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# A POSITIVE MASS THEOREM FOR ASYMPTOTICALLY HYPERBOLIC MANIFOLDS WITH INNER BOUNDARY

OUSSAMA HIJAZI, SEBASTIÁN MONTIEL, AND SIMON RAULOT

ABSTRACT. In this note, we prove an optimal Positive Mass Theorem for Asymptotically Hyperbolic spin manifolds with compact inner boundary. This improves a previous result of Chruściel and Herzlich [CH].

## 1. INTRODUCTION

A Riemannian manifold  $(M^n, g)$  is called Asymptotically Flat (AF) of order  $\tau > 0$  if there exists a compact  $K \subset M$  and a diffeomorphism  $\Phi : M \setminus K \rightarrow \mathbb{R}^n \setminus \overline{B}(r_0)$  where  $\overline{B}(r_0)$  is the closed Euclidean ball of radius  $r_0 > 0$  such that, in this chart, the metric satisfies

$$g_{ij} - \delta_{ij} = O(|\Phi|^{-\tau}), \quad \frac{\partial g_{ij}}{\partial \Phi^k} = O(|\Phi|^{-\tau-1}), \quad \frac{\partial^2 g_{ij}}{\partial \Phi^k \partial \Phi^l} = O(|\Phi|^{-\tau-2}).$$

From the work of Bartnik [B], it is known that if  $\tau > \frac{n-2}{2}$  and if the scalar curvature  $R_g$  of  $(M^n, g)$  is integrable, we can assign to such a manifold a geometrical invariant called the ADM mass by letting

$$m_{ADM}(g) := \frac{1}{2(n-1)\omega_{n-1}} \lim_{R \rightarrow \infty} \int_{\mathbb{S}_R} \left( \operatorname{div}^\delta e - d(\operatorname{tr}^\delta e) \right) (\nu_R) d\mathbb{S}_R.$$

Here  $\mathbb{S}_R$  is a coordinate sphere of radius  $R$  with  $\nu_R$  as its unit normal vector field pointing towards infinity,  $\delta$  the Euclidean metric and  $\omega_{n-1}$  is the volume of the standard  $(n-1)$ -sphere in  $\mathbb{R}^n$ . Moreover,  $\operatorname{div}^\delta$  (resp.  $\operatorname{tr}^\delta$ ) is the divergence (resp. the trace) with respect to  $\delta$  and  $e := \Phi_* g - \delta$ . The standard positive mass conjecture asserts that

**(AF)-Positive Mass Conjecture:** *Under the previous assumptions, if  $R_g \geq 0$  then  $m_{ADM}(g) \geq 0$ . Moreover the mass is zero if and only if  $(M^n, g)$  is isometric to the flat Euclidean space.*

By Schoen and Yau [SY1, SY2] this conjecture holds for  $3 \leq n \leq 7$  and by Witten [Wi] for spin manifolds. Now, if furthermore we assume the existence of a compact minimal hypersurface in  $M$ , heuristic arguments

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which rely heavily on the cosmic censorship conjecture, lead Penrose [P] to the following conjecture

**(AF)-Penrose Inequality Conjecture:** *Assume that the assumptions of the (AF)-Positive Mass Conjecture are fulfilled and that  $\Sigma$  is the outermost compact minimal hypersurface, then*

$$(1) \quad m_{ADM}(g) \geq \frac{1}{2} \left( \frac{\text{Vol}_g(\Sigma)}{\omega_{n-1}} \right)^{\frac{n-2}{n-1}}.$$

Moreover equality occurs if and only if the region in  $(M^n, g)$  outside its outermost minimal hypersurface  $\Sigma$  is isometric to the domain  $r > R_S$  of the Schwarzschild space

$$\left( \mathbb{R}^n \setminus \{0\}, \left( 1 + \left( \frac{R_S}{r} \right)^{n-2} \right)^{\frac{4}{n-2}} \delta \right) \quad \text{with} \quad R_S = \left( \frac{m}{2} \right)^{\frac{1}{n-2}}.$$

Here  $\text{Vol}_g(\Sigma)$  denotes the volume of  $\Sigma$  with respect to the metric induced by  $g$  and  $m$  is a positive real number which corresponds to the ADM mass of the Schwarzschild space. Moreover by outermost minimal hypersurface we mean that  $\Sigma$  contains all other minimal hypersurfaces in its interior. Note that  $\Sigma$  may be disconnected. Two major breakthroughs in the proof of this inequality were obtained for 3-dimensional manifolds by Huisken and Ilmanen [HI] for connected  $\Sigma$  and by Bray [Br] for general  $\Sigma$ . The technique used in Bray has recently been extended in [BrL] to  $n$ -dimensional manifolds with  $3 \leq n \leq 7$ . It should be noticed that more recently, the case of  $n$ -dimensional manifolds which are the graph of a smooth function over  $\mathbb{R}^n$  is treated in [L].

Another interesting approach has been developed by Herzlich [He1, He2] using the spin framework *à la Witten*. More precisely, he proves a Penrose-type inequality valid in all dimensions under the additional assumption that the manifold is spin. His result does not give the full Penrose inequality since the assumptions are weaker: indeed, the minimal hypersurface is not supposed to be the outermost. However it is sharp since the equality case is only reached by the Schwarzschild metric. One of the main tools used to obtain this inequality is an optimal positive mass theorem for asymptotically flat manifolds with compact inner boundary generalizing (in a certain sense) a result of Gibbons, Hawking, Horowitz and Perry [GHHP].

The (AF)-Penrose Inequality Conjecture is in fact a particular case (the time-symmetric case) of the full Penrose inequality which is conjectured to hold for initial data sets  $(M^n, g, k)$  of the Einstein equations with zero cosmological constant satisfying the dominant energy condition (see [Mar] for more details). Up to now, only few results are known in this direction. It should be pointed out that for  $n = 3$ , Khuri [K1] uses the approach developed by Herzlich and obtains an interesting inequality in this situation.

The aim of this paper is to extend Herzlich's proof of the positive mass theorem to the asymptotically hyperbolic setting.

## 2. ASYMPTOTICALLY HYPERBOLIC RIEMANNIAN MANIFOLDS

The Positive Mass and the Penrose Conjectures can be generalized to spacetime manifolds with negative cosmological constant. In this context, the first difficulty is to define an appropriate notion of mass and energy-momentum. We recall here the time-symmetric version of these concepts developed by Chruściel and Herzlich [CH] generalizing a previous work of Wang [Wa].

Let  $\mathbb{H}^n$  be the  $n$ -dimensional hyperbolic space with coordinates  $(r, \theta) \in \mathbb{R}^+ \times \mathbb{S}^{n-1}$ , endowed with the metric

$$b = dr^2 + \sinh^2(r) g_{\mathbb{S}^{n-1}},$$

where  $r$  is the geodesic distance to a chosen origin corresponding to  $r = 0$  and  $g_{\mathbb{S}^{n-1}}$  is the round metric on the  $(n-1)$ -sphere. A Riemannian manifold  $(M^n, g)$  is called Asymptotically Hyperbolic (AH) if there exists a compact subset  $K \subset M$  and a diffeomorphism

$$\Psi : M \setminus K \longrightarrow \mathbb{H}^n \setminus \overline{B}(r_0)$$

where  $\overline{B}(r_0)$  is a closed ball of radius  $r_0 > 0$  in  $\mathbb{H}^n$  for which  $\Psi_*g$  and  $b$  are uniformly equivalent on  $\mathbb{H}^n \setminus \overline{B}(r_0)$ ,

$$(2) \quad \int_{\mathbb{H}^n \setminus \overline{B}(r_0)} (|e|^2 + |\nabla^b e|^2) \cosh r \, dv_b < +\infty$$

and

$$(3) \quad \int_{\mathbb{H}^n \setminus \overline{B}(r_0)} |R_g + n(n-1)| \cosh r \, dv_b < +\infty$$

where  $\nabla^b$  stands for the Levi-Civita connection with respect to  $b$  and  $e := \Psi_*g - b$ . As for the mass of an (AF) manifold, the mass and the energy-momentum invariants in this setting appear from a Hamiltonian formulation of general relativity (see [BI] for example). In particular they are related to symmetries of the background metric which is considered (the hyperbolic space in this situation). More precisely, the set defined by

$$\mathcal{N} := \{V \in C^\infty(\mathbb{H}^n) / \text{Hess}^b V = Vb\}$$

is an  $(n+1)$ -dimensional vector space generated by the following basis of functions:

$$V_{(0)} = \cosh r, \quad V_{(1)} = x^1 \sinh r, \dots, \quad V_{(n)} = x^n \sinh r,$$

where  $x^1, \dots, x^n$  are the coordinate functions on  $\mathbb{R}^n$  restricted to  $\mathbb{S}^{n-1}$ . In the upper unit hyperboloid model of  $\mathbb{H}^n$ , the functions  $V_{(i)}$  are the restrictions to  $\mathbb{H}^n$  of the coordinate functions on the Minkowski space-time  $\mathbb{R}^{n,1}$ . The space  $\mathcal{N}$  is naturally endowed with a Lorentzian metric  $\eta$ , with signature  $(+, -, \dots, -)$ , coming from the action of the group of isometries  $O^+(1, n)$

of the hyperbolic metric. This metric is characterized by the condition that the above basis is orthonormal that is

$$\eta(V_{(0)}, V_{(0)}) = 1 \quad \text{and} \quad \eta(V_{(i)}, V_{(i)}) = -1$$

for all  $i = 1, \dots, n$ . A time-orientation on  $\mathcal{N}$  is provided by declaring that a time-like vector  $X = X^{(\mu)}V_{(\mu)} \in \mathcal{N}$  is future-directed if  $X^{(0)} > 0$ .

Consider now an asymptotically hyperbolic manifold  $(M^n, g)$  with a chart at infinity denoted by  $\Psi$  such that the asymptotic decay conditions (2) and (3) hold. The mass functional of  $(M^n, g)$  with respect to  $\Psi$  is the linear form on  $\mathcal{N}$  defined for  $V \in \mathcal{N}$  by

$$(4) \quad H_\Psi(V) := \frac{1}{2(n-1)\omega_{n-1}} \lim_{R \rightarrow \infty} \int_{\mathbb{S}_R} \mathbb{U}(V, \Psi_*g - b)(\nu_R) d\mathbb{S}_R$$

where the integrand term is given by

$$\mathbb{U}(V, e) := V(\operatorname{div}^b e - d(\operatorname{tr}^b e)) + (\operatorname{tr}^b e)dV - i_{\nabla^b V} e.$$

Here  $i_X$  denotes the contraction by a vector field  $X$ . It is important to note that the assumptions (2) and (3) ensure that the limit (4) exists and is finite. On the other hand, if  $\mathcal{I}$  is an isometry of the hyperbolic space, the map  $\mathcal{I} \circ \Psi$  also defines a chart at infinity with the appropriate decay conditions. The mass functional with respect to these charts are related by

$$(5) \quad H_{\mathcal{I} \circ \Psi} V = H_\Psi(V \circ \mathcal{I}^{-1}).$$

Moreover, if  $\Phi$  is another chart at infinity as above, one can check that there is an isometry  $\mathcal{I}$  of  $b$  such that  $\Phi = \mathcal{I} \circ \Psi$  modulo lower order terms which do not affect the mass functional. From this and since the action of the isometry group of  $\mathbb{H}^n$  on  $\mathcal{N}$  (appearing on the right-hand side of (5)) preserves the Lorentzian metric  $\eta$ , it follows that if  $p_\mu := H_\Psi(V_{(\mu)})$  then the real number  $m_{HYP}(g)$  defined (up to a sign) by

$$m_{HYP}(g)^2 := \eta(\mathbf{p}_g, \mathbf{p}_g) = \left| p_{(0)}^2 - \sum_{j=1}^n p_{(j)}^2 \right|$$

does not depend on the chart  $\Psi$  and is called the mass of  $(M^n, g)$ . Here  $\mathbf{p}_g$  is the vector with components  $p_{(\mu)}$  and is called the energy-momentum vector of  $(M^n, g)$ . It is important to point out that the causal character as well as the future/past pointing nature of  $\mathbf{p}_g$  do not depend on the chart  $\Psi$  provided that the suitable decay properties (2) and (3) are satisfied. Then if  $\mathbf{p}_g$  is causal and future-directed, it is natural to choose

$$m_{HYP}(g) := \sqrt{\eta(\mathbf{p}_g, \mathbf{p}_g)}.$$

In this setting, the positive mass conjecture takes the following form

**(AH)-Positive Mass Conjecture:** *Let  $(M^n, g)$  be a complete  $n$ -dimensional Riemannian, asymptotically hyperbolic manifold. Assume that the scalar curvature  $R_g$  of  $g$  satisfies  $R_g \geq -n(n-1)$ . Then the energy-momentum*

vector  $\mathbf{p}_g$  is time-like future-directed, unless  $(M^n, g)$  is isometric to the hyperbolic space  $\mathbb{H}^n$ .

This conjecture is proved for spin manifolds (see [CH] and [Wa]) whereas only partial results are known for non-spin manifolds (see for example [ACG]).

Now if we assume the existence of a minimal hypersurface in  $M$ , an inequality similar to the (AF)-Penrose inequality (1) is conjectured

**(AH)-Penrose Inequality Conjecture:** *Assume that  $(M^n, g)$  satisfies the assumptions of the (AH)-Positive Mass Conjecture and that  $M$  carries a compact, outermost minimal boundary  $\Sigma$ , then*

$$(6) \quad m_{HYP}(g) \geq \frac{1}{2} \left( \left( \frac{\text{Vol}_g(\Sigma)}{\omega_{n-1}} \right)^{\frac{n-2}{n-1}} + \left( \frac{\text{Vol}_g(\Sigma)}{\omega_{n-1}} \right)^{\frac{n}{n-1}} \right).$$

Moreover equality occurs if and only if  $(M^n, g)$  is isometric to the exterior of the Anti-de-Sitter Schwarzschild space.

Recall that the  $n$ -dimensional Anti-de Sitter Schwarzschild space arises as the slice  $t = 0$  of the Anti-de Sitter Schwarzschild spacetime of dimension  $n + 1$  and mass  $m > 0$  whose metric is given by

$$\tilde{g}_{\text{AdS-Schw}} = - \left( 1 + \rho^2 - \frac{2m}{\rho^{n-2}} \right) dt^2 + \frac{d\rho^2}{1 + \rho^2 - \frac{2m}{\rho^{n-2}}} + \rho^2 g_{\mathbb{S}^{n-1}}.$$

Up to now, even the 3-dimensional case of this conjecture is still open. However, several results [DGS, dLG] confirm the validity of this conjecture by proving it for (AH) graphical manifolds. Another result which goes in this direction is the following positive mass theorem for (AH) manifolds with a compact inner boundary proved by Chruściel and Herzlich [CH] :

**Theorem 1.** [CH] *Let  $(M^n, g)$  be a complete spin (AH) manifold with a compact nonempty inner boundary of mean curvature  $H \leq 1$ , and with scalar curvature satisfying  $R_g \geq -n(n-1)$ . Then the energy-momentum vector  $\mathbf{p}_g$  is time-like future-directed.*

In particular, this result implies that if  $M$  has a minimal compact inner boundary, then the mass is far from zero. As noticed in the proof of this theorem, equality is never reached. Our goal in this paper is to generalize this result in such a way that equality can be achieved.

### 3. A SHARP POSITIVE MASS THEOREM FOR ASYMPTOTICALLY HYPERBOLIC MANIFOLDS WITH COMPACT INNER BOUNDARY

In this section, we state and give the proof of our main result. For the sake of clarity and since the assumptions and the conclusions slightly differ,

we choose to treat the 3-dimensional case separately. Indeed, since a 3-dimensional manifold is automatically spin, we have

**Theorem 2.** *Let  $(M^3, g)$  be a 3-dimensional complete (AH) manifold with scalar curvature satisfying  $R_g \geq -6$  and compact inner boundary  $\Sigma$  homeomorphic to a 2-sphere whose mean curvature is such that*

$$(7) \quad H \leq \sqrt{\frac{4\pi}{\text{Area}_g(\Sigma)} + 1}.$$

*Then the energy-momentum vector  $\mathbf{p}_g$  is time-like future-directed or zero. Moreover, if it vanishes then  $(M^3, g)$  is isometric to the complement of a geodesic ball in the hyperbolic space  $\mathbb{H}^3$ .*

If the dimension of  $M$  is at least 4, the mean curvature has to be controlled by a quantity involving the Yamabe invariant  $Y(\Sigma)$  of  $\Sigma$  for the induced metric (see for example [Hi2] for a precise definition). More precisely

**Theorem 3.** *Let  $(M^n, g)$  be an  $n$ -dimensional ( $n \geq 4$ ) complete (AH) spin manifold with scalar curvature satisfying  $R_g \geq -n(n-1)$ , compact inner boundary  $\Sigma$  of positive Yamabe invariant  $Y(\Sigma)$  and of mean curvature*

$$(8) \quad H \leq \sqrt{\frac{Y(\Sigma)}{(n-1)(n-2)} \text{Vol}_g(\Sigma)^{-\frac{2}{n-1}} + 1}.$$

*Then the energy-momentum vector  $\mathbf{p}_g$  is time-like future-directed or zero. If it is zero, then  $(M^n, g)$  has an imaginary Killing spinor field and  $\Sigma$  is a totally umbilical hypersurface with constant mean curvature carrying a real Killing spinor.*

The existence of an imaginary (resp. real) Killing spinor on  $M$  (resp.  $\Sigma$ ) imposes strong restrictions on the geometry of  $M$  (resp.  $\Sigma$ ): in particular, it has to be an Einstein manifold with negative (resp. positive) scalar curvature. The simplest example of an asymptotically hyperbolic manifold with zero energy-momentum vector and which satisfies the assumptions of our result is the complement of a geodesic ball in the hyperbolic space  $\mathbb{H}^n$ . Indeed, a straightforward calculation shows that for such a manifold one has equality in (8) and also zero mass so that it is a limiting manifold of our positive mass theorem. An interesting open question is to know whether it is unique.

**Remark 1.** In [C], Cortier asserts that Theorem 1 is somehow optimal. Indeed, as explained in this paper, if in Theorem 1 one modifies the condition on the mean curvature of the inner boundary by the condition  $H \leq \alpha$  with  $\alpha > 1$ , then one can construct asymptotically hyperbolic metrics of constant scalar curvature equals to  $-n(n-1)$  with a compact inner boundary satisfying  $H \leq \alpha$  and an arbitrary energy-momentum tensor. In particular, the energy-momentum tensor can be chosen to be space-like and this should contradicts our Theorem 2 and 3. In fact this is not the case : in Cortier's

construction, the number  $\alpha > 1$  is fixed and then the “counter-example” is produced. However, in our statement, the fact is that the bound on the mean curvature depends on the geometry of the boundary. This is why these counter-examples do not satisfy our assumptions.

**3.1. The Schrödinger-Lichnerowicz formula.** The first main ingredient for the proof of Theorem 2 and 3 is the integral version of the Schrödinger-Lichnerowicz formula for (AH) manifolds. First, let us recall briefly some basic facts about spin geometry and fix the notations. For more details, we refer for example to [LM, BHMMM, Gin].

Let  $(M^n, g)$  be an  $n$ -dimensional Riemannian spin (AH) manifold, supposed connected from now on, and denote by  $\nabla$  the Levi-Civita connection on its tangent bundle  $TM$ . We also denote by  $\nabla$  the Levi-Civita connection associated with the induced metric on the inner boundary  $\Sigma$  of  $M$ . Then the Gauß formula states that

$$\nabla_X Y = \nabla_X Y + g(AX, Y)N,$$

where  $X, Y$  are vector fields tangent to the hypersurface  $\Sigma$ , the vector field  $N$  is the inner unit field normal to  $\Sigma$  and  $A$  stands for the shape operator defined by

$$(9) \quad \nabla_X N = AX, \quad \forall X \in \Gamma(T\Sigma).$$

The mean curvature of  $\Sigma$  is then obtained by taking the normalized trace of  $A$ , that is  $(n-1)H = \text{tr}A$ , in such a way that the inner round boundary of  $\mathbb{H}^n \setminus B(r_0)$  has positive mean curvature.

On the other hand, since the manifold is spin, we choose a spin structure on  $M$  and consider the corresponding spinor bundle  $\mathbb{S}M$ . Equipped with  $\gamma$ , the Clifford multiplication,  $\mathbb{S}M$  becomes a bundle of complex left modules over the complex Clifford bundle  $\mathcal{C}\ell(M)$  of the manifold  $M$ . On the spinor bundle  $\mathbb{S}M$ , one also has a natural Hermitian metric, denoted by  $\langle \cdot, \cdot \rangle$ , and the spinorial Levi-Civita connection  $\nabla$  acting on spinor fields. The Hermitian metric and  $\nabla$  are compatible with the Clifford multiplication and compatible with each other, in such a way that  $(\mathbb{S}M, \gamma, \langle \cdot, \cdot \rangle, \nabla)$  is a Dirac bundle in the sense of [LM]. The Dirac operator  $D$  on  $\mathbb{S}M$  is the first order elliptic differential operator locally given by

$$D = \sum_{i=1}^n \gamma(e_i) \nabla_{e_i},$$

where  $\{e_1, \dots, e_n\}$  is a local orthonormal frame of  $TM$ .

Consider now the restriction

$$\mathbb{S}\Sigma := \mathbb{S}M|_{\Sigma}$$

which is a left module over  $\mathcal{C}\ell(\Sigma)$  for the induced Clifford multiplication defined by

$$\not\!X\psi = \gamma(X)\gamma(N)\psi$$



for every  $\psi \in \Gamma(\mathcal{S}\Sigma)$  and  $X \in \Gamma(T\Sigma)$ . We endow  $\mathcal{S}\Sigma$  with the Hermitian metric  $\langle \cdot, \cdot \rangle$  induced from that of  $\mathbb{S}M$  as well as the spin Levi-Civita connection

$$(10) \quad \nabla_X \psi = \nabla_X \psi + \frac{1}{2} \not\gamma(AX) \psi = \nabla_X \psi + \frac{1}{2} \gamma(AX) \gamma(N) \psi$$

for every  $\psi \in \Gamma(\mathcal{S}\Sigma)$  and  $X \in \Gamma(T\Sigma)$ . With these constructions,  $(\mathcal{S}\Sigma, \not\gamma, \langle \cdot, \cdot \rangle, \nabla)$  is also a Dirac bundle over  $\Sigma$ . We will denote by  $\not{D} : \Gamma(\mathcal{S}\Sigma) \rightarrow \Gamma(\mathcal{S}\Sigma)$  the Dirac operator associated with the Dirac bundle  $\mathcal{S}\Sigma$ . It is a well-known fact that  $\not{D}$  is a first order elliptic differential operator which is formally  $L^2$ -selfadjoint. By (10), we have for any spinor field  $\psi \in \Gamma(\mathbb{S}M)$ ,

$$\not{D}\psi = \sum_{j=1}^{n-1} \not\gamma(e_j) \nabla_{e_j} \psi = -\frac{n-1}{2} H\psi - \gamma(N) D\psi - \nabla_N \psi,$$

where  $\{e_1, \dots, e_{n-1}\}$  is a local orthonormal frame of  $T\Sigma$ . Moreover, from (9) and (10), it is straightforward to see that the skew-commutativity rule

$$(11) \quad \not{D}(\gamma(N)\psi) = -\gamma(N)\not{D}\psi$$

holds for any spinor field  $\psi \in \Gamma(\mathcal{S}\Sigma)$ . From this fact, we observe that *the spectrum of  $\not{D}$  is always symmetric with respect to zero*, while this is the case for the Dirac operator  $D_\Sigma$  of the intrinsic spinor bundle *when  $n$  is odd*. If  $n$  is even, the spectrum of  $D_\Sigma$  is not necessarily symmetric. In fact, in this case, the spectrum of  $\not{D}$  is just the symmetrization of the spectrum of  $D_\Sigma$ . Consequently, any assumptions on the spectrum of  $D_\Sigma$  automatically translates to assumptions on the spectrum of  $\not{D}$ .

In the hyperbolic setting, we have to consider another Dirac-type operator which arises as associated to the following modification of the spin Levi-Civita connection

$$(12) \quad \nabla_X^\pm := \nabla_X \pm (i/2)\gamma(X)$$

for  $X \in \Gamma(TM)$ . Note that  $\nabla^\pm$ -parallel spinor fields correspond to imaginary Killing spinors with Killing number  $\mp(i/2)$ . The associated Dirac operators are thus defined for all  $\psi \in \Gamma(\mathbb{S}M)$  by

$$D^\pm \psi := \sum_{j=1}^n \gamma(e_j) \nabla_{e_j}^\pm \psi = D\psi \mp \frac{n}{2} i\psi.$$

Since  $D^\pm$  are zero order modifications of the Dirac operator  $D$ , they also define first order elliptic differential operators. From the classical Schrödinger-Lichnerowicz formula

$$D^2 = \nabla^* \nabla + \frac{1}{4} R_g$$

and an integration by parts on the manifold  $M_R = M \setminus \{r \geq R\} \subset M$ , we get

$$(13) \quad \int_{M_R} \langle D\psi, \varphi \rangle dM = \int_{M_R} \langle \psi, D\varphi \rangle dM - \int_{\Sigma_R} \langle \gamma(\tilde{N}_R)\psi, \varphi \rangle d\Sigma_R$$

where  $\tilde{N}_R$  is the inner unit normal vector field to  $\Sigma_R = \partial M_R = \Sigma \amalg \mathbb{S}_R$  (which is given by  $N$  on  $\Sigma$  and  $N_R$  on  $\mathbb{S}_R$ ). Now we can compute the corresponding formulae for the modified connections  $\nabla^\pm$  and the corresponding Dirac operators  $D^\pm$  (see the proof of Theorem 4.7 in [CH]):

**Lemma 2.** *For any smooth spinor field  $\psi$  on  $M_R$ , we have*

$$\begin{aligned} - \int_{\mathbb{S}_R} \langle \nabla_{N_R}^\pm \psi + \gamma(N_R) D^\pm \psi, \psi \rangle d\mathbb{S}_R &= - \int_{\Sigma} \langle \mathcal{D}^\pm \psi + \frac{n-1}{2} H \psi, \psi \rangle d\Sigma \\ &+ \int_{M_R} \left( |\nabla^\pm \psi|^2 + \frac{1}{4} (R_g + n(n-1)) |\psi|^2 - |D^\pm \psi|^2 \right) dM. \end{aligned}$$

Here, the operators  $\mathcal{D}^\pm$  are given for any  $\psi \in \Gamma(\mathcal{S}\Sigma)$  by

$$\mathcal{D}^\pm \psi := \mathcal{D} \psi \pm \frac{n-1}{2} i \gamma(N) \psi.$$

An important feature for our concern is the study their spectra.

**3.2. On the spectra of  $\mathcal{D}^\pm$ .** Here we relate the spectra of the operators  $\mathcal{D}^\pm$  to the spectrum of the Dirac operator  $\mathcal{D}$ . Then, from well-known eigenvalue estimates on the first eigenvalue of  $\mathcal{D}$ , we deduce a lower bound for their eigenvalues (see Proposition 4).

**Proposition 3.** *The operators  $\mathcal{D}^\pm$  are first order elliptic and self-adjoint differential operators. Moreover, if the Dirac operator  $\mathcal{D}$  is invertible, then*

$$\text{Spec}(\mathcal{D}^+) = \text{Spec}(\mathcal{D}^-) = \left\{ \tilde{\lambda} = \pm \sqrt{\lambda^2 + \frac{(n-1)^2}{4}} / \lambda \in \text{Spec}(\mathcal{D}) \right\}$$

and for any  $\lambda \in \text{Spec}(\mathcal{D}^\pm)$  the map  $\gamma(N) : E_\lambda(\mathcal{D}^\pm) \rightarrow E_{-\lambda}(\mathcal{D}^\mp)$  is an isomorphism. Here  $E_\lambda(\mathcal{D}^\pm)$  denote the eigenspaces associated with the eigenvalue  $\lambda$  for the corresponding operators.

*Proof :* Since  $\mathcal{D}^\pm$  are zero order modifications of the extrinsic Dirac operator it is obvious to see that they are elliptic first order differential operators. Moreover, since  $\mathcal{D}$  is self-adjoint with respect to the  $L^2$ -scalar product on  $\Sigma$  and since the map  $i\gamma(N)$  is pointwise Hermitian, it is straightforward to check that  $\mathcal{D}^\pm$  are also self-adjoint for the  $L^2$ -scalar product. In particular, since  $\Sigma$  is compact, their spectra consist of unbounded sequence of real numbers  $(\mu_k^\pm)_{k \in \mathbb{Z}}$ . Note that it is clear from (11) that

$$(14) \quad \mathcal{D}^\pm(\gamma(N)\psi) = -\gamma(N)\mathcal{D}^\mp\psi$$

for all  $\psi \in \Gamma(\mathcal{S}\Sigma)$ . Since by (11) the operator  $\mathcal{D}$  anticommutes with  $i\gamma(N)$ , the operators  $\mathcal{D}^\pm$  leave invariant the finite-dimensional space  $E_\lambda(\mathcal{D}) \oplus E_{-\lambda}(\mathcal{D})$  and they act on as

$$\mathcal{D}^\pm = \begin{pmatrix} \lambda & \pm \frac{n-1}{2} \\ \pm \frac{n-1}{2} & -\lambda \end{pmatrix}.$$

We conclude by noting that these two matrices have the same characteristic polynomials. q.e.d.

This result implies in particular that the spectra of  $\mathcal{D}^+$  and  $\mathcal{D}^-$  coincide and are symmetric with respect to zero. For this reason we denote their spectra by  $(\mu_k)_{k \in \mathbb{Z}^*}$  in such a way that

$$\mu_k > 0 \quad \text{for } k > 0 \quad \text{and} \quad \mu_k = -\mu_{-k} \quad \text{for } k \in \mathbb{Z}^*.$$

Now from well-known estimates of the first author [Hi1, Hi2] and C. Bär [Bä] on the first eigenvalue of the Dirac operator and from the previous characterization of the spectrum of  $\mathcal{D}^\pm$ , we immediately get:

**Proposition 4.** *Let  $(\Sigma^{n-1}, g)$  be an oriented compact hypersurface in an  $n$ -dimensional Riemannian spin manifold. The first positive eigenvalue  $\mu_1$  of  $\mathcal{D}^\pm$  satisfies:*

(1) *If  $n = 3$  and  $\Sigma^2$  is homeomorphic to a 2-sphere then*

$$(15) \quad \mu_1^2 \text{Area}_g(\Sigma) \geq 4\pi + \text{Area}_g(\Sigma).$$

*Moreover equality occurs if and only if  $(\Sigma^2, g)$  is isometric to a round sphere.*

(2) *If  $n \geq 4$  and  $(\Sigma^{n-1}, g)$  has a positive Yamabe invariant  $Y(\Sigma)$  then*

$$(16) \quad \mu_1^2 \text{Vol}_g(\Sigma)^{\frac{2}{n-1}} \geq \frac{n-1}{4(n-2)} Y(\Sigma) + \frac{(n-1)^2}{4} \text{Vol}_g(\Sigma)^{\frac{2}{n-1}}.$$

*Moreover equality occurs if and only if  $(\Sigma^{n-1}, g)$  carries a non-trivial real Killing spinor.*

This estimate on  $\mu_1$  will appear to be crucial in the proof of Theorem 2 and 3 because of the choice of the boundary condition we describe now.

**3.3. An Atiyah-Patodi-Singer boundary condition.** In this section, we introduce the boundary condition which we will use to prove Theorem 2 and 3. This condition appears naturally by looking at the  $\Sigma$ -boundary term in the formula of Lemma 2. Note that it differs from the one used in [CH] to prove Theorem 1. This condition has been used by the first two authors and A. Roldán in [HMR] to prove lower bounds for the first eigenvalue of the Dirac operator on compact manifolds bounding compact domains with  $R_g \geq -n(n-1)$ . It belongs to the broad class of spectral conditions introduced by Atiyah, Patodi and Singer [APS] to prove the index theorem for manifolds with boundary.

From Proposition 3, we know that the first order operators  $\mathcal{D}^\pm$  are self-adjoint, elliptic and their spectra are unbounded discrete sets of real numbers  $(\mu_k)_{k \in \mathbb{Z}}$ . Moreover, there exists a complete orthonormal basis of the Hilbert space  $L^2(\mathcal{S}\Sigma)$  consisting of associated smooth eigenspinors. Now let

$$\mathbf{P}_\pm^\pm : L^2(\mathcal{S}\Sigma) \longrightarrow L^2(\mathcal{S}\Sigma)$$

be the  $L^2$ -orthogonal projections onto the subspace spanned by the eigenspinors corresponding to the positive eigenvalues of  $\mathcal{D}^\pm$ . It is shown in [BC] that such maps  $\mathbf{P}_\pm^\pm$  provide (global) elliptic boundary conditions for the Dirac-type operator  $D^\pm$ . As a direct consequence of Formula (13) and

Proposition 3, we obtain that the adjoint of the operators  $D^\pm$  under the boundary condition  $\mathbf{P}_\pm^\pm$  are the operators  $D^\mp$  under the boundary condition  $\mathbf{P}_\pm^\mp$ .

On the hyperbolic space  $(\mathbb{H}^n, b)$ , we have a set of distinguished spinor fields, the imaginary Killing spinor fields. These are solutions  $\varphi$  of the overdetermined partial differential equation

$$\nabla_X^b \varphi = \mp \frac{i}{2} \gamma(X) \varphi$$

for all  $X \in \Gamma(T\mathbb{H}^n)$  and where  $\nabla^b$  is the Levi-Civita connection associated with the hyperbolic metric  $b$ . Using the ball model  $(\mathbb{B}^n, \omega^2 \delta)$  of the hyperbolic space  $\mathbb{H}^n$  with  $\omega(x) = 2/(1 - |x|^2)$ , Baum [Bm] gave a nice and simple description of these particular fields. Indeed, under a conformal change of the metric, we can identify the spinor bundle over  $(\mathbb{B}^n, \delta)$  with the one over  $(\mathbb{H}^n, b)$  in such a way that to any constant spinor  $u \in \Gamma(\mathbb{S}\mathbb{B}^n)$  corresponds an imaginary Killing spinor fields  $\varphi_u^\pm$  on  $\mathbb{H}^n$  by letting

$$\varphi_u^\pm(x) = \sqrt{\frac{2}{1 - |x|^2}} (1 \mp i\gamma_\delta(x)) u.$$

Here we omit the identification between  $\mathbb{S}\mathbb{B}^n$  and  $\mathbb{S}\mathbb{H}^n$  and  $\gamma_\delta$  denotes the Euclidean Clifford multiplication. A straightforward calculation shows that for any such  $\varphi_u$ , the function  $V_u = |\varphi_u|_b^2$  is always an element of  $\mathcal{N}$ . More importantly again is the fact that any future-directed combination  $V_{(0)} - \sum a_i V_{(i)}$ , with  $(a_i) \in \mathbb{S}^{n-1}$ , can be obtained as a  $V_u$  for some imaginary Killing spinor  $\varphi_u$ . The choice of the boundary conditions  $\mathbf{P}_\pm^\pm$  are due to the fact that imaginary Killing spinors  $\varphi_u^\pm$  are solutions of  $D^\pm \varphi_u^\pm = 0$  with our boundary conditions on  $\mathbb{H}^n \setminus B(r_0)$  where  $B(r_0) \subset \mathbb{H}^n$  is a geodesic ball of radius  $r_0 > 0$ . This was not the case for the boundary condition used in [CH] to prove Theorem 1.

**3.4. Proof of Theorem 2 and 3.** We closely follow the analytic setting developed by Bartnik and Chruściel in [BC]. For this, we observe that under our assumptions, we have:

- *A weighted Poincaré inequality* (Definition 8.2 in [BC]), that is there exists  $w \in L_{loc}^1(M)$  with  $\text{ess inf}_\Omega w > 0$  for all relatively compact  $\Omega$  in  $M$  such that for all  $\psi \in C_c^1(M)$ , the space of  $C^1$  compactly supported spinor fields on  $M$ , we have

$$(17) \quad \int_M |\psi|^2 w \, dM \leq \int_M |\nabla^\pm \psi|^2 \, dM.$$

Indeed, it is easy to see that  $\Gamma^\pm$ , the symmetric parts of the connections  $\nabla^\pm$  are given by  $\Gamma_X^\pm = \mp (i/2) \gamma(X)$  so that we can check that the assumptions of Theorem 9.10 in [BC] are fulfilled.

• *A Schrödinger-Lichnerowicz estimate for the pairs  $(D^\pm, \mathbf{P}_\pm^\pm)$*  (Definition 8.4 in [BC]): indeed first note that for  $\psi \in C_c^1(M)$ , we can write

$$\psi|_\Sigma = \sum_{k \in \mathbb{Z}^*} A_k \Phi_k^\pm = \sum_{k < 0} A_k \Phi_k^\pm + \mathbf{P}_+^\pm \psi|_\Sigma$$

with  $(A_k)_{k \in \mathbb{Z}^*} \subset \mathbb{C}$  and where  $(\Phi_k^\pm)_{k \in \mathbb{Z}^*}$  is a complete  $L^2$ -orthonormal basis of  $\mathcal{S}\Sigma$  consisting of smooth eigenspinors for  $\mathcal{D}^\pm$  associated with the eigenvalue  $\mu_k$ . Then the boundary term (on  $\Sigma$ ) in the Schrödinger-Lichnerowicz formula of Lemma 2 can be bounded as follows

$$\begin{aligned} \int_\Sigma \langle \mathcal{D}^\pm \psi + \frac{n-1}{2} H \psi, \psi \rangle d\Sigma &\leq \sum_{k > 0} (\mu_k + \frac{n-1}{2} \sup_\Sigma(H)) |A_k|^2 \\ &\quad + (\frac{n-1}{2} \sup_\Sigma(H) - \mu_1) \sum_{k < 0} |A_k|^2. \end{aligned}$$

The assumptions (7) and (8) on the mean curvature combined with the lower bounds on  $\mu_1$  given by (15) and (16) allow to show that the second term in the right-hand side of the previous inequality is non-positive and to bound the first term in such a way that we finally get

$$\int_\Sigma \langle \mathcal{D}^\pm \psi + \frac{n-1}{2} H \psi, \psi \rangle d\Sigma \leq \int_\Sigma |\mathcal{J}^\pm \mathbf{P}_+^\pm \psi|_\Sigma|^2 d\Sigma.$$

Here  $\mathcal{J}^\pm := (|\mathcal{D}^\pm| + \frac{n-1}{2} \sup_\Sigma(H))^{1/2}$  is the pseudo-differential operator defined by

$$\mathcal{J}^\pm \psi|_\Sigma = \sum_{k \in \mathbb{Z}^*} \left( |\mu_k| + \frac{n-1}{2} \sup_\Sigma(H) \right)^{1/2} A_k \Phi_k^\pm.$$

From this estimate, the Schrödinger-Lichnerowicz formula of Lemma 2 and the fact that  $R_g \geq -n(n-1)$  we conclude that the pairs  $(D^\pm, \mathbf{P}_\pm^\pm)$  satisfy a Schrödinger-Lichnerowicz estimate, that is for all  $\psi \in C_c^1(M)$ , we have

$$(18) \quad \int_M |\nabla^\pm \psi|^2 dM \leq \int_M |D^\pm \psi|^2 dM + \int_\Sigma |\mathcal{J}^\pm \mathbf{P}_+^\pm \psi|_\Sigma|^2 d\Sigma.$$

Now consider the space

$$C_c^{1,\pm}(M) := \{\psi \in C_c^1(M) / \mathbf{P}_+^\pm \psi|_\Sigma = 0\}$$

on which a norm is given by

$$\|\psi\|_{\mathcal{H}^\pm}^2 := \int_M \left( |\nabla^\pm \psi|^2 + \frac{1}{4} (R_g + n(n-1)) |\psi|^2 \right) dM$$

since we assumed  $R_g \geq -n(n-1)$ . Therefore the space

$$\mathcal{H}^\pm := \|\cdot\|_{\mathcal{H}^\pm} - \text{completion of } C_c^{1,\pm}(M)$$

is a Hilbert space. The Poincaré inequality (17) ensures that  $\mathcal{H}^\pm$  embeds continuously in  $H_{loc}^1$ . In particular, it implies that any  $\psi \in \mathcal{H}^\pm$  can be

represented by a spinor field in  $H_{loc}^1$  with  $\mathbf{P}_+^\pm \psi|_\Sigma = 0$ . Consider now the bilinear form defined by

$$\alpha(\varphi, \psi) := \int_M \langle D^+ \varphi, D^+ \psi \rangle dM$$

for  $\varphi, \psi \in \mathcal{H}^+$ . From Lemma 8.5 in [BC], we get that the map  $\psi \in \mathcal{H}^+ \mapsto D^+ \psi \in L^2$  is continuous and so  $\alpha$  is also continuous on  $\mathcal{H}^+ \times \mathcal{H}^+$ . Moreover, using the weighted Poincaré inequality (17) and the Schrödinger-Lichnerowicz estimate (18), we immediately observe that  $\alpha$  is coercive on  $\mathcal{H}^+$ . So if for  $\chi \in L^2$ , we define the continuous linear form

$$\mathcal{L}_\chi(\psi) = \int_M \langle \chi, D^+ \psi \rangle dM$$

on  $\mathcal{H}^+$ , the Lax-Milgram theorem implies that there exists a unique  $\xi_0 \in \mathcal{H}^+$  satisfying for all  $\psi \in \mathcal{H}^+$

$$\int_M \langle \zeta, D^+ \psi \rangle dM = 0$$

with  $\zeta := \mathcal{D}^+ \xi_0 - \chi$ . This leads to the fact that  $\zeta \in L^2$  is a weak solution of the boundary value problem

$$\begin{cases} (D^+)^* \zeta = 0 & \text{on } M, \\ (\mathbf{P}_+^+)^* \zeta|_\Sigma = 0 & \text{along } \Sigma \end{cases}$$

which from Section 3.3 reads as

$$\begin{cases} D^- \zeta = 0 & \text{on } M, \\ \mathbf{P}_+^- \zeta|_\Sigma = 0 & \text{along } \Sigma. \end{cases}$$

From the ellipticity of the Dirac-type operator  $D^-$  and the boundary condition  $\mathbf{P}_+^-$ , we conclude that  $\zeta \in \mathcal{H}^- \cap L^2$  is in fact a strong solution of this boundary value problem. Now it follows from standard arguments (see [AD, BC]) that, under the assumptions of Theorem 2 and 3, the operator  $D^-$  defined on  $\mathcal{H}^-$  has a trivial  $L^2$ -kernel. This implies in particular that  $\zeta \equiv 0$  and so  $\xi_0 \in \mathcal{H}^+$  is the unique solution of the boundary value problem

$$\begin{cases} D^+ \xi_0 = \chi & \text{on } M, \\ \mathbf{P}_+^+ \xi_0|_\Sigma = 0 & \text{along } \Sigma. \end{cases}$$

To summarize, we proved the following existence result:

**Proposition 5.** *Under the assumptions of Theorem 2 and 3, for all  $\chi \in L^2$  there exists a unique  $\xi_0 \in \mathcal{H}^\pm$  such that  $D^\pm \xi_0 = \chi$ .*

Now we can apply the classical Witten's argument to conclude. To every constant spinor  $u \in \Gamma(\mathbb{S}\mathbb{B}^n)$  is associated an imaginary Killing spinor  $\varphi_u \in \Gamma(\mathbb{S}\mathbb{H}^n)$  with Killing number  $-(i/2)$  (see Section 3.3). Since  $(M^n, g)$  is asymptotically hyperbolic, we can identify the spinor bundle over  $M_{ext} = M \setminus K$  with the one over  $\mathbb{H}^n \setminus B$  (by using the chart at infinity). So, through this identification, the field  $\varphi_u$  on  $\mathbb{H}^n$  is mapped to a spinor field over  $M_{ext}$ , also denoted by  $\varphi_u$ . Let  $\eta$  be a cut-off function that vanishes outside of  $M_{ext}$

and is equal to 1 for large  $r$  and consider the spinor field  $\eta\varphi_u$ . It is now a classical fact [AD] that  $D^+(\eta\varphi_u) \in L^2$  and so Proposition 5 applies to get the existence of  $\psi_u \in \mathcal{H}^+$  such that  $D^+\psi_u = -D^+(\eta\varphi_u)$ . In other words, the spinor field  $\xi_u = \psi_u + \eta\varphi_u$  is a solution of the boundary value problem

$$\begin{cases} D^+\xi_u = 0 & \text{on } M, \\ \mathbf{P}_+^+\xi_u|_\Sigma = 0 & \text{along } \Sigma. \end{cases}$$

If  $\psi_u$  was an element of  $C_c^{1,+}(M)$  one could directly apply the Schrödinger-Lichnerowicz formula of Lemma 2 and calculations in [CH] to conclude that

$$(19) \quad \begin{aligned} \frac{1}{4}H_\Psi(V_u) &= \int_M \left( |\nabla^+\xi_u|^2 + \frac{1}{4}(R_g + n(n-1))|\xi_u|^2 \right) \\ &\quad - \int_\Sigma \langle \mathcal{D}^+\psi_u + \frac{n-1}{2}H\psi_u, \psi_u \rangle d\Sigma \end{aligned}$$

and so  $H_\Psi(V_u) \geq 0$  for  $R_g \geq -n(n-1)$  and  $H$  satisfying (7) and (8). Actually, it is not difficult to show that the previous equality holds for  $\psi_u \in \mathcal{H}^\pm$ . This is a simple consequence of the fact that the right-hand side of (19) is continuous on  $\mathcal{H}^+$  (and  $C_c^{1,+}(M)$  is dense in  $\mathcal{H}^+$ ). Now, as mentioned in Section 3.3, any future-directed null  $V \in \mathcal{N}$  is obtained as  $V_u$ , the squared norm of an imaginary Killing spinor field. We then deduce that the mass functional is non-negative on any such vectors  $V \in \mathcal{N}$  and this implies that  $H_\Psi$  is causal and future-directed.

Assume now that the mass functional  $H_\Psi$  is null or zero. From the Schrödinger-Lichnerowicz formula (19), the spinor field  $\xi_u$  has to be an imaginary Killing spinor field on  $M$  and so  $M$  is an Einstein manifold with constant negative scalar curvature equals to  $-n(n-1)$ . Moreover, we also have equality in the estimates (15) and (16) depending on whether  $n = 3$  or  $n \geq 4$ .

Suppose first that  $n = 3$ . First since  $(M^3, g)$  is Einstein, it implies that  $M$  has constant sectional curvature  $-1$ . Moreover, equality in (15) implies that  $(\Sigma^2, g)$  is isometric to a round sphere. On the other hand, integrating the Gauß formula on  $\Sigma$  with the help of the Gauß-Bonnet formula gives

$$8\pi = 2 \int_\Sigma (H^2 - 1) d\Sigma - \int_\Sigma |\mathcal{O}|^2 d\Sigma$$

where  $\mathcal{O}$  is the traceless part of the second fundamental form. This identity combined with the upper bound (7) on the mean curvature of  $\Sigma$  implies that  $(\Sigma^2, g)$  is isometric to a totally umbilical round sphere with constant mean curvature. Moreover, the second fundamental form of  $\Sigma$  in  $M$  agrees with the one of  $\Sigma$  in  $\mathbb{H}^3$ . We can then glue along  $\Sigma$  in  $M$  a geodesic ball of the hyperbolic space to obtain a smooth and complete Riemannian manifold  $\widetilde{M}$  with constant negative sectional curvature in which a round sphere is embedded and so  $\widetilde{M}$  is isometric to  $\mathbb{H}^3$  (see Corollary 7 in [HMR] for example). We finally deduce that  $M$  is isometric to the complement of a geodesic ball in  $\mathbb{H}^3$ .

For  $n \geq 4$ , the previous method cannot be applied since we only have the existence of one imaginary Killing spinor  $\xi_u \in \Gamma(\mathbb{S}M)$  so that we can only deduce that  $(M^n, g)$  is Einstein with negative scalar curvature  $-n(n-1)$ . On the other hand, the restriction of  $\xi_u$  to  $\Sigma$ , which is the spinor field  $\psi_u$ , has to be an eigenspinor for the operator  $\mathcal{D}^+$  associated with the eigenvalue  $-\mu_1$ . Moreover, as observed in Proposition 4, this implies the existence of a real Killing spinor on  $\Sigma$ . However, from the relation between the eigenspaces of  $\mathcal{D}^+$  and  $\mathcal{D}$  established in the proof of Lemma 3, it is straightforward to observe that the spinor field

$$(20) \quad \tilde{\psi}_u := \frac{n-1}{2}\gamma(N)\psi_u - i(\mu_1 - \lambda_1)\psi_u$$

has to be a real Killing spinor that is

$$(21) \quad \nabla_X \tilde{\psi}_u = -\frac{\lambda_1}{n-1}\gamma(X)\gamma(N)\tilde{\psi}_u$$

for all  $X \in \Gamma(T\Sigma)$ . Here  $\lambda_1 = \sqrt{\mu_1^2 - \frac{(n-1)^2}{4}}$  is the first positive eigenvalue of the Dirac operator  $\mathcal{D}$ . Now differentiating equality (20) along  $\Sigma$  using the fact that  $\psi_u$  is an imaginary Killing spinor and the spin Gauß formula (10) gives

$$(22) \quad \begin{aligned} \nabla_X \tilde{\psi}_u &= \frac{n-1}{4}i\gamma(X)\gamma(N)\psi_u - \frac{1}{2}(\mu_1 - \lambda_1)\gamma(X)\psi_u \\ &+ \frac{n-1}{4}\gamma(AX)\psi_u - \frac{i}{2}(\mu_1 - \lambda_1)\gamma(AX)\gamma(N)\psi_u \end{aligned}$$

for all  $X \in \Gamma(T\Sigma)$ . On the other hand, replacing  $\tilde{\psi}_u$  in the right-hand side of (21) by its expression (20) gives, for all  $X \in \Gamma(T\Sigma)$ :

$$(23) \quad \nabla_X \tilde{\psi}_u = \frac{\lambda_1}{2}\gamma(X)\psi_u + i\frac{\lambda_1}{n-1}(\mu_1 - \lambda_1)\gamma(X)\gamma(N)\psi_u.$$

Hence, the last two equations, lead to

$$\gamma\left(\frac{1}{2}AX - \frac{\mu_1}{n-1}X\right)\left(\frac{n-1}{2}\psi_u - i(\mu_1 - \lambda_1)\gamma(N)\psi_u\right) = 0$$

for all  $X \in \Gamma(T\Sigma)$ . Now since  $\psi_u$  is an imaginary Killing spinor it has no zero, we conclude that the same property holds for the spinor field  $\frac{n-1}{2}\psi_u - i(\mu_1 - \lambda_1)\gamma(N)\psi_u$  by invertibility of the matrix

$$\begin{pmatrix} \frac{n-1}{2} & -(\mu_1 - \lambda_1) \\ -(\mu_1 - \lambda_1) & \frac{n-1}{2} \end{pmatrix}.$$

This allows to conclude that  $\Sigma$  is totally umbilical with constant mean curvature equal to  $\frac{2\mu_1}{n-1}$ .



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