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► **To cite this version:**

van Hao Can. Exponential extinction time of the contact process on rank-one inhomogeneous random graphs . 2017. hal-01267122v2

HAL Id: hal-01267122

<https://hal.science/hal-01267122v2>

Preprint submitted on 19 Sep 2017

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EXPONENTIAL EXTINCTION TIME OF THE CONTACT PROCESS ON RANK-ONE INHOMOGENEOUS RANDOM GRAPHS

VAN HAO CAN

ABSTRACT. We show that the contact process on the rank-one inhomogeneous random graphs and Erdos-Rényi graphs with mean degree large enough survives a time exponential in the size of these graphs for any positive infection rate. In addition, a metastable result for the extinction time is also proved.

1. INTRODUCTION

The contact process was introduced in [10] by T. E. Harris and is also often interpreted as a model for the spread of a virus in a population. Given a locally finite graph $G = (V, E)$ and $\lambda > 0$, the contact process on G with infection rate λ is a Markov process $(\xi_t)_{t \geq 0}$ on $\{0, 1\}^V$. Vertices of V (also called sites) are regarded as individuals which are either infected (state 1) or healthy (state 0). By considering ξ_t as a subset of V via $\xi_t \equiv \{v : \xi_t(v) = 1\}$, the transition rates are given by

$$\begin{aligned} \xi_t &\rightarrow \xi_t \setminus \{v\} \text{ for } v \in \xi_t \text{ at rate } 1, \text{ and} \\ \xi_t &\rightarrow \xi_t \cup \{v\} \text{ for } v \notin \xi_t \text{ at rate } \lambda |\{w \in \xi_t : \{v, w\} \in E\}|. \end{aligned}$$

Given $A \subset V$, we denote by $(\xi_t^A)_{t \geq 0}$ the contact process with initial configuration A and if $A = \{v\}$ we simply write (ξ_t^v) .

Since the contact process is monotone in λ , we can define the critical value

$$\lambda_c(G) = \inf\{\lambda : \mathbb{P}(\xi_t^v \neq \emptyset \forall t) > 0\}.$$

If G is connected, this definition does not depend on the choice of v . For integer lattices, it has been proved that $\lambda_c(\mathbb{Z}^d)$ is positive. Interestingly, the contact process on finite boxes $\llbracket 0, n \rrbracket^d$ exhibits a phase transition at the same critical value. More precisely, if we define the extinction time of the process (ξ_t^1) starting from full occupancy by

$$\tau_n = \inf\{t \geq 0 : \xi_t^1 = \emptyset\},$$

then with high probability (w.h.p.) τ_n is of logarithmic order when $\lambda < \lambda_c(\mathbb{Z}^d)$, and of exponential order when $\lambda > \lambda_c(\mathbb{Z}^d)$, see e.g. [12] Section I.3.

Recently, this phenomena has been observed in a few of other situations: the random regular graphs and their limit in the sense of the Benjamini-Schramm's local weak convergence of graphs [4] - the homogeneous tree, see [13, 16]; the finite homogeneous trees and their limit - the canopy tree, see [5, 16, 20]; the configuration models with heavy tail degree distributions and their limit - the Galton-Watson tree, see [6, 7, 14, 15, 19]; the preferential attachment graphs and their limit - the Pólya-point random graph, see [2, 8]; random geometric graphs, see [9, 17].

The goal of this paper is to explore the case of rank-one inhomogeneous graphs. An inhomogeneous random graph (IRG), $G_n = (V_n, E_n)$, is defined as follows. Let $V_n = \{v_1, \dots, v_n\}$ be the vertex set and let (w_i) be a sequence of i.i.d. positive random variables

with the same law as w . Then for any $1 \leq i \neq j \leq n$, we independently draw an edge between v_i and v_j with probability

$$p_{i,j} = 1 - \exp(-w_i w_j / \ell_n),$$

where

$$\ell_n = \sum_{i=1}^n w_i.$$

It is shown in [11, Vol. 2, Chapter 3] that when $\mathbb{E}(w)$ is finite, G_n converges weakly to a two-stages Galton-Watson tree. In this tree, the reproduction law of the root is (p_k) and the one of other vertices is (g_k) with

$$(1) \quad p_k = \mathbb{P}(\text{Poi}(w) = k) = \mathbb{E} \left(e^{-w} \frac{w^k}{k!} \right)$$

and

$$(2) \quad g_k = \mathbb{P}(\text{Poi}(w^*) = k) = \frac{1}{\mathbb{E}(w)} \mathbb{E} \left(e^{-w} \frac{w^{k+1}}{k!} \right),$$

where w^* is the size-bias distribution of w . We also assume in addition that

(H1) $w \geq 1$ a.s. and $\mathbb{E}(w) < \infty$,

(H2) the limiting tree is super critical, or equivalently

$$\nu =: \mathbb{E}(g) = \frac{\mathbb{E}(w^2)}{\mathbb{E}(w)} > 1,$$

(H3) there exists a function $\varphi(k)$ increasing to infinity, such that

$$\limsup_{k \rightarrow \infty} g_k e^{k/\varphi(k)} \geq 1.$$

Theorem 1.1. *Let τ_n be the extinction time of the contact process on inhomogeneous random graphs with the weight w satisfying the hypotheses (H1) – (H3), starting from full occupancy. Then for any $\lambda > 0$, there exist positive constants c and C , such that*

$$\mathbb{P}(\exp(Cn) \geq \tau_n \geq \exp(cn)) \rightarrow 1 \quad \text{as } n \rightarrow \infty.$$

Moreover,

$$\frac{\tau_n}{\mathbb{E}(\tau_n)} \xrightarrow[n \rightarrow \infty]{(\mathcal{D})} \mathcal{E}(1),$$

with $\mathcal{E}(1)$ the exponential random variable with mean 1.

For simplicity, we will replace the hypothesis (H3) by the following stronger version: there exists a function $\varphi(k)$ increasing to infinity, such that

$$(H3') \quad g_k e^{k/\varphi(k)} \geq 1 \quad \text{for all } k \geq 1.$$

We will see in Section 3 that this assumption does not change anything to the proof.

We note that (H1) is necessary for the weak convergence, (H2) is essential since without it w.h.p. all components have size $o(n)$ and (H3) is the key hypothesis in our proof.

We also remark that under (H2) and (H3), Pemantle proves in [19] that $\lambda_c(GW(g))$ is zero, with $GW(g)$ the Galton-Watson tree with reproduction law g .

It is worth noting that studying $\lambda_c(GW(g))$ when $g_k \asymp \exp(-ck)$ is still a challenge. An equivalently interesting problem is to study the extinction time of the contact process on

a super-critical Erdos-Rényi graph, which is a special inhomogeneous graph and converges weakly to a Galton-Watson tree with Poisson reproduction law.

Let us make some comments on the proof of Theorem 1.1. The upper bound on τ_n follows from a general result in [9]. To prove the lower bound, we will show that G_n contains a sequence of disjoint star graphs with large degree, whose total size is of order n . Moreover, the distance between two consecutive star graphs is not too large, so that the virus starting from a star graph can infect the other one with high probability. Then by comparing with an oriented percolation with density close to 1, we get the lower bound. The convergence in law can be proved similarly as in [9].

The paper is organized as follows. In Section 2, we prove some preliminary results to describe the neighborhood of a vertex in the graph. In Section 3, by defining some exploration process of the vertices we prove the existence of the sequence of star graphs mentioned above. Then we prove our main theorem. In Section 4, we prove a similar result for the Erdos-Rényi graph: for any $\lambda > 0$, if the mean degree of the Erdos-Rényi graph is larger than some explicit function of λ , then the extinction time is also of exponential order.

Now we introduce some notation. We denote the indicator function of a set E by $\mathbf{1}(E)$. For any vertices v and w we write $v \sim w$ if there is an edge between them. We call size of a graph G the cardinality of its set of vertices, and we denote it by $|G|$. A graph in which all vertices have degree one, except one which is connected to all the others is called a *star graph*. The only vertex with degree larger than one is called the center of the star graph, or central vertex.

Furthermore we denote by $\text{Bin}(n, p)$ the binomial distribution with parameters n and p and denote by $\text{Poi}(\mu)$ the Poisson distribution with mean μ . Let X and Y be two random variables or two distributions, we write $X \preceq Y$ if X is stochastically dominated by Y . If f and g are two real functions, we write $f = \mathcal{O}(g)$ if there exists a constant $C > 0$, such that $f(x) \leq Cg(x)$ for all x ; $f \asymp g$ if $f = \mathcal{O}(g)$ and $g = \mathcal{O}(f)$; $f = o(g)$ if $f(x)/g(x) \rightarrow 0$ as $x \rightarrow \infty$. Finally for a sequence of random variables (X_n) and a function $f : \mathbb{N} \rightarrow (0, \infty)$, we say that $X_n \asymp f(n)$ holds w.h.p. if there exist positive constants c and C , such that $\mathbb{P}(cf(n) \leq X_n \leq Cf(n)) \rightarrow 1$, as $n \rightarrow \infty$.

2. PRELIMINARIES

2.1. A preliminary result on the sequence of weights.

Lemma 2.1. *Let (w_i) be the sequence of i.i.d. weights as in the definition of IRGs. Then for any $\delta > 0$, there exists $\varkappa_1 = \varkappa_1(\delta) \in (0, 1)$, such that*

$$\mathbb{P} \left(\sum_{i \in U} w_i \geq (1 - \delta) \sum_{i=1}^n w_i \text{ for all } U \subset \{1, \dots, n\} \text{ with } |U| \geq n(1 - \varkappa_1) \right) \rightarrow 1.$$

Proof. Using the law of large numbers, we get

$$\mathbb{P} \left(\sum_{i=1}^n w_i = n(\mu + o(1)) \right) \rightarrow 1,$$

with

$$\mu = \mathbb{E}(w).$$

Observe that the worst case of the sum $\sum_{i \in U} w_i$ is where U contains the first $[n(1 - \varkappa)n]$ order statistics. Hence, it is sufficient to show that for any $\delta > 0$, there exists $\varkappa_1 \in (0, 1)$, such that

$$(3) \quad \mathbb{P} \left(\frac{1}{n} \sum_{i=n-[\varkappa_1 n]}^n w_{(i)} \geq \mu\delta \right) \rightarrow 0,$$

where $w_{(1)} \leq w_{(2)} \leq \dots \leq w_{(n)}$ is the order statistics of the sequence (w_i) . Clearly, if w is bounded (and thus the sequence (w_i) is also), i.e. there exists a positive constant K , such that $\mathbb{P}(w \leq K) = 1$, then (3) easily follows by taking $\varkappa_1 = (\mu\delta)/(2K)$. We assume that w is unbounded. Since $\mathbb{E}(w) \in (0, \infty)$, we can find a number $x \in (0, \infty)$, such that

$$\mathbb{E}(w1(w > x)) \leq (\mu\delta)/2.$$

Let us define

$$\Lambda = \{i : X_i > x\}.$$

Since w is unbounded,

$$\varkappa_1 := \frac{\mathbb{P}(w > x)}{2} \in (0, 1).$$

By law of large number, we have

$$(4) \quad \frac{|\Lambda|}{n} \xrightarrow{\text{a.s.}} 2\varkappa_1,$$

$$(5) \quad \frac{1}{n} \sum_{i \in \Lambda} w_i = \frac{1}{n} \sum_{i=1}^n w_i 1(w_i > x) \xrightarrow{\text{a.s.}} \mathbb{E}(w1(w > x)) \in (0, \mu\delta/2).$$

We observe that

$$(6) \quad \sum_{i \in \Lambda} w_i = \sum_{i=n-|\Lambda|+1}^n w_{(i)}.$$

Now, we can deduce (3) from (4), (5) and (6). \square

2.2. Coupling of an IRG with a Galton-Watson tree. We describe here the neighborhood of a vertex in a set. Let $U \subset V_n$ and $v \in V_n \setminus U$, and let R be a positive integer. We denote by $B_R(v, U)$ the graph containing all vertices in U at distance less than or equal to R from v . We adapt the construction in [11, Vol. 2, Section 3.4] to make a coupling between $B_R(v, U)$ and a *marked mixed-Poisson Galton-Watson tree*.

Conditionally on the weights (w_i) , we define a random variable M_U as follows

$$(7) \quad \mathbb{P}(M_U = m) = w_m/\ell_U \quad \text{for all indices } m \text{ such that } v_m \in U,$$

with $\ell_U = \sum w_i 1(v_i \in U)$. Note that $\ell_n = \ell_{V_n}$.

We define a random tree with root o as follows. We first define the mark of the root as $M_o = m$, with m the index such that $v_m = v$. Then o has X_o children, with

$$X_o \sim \text{Poi}(w_{M_o} \ell_U / \ell_n).$$

Each child of the root, say x , is assigned an independent mark M_x , with the same distribution as M_U . Conditionally on M_x , the number of children of x has distribution $\text{Poi}(w_{M_x} \ell_U / \ell_n)$.

Suppose that all the vertices at height smaller than or equal to i are defined. We determine the vertices at height $(i + 1)$ as follows. Each vertex at height i , say y , has an independent mark M_y with the same distribution as M_U and it has X_y children, where X_y is a Poisson random variable with mean $w_{M_y}\ell_U/\ell_n$.

We denote the resulted tree by $\mathbb{T}(v, U)$ and call it the *marked mixed-Poisson Galton-Watson tree* associated to (v, U) . In order to make a relation between $B_R(v, U)$ and $\mathbb{T}(v, U)$, we define a *thinning* procedure on $\mathbb{T}(v, U)$ as follows.

For a vertex y different from the root, we thin y when either one of the vertices on the unique path between the root and y has been thinned, or when $M_y = M_{y'}$, for some vertex y' on this path.

We denote by $\tilde{\mathbb{T}}(v, U)$ the tree resulting from the thinning on $\mathbb{T}(v, U)$.

Proposition 2.2. [11, Vol. II, Proposition 3.10] *Conditionally on (w_i) , the set of vertices in U at distance k from v (considering the graph induced in $U \cup \{v\}$) has the same distribution as*

$$\left(\{v_{M_x} : x \in \tilde{\mathbb{T}}(v, U) \text{ and } |x| = k\} \right)_{k \geq 0},$$

with $|x|$ the height of x . Moreover, $B_R(v, U)$ contains a subgraph which has the same law as $\tilde{\mathbb{T}}_R(v, U)$ - the graph containing all vertices in $\tilde{\mathbb{T}}(v, U)$ whose heights are smaller than or equal to R .

We note that in [11], the author only proves this proposition for $U = V \setminus \{v\}$. The proof for any subset of $V \setminus \{v\}$ is essentially the same, so we do not present here.

The law of the marked-mixed Poisson Galton-Watson tree. The offspring distribution of the root is given by

$$p_k^U = \mathbb{P}(\text{Poi}(w_{M_o}\ell_U/\ell_n) = k) \quad \text{for } k \geq 0.$$

The individuals of the second and further generations have the same offspring distribution, denoted by (g_k^U) . It is given as follows: for all $k \geq 0$

$$g_k^U = \mathbb{P}(\text{Poi}(w_{M_U}\ell_U/\ell_n) = k) = \sum_{v_i \in U} \mathbb{P}(\text{Poi}(w_i\ell_U/\ell_n) = k)w_i/\ell_U.$$

If $U = V_n$, we write $g_k^{(n)}$ for g_k^U . Hence

$$g_k^{(n)} = \sum_{i=1}^n \mathbb{P}(\text{Poi}(w_i) = k)w_i/\ell_n = \mathbb{P}(\text{Poi}(W_n^*) = k),$$

with W_n^* the size-bias distribution of the empirical mean weight $W_n = (w_1 + \dots + w_n)/n$. It is shown in [11] that since the (w_i) are i.i.d. with the same law as w and $\mathbb{E}(w)$ is finite,

$$W_n \xrightarrow[n \rightarrow \infty]{(\mathcal{D})} w \quad \text{and} \quad W_n^* \xrightarrow[n \rightarrow \infty]{(\mathcal{D})} w^*,$$

with w^* the size-bias distribution of w . Therefore, we have the following convergence.

Lemma 2.3. [11, Vol. II, Lemma 3.12]. *For all $k \geq 0$,*

$$\lim_{n \rightarrow \infty} g_k^{(n)} = g_k,$$

with (g_k) as in (2).

Using Lemmas 2.1 and 2.3, we will show that the distribution (g_k^U) approximates (g_k) , provided $|U|$ is large enough.

Lemma 2.4. *For any $\varepsilon > 0$ and $K \in \mathbb{N}$, there exists a constant $\varkappa_2 = \varkappa_2(\varepsilon, K) \in (0, \varkappa_1(1/2))$, such that*

$$\mathbb{P}(g_k^U \geq (1 - \varepsilon)g_k \text{ for all } 0 \leq k \leq K \text{ and } U \subset V_n \text{ with } |U| \geq (1 - \varkappa_2)n) \rightarrow 1.$$

Proof. If $g_k = 0$, then $g_k^U \geq (1 - \varepsilon)g_k$. Assume that $k \leq K$ and $g_k > 0$. We have

$$|g_k^U - g_k| \leq |g_k^U - g_k^{(n)}| + |g_k^{(n)} - g_k|.$$

Lemma 2.3 implies that for all n large enough

$$(8) \quad |g_k^{(n)} - g_k| \leq \varepsilon g_k / 2.$$

On the other hand,

$$\begin{aligned} |g_k^U - g_k^{(n)}| &= \left| \sum_{v_i \in U} \mathbb{P}(\text{Poi}(w_i \ell_U / \ell_n) = k) \frac{w_i}{\ell_U} - \sum_{v_i \in V_n} \mathbb{P}(\text{Poi}(w_i) = k) \frac{w_i}{\ell_n} \right| \\ &\leq \sum_{v_i \in U} \left| \mathbb{P}(\text{Poi}(w_i \ell_U / \ell_n) = k) \frac{w_i}{\ell_U} - \mathbb{P}(\text{Poi}(w_i) = k) \frac{w_i}{\ell_n} \right| + \sum_{v_i \notin U} \frac{w_i}{\ell_n} \\ &\leq \sum_{v_i \in U} \left| \mathbb{P}(\text{Poi}(w_i \ell_U / \ell_n) = k) - \mathbb{P}(\text{Poi}(w_i) = k) \right| \frac{w_i}{\ell_U} \\ &\quad + \sum_{v_i \in U} \mathbb{P}(\text{Poi}(w_i) = k) \left(\frac{w_i}{\ell_U} - \frac{w_i}{\ell_n} \right) + \sum_{v_i \notin U} \frac{w_i}{\ell_n} \\ &= S_1 + S_2 + S_3. \end{aligned}$$

Here, we have used that $|x_i y_i - a_i b_i| \leq |x_i - a_i| y_i + |y_i - b_i| a_i$ for all $x_i, y_i, a_i, b_i \geq 0$.

We now define

$$f_k(x) = \mathbb{P}(\text{Poi}(x) = k) = \frac{e^{-x} x^k}{k!}.$$

By the mean value theorem, for any $x < y$

$$|f_k(x) - f_k(y)| \leq \max_{x \leq u \leq y} |f_k'(u)| |x - y|.$$

If $k = 0$, then

$$|f_0(x) - f_0(y)| \leq e^{-x} (y - x).$$

If $k \geq 1$ then for $u > 0$,

$$\begin{aligned} f_k'(u) &= e^{-u} (k - u) \frac{u^{k-1}}{k!}, \\ f_k''(u) &= e^{-u} (u^2 - 2ku + k(k-1)) \frac{u^{k-2}}{k!}. \end{aligned}$$

Thus $f_k'(u) \leq 0 \leq f_k''(u)$ for all $u \geq 2k$. Hence, for $2k \leq x \leq u$,

$$|f_k'(u)| \leq |f_k'(x)| \leq f_k(x).$$

On the other hand, $|f_k'(u)| \leq 1$ for all $u \geq 0$. Therefore,

$$\max_{x \leq u \leq y} |f_k'(u)| \leq 1(x \leq 2k) + f_k(x).$$

In summary, for all k and $0 \leq x \leq y$

$$(9) \quad |f_k(x) - f_k(y)| \leq (1(x \leq 2k) + f_k(x))(y - x).$$

Applying (9), we get that if $\ell_U \geq \ell_n/2$ then

$$\begin{aligned} |f_k(w_i \ell_U / \ell_n) - f_k(w_i)| &\leq (1(w_i \ell_U / \ell_n \leq 2k) + f_k(w_i \ell_U / \ell_n)) \frac{w_i(\ell_n - \ell_U)}{\ell_n} \\ &\leq (1(w_i \leq 4k) + e^{-w_i/2} w_i^k / k!) \frac{w_i(\ell_n - \ell_U)}{\ell_n}. \end{aligned}$$

Therefore,

$$\begin{aligned} S_1 &= \sum_{v_i \in U} |f_k(w_i \ell_U / \ell_n) - f_k(w_i)| \frac{w_i}{\ell_U} \\ &\leq \sum_{i=1}^n \frac{w_i^2(\ell_n - \ell_U)}{\ell_U \ell_n} 1(w_i \leq 4k) + \sum_{i=1}^n \frac{e^{-w_i/2} w_i^{k+2} (\ell_n - \ell_U)}{k! \ell_U \ell_n} \\ &\leq \frac{4k(\ell_n - \ell_U)}{\ell_U} + \left(\frac{1}{\ell_n} \sum_{i=1}^n \frac{e^{-w_i/2} w_i^{k+2}}{k!} \right) \left(\frac{\ell_n - \ell_U}{\ell_U} \right). \end{aligned}$$

Observe that

$$\frac{1}{n} \sum_{i=1}^n \frac{e^{-w_i/2} w_i^{k+2}}{k!} \rightarrow \mathbb{E}(e^{-w/2} w^{k+2} / k!) < \infty.$$

On the other hand, $\ell_n \asymp n$. Therefore,

$$\frac{1}{\ell_n} \sum_{i=1}^n \frac{e^{-w_i/2} w_i^{k+2}}{k!} = \mathcal{O}(1).$$

Hence

$$S_1 = \mathcal{O}\left(\frac{k(\ell_n - \ell_U)}{\ell_U}\right).$$

Moreover,

$$\begin{aligned} S_2 &= \sum_{v_i \in U} \mathbb{P}(\text{Poi}(w_i) = k) \left| \frac{w_i}{\ell_U} - \frac{w_i}{\ell_n} \right| \\ &\leq \sum_{i=1}^n \frac{w_i(\ell_n - \ell_U)}{\ell_U \ell_n} = \frac{\ell_n - \ell_U}{\ell_U}, \end{aligned}$$

and

$$S_3 = \sum_{v_i \notin U} \frac{w_i}{\ell_n} \leq \frac{\ell_n - \ell_U}{\ell_U}.$$

In conclusion, if $\ell_U \geq \ell_n/2$ and $\ell_n \asymp n$ then

$$|g_k^U - g_k^{(n)}| \leq S_1 + S_2 + S_3 = \mathcal{O}\left(\frac{k(\ell_n - \ell_U)}{\ell_U}\right).$$

Therefore, there exists $\delta_k = \delta_k(k, \varepsilon, g_k) > 0$, such that if $\ell_U \geq (1 - \delta_k)\ell_n$ then

$$(10) \quad |g_k^U - g_k^{(n)}| \leq \varepsilon g_k / 2.$$

Define $\delta = \min\{\delta_k : k \leq K \text{ and } g_k > 0\}$. Then we have $\delta > 0$. Now, by Lemma 2.1 there exists $\varkappa_2 = \varkappa_1(\delta) \wedge \varkappa_1(1/2) \in (0, 1)$, such that w.h.p. for all $U \subset V_n$ with $|U| \geq (1 - \varkappa_2)n$,

$$\ell_U \geq (1 - \delta)\ell_n.$$

Thus by (8) and (10), w.h.p. for all $U \subset V_n$ with $|U| \geq (1 - \varkappa_2)n$,

$$g_k^U \geq (1 - \varepsilon)g_k \text{ for all } k \leq K,$$

which proves the result. \square

For any $\varepsilon \in (0, 1)$ and $K \in \mathbb{N}$, we define a distribution $(g^{\varepsilon, K})$ as follow:

$$\begin{aligned} g_k^{\varepsilon, K} &= 0 & \text{if } k &\geq K + 1, \\ g_k^{\varepsilon, K} &= (1 - \varepsilon)g_k & \text{if } 1 \leq k &\leq K, \\ g_0^{\varepsilon, K} &= 1 - (1 - \varepsilon) \sum_{k=1}^K g_k. \end{aligned}$$

The following result is a direct consequence of Lemma 2.4.

Lemma 2.5. *For any $\varepsilon > 0$ and $K \in \mathbb{N}$,*

$$\mathbb{P}((g^{\varepsilon, K}) \preceq (g^U) \text{ for all } U \subset V_n \text{ with } |U| \geq (1 - \varkappa_2)n) \rightarrow 1,$$

with \varkappa_2 as in Lemma 2.4.

Observe that $(g^{\varepsilon, K})$ stochastically increases (resp. decreases) in K (resp. ε). Moreover, it converges to (g) as $\varepsilon \rightarrow 0$ and $K \rightarrow \infty$. Therefore, by the hypothesis (H_2) , there are positive constants ε_0 and K_0 , such that for all $\varepsilon \leq \varepsilon_0$ and $K \geq K_0$,

$$(11) \quad \nu_{\varepsilon, K} := \sum_{k=0}^{\infty} k g_k^{(\varepsilon, K)} \geq \bar{\nu},$$

where

$$\bar{\nu} = \frac{1 + \nu}{2} \in (1, \nu).$$

Define for $K \geq K_0$,

$$\mathcal{E}(K) = \{(g^{\varepsilon_0, K}) \preceq (g^U) \text{ and } \ell_U \geq \ell_n/2 \text{ for all } U \subset V_n \text{ with } |U| \geq (1 - \varkappa_2)n\},$$

with $\varkappa_2 = \varkappa_2(\varepsilon_0, K)$ as in Proposition 2.5. Using this proposition and Lemma 2.1 with the fact that $\varkappa_2 \leq \varkappa_1(1/2)$, we obtain

$$(12) \quad \lim_{n \rightarrow \infty} \mathbb{P}(\mathcal{E}(K)) \rightarrow 1.$$

We call $T_{\varepsilon, K}$ the Galton-Watson tree with reproduction law $(g^{\varepsilon, K})$. Then (11) implies that $T_{\varepsilon, K}$ is super critical when $\varepsilon \leq \varepsilon_0$ and $K \geq K_0$. From now on, we set

$$\psi_1(K) = \lceil K / \sqrt{\varphi(3K)} \rceil,$$

with the function φ as in the hypothesis $(H3')$.

We prove here a key lemma saying that when U is large enough, with positive probability there exists a vertex in $\tilde{\mathbb{T}}(v, U)$ at distance less than $\psi_1(K)$ from the root having more than $3K$ children (which implies that there exists a vertex with degree larger than $3K$ in $B_{\psi_1(K)}(v, U)$ with positive probability).

Lemma 2.6. *There are positive constants θ_1 and K_1 , such that for all $K \geq K_1$ and $U \subset V_n$ with $|U| \geq (1 - \varkappa_2)n$ and n large enough,*

$$\mathbb{P}\left(\exists x \in \tilde{\mathbb{T}}(v, U) : |x| \leq \psi_1(K), \deg(x) \geq 3K + 1 \mid \mathcal{E}(K)\right) \geq \theta_1,$$

with \varkappa_2 as in Lemma 2.5.

Proof. If w_v -the weight of v - is larger than $10K$, then $\deg(o)$ is larger than $3K$ with positive probability. Indeed, $\deg(o)$ is a Poisson random variable with parameter $w_v \ell_U / \ell_n$. Moreover, on $\mathcal{E}(K)$, we have $\ell_U \geq \ell_n / 2$. Therefore,

$$\begin{aligned} \mathbb{P}(\deg(o) \geq 3K \mid \mathcal{E}(K)) &= \mathbb{P}(\text{Poi}(w_v \ell_U / \ell_n) \geq 3K \mid \mathcal{E}(K)) \\ &\geq \mathbb{P}(\text{Poi}(5K) \geq 3K) > 0. \end{aligned}$$

Hence, the result follows. We now suppose that $w_v \leq 10K$. Then, in the proof of [11, Vol. II, Corollary 3.13], it is shown that for any ℓ

$$(13) \quad \mathbb{P}(\mathbb{T}_\ell(v, U) \equiv \tilde{\mathbb{T}}_\ell(v, U)) \rightarrow 1 \quad \text{as } n \rightarrow \infty.$$

We denote by

$$Z_\ell^{v, U} = |\{x \in \mathbb{T}(v, U), |x| = \ell\}|.$$

Then by Lemma 2.5 conditionally on $\deg(o) \geq 1$,

$$(14) \quad Z_{\ell+1}^{v, U} \succeq Z_\ell^{\varepsilon_0, K},$$

with $Z_\ell^{\varepsilon_0, K}$ the number of individuals at the ℓ^{th} generation of $T_{\varepsilon_0, K}$. We remark that

$$\mathbb{P}(Z_\ell \geq m^{\ell/2} \mid Z_\ell \geq 1) \rightarrow 1 \quad \text{as } \ell \rightarrow \infty,$$

where Z_ℓ is the number of individuals at the ℓ^{th} generation of a Galton-Watson tree T with mean $m > 1$. Therefore, for all ℓ large enough

$$\mathbb{P}(Z_\ell \geq m^{\ell/2}) \geq \mathbb{P}(|T| = \infty) / 2.$$

Hence, for $K \geq K_0$

$$(15) \quad \mathbb{P}(Z_\ell^{\varepsilon_0, K} \geq \bar{\nu}^{\ell/2}) \geq \mathbb{P}(Z_\ell^{\varepsilon_0, K_0} \geq \bar{\nu}^{\ell/2}) \geq \mathbb{P}(|T_{\varepsilon_0, K_0}| = \infty) / 2.$$

It follows from (14) and (15) that for all ℓ large enough,

$$(16) \quad \begin{aligned} \mathbb{P}(Z_{\ell+1}^{v, U} \geq \bar{\nu}^{\ell/2} \mid \mathcal{E}(K)) &\geq \mathbb{P}(\deg(o) \geq 1 \mid \mathcal{E}(K)) \times \mathbb{P}(|T_{\varepsilon_0, K_0}| = \infty) / 2 \\ &\geq (1 - e^{-w_v/2}) \times \mathbb{P}(|T_{\varepsilon_0, K_0}| = \infty) / 2 \\ &\geq \mathbb{P}(|T_{\varepsilon_0, K_0}| = \infty) / 8. \end{aligned}$$

Here, we have used that $w_v \geq 1$. Now, we set

$$\ell = \psi_1(K) - 1.$$

Then we have

$$(17) \quad \begin{aligned} \mathbb{P}\left(\exists x \in \mathbb{T}(v, U) : |x| = \ell + 1, \deg(x) \geq 3K + 1 \mid Z_{\ell+1}^{v, U} \geq \bar{\nu}^{\ell/2}\right) \\ \geq \mathbb{P}(\text{Bin}([\bar{\nu}^{\ell/2}], g_{3K}) \geq 1) \rightarrow 1 \quad \text{as } K \rightarrow \infty, \end{aligned}$$

since under $(H3')$,

$$\bar{\nu}^{\ell/2} g_{3K} \geq \bar{\nu}^{\ell/2} \exp(-3K/\varphi(3K)) \rightarrow \infty \quad \text{as } K \rightarrow \infty.$$

Now, the result follows from (13), (16) and (17). \square

3. PROOF OF THEOREM 1.1

3.1. Structure of the proof. For $\ell, M \in \mathbb{N}$, we define the class $\mathcal{S}(\ell, M)$ as the set of all graphs containing a sequence of ℓ disjoint star graphs of size M with centers $(x_i)_{i \leq \ell}$, such that $d(x_i, x_{i+1}) \leq \psi_1(M) + 1$ for all $i \leq \ell - 1$.

The proof of Theorem 1.1 relies on the following propositions.

Proposition 3.1. *For any positive integer M , there exist positive constants c and K , such that $K \geq M$ and w.h.p. G_n belongs to the class $\mathcal{S}([cn], K)$.*

Proposition 3.2. *Let $\tau_{\ell, M}$ be the extinction time of the contact process on a graph of the class $\mathcal{S}(\ell, M)$ starting from full occupancy. Then there exist positive constants c and C independent of λ , such that if $h(\lambda)M \geq C\psi_1(M)$, then*

$$(18) \quad \mathbb{P}(\tau_{\ell, M} \geq \exp(c\lambda^2 \ell M)) \rightarrow 1 \quad \text{as } \ell \rightarrow \infty,$$

with $h(\lambda) = \bar{\lambda}^2 / |\log \bar{\lambda}|$ and $\bar{\lambda} = \lambda \wedge 1/2$.

Proposition 3.3. *Let (G_n^0) be a sequence of connected graphs, such that $|G_n^0| \leq n$ and G_n^0 belongs to the class $\mathcal{S}(k_n, M)$, for some sequence (k_n) . Let τ_n denote the extinction time of the contact process on G_n^0 starting from full occupancy. Then there exists a positive constant C , such that if $h(\lambda)M \geq C\psi_1(M)$ with $h(\lambda)$ as in Proposition 3.2 and*

$$(19) \quad \frac{k_n}{d_n \vee \log n} \rightarrow \infty,$$

with d_n the diameter of G_n^0 , then

$$\frac{\tau_n}{\mathbb{E}(\tau_n)} \xrightarrow[n \rightarrow \infty]{(\mathcal{D})} \mathcal{E}(1).$$

Proof of Theorem 1.1. Observe that Proposition 3.1 and Lemma 3.2 imply the lower bound on τ_n . On the other hand, the upper bound follows from Lemma 3.4 in [9] and the fact that $|E_n| \asymp n$ w.h.p., see [11, Vol. I, Theorem 6.6]. Finally, the convergence in law of $\tau_n / \mathbb{E}(\tau_n)$ can be proved similarly as in the proof of Theorem 1.1 in [9] by using Propositions 3.1, 3.3 and the following:

- w.h.p. $d_n = \mathcal{O}(\log n)$ with d_n the diameter of the largest component of the IRG,
- w.h.p. the size of the second largest component in the IRG is $\mathcal{O}(\log n)$.

These claims are proved in Theorems 3.12 and 3.16 in [3] for a general model of IRG. \square

Proof of Proposition 3.2. Similarly to Lemma 3.2 in [9], we can prove (18) by using a comparison between the contact process and an oriented percolation on $\llbracket 1, n \rrbracket$ with density close to 1. Note that here, we use a mechanism of infection between star graphs instead of complete graphs as it was the case in [9]. The mechanism for star graphs is described in Lemmas 3.1 and 3.2 in [15] and the function $h(\lambda)$ is chosen appropriately to apply these results. \square

Proof of Proposition 3.3. The proof is the same as for Lemma 3.3 in [9]. \square

3.2. Proof of Proposition 3.1. This subsection is divided into four parts. In the first part, we define a preliminary process, called an exploration, which uses Proposition 2.2 and Lemma 2.6 to discover the neighborhood of a vertex. In Parts two and three, we describe the two main tasks, and the last part gives the conclusion.

3.2.1. *Exploration process.* For a given vertex v and a set $U \subset V_n$, we denote by $\mathcal{N}(v, U)$ the set of neighbors of v in U , i.e.

$$\mathcal{N}(v, U) = \{w \in U : w \sim v\}.$$

Moreover, we will define an **exploration** of v in U of type K (and denote it by $E_K(v, U)$ and call U the *source set* of the exploration). The aim of this exploration is to find a vertex x in U with degree larger than $3K$ at distance less than $\psi_1(K)$ from v , by exploring at most $\psi_2(K)$ vertices in U , with

$$\psi_2(K) = (3K)^{\psi_1(K)+1}.$$

First, we set $x_0 = v$, $U_0 = U$ and $W_0 = \{x_0\}$ and call it the *waiting set*. We define a sequence of trees $(T^k(v, U))_{k \geq 0}$ as the record of the exploration, starting with $T^0(v, U) = \{x_0\}$.

- If $\mathcal{N}(x_0, U_0) = \emptyset$, we define $U_1 = U_0$ and $W_1 = W_0 \setminus \{x_0\}$ and $T^1(v, U) = T^0(v, U)$.
- If $|\mathcal{N}(x_0, U_0)| \geq 3K$, we arbitrarily choose $3K$ vertices in $\mathcal{N}(v, U)$ to form three *seed sets* of size K denoted by $F_{v,1}$, $F_{v,2}$ and $F_{v,3}$. Then we declare that $E_K(v, U)$ is *successful*; we stop the exploration and define

$$U_1 = U \setminus (F_{v,1} \cup F_{v,2} \cup F_{v,3}).$$

- If $1 \leq |\mathcal{N}(x_0, U_0)| < 3K$, we define

$$\begin{aligned} U_1 &= U_0 \setminus \mathcal{N}(x_0, U_0), \\ T^1(v, U) &= T^0(v, U) \cup \mathcal{N}(x_0, U_0) \text{ together with the edges} \\ &\hspace{15em} \text{between } x_0 \text{ and } \mathcal{N}(x_0, U_0), \\ W_1 &= (W_0 \setminus \{x_0\}) \cup \{x \in \mathcal{N}(x_0, U_0) : d_{T^1(v, U)}(x_0, x) \leq \psi_1(K)\}, \end{aligned}$$

with $d_T(x, y)$ the graph distance between x and y in a tree T .

Then we chose an arbitrary vertex x_1 in W_1 and repeat this step with x_1 and U_1 in place of x_0 and U_0 .

We continue like this until the waiting set is empty, or we succeed at some step.

Note that after the k^{th} step, we define

$$T^{k+1}(v, U) = \begin{cases} T^k(v, U) \cup \mathcal{N}(x_k, U_k) \text{ together with the edges} \\ \hspace{10em} \text{between } x_k \text{ and } \mathcal{N}(x_k, U_k) & \text{if } |\mathcal{N}(x_k, U_k)| < 3K \\ T^k(v, U) & \text{if } |\mathcal{N}(x_k, U_k)| \geq 3K, \end{cases}$$

and

$$U_{k+1} = \begin{cases} U_k \setminus \mathcal{N}(x_k, U_k) & \text{if } |\mathcal{N}(x_k, U_k)| < 3K \\ U_k \setminus (F_{x_k,1} \cup F_{x_k,2} \cup F_{x_k,3}) & \text{if } |\mathcal{N}(x_k, U_k)| \geq 3K, \end{cases}$$

and

$$W_{k+1} = (W_k \setminus \{x_k\}) \cup \{x \in \mathcal{N}(x_k, U_k) : d_{T^{k+1}(v, U)}(x_0, x) \leq \psi_1(K)\}.$$

If the process stops after k_0 steps, we define the remaining source set

$$\tilde{U} = U_{k_0+1}.$$

When an exploration is successful, its outputs are the set \tilde{U} and a vertex, say u , with three seed sets $F_{u,1}$, $F_{u,2}$ and $F_{u,3}$ of size K . Otherwise, the output is simply \tilde{U} .

Lemma 3.4. *The following statements hold.*

(i) For all v and U ,

$$|\tilde{U}| \geq |U| - \psi_2(K),$$

with $\psi_2(K) = (3K)^{\psi_1(K)+1}$.

(ii) For all $K \geq K_1$, we have

$$\lim_{n \rightarrow \infty} \mathbb{P}(E_K(v, U) \text{ is successful} \mid \mathcal{E}(K), |U| \geq (1 - \varkappa_2)n) \geq \theta_1,$$

with \varkappa_2, θ_1 and K_1 as in Lemmas 2.5 and 2.6.

Proof. Part (i) follows from the facts that at each step we remove from the source set at most $3K$ vertices, and that we only explore the vertices at distance less than or equal to $\psi_1(K)$ from v .

We now prove (ii). Similarly to Lemma 2.6, if w_v -the weight of v - is larger than $10K$,

$$\mathbb{P}(\deg(v) \geq 3K) \geq \theta_1,$$

and thus (ii) follows. Suppose that $w_v \leq 10K$. Then using the same argument in Lemma 2.6, or [11, Vol. II, Corollary 3.13] (for showing that $\mathbb{T}_\ell(v, U) \equiv \tilde{\mathbb{T}}_\ell(v, U)$ w.h.p. when w_v is bounded), we get

$$(20) \quad \mathbb{P}(B_{\psi_1(K)}(v, U) \text{ has no cycle}) \rightarrow 1 \quad \text{as } n \rightarrow \infty.$$

Suppose that $B_{\psi_1(K)}(v, U)$ is a tree (or has no cycle). Then the order of explorations of vertices in the waiting set does not effect to the outcome of $E_K(v, U)$. Thus

$$\{E_K(v, U) \text{ is successful}\} \supset \{\exists x \in B_{\psi_1(K)}(v, U) : \deg(x) \geq 3K+1\} \cap \{B_{\psi_1(K)}(v, U) \text{ is a tree}\}.$$

Therefore, the result follows from Proposition 2.2, Lemma 2.6 and (20). \square

3.2.2. Task I. The goal of this task is to show that w.h.p. by discovering $o(\log n)$ vertices, we can find in G_n a subgraph belonging to the class $\mathcal{S}(L_n, 3K)$ with

$$L_n = \lceil \log \log \log n \rceil.$$

The proof of this task contains two parts. First, by introducing a process called *trail*, we show that given a large source set U , with positive probability by exploring at most $o(\log n)$ vertices in U , we can find a graph of class $\mathcal{S}(L_n, 3K)$. Secondly, we continuously operate trails. By the first part, the time to the first success is stochastically dominated by a geometric random variable with positive mean, thus it is finite with high probability. Let us start with the definition of trail.

For $K \geq K_1$ and $U \subset V_n \setminus \{v\}$, with $|U| \geq (1 - \varkappa_2/2)n$ and $\varkappa_2 = \varkappa_2(\varepsilon_0, K)$ as in Lemma 2.5, we define a **trial** $Tr(U, L_n, K)$ as follows. First, we take arbitrarily a vertex $v \in U$.

At *level 0*, we define $\tilde{W}_0 = \{v\}$ and call it the waiting set at level 0. Then we perform $E_K(v, U)$ and call \tilde{U}_v the source set after this exploration. If it fails, we declare that $Tr(U, L_n, K)$ *fails*. Otherwise, we are now in *level 1* and continue as follows.

Let x_1 be the vertex with degree larger than $3K + 1$ which makes $E_K(v, U)$ successful and let $F_{x_1,1}, F_{x_1,2}$ and $F_{x_1,3}$ be its three seed sets of size K . We denote $\tilde{W}_1 = F_{x_1,3}$ and call it the waiting set at level 1 ($F_{x_1,1}$ and $F_{x_1,2}$ are reserved for Task II). We sequentially perform explorations of vertices in \tilde{W}_1 . More precisely, we choose arbitrarily a vertex, say y_1 in \tilde{W}_1 and operate $E_K(y_1, \tilde{U}_v)$ and get a new source set \tilde{U}_{y_1} . Then we operate $E_K(y_2, \tilde{U}_{y_1})$ with y_2 chosen arbitrarily from $\tilde{W}_1 \setminus \{y_1\}$, and so on.

If none of these explorations is successful, we declare that the trial *fails*.

If some of those are successful, we are in *level 2* and get some triples of seed sets $F_{.,1}$, $F_{.,2}$ and $F_{.,3}$. Denote by \tilde{W}_2 the waiting set at level 2, which is the union of all the seed sets of the third type $F_{.,3}$. Then we sequentially perform the explorations of vertices in \tilde{W}_2 .

We continue this process until either we explore all vertices in waiting sets, or when we exceed to the L_n -th level.

We declare that $Tr(U, L_n, K)$ is *successful* if we can access to the L_n -th level, or that it has failed otherwise.

The output of a successful trial is a sequence of L_n vertices associated with L_n pairs of disjoint seed sets and a remained source set \bar{U} , with $|\bar{U}| \geq |U| - o(\log n)$. The output of a failed trail is simply \bar{U} .

Lemma 3.5. *The following statements hold.*

- (i) *A trial discovers at most $K^{L_n+1}\psi_2(K)$ vertices.*
- (ii) *There exist constants $\theta_2 \in (0, 1)$ and $K_2 \geq K_1$, such that for any $K \geq K_2$ and $U \subset V_n$ satisfying $|U| \geq (1 - \varkappa_2/2)n$, we have*

$$\lim_{n \rightarrow \infty} \mathbb{P}(Tr(U, L_n, K) \text{ is successful} \mid \mathcal{E}(K)) \geq \theta_2,$$

with \varkappa_2 and K_1 as in Lemmas 2.5 and 2.6.

Proof. For (i), we observe that an exploration creates at most one third type seed set of size K . Then in a trial, we operate at most

$$1 + K + K^2 + \dots + K^{L_n} \leq K^{L_n+1} \quad \text{explorations.}$$

Moreover, an exploration uses at most $\psi_2(K)$ vertices. Therefore, a trial discovers at most

$$K^{L_n+1}\psi_2(K) = o(\log n) \text{ vertices.}$$

We now prove (ii). As a trial uses at most $o(\log n)$ vertices, and initially $|U| \geq (1 - \varkappa_1/2)n$, during the trial $Tr(U, L_n, K)$ the source sets of explorations always have cardinality larger than $(1 - \varkappa_2)n$.

Hence, by Lemma 3.4 (ii), for all n and K large enough, on $\mathcal{E}(K)$ each exploration in $Tr(U, L_n, K)$ is successful with probability larger than $\theta_1/2$.

On the other hand, each successful exploration creates K new vertices in the next level. Hence $(|\tilde{W}_i|)_{i \leq L_n}$ - the numbers of vertices to explore up to the L_n -th level in the trial stochastically dominate a branching process $(\eta_i)_{i \leq L_n}$ starting from $\eta_0 = 1$ with reproduction law η given by

$$\begin{aligned} \mathbb{P}(\eta = K) &= \theta_1/2 \\ \mathbb{P}(\eta = 0) &= 1 - \theta_1/2. \end{aligned}$$

We choose K large enough, such that $K\theta_1 > 2$. Then (η_i) is super critical, and thus

$$\mathbb{P}(Tr(x, W, L_n, K) \text{ is successful} \mid \mathcal{E}(K)) \geq \mathbb{P}(\eta_{L_n} \geq 1) > 0,$$

which proves the result. \square

We can now define **Task I** as follows. Let $U_1 = V_n$ and operate $Tr(U_1, L_n, K)$.

- If $Tr(U_1, L_n, K)$ is successful, we declare that Task I is *successful*.
- Otherwise, we call \bar{U}_1 the source set after this trial. We then perform $Tr(U_2, L_n, K)$ with $U_2 = \bar{U}_1$.

We continuously operate trials. We stop the task I if we get a success or we operate more than L_n trails.

We declare that Task I is *successful* if there is a successful trial and that it *fails* otherwise.

Proposition 3.6. *The following statements hold.*

(i) *Task I uses at most $o(\log n)$ vertices.*

(ii) *We have*

$$\lim_{n \rightarrow \infty} \mathbb{P}(\text{Task I is successful} \mid \mathcal{E}(K)) = 1.$$

Proof. By Lemma 3.5 (i) in this task, we discover at most

$$L_n K^{L_n+1} \psi_2(K) = o(\log n) \text{ vertices.}$$

Hence, the cardinality of the source set is always larger than $n - o(\log n)$. Therefore, by Lemma 3.5 (ii), each trial is successful with probability larger than $\theta_2/2$ for n large enough.

We define \mathcal{I} the first index of the first successful trail (if there is no such index, we let $\mathcal{I} = \infty$). Then conditionally on $\mathcal{I} \leq L_n$, it is stochastically dominated by a geometric random variable with parameter $\theta_2/2$. Therefore,

$$\mathbb{P}(\mathcal{I} = \infty) \leq (1 - \theta_2/2)^{L_n} = o(1).$$

In other words, Task I is successful w.h.p. □

3.2.3. Task II. Suppose that Task I is successful. Then there is a sequence of vertices $\{u_1, \dots, u_{L_n}\}$, such that $d(u_{i-1}, u_i) \leq \psi_1(K)$ for all $2 \leq i \leq L_n$, together with L_n pairs of disjoint seed sets $(F_{u_1,1}, F_{u_1,2}), \dots, (F_{u_{L_n},1}, F_{u_{L_n},2})$ attached respectively to (u_i) . Moreover, the cardinality of U^* -the source set after Task I- is larger than $n - o(\log n)$. Importantly, we have not yet discovered the vertices in

$$U^+ = U^* \cup \bigcup_{1 \leq i \leq L_n} (F_{u_i,1} \cup F_{u_i,2}).$$

We will show that from (u_i) and U^+ , we can find a sequence of $\mathcal{O}(n)$ good vertices which are associated with seed sets. This particularly shows the existence of a graph of class $\mathcal{S}(\mathcal{O}(n), K)$. To achieve this goal, the strategy is as follows. First, by introducing *experiments*, we show that with probability larger than $2/3$ starting from a good vertex, we can find another good vertex at distance less than $\psi_2(K)$. Moreover, each experiment only explores at most $\psi_3(K)$ vertices with

$$\psi_3(K) = K\psi_2(K).$$

Secondly, by the first step the number of good vertices stochastically dominates a random walk biased to the right. By a standard argument for random walk, we deduce that with high probability there are $\mathcal{O}(n)$ good vertices.

For any set $F \subset U^+$ of size K and $S \subset U^* \setminus F$ we define an **experiment** $Ep(F, S, K)$ as follows.

We write F as $\{z_1, \dots, z_K\}$. Then we sequentially operate explorations $E_K(z_1, S_{z_0}), \dots, E_K(z_K, S_{z_{K-1}})$, where S_{z_i} is the source set after the exploration $E_K(z_i, S_{z_{i-1}})$ for $1 \leq i \leq K$ with $S_{z_0} = S$.

If none of these explorations is successful, we declare that $Ep(F, S, K)$ *fails*, otherwise we say that it is *successful*. In the latter case, there is a vertex u with $d(u, F) \leq \psi_1(K)$

together with two seed sets $F_{u,1}$ and $F_{u,2}$ of size K (in fact, we even have three sets, but we will only use two of them).

Lemma 3.7. *We have*

- (i) *the number of vertices used in an experiment is at most $\psi_3(K) = K\psi_2(K)$,*
- (ii) *there exists a positive constant $K_3 \geq K_2$, such that for all $K \geq K_3$, and n large enough*

$$\mathbb{P}(Ep(F, S, K) \text{ is successful} \mid \mathcal{E}(K), |S| \geq (1 - \varkappa_2/2)n) \geq 2/3,$$

with \varkappa_2 as in Lemma 2.5.

Proof. Part (i) is immediate, since in an experiment, we perform K explorations and each exploration uses at most $\psi_2(K)$ vertices. For (ii), we note that by (i) and the assumption $|S| \geq (1 - \varkappa_2/2)n$, the source set S_{z_i} has more than $(1 - \varkappa_2)n$ vertices for all i . Hence, by Lemma 3.4 (ii), for all $1 \leq i \leq K$ and n large enough

$$\mathbb{P}(E_K(z_i, S_{z_{i-1}}) \text{ is successful} \mid \mathcal{E}(K)) \geq \theta_1/2.$$

Thus on $\mathcal{E}(K)$, the probability that the experiment fails is less than

$$(1 - \theta_1/2)^K \leq 1/3,$$

provided K is large enough. □

We define **Task II** as follows. First, we fix a constant $K \geq K_3$ and let

$$\varepsilon_1 = \varkappa_2/(3\psi_3(K)).$$

We label the seed sets *active* and make an order as follows

$$F_{u_1,1} < F_{u_1,2} < F_{u_2,1} < \dots < F_{u_{L_n},1} < F_{u_{L_n},2}.$$

We perform $Ep(F_{u_{L_n},2}, U^*, K)$ and let $F_{u_{L_n},2}$ be inactive. If the experiment is successful, we find a vertex u at distance smaller than $\psi_1(K) + 1$ from u_{L_n} and two seed sets $F_{u,1}$ and $F_{u,2}$ of size K . We now add u in the sequence: $u_{L_{n+1}} = u$, label these sets $F_{u_{L_{n+1}},1}$ and $F_{u_{L_{n+1}},2}$ active, and make an order

$$F_{u_1,1} < \dots < F_{u_{L_n},1} < F_{u_{L_{n+1}},1} < F_{u_{L_{n+1}},2}.$$

We then perform the experiment of the newest active set i.e. the active set with the largest order. After an experiment of an active set, we let it be inactive and either get a new vertex with two active sets attached on it (if the experiment is successful), or get nothing (otherwise).

Continue this procedure until one of the three following conditions is satisfied.

- There is no more active set. We declare that Task II fails.
- We do more than $[\varepsilon_1 n]$ experiments. We declare that Task II fails.
- We have more than $[\varepsilon_1 n/4]$ active sets. We declare that Task II is successful.

Proposition 3.8. *For all $K \geq K_3$, we have*

$$\lim_{n \rightarrow \infty} \mathbb{P}(\text{Task II is successful} \mid \mathcal{E}(K), \text{Task I is successful}) = 1.$$

Proof. By Lemma 3.7 (i), the first $[\varepsilon_1 n]$ experiments use at most $[\varkappa_2 n/3]$ vertices. Therefore, during Task II, the source set always has cardinality larger than $(1 - \varkappa_2/2)n$. Thus by Lemma 3.7 (ii), on $\mathcal{E}(K)$ during the time to perform this task, each experiment is successful with probability larger than $2/3$.

Therefore, the number of active sets stochastically dominates a random walk (R_i) satisfying $R_0 = 2L_n$ and

$$\begin{aligned} R_{i+1} &= R_i + 1 \text{ with prob. } 2/3 \\ R_{i+1} &= R_i - 1 \text{ with prob. } 1/3. \end{aligned}$$

Define

$$T_0 = \inf\{i : R_i = 0\} \quad \text{and} \quad T_1 = \inf\{i : R_i \geq \lceil \varepsilon_1 n / 4 \rceil\}.$$

Then using the optional stopping theorem, we get

$$\mathbb{P}(T_1 \leq T_0) = 1 - o(1).$$

On the other hand, the law of large numbers gives that

$$\mathbb{P}(T_1 \leq \lceil \varepsilon_1 n \rceil) \geq \mathbb{P}(R_{\lceil \varepsilon_1 n \rceil} \geq \lceil \varepsilon_1 n / 4 \rceil) = 1 - o(1).$$

It follows from the last two inequalities that

$$\begin{aligned} &\mathbb{P}(\text{Task II is successful} \mid \mathcal{E}(K), \text{Task I is successful}) \\ &\geq \mathbb{P}(T_1 \leq \min\{T_0, \lceil \varepsilon_1 n \rceil\}) = 1 - o(1), \end{aligned}$$

which proves the result. \square

3.2.4. Proof of Proposition 3.1. By Propositions 3.6, 3.8 and (12), we can assume that both Tasks I and II are successful. Then we have more than $\lceil \varepsilon_1 n / 4 \rceil$ active sets. Observe that a vertex is attached to at most two active sets. Then the number of vertices having at least one active set is larger than $\lceil \varepsilon_1 n / 8 \rceil$. Therefore, w.h.p. G_n belongs to the class $\mathcal{S}(\lceil cn \rceil, 2K)$ with $c = \varepsilon_1 / 8$. Moreover, K can be chosen arbitrarily large, so Proposition 3.1 has been proved.

4. CONTACT PROCESS ON ERDOS-RÉNYI RANDOM GRAPHS

We recall the definition of $ER(n, p/n)$ -the Erdos-Rényi graph with parameter p . Let $V_n = \{v_1, \dots, v_n\}$ be the vertex set. Then for $1 \leq i \neq j \leq n$, we independently draw an edge between v_i and v_j with probability p/n .

Proposition 4.1. *Let τ_n be the extinction time of the contact process on $ER(n, p/n)$ starting with all sites infected. There exists a positive constant C , such that for any $\lambda > 0$ and $p \geq \lceil C/h(\lambda) \rceil$,*

$$\mathbb{P}(\tau_n \geq \exp(cn)) = 1 - o(1),$$

with $h(\lambda)$ as in Proposition 3.2 and $c = c(\lambda)$ a positive constant. Moreover, in this setting

$$\frac{\tau_n}{\mathbb{E}(\tau_n)} \xrightarrow[n \rightarrow \infty]{(D)} \mathcal{E}(1).$$

Let us denote by $S(\ell, M)$ the graph obtained by gluing to each vertex in $\llbracket 1, n \rrbracket$ a star graph of size M . Similarly to Proposition 3.2, we get that if $h(\lambda)M$ is large enough, then the extinction time of the contact process on $S(\ell, M)$ is exponential in $\ell \times M$ w.h.p. Therefore, similarly to Theorem 1.1, Proposition 4.1 follows from the following lemma.

Lemma 4.2. *For any M , there exists a positive constant c , such that if $p \geq 16M!$ then w.h.p. $ER(n, p/n)$ contains as a subgraph a copy of $S(\lceil cn \rceil, M)$.*

To prove Lemma 4.2, we will use the following.

Lemma 4.3. [1, Theorem 2] *If $p > 1$ then w.h.p. $ER(n, p/n)$ contains a path of length $\varkappa n$ with some positive constant $\varkappa = \varkappa(p)$.*

Proof of Lemma 4.2. We first define

$$n_1 = \lfloor n/2 \rfloor, \quad A = \{v_1, \dots, v_{n_1-1}\} \quad \text{and} \quad A^c = V_n \setminus A.$$

For $v_i \in A$ and $v_j \in A^c$, define

$$Y_{i,j} = 1(v_i \sim v_j).$$

Then $(Y_{i,j})$ are i.i.d. Bernoulli random variables with mean p/n . We set $B_0 = \emptyset$ and

$$\sigma_1 = \inf \left\{ j \leq n : \sum_{k=n_1}^j Y_{1,k} \geq M \right\},$$

with the convention $\inf \emptyset = \infty$. Define

$$B_1 = \begin{cases} \{k : k \leq \sigma_1, Y_{1,k} = 1\} & \text{if } \sigma_1 < \infty, \\ B_0 & \text{if } \sigma_1 = \infty. \end{cases}$$

Suppose that σ_i and B_i have been already defined. Then we set

$$\sigma_{i+1} = \inf \left\{ j \leq n : \sum_{k=n_1}^j Y_{i+1,k} 1(k \notin B_i) \geq M \right\},$$

and

$$B_{i+1} = \begin{cases} B_i \cup \{k : k \leq \sigma_{i+1}, k \notin B_i, Y_{i+1,k} = 1\} & \text{if } \sigma_{i+1} < \infty, \\ B_i & \text{if } \sigma_{i+1} = \infty. \end{cases}$$

Then

$$|B_{i+1}| = \begin{cases} |B_i| + M & \text{if } \sigma_{i+1} < \infty, \\ |B_i| & \text{if } \sigma_{i+1} = \infty. \end{cases}$$

Hence, for all $i \leq \lfloor n/M \rfloor$,

$$(21) \quad |B_i| \leq iM.$$

We now define

$$Y_i = 1(\sigma_i < \infty).$$

Then

$$\begin{aligned} \mathbb{P}(Y_{i+1} = 1 \mid B_i) &= \mathbb{P} \left(\sum_{k=n_1}^n Y_{i+1,k} 1(k \notin B_i) \geq M \mid B_i \right) \\ &= \mathbb{P}(\text{Bin}(n - n_1 + 1 - |B_i|, p/n) \geq M \mid B_i). \end{aligned}$$

It follows from (21) that when $i \leq \lfloor n/4M \rfloor$,

$$n - n_1 + 1 - |B_i| \geq \lfloor n/4 \rfloor.$$

Thus

$$(22) \quad \begin{aligned} \mathbb{E}(Y_{i+1} \mid B_i) &= \mathbb{P}(Y_{i+1} = 1 \mid B_i) \geq \mathbb{P}(\text{Bin}(\lfloor n/4 \rfloor, p/n) \geq M) \\ &\geq \mathbb{P}(\text{Poi}(p/4) \geq M)/2 \geq 1/(M-1)! \end{aligned}$$

with $p \geq 8$ and n large enough. Now we set

$$\Gamma = \{i \leq \lfloor n/4M \rfloor : Y_i = 1\},$$

and let

$$(23) \quad Z_k = \sum_{i=1}^k (Y_i - \mathbb{E}(Y_i | B_{i-1})).$$

Then (Z_k) is a $(\sigma(B_k))$ -martingale satisfying $|Z_k - Z_{k-1}| \leq 1$ for all $1 \leq k \leq [n/4M]$. Thus it follows from Doob's martingale inequality that

$$\mathbb{P} \left(\max_{k \leq \ell} |Z_k| \geq x \right) \leq \frac{\mathbb{E}(Z_\ell^2)}{x^2} \leq \frac{\ell}{x^2} \quad \text{for all } \ell \leq [n/4M], x > 0.$$

In particular, w.h.p.

$$(24) \quad |Z_{[n/4M]}| = o(n).$$

It follows from (22), (23) and (24) that w.h.p.

$$\begin{aligned} |\Gamma| &= \sum_{i=1}^{[n/4M]} Y_i = Z_{[n/4M]} + \sum_{i=1}^{[n/4M]} \mathbb{E}(Y_i | B_{i-1}) \\ &\geq Z_{[n/4M]} + [n/4M]/(M-1)! \geq n/(8M!). \end{aligned}$$

Conditionally on $|\Gamma| \geq n/(8M!)$, the graph induced in Γ contains a Erdos-Rényi graph, say H_n , of size $[n/(8M!)]$ with probability of connection p/n . Observe that H_n is super critical when

$$p/(8M!) \geq 2.$$

Therefore, by Lemma 4.3, w.h.p. H_n contains a path of length $\varkappa n$, with $\varkappa = \varkappa(p, M)$ if $p \geq 16M!$.

On the other hand, each vertex in Γ has at least M disjoint neighbors in A^c . Combining these facts proves the result. \square

Remark 4.4. We can use Proposition 4.1 to prove the lower bound on the extinction time in Theorem 1.1 for a small class of weights. Indeed, for any p we define

$$A_p = \{v_i : w_i \geq \sqrt{4p\mathbb{E}(w)}\}.$$

Then

$$|A_p| = (\gamma_p + o(1))n,$$

with

$$\gamma_p = \mathbb{P}(w \geq \sqrt{4p\mathbb{E}(w)}).$$

On the other hand, for any v_i and v_j in A_p ,

$$\begin{aligned} \mathbb{P}(v_i \sim v_j) &= 1 - \exp(-w_i w_j / \ell_n) \\ &\geq \frac{w_i w_j}{2\ell_n} 1(w_i w_j \leq \ell_n) + \frac{1}{2} 1(w_i w_j > \ell_n) \\ &\geq \frac{4p\mathbb{E}(w)}{2\ell_n} \geq \frac{p}{n}, \end{aligned}$$

since $\ell_n = n(\mathbb{E}(w) + o(1))$. Therefore, the graph induced in A_p contains a Erdos-Rényi graph $ER([\gamma_p n], p/n)$. By proposition 4.1, if $p\gamma_p \rightarrow \infty$, for all $\lambda > 0$, w.h.p. the extinction time of the contact process on $ER([\gamma_p n], p/n)$ is exponential in n . Hence, when

$$(25) \quad \lim_{p \rightarrow \infty} p\mathbb{P}(w \geq \sqrt{4p\mathbb{E}(w)}) = \infty,$$

w.h.p. the extinction time on the $\text{IRG}(w)$ is exponential in n for all $\lambda > 0$. The condition (25) is satisfied for some weight w , for example the power-law distribution with exponent between 2 and 3.

Acknowledgments. I am grateful to Bruno Schapira for his help and suggestions during the preparation of this work. I wish to thank also the anonymous referee for carefully reading the manuscript and many valuable comments. This work is supported by the Vietnam National Foundation for Science and Technology Development (NAFOSTED) under Grant number 101.03–2017.07.

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