmatrix} t^{n+1} & 0 \\ 0 & 1 \end{pmatrix}$ is mapped

to $(x_n, x_{n+1}) \in E(X)$. Let λ be the Haar measure on G normalized by $\lambda(K) = 1$. We have a decomposition of $\Gamma\backslash G$ as disjoint union

$$\Gamma\backslash G = \coprod_{n\geq 0} x_n K,$$

where $x_n K \subset \Gamma\backslash G$ is the K-orbit of x_n in $\Gamma\backslash G$. So, $x_n K \cong (x_n^{-1}\Gamma_n x_n)\backslash K$, where $\Gamma_n = \Gamma \cap x_n K x_n^{-1}$ is the stabilizer in Γ of x_n for the action $\Gamma \curvearrowright T$. One checks that $|\Gamma_0| = q(q^2-1)$ and $|\Gamma_n| = q^{n+1}(q-1)$ for $n \geq 1$. Let $\overline{m}$ be the measure on the set X defined by

$$\overline{m}(x_0) = \frac{1}{|\Gamma_0|} = \frac{1}{q(q^2-1)}, \qquad \overline{m}(x_n) = \frac{1}{|\Gamma_n|} = \frac{1}{q^{n+1}(q-1)} \quad \text{for} \qquad n \geq 1.$$

Since $\sum_{n=1}^{\infty} \dfrac{1}{q^{n+1}(q-1)} < \infty$, X has finite measure and hence we have

$$\mathrm{v}ol(\Gamma\backslash G) = \lambda\left(\coprod_{n\geq 0} x_n K\right) = \overline{m}(X) < \infty,$$

showing that Γ is a (non-uniform) lattice in G.

The simple random walk on T, which is given by the transition probabilities

$$\mu(x,y) = \begin{cases} \dfrac{1}{q+1} & \text{if } (x,y) \in E(T) \\ 0 & \text{otherwise,} \end{cases}$$

is reversible, with stationary measure $m : x \longmapsto 1$. The associated projected random on X is given by the transition probabilities $\overline{\mu}(x_0, x_1) = 1$ and

$$\overline{\mu}(x_n, x_{n+1}) = \frac{1}{q+1},\ \overline{\mu}(x_n, x_{n-1}) = \frac{q}{q+1} \quad \text{for} \quad n \geq 1.$$

As is easily checked, one has

$$\overline{m}(x_n)\overline{\mu}(x_n, x_{n-1}) = \overline{m}(x_{n-1})\overline{\mu}(x_{n-1}, x_n)$$

for all $n \geq 1$, which means that $\overline{m}$ is a stationary measure for $\overline{\mu}$. The Markov operator M_T on $\ell^2(T, m)$ for the random walk on T, which is defined by

$$M_T f(x) = \frac{1}{q+1} \sum_{(x,y)\in E(T)} f(y), \qquad \text{for all} \quad f \in \ell^2(T),$$

commutes with the Γ-action; it induces the Markov operator M_X on $\ell^2(X, \overline{m})$ corresponding to the projected random walk on X and is given by

$$M_X f(x_n) = \begin{cases} \dfrac{q}{q+1} f(x_{n-1}) + \dfrac{1}{q+1} f(x_{n+1}) & \text{for } n \geq 1 \\ f(x_1) & \text{for } n = 0 \end{cases} \qquad f \in \ell^2(X, \overline{m}).$$

The operator M_X is self-adjoint; the corresponding Laplacian operator $\Delta_X = \mathrm{Id}_X - M_X$ is a non-negative operator with spectrum contained $[0,2]$. It is easy to show that $G \curvearrowright \Gamma\backslash G$ has the Spectral Gap Property if and only if 1 does not belong to the spectrum of the restriction of M_X to $\ell^2_0(X)$, the orthogonal space to the constants (see Proposition 6 in [BeLu11]; here the compactness of K is crucial). So, by Theorem 8.4, it suffices to show that the Cheeger constant $h(X)$ of the random walk on X is strictly positive. This is indeed the case.

Proposition 8.8. *We have* $h(X) \geq \min\left\{\dfrac{q-1}{q+1}, \dfrac{4q^2}{(q+1)(q^2-1)}\right\} > 0$. *Hence, the action* $G \curvearrowright \Gamma\backslash G$ *has the Spectral Gap Property.*

Proof Recall that

$$h(X) := \inf_{S\subset X, S\neq\emptyset} \frac{\widetilde{\mu}(S\times S^c)}{\overline{m}(S)\overline{m}(S^c)},$$

where $\widetilde{\mu}(x,y) = \overline{m}(x)\overline{\mu}(x,y)$. In order to simplify the computations, we rescale $\overline{m}$ by

$$\overline{m}(x_0) = \frac{1}{q+1}, \qquad \overline{m}(x_n) = \frac{1}{q^n} \qquad \text{for all} \quad n \geq 1.$$

Let S be a non empty subset of X; replacing S by S^c if necessary, we can assume that $\overline{m}(S) \leq m(X)/2$.

One checks that $\overline{m}(X) = \dfrac{2q}{q^2-1}$ and that

$$\overline{m}(x_0) + \overline{m}(x_1) = \frac{2q+1}{q(q+1)} > \overline{m}(X)/2,$$

since $q \geq 2$. So, two cases may occur.

•*First case:* $x_0 \in S$. Then $x_1 \notin S$ and hence $(x_0, x_1) \in S \times S^c$. Therefore,

$$\frac{\widetilde{\mu}(S\times S^c)}{\overline{m}(S)\overline{m}(S^c)} \geq 2\frac{\widetilde{\mu}(x_0,x_1)}{\overline{m}(X)^2} = \frac{2}{(q+1)\overline{m}(X)^2}.$$

•*Second case:* $x_0 \notin S$. Let $n \geq 0$ be minimal with the property that $x_{n+1} \in S$. Then $(x_{n+1}, x_n) \in S \times S^c$. Hence,

$$\overline{m}(S) \leq \sum_{k\geq n+1} \overline{m}(x_k) = \frac{1}{q^n(q-1)}$$

and so

$$\begin{aligned}\frac{\widetilde{\mu}(S\times S^c)}{\overline{m}(S)\overline{m}(S^c)} &\geq \frac{\widetilde{\mu}(x_{n+1},x_n)}{\overline{m}(X)\sum_{k\geq n+1}\overline{m}(x_k)} = \frac{1/q^n(q+1)}{\overline{m}(X)/q^n(q-1)} \\ &= \frac{q-1}{(q+1)\overline{m}(X)}.\end{aligned}$$

Normalizing $\overline{m}$ to a probability measure, we obtain

$$h(X) \geq \min\{\frac{q-1}{q+1}, \frac{2}{(q+1)\overline{m}(X)}\} = \min\{\frac{q-1}{q+1}, \frac{4q^2}{(q+1)(q^2-1)}\}.\blacksquare$$

Remark 8.9. The precise spectral decomposition of M_X (or Δ_X) acting on $\ell^2(X)$ is determined in [Efra91]. In particular, it is shown there that the spectrum of M_X is $\left[-2\sqrt{q}/(q+1), 2\sqrt{q}/(q+1)\right] \cup \{\pm 1\}$. (Observe that -1 is indeed an eigenvalue of M_X with eigenfunction $f : X \to \mathbf{R}$ defined by $f(x_n) = (-1)^n$ for all $n \geq 0$; this is related to the fact that $G = PGL_2(\mathbf{F}_p((t^{-1}))$ has a a one dimensional character $\neq 1_G$, as its abelianization $PGL_2(\mathbf{F}_p((t^{-1})))/PSL_2(\mathbf{F}_p((t^{-1})))$ is a group of order 2.)

8.4 Lattices without the Spectral Gap Property

In view of Theorem 8.3, one might think that $G \curvearrowright G/\Gamma$ has the Spectral Gap Property for any locally compact group G and any lattice Γ in G. This is however not the case, as the following result from [BeLu11] shows.

Theorem 8.10. ***([BeLu11])*** *For an integer $k > 2$, let T_k be the k–regular tree and $G = \mathrm{Aut}(T_k)$. Then G contains a lattice Γ such that $G \curvearrowright G/\Gamma$ does not have the Spectral Gap Property.*

The proof here is based on the (easy) inequality $\lambda_1 \leq 2h(X)$ between the Cheeger constant and the bottom of the spectrum of the Laplacian λ_1 of an appropriate random walk which is determined as follows. We can find a reversible random walk on a countable graph X with transition probability $\overline{\mu} : X \times X \to \mathbf{R}$ and stationary measure $\overline{m}$ with the following properties: $\overline{m}(X) < \infty$, the Cheeger constant $h(X)$ of X is 0 and, moreover, $k\overline{\mu}(x,y)$ and $1/\overline{m}(x)$ are integers for all x, y. By the "inverse Bass–Serre theory" of groups acting on trees (see [BaLu01]), there exists a lattice Γ in $G = \mathrm{Aut}(T_k)$ such that the projection of the standard random walk on $\Gamma\backslash T_k$ can be identified with $(X, \overline{\mu}, \overline{m})$. For more details, see [BeLu11].

9 Actions on tori and nilmanifolds with the Spectral Gap Property

We review the main results from [BeGu15], concerning actions of a (countable) group Γ by automorphisms (or affine transformation) on a compact nilmanifold. First, we state the result for actions on the torus.

Let $T = \mathbf{R}^d/\mathbf{Z}^d$ be the d-dimensional torus. Observe that $\mathrm{A}ut(T)$ can be identified with $GL_d(\mathbf{Z})$. Set $V = \mathbf{R}^d$ and $\Lambda = \mathbf{Z}^d$. If W is a rational linear subspace of V, then $S = W/(W \cap \Lambda)$ is a subtorus of T and we have a torus factor $\overline{T} = T/S$. Let Γ be a subgroup of $\mathrm{A}ut(T)$ and assume that W is Γ-invariant. Then Γ leaves S invariant and the induced action of Γ on $\overline{T}$ is a factor of the action of Γ on T. We will say that $\overline{T}$ is a Γ-invariant torus factor of T. Here is our main result for actions by torus automorphisms.

Theorem 9.1. *Let Γ be a subgroup of $GL_d(\mathbf{Z})$. The following properties are equivalent.*

(i) The action of $\Gamma \curvearrowright T = \mathbf{R}^d/\mathbf{Z}^d$ has the Spectral Gap Property.

(ii) There exists no non-trivial Γ-invariant torus factor $\overline{T}$ such that the projection of Γ on $\mathrm{A}ut(\overline{T})$ is amenable.

(ii) There exists no non-trivial Γ-invariant torus factor $\overline{T}$ of T such that the projection of Γ on $\mathrm{A}ut(\overline{T})$ is a virtually abelian group (that is, it contains an abelian subgroup of finite index).

The following corollary gives a large class of examples of groups of automorphisms of the torus with the Spectral Gap Property.

Corollary 9.2. *Let Γ be a subgroup of $GL_d(\mathbf{Z})$. Assume that is Γ is not virtually abelian and that Γ acts $\mathbf{Q}$-irreducibly on $\mathbf{R}^d$ (that is, there are no non trivial $\mathbf{Q}$-rational subspaces which are invariant under Γ). Then $\Gamma \curvearrowright T = \mathbf{R}^d/\mathbf{Z}^d$ has the Spectral Gap Property.*

Remark 9.3. The previous corollary was obtained in [FuSh99] under the stronger assumption that Γ acts $\mathbf{R}$-irreducibly on $\mathbf{R}^d$.

Example 9.4. Let Γ be a subgroup of $GL_d(\mathbf{Z}) = \mathrm{A}ut(T)$. We identity the dual group of $T = \mathbf{R}^d/\mathbf{Z}^d$ with $\mathbf{Z}^d$ in the usual way. As in the proof of Proposition 2.3, the Fourier transform sets up a unitary equivalence between the Koopmann representation π_T on $L^2_0(T)$ and the natural representation of Γ on $\ell^2\left(\mathbf{Z}^d \setminus \{0\}\right)$ defined by the dual action of Γ on $\mathbf{Z}^d$, which is given by $(\gamma, x) \mapsto (\gamma^t)^{-1}x$. So, $\Gamma \curvearrowright T$ has the Spectral Gap Property if and only if the action $\Gamma \curvearrowright \mathbf{Z}^d \setminus \{0\}$ is not co-amenable.

Choose a set of representatives S for the Γ-orbits in $\mathbf{Z}^d \setminus \{0\}$; we have a direct sum decomposition

$$\ell^2\left(\mathbf{Z}^d \setminus \{0\}\right) = \bigoplus_{s\in S} \ell^2(\Gamma/\Gamma_s),$$

into Γ-invariant subspaces, where Γ_s is the stabilizer of s. Therefore,

$$\|\pi_T(\mu)\| = \sup_{s\in S} \|\pi_{\Gamma/\Gamma_s}(\mu)\|$$

for every probability measure μ on Γ.

Let $d = 2$. Then every subgroup Γ_s is amenable, as it is conjugate inside $GL_2(\mathbf{R})$ to a subgroup of $N = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbf{R} \right\}$. Hence, $\|\pi_{\Gamma/\Gamma_s}(\mu)\| = \|\pi_\Gamma(\mu)\|$ and so $\|\pi_T(\mu)\| = \|\pi_\Gamma(\mu)\|$. In particular, $\Gamma \curvearrowright T$ has the Spectral Gap Property if and only if Γ is not amenable.

Actions by automorphisms on nilmanifolds are a natural generalization of actions by torus automorphisms. The setting is as follows.

Let N be a connected and simply connected nilpotent Lie group. Let Λ be a lattice in N; the associated nilmanifold N/Λ is known to be compact. Observe that not every nilpotent Lie group has a lattice: a necessary and sufficient condition for this to happen is that N is an algebraic group defined over $\mathbf{Q}$ (see [Ragh72]).

Let $\mathrm{A}ut(N)$ be the group of continuous automorphisms of N and denote by $\mathrm{A}ut(N/\Lambda)$ the subgroup of continuous automorphisms g of N such that $g(\Lambda) =$

Λ. Every $g \in \mathrm{A}ut(N/\Lambda)$ preserves the translation invariant probability measure m on N/Λ induced by a Haar measure on N. The nilsystem $(N/\Lambda, m)$ has a natural maximal torus factor (T, m'); every automorphism $g \in \mathrm{A}ut(N/\Lambda)$ induces a torus automorphism $\overline{g} \in \mathrm{A}ut(T)$ and the mapping $g \mapsto \overline{g}$ is a homomorphism $\mathrm{A}ut(N/\Lambda) \to \mathrm{A}ut(T)$; see [Parr69] for this, as well as for other results on the ergodic theory of automorphisms of nilmanifolds.

The following theorem reduces the question of the Spectral Gap Property for groups of automorphism of nilmanifolds to the same question for groups acting on tori.

Theorem 9.5. *Let N/Λ be a compact nilmanifold, with associated maximal torus factor T. Let Γ be a subgroup of $\mathrm{A}ut(N/\Lambda)$. The following properties are equivalent.*

(i) The action of $\Gamma \curvearrowright N/\Lambda$ has the Spectral Gap Property.

(ii) The action of $\Gamma \curvearrowright T$ has the Spectral Gap Property.

Example 9.6. Let $N = H_3(\mathbf{R})$ be the 3–dimensional real Heisenberg group. Recall that N can be realized as the group with underlying set $\mathbf{R}^3 = \mathbf{R}^2 \times \mathbf{R}$ and product

$$((x,y),s)((x',y'),t) = ((x+x',y+y'),s+t+xy'-x'y))\,.$$

The Lebesgue measure m on $\mathbf{R}^3$ is a (left and right) Haar measure on N. The group N is a two-step nilpotent Lie group; its centre Z coincides with its commutator subgroup and is given by $Z = \{((0,0),s) \ : \ s \in \mathbf{R}\}$. The group $SL_2(\mathbf{R})$ acts by automorphisms on N, via

$$g((x,y),t) = (g(x,y),t) \qquad \text{for all} \quad g \in SL_2(\mathbf{R}),\ (x,y) \in \mathbf{R}^2,\ t \in \mathbf{R}.$$

The discrete subgroup $\Lambda = \{((x,y),s) \ : \ x,y \in \mathbf{Z}^2, s \in \mathbf{Z}\}$ is a lattice in N. The group $SL_2(\mathbf{Z}) \subset \mathrm{A}ut(N)$ preserves Λ and acts therefore on the Heisenberg nilmanifold $X := N/\Lambda$.

The maximal torus factor T can be identified with $\mathbf{R}^2/\mathbf{Z}^2$, via the $SL_2(\mathbf{Z})$-equivariant mapping $X \to \mathbf{R}^2/\mathbf{Z}^2, [((x,y),s)] \mapsto [(x,y)]$. Identifying $L^2(T)$ with a closed $\pi_X(\Gamma)$-invariant subspace of $L^2(X)$, we have a decomposition

$$L^2(X) = L^2(T) \oplus \mathcal{H}$$

into $\pi_X(\Gamma)$-invariant subspaces, where $\mathcal{H}$ the orthogonal complement of $L^2(T)$ in $L^2(X)$.

Let μ be a symmetric probability measure on $SL_2(\mathbf{Z})$ and $\Gamma = \Gamma(\mu)$ the subgroup generated by the support of μ. By Example 9.4, the restriction of $\pi_X(\mu)$ to $L^2_0(T)$ has norm

$$\|\pi_X(\mu)\|_{L^2_0(T)} = \|\pi_\Gamma(\mu)\|.$$

On the other hand, it can be shown that the restriction of $\pi_X(\mu)$ to $\mathcal{H}$ has norm

$$\|\pi_X(\mu)\|_{\mathcal{H}} \leq \|\pi_\Gamma(\mu)\|^{1/4}.$$

The proof of this inequality involves the consideration of the so-called Weil representation of (a two-fold cover of) the simple Lie group $SL_2(\mathbf{R})$ and estimates of its matrix coefficients, much in the spirit of Section 6. Summarizing, we have

$$\|\pi_X(\mu)\| \leq \|\pi_\Gamma(\mu)\|^{1/4}.$$

In particular, $\Gamma \curvearrowright X$ has the Spectral Gap Property if and only if Γ is non-amenable. For more details on this example, see [BeHe11].

10 Some applications of the Spectral Gap Property

We give two applications of the Spectral Gap Property of group actions, one to the construction of expander graphs and the other to the escape rate of random walks on linear groups.

10.1 Expander graphs

Let $\mathcal{G} = (X, \mathbf{E})$ be a finite k-regular graph, where X is the set of vertices and $\mathbf{E} \subset X \times X$ the set of edges of $\mathcal{G}$. We consider the simple random walk on X defined by the transition probabilities $p : X \times X \to \mathbf{R}$ given by $p(x, y) = \frac{1}{k}$ if $(x, y) \in \mathbf{E}$ and $p(x, y) = 0$ otherwise. The map $m : X \to \mathbf{R}, x \mapsto 1/|X|$ is a stationary measure for μ. The Cheeger constant (also known as expanding constant) of X or $\mathcal{G}$, as defined in Section 8.3.1, is

$$h(X) = \frac{|X|}{k} \min\left\{ \frac{|\partial S|}{|S||S|^c} \ : S \subset X, S \neq \emptyset \right\},$$

where the boundary ∂S of a subset S of X is the set of edges (x, y) with $x \in S$ and $y \notin S$. A more commonly used constant is the so-called expanding or isoperimetric constant of the graph, defined by

$$\widetilde{h}(X) = \min\left\{ \frac{|\partial S|}{|S|} \ : 0 < |S| < |X|/2 \right\};$$

$\widetilde{h}(X)$ and $h(X)$ are related by $\widetilde{h}(X)/k \leq h(X) \leq 2\widetilde{h}(X)/k$.

Expander graphs are families of graphs which are both sparse and strongly connected. More precisely, a sequence of finite k-regular graphs $\mathcal{G}_n = (X_n, \mathbf{E}_n)$ with $\lim_{n\to\infty} |X_n| = \infty$ is a family of *expanders* if $\inf_n h(X_n) > 0$. A constant $\varepsilon > 0$ with $\inf_n h(X_n) \geq \varepsilon$ is called an *expanding constant* for the sequence of expanders $(\mathcal{G}_n)_n$.

Let $\mathcal{G}_n = (X_n, \mathbf{E}_n)$ be a family of finite k-regular graphs with $\lim_{n\to\infty} |X_n| = \infty$. Let Δ_n be the corresponding Laplacian on X_n; recall that Δ_n is defined on $\ell^2(X_n)$ by

$$\Delta_n f(x) = f(x) - \frac{1}{k} \sum_{(x,y)\in\mathbf{E}_n} f(y).$$

Let $\lambda_1^{(n)}$ denote the smallest eigenvalue $\neq 0$ of Δ_n. In view Cheeger's inequalities (Theorem 8.4 and Remark 8.7), we see that $(X_n)_n$ is a family of expanders if and only if $\inf_n \lambda_1^{(n)} > 0$.

The existence of expander graphs is settled by elementary counting arguments (see [Lubo94, Proposition 1.2.1]). However, their constructions seem to require sophisticated mathematical tools. We will give the explicit construction of a family of expander graphs using Kazhdan's Property (T), following the original idea of Margulis; recently, families of expanders have been found using the so-called zig-zag construction (see [RVW02], [ALW01]).

Let G be a finitely generated group, with a fixed finite generating set S with $S^{-1} = S$. The Cayley graph $\mathcal{G}(G, S)$ is a connected k-regular graph for $k = |S|$ (see Section 4.1).

We assume now that G is a *finite* group. Let π_G be the right regular representation of G on $\ell^2(G)$. Denote by π_G^0 the corresponding representation of G on the G-invariant subspace $\ell_0^2(G) = \{\mathbf{1}_G\}^\perp$.

Let μ be the probability measure $\mu = \frac{1}{|S|}\sum_{s\in S}\delta_s$ on G. The following crucial lemma establishes a link between the norm of $\pi_G^0(\mu)$ and the smallest eigenvalue λ_1 in $\ell_0^2(G)$ of the Laplace operator Δ associated to the simple random walk on $\mathcal{G}(G, S)$.

Lemma 10.1. *We have* $\lambda_1 \geq \dfrac{1}{2}(1 - \|\pi_G^0(\mu)\|)^2$.

Proof Let $f \in \ell_0^2(G)$ be an eigenfunction of Δ with $\|f\| = 1$, for the eigenvalue λ_1. Denoting by $\mathbf{E}$ the set of edges of $\mathcal{G}(G, S)$, and using Lemma 8.5 as well as Jensen's inequality, we have

$$\begin{aligned} 2\lambda_1 = 2\langle \Delta f, f\rangle &= \frac{1}{k}\sum_{(x,y)\in\mathbf{E}} |f(y) - f(x)|^2 = \frac{1}{k}\sum_{s\in S}\sum_{x\in G} |f(xs) - f(x)|^2 \\ &= \frac{1}{k}\sum_{s\in S} \|\pi_G^0(s)f - f\|^2 \geq \left(\frac{1}{k}\sum_{s\in S}\|\pi_G^0(s)f - f\|\right)^2 \\ &\geq \left\|\frac{1}{k}\sum_{s\in S}(\pi_G^0(s)f - f)\right\|^2 = \|\pi_G^0(\mu)f - f\|^2 \geq (1 - \|\pi_G^0(\mu)\|)^2. \blacksquare \end{aligned}$$

As a consequence of the previous lemma, we obtain a construction scheme for expanders.

Theorem 10.2. *Let Γ be a group with Kazhdan's Property (T), and S a finite generating set of Γ with $S^{-1} = S$. Let $(N_n)_n$ be a sequence of normal subgroups of Γ of finite index with $\lim_n |\Gamma/N_n| = \infty$. Then the sequence of the Cayley graphs $\mathcal{G}(\Gamma/N_n, \varphi_n(S))$ is an expander family, where $\varphi_n(S)$ is the image of S under the canonical projection $\varphi_n : \Gamma \to \Gamma/N_n$.*

Proof Let $\mu = \frac{1}{|S|}\sum_{s\in S}\delta_s$. By Proposition 3.13, that there exists a constant $\delta > 0$ such that $1 - \|\pi(\mu)\| \geq \delta$ for every unitary representation π of Γ without non-zero invariant vectors. This holds in particular for the representations $\pi^0_{\Gamma/N_n} \circ \varphi_n$ acting on $\ell_0^2(\Gamma/N_n)$. The result follows now from the previous lemma. $\blacksquare$

Remark 10.3. With the notation as in the previous theorem, let $\lambda_1^{(n)}$ denote the first non-zero eigenvalue of the Laplacian on $X_n = \Gamma/N_n$. Since $h(\Gamma/N_n) \geq \dfrac{\lambda_1^{(n)}}{2}$, we see that an expanding constant for the family $\mathcal{G}(\Gamma/N_n, p_n(S))$ is $\varepsilon = \delta^2/2$, where $\delta = \inf_\pi(1 - \|\pi(\mu)\|)$ with π running over all unitary representations of Γ without non-zero invariant vectors.

Example 10.4. Let $\Gamma = SL_3(\mathbf{Z})$. Then $S = \{E_{ij}^{\pm 1} \ : \ 1 \leq i, j \leq 3, \ i \neq j\}$ is a generating set of Γ, where the E_{ij}'s are the usual elementary matrices. For every prime integer p, let $\Gamma(p)$ be the so-called principal congruence subgroup, that is,

$$\Gamma(p) = \{A \in \Gamma \ : \ A \equiv I \mod p\}$$

is the kernel of the surjective homomorphism $\varphi_p : SL_3(\mathbf{Z}) \to SL_3(\mathbf{Z}/p\mathbf{Z})$ given by reduction modulo p. Since $\Gamma/\Gamma(p) \cong SL_3(\mathbf{Z}/p\mathbf{Z})$, the subgroup $\Gamma(p)$ has finite index $p^3(p^3-1)(p^2-1) \approx p^8$. The family of Cayley graphs $(\mathcal{G}(\Gamma/\Gamma(p), \varphi_p(S))_p$ is a family of k-regular expanders with $k = 12$. It can be shown that $\varepsilon \approx \dfrac{10^{-6}}{4}$ is an expanding constant for this family (see Example 6.1.11 in [BHV]).

For a comprehensive account on expander graphs and their applications, see [HLW06]. An overview of recent developments in this subject is given in [Breu14].

10.2 Growth of products of random matrices

We now give an application of the Spectral Gap Property to a result of Furstenberg from [Furs63] about random walks on linear groups. The setting is as follows.

Let μ be a probability measure on the special linear group $G = SL_d(\mathbf{R})$. We set $V = \mathbf{R}^d$. We will consider the operator norm on $End(V) = M_d(\mathbf{R})$ associated to the Euclidean norm $\|\cdot\|$ on V.

Let $S_n(\omega) = X_n(\omega) \cdots X_1(\omega)$ be a sequence of random products, where $(X_n)_{n\geq 1}$ is a sequence of independent random variables defined on a common probability space $(\Omega, \mathbf{P})$, with values in G and identically distributed according to μ. One is interested in a non-commutative analogue of the *Law of Large Numbers* describing the top Lyapunov exponent $\lambda_1(\mu)$ of the random matrix products which is defined as follows.

Assume that μ has finite first moment, that is, $\int_G \log \|g\| d\mu(g) < \infty$. It follows from Kingman's subadditive ergodic theorem (see Theorem 10.1 in [Walte82]) that

$$\lim_{n\to\infty} \frac{1}{n} \log \|S_n(\omega)\| = \lim_{n\to\infty} \frac{1}{n} \log \|X_n(\omega) \cdots X_1(\omega)\|$$

exists $\mathbf{P}$-almost surely and is $\mathbf{P}$-almost everywhere constant; we denote by $\lambda_1(\mu)$ this limit, which may also be computed as

$$\lambda_1(\mu) = \lim_{n\to\infty} \frac{1}{n} \int_G \log \|g\| d\mu^n(g) = \inf_{n\geq 1} \frac{1}{n} \int_G \log \|g\| d\mu^n(g).$$

For more details on random matrix products, see the survey [Furm02].

We give in the following proposition a lower bound for $\lambda_1(\mu)$ in terms of a Spectral Gap Property. For this, we follow [Guiv15, Corollary 2.2] (see also the proof of Corollaire 1 in [Guiv15] and Theorem 1.19 in [Furm02]).

Let π_V be the unitary representation of $G = SL_d(\mathbf{R})$ on $L^2(V, m)$ given by the natural action of G on V equipped with the Lebesgue measure m.

Proposition 10.5. *The following inequality holds:*

$$\lambda_1(\mu) \geq \frac{1}{d} \log \left(\frac{1}{r_{spec}(\pi_V(\mu))} \right).$$

Proof For $n \geq 1$, set $u_n = \int_G \log \|g\| d\mu^n(g)$. So, $\lambda_1(\mu) = \lim_n u_n/n$.

Since $1 = \det g \leq \|g\|^d$, we have $\|g\| \geq 1$ for every $g \in G$. Fix $\varepsilon > 0$. Let $f_\varepsilon \in L^2(V)$ be defined by $f_\varepsilon(x) = 1$ if $\|x\| \leq 1$ and $f_\varepsilon(x) = \dfrac{1}{\|x\|^{d+\varepsilon}}$ if $\|x\| \geq 1$. For $A = \{x \in V \ : \ 1 \leq \|x\| \leq 2\}$ and $n \geq 1$, we have $\mathbf{1}_A \in L^2(V)$ and

$$\begin{aligned}
\langle \pi_V(\mu^n)\mathbf{1}_A, f_\varepsilon \rangle &= \langle \pi_V(\check{\mu}^n) f_\varepsilon, \mathbf{1}_A \rangle \\
&= \int_A \int_G \frac{1}{\|gx\|^{d+\varepsilon}} dm(x) d\mu^n(g) \\
&\geq \int_A \int_G \frac{1}{(\|g\|\|x\|)^{d+\varepsilon}} dm(x) d\mu^n(g) \\
&\geq m(A) \frac{1}{2^{d+\varepsilon}} \int_G \frac{1}{\|g\|^{d+\varepsilon}} dm(x) d\mu^n(g).
\end{aligned}$$

Hence, by concavity of the logarithm, we obtain

$$\log(\langle \pi_V(\mu^n)\mathbf{1}_A, f_\varepsilon \rangle) \geq \log \left(\frac{m(A)}{2^{d+\varepsilon}} \right) - (d+\varepsilon) \int_G \log \|g\| dm(x) d\mu^n(g),$$

that is,

$$(d+\varepsilon) u_n \geq -\log(\langle \pi_V(\mu^n)\mathbf{1}_A, f_\varepsilon \rangle + \log \left(\frac{m(A)}{2^{d+\varepsilon}} \right).$$

Since

$$\limsup_n |\langle \pi_V(\mu^n)\mathbf{1}_A, f_\varepsilon \rangle|^{1/n} \leq \limsup_n \|\pi_V(\mu^n)\|^{1/n} (\|\mathbf{1}_A\| |f_\varepsilon\|)^{1/n} = r_{spec}(\pi_V(\mu)),$$

we have therefore

$$(d+\varepsilon) \lim_n \frac{u_n}{n} \geq -\log r_{spec}(\pi_V(\mu)).$$

Letting $\varepsilon \to 0$, we obtain the claim.■

The following result is an immediate consequence of the previous proposition in combination with Theorem 5.7.

Corollary 10.6. *Let μ be probability measure on $SL_d(\mathbf{R})$ and denote by $\Gamma(\mu)$ the subgroup generated by the support of μ. Assume that the linear action of $\Gamma(\mu)$ on $V = \mathbf{R}^d$ is not co-amenable. Then $\lambda_1(\mu) > 0$.*

So, under the assumption of the previous corollary, the norm $\|S_n(\omega)\|$ grows exponentially almost surely, where $S_n(\omega) = X_n(\omega)\cdots X_1(\omega)$ is the product of independent random unimodular matrices identically distributed according to μ.

Applying Theorem 5.19, we recover the following result of Furstenberg (Theorem 8.6 in [Furs63]).

Corollary 10.7. ***([Furs63])*** *Assume that $\Gamma(\mu)$ is not bounded and that the action of $\Gamma(\mu)$ on V is totally irreducible. Then $\lambda_1(\mu) > 0$.*

Example 10.8. Let μ be the probability measure on $G = SL_2(\mathbf{R})$ given by $\mu = \frac{1}{4}(\delta_a + \delta_b + \delta_{a^{-1}} + \delta_{b^{-1}})$ for the matrices

$$a = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad b = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}.$$

We claim that

$$\lambda_1(\mu) \geq \frac{1}{2}\log\left(\sqrt{\frac{2}{\sqrt{3}}}\right) \approx 0.015617$$

for the corresponding top Lyapunov exponent $\lambda_1(\mu)$.

Indeed, the subgroup Γ generated by the support of μ is the free group on a and b and is a subgroup (of index 12) of $SL_2(\mathbf{Z})$. Moreover, the representation π_V of $G = SL_2(\mathbf{R})$ on $L^2(V)$, for $V = \mathbf{R}^2$, is weakly contained in the regular representation π_G; indeed, G acts transitively on $V\backslash\{0\}$ with stabilizers conjugated to $N = \left\{\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}\right\}$. It follows that π_V is equivalent to the quasi representation $\pi_{G/N} \cong \mathrm{Ind}_N^G(1_N)$ on $L^2(G/N)$. Since N is amenable, π_V is weakly contained in π_G (see Theorem F.3.5 in [BHV]). So, π_V is strongly L^p for every $p > 2$ and hence (see Proposition 6.1) $\pi_V \otimes \overline{\pi_V}$ is contained in a multiple of π_G. Therefore, by Proposition 6.5,

$$\|\pi_V(\mu)\| \leq \|\pi_G(\mu)\|^{1/2}.$$

Now, since Γ is a discrete subgroup of G, the restriction of π_G to Γ is a multiple of the regular representation π_Γ and so

$$\|\pi_V(\mu)\| \leq \|\pi_\Gamma(\mu)\|^{1/2}.$$

Finally, as Γ is a free group on two generators, we have $\|\pi_\Gamma(\mu)\| = \sqrt{3}/2$ (see Section 4.2) and the claim follows from Proposition 10.5.

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