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WEIGHTED MULTIPLE ERGODIC AVERAGES AND CORRELATION SEQUENCES

NIKOS FRANTZIKINAKIS AND BERNARD HOST

ABSTRACT. We study mean convergence results for weighted multiple ergodic averages defined by commuting transformations with iterates given by integer polynomials in several variables. Roughly speaking, we prove that a bounded sequence is a good universal weight for mean convergence of such averages if and only if the averages of this sequence times any nilsequence converge. Key role in the proof play two decomposition results of independent interest. The first states that every bounded sequence in several variables satisfying some regularity conditions is a sum of a nilsequence and a sequence that has small uniformity norm (this generalizes a result of the second author and B. Kra); and the second states that every multiple correlation sequence in several variables is a sum of a nilsequence and a sequence that is small in uniform density (this generalizes a result of the first author). Furthermore, we use the previous results in order to establish mean convergence and recurrence results for a variety of sequences of dynamical and arithmetic origin and give some combinatorial implications.

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1. INTRODUCTION

Since the early 80's a lot of effort has been put in the study of the limiting behavior of multiple ergodic averages. This study was partly motivated by combinatorial implications, since positiveness properties of such averages imply various far reaching extensions of the celebrated theorem of Szemerédi on arithmetic progressions. After a long series of partial results, most notably those in [7, 21, 22, 35, 45, 46, 47, 54, 62, 64, 68], M. Walsh [66], building on previous work of T. Tao [64], proved the following mean convergence result:

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Theorem 1.1 ([66]¹). *Let $d, \ell, s \in \mathbb{N}$, (X, \mathcal{X}, μ) be a probability space, and $T_1, \dots, T_\ell: X \rightarrow X$ be invertible commuting measure preserving transformations. Then for every Følner sequence $(I_k)_{k \in \mathbb{N}}$ of subsets of \mathbb{N}^d , polynomials $p_{i,j}: \mathbb{N}^d \rightarrow \mathbb{Z}$, $i = 1, \dots, \ell$, $j = 1, \dots, s$, and functions $f_1, \dots, f_s \in L^\infty(\mu)$, the averages*

$$(1.1) \quad \frac{1}{|I_k|} \sum_{\mathbf{n} \in I_k} f_1\left(\prod_{i=1}^{\ell} T_i^{p_{i,1}(\mathbf{n})} x\right) \cdot \dots \cdot f_s\left(\prod_{i=1}^{\ell} T_i^{p_{i,s}(\mathbf{n})} x\right),$$

converge in $L^2(\mu)$ as $k \rightarrow +\infty$.

Remark. In [66] the previous result was established under the weaker hypothesis that the transformations T_1, \dots, T_ℓ generate a nilpotent group. We believe that our results can be extended to this more general setup but we restrict in this article to the case of commuting transformations.

One of the main purposes of this article is to study mean convergence for weighted versions of the averages (1.1), that is, averages of the form

$$(1.2) \quad \frac{1}{|I_k|} \sum_{\mathbf{n} \in I_k} w(\mathbf{n}) f_1\left(\prod_{i=1}^{\ell} T_i^{p_{i,1}(\mathbf{n})} x\right) \cdot \dots \cdot f_s\left(\prod_{i=1}^{\ell} T_i^{p_{i,s}(\mathbf{n})} x\right),$$

where $w: \mathbb{N}^d \rightarrow \mathbb{C}$ is a bounded sequence. We call a sequence w for which the previous averages converge for all choices of systems, functions, and polynomials, a *good universal weight for mean convergence* of the averages (1.2).

When $d = 1$, examples of good universal weights for some multiple ergodic averages can be found in [1, 2, 3, 4, 19, 26, 30, 32, 48, 70]. Most of these results deal with the case where $\ell = 1$ and are based on the theory of characteristic factors that was pioneered by H. Furstenberg. They depend crucially on the work of B. Host and B. Kra [46] and subsequent developments in [47, 54], which, in the case where all the transformations are equal, gives a characterization in terms of nilsystems of the smallest factor of the system that controls the limiting behavior of the averages (1.2). Unfortunately, no such characterization is known in the case of general commuting transformations (but see [8, 9] for related progress), which is the reason why this method is not applicable for our more general setup. Moreover, the method used by M. Walsh in [66] does not seem applicable to weighted averages and no general criterion suitable for checking mean convergence of averages of the form (1.2) is known. We fill this gap by showing in Theorem 2.2 that a bounded sequence $w: \mathbb{N}^d \rightarrow \mathbb{C}$ is a good universal weight for mean convergence of the averages (1.2) if and only if the averages

$$(1.3) \quad \frac{1}{|I_k|} \sum_{\mathbf{n} \in I_k} w(\mathbf{n}) \cdot \psi(\mathbf{n})$$

converge for every nilsequence ψ in d variables and Følner sequence $(I_k)_{k \in \mathbb{N}}$ in \mathbb{N}^d . Furthermore, when one replaces throughout the averages $\frac{1}{|I_k|} \sum_{\mathbf{n} \in I_k}$ by the Cesàro averages $\frac{1}{N^d} \sum_{\mathbf{n} \in [1, N]^d}$, we prove in Theorem 2.4 that a similar criterion holds for weak convergence, and a condition somewhat stronger than (1.3) suffices for mean convergence. Even for single variable sequences the mean convergence criterion is new and its proof (strangely) depends on decomposition results for sequences in two variables. Prior to this work, only the case $d = \ell = 1$ was treated in [19] for mean convergence, while for weak convergence the case where $d = 1$ and $\ell \in \mathbb{N}$ is arbitrary was treated in [30].

To prove these results we use the mean convergence result of M. Walsh as a black box and two decomposition results of independent interest. These are Theorems 3.9 and

¹The argument in [66] is given for Cesàro averages and $d = 1$ but the same proof works in this more general case (see [69] for details).

3.10 which extend similar results for single variable sequences from [48, Theorem 2.19] and [30, Theorem 1.2]. Roughly speaking, they state the following:

- (i) If the averages (1.3) converge for every nilsequence $\psi: \mathbb{N}^d \rightarrow \mathbb{C}$ and every Følner sequence $(I_k)_{k \in \mathbb{N}}$ in \mathbb{N}^d , then the sequence $w \in \ell^\infty(\mathbb{N}^d)$ is a sum of a nilsequence and a sequence that has small uniformity norm.
- (ii) Any sequence of the form

$$\int \prod_{i=1}^{\ell} f_1(T_i^{p_{i,1}(\mathbf{n})} x) \cdot \dots \cdot \prod_{i=1}^{\ell} f_s(T_i^{p_{i,s}(\mathbf{n})} x) d\mu, \quad \mathbf{n} \in \mathbb{N}^d,$$

is the sum of a nilsequence and a sequence that is small in uniform density.

Regarding the second decomposition, Theorem 2.6 gives more precise information when the iterates are linear, it implies for example that the sequences

$$(1.4) \quad \int f \cdot T_1^m f \cdot T_2^n f \cdot T_3^r f d\mu, \quad \int f \cdot T_1^m f \cdot T_2^n f \cdot T_3^{m+n} f d\mu, \quad \int f \cdot T_1^n f \cdot T_2^n f \cdot T_3^n f d\mu,$$

are 1-step, 2-step, and 3-step nilsequences respectively modulo small errors in uniform density (simple examples show that the degree of nilpotency is optimal).

Using the previous criteria we prove mean convergence results for weighted ergodic averages with weights given by various sequences of dynamical origin, bounded multiplicative functions, generalized polynomials, and Hardy field sequences (see Sections 2.2, 2.3, 2.6). We deduce some multiple recurrence results and combinatorial consequences; showing for example that every set of integers with positive upper density contains arbitrarily long arithmetic progressions with common difference of the form $m^2 + n^2$ where m, n have an odd (or an even) number of distinct prime factors (see Theorems 2.12, 2.13, 2.14) or m, n are taken from the set $\{k \in \mathbb{N}: \|k^a\| \in [1/2, 3/4]\}$ where a is any positive non-integer (see Theorems 9.1, 9.5). We also establish multidimensional variants of these results regarding patterns in positive density subsets of \mathbb{Z}^ℓ .

In the next section, we give the precise formulation of our main results and define some concepts used throughout the article.

2. PRECISE STATEMENT OF MAIN RESULTS

2.1. Notation and definitions. We first introduce some notation that is going to facilitate our presentation.

2.1.1. Ergodic theory. Following for example [36] we say that a probability space (X, \mathcal{X}, μ) is a *Lebesgue space*, if X can be given the structure of a Polish space (i.e. metrizable, separable, complete) such that \mathcal{X} is its Borel σ -algebra. Throughout the article we make the standard assumption that *all probability spaces considered are Lebesgue*.

By a system $(X, \mathcal{X}, \mu, T_1, \dots, T_\ell)$ we mean a Lebesgue probability space (X, \mathcal{X}, μ) endowed with ℓ invertible commuting measure preserving transformations. For $\vec{n} = (n_1, \dots, n_\ell) \in \mathbb{Z}^\ell$ we write $T_{\vec{n}} = T_1^{n_1} \cdot \dots \cdot T_\ell^{n_\ell}$. Sometimes we denote by \vec{T} the action of \mathbb{Z}^ℓ on X and write the system as $(X, \mathcal{X}, \mu, \vec{T})$. In the sequel, we generally omit the σ -algebra \mathcal{X} from our notation.

2.1.2. Nilmanifolds and nilsequences. Let $s \in \mathbb{N}$, G be an s -step nilpotent Lie group, and Γ be a discrete cocompact subgroup of G . Then the quotient space $X = G/\Gamma$ is called an *s -step nilmanifold*. We prefer to denote the elements of X as points x, y, \dots , not as cosets. The point e_X is the image in X of the unit element of G . The natural action of G on X is written $(g, x) \mapsto g \cdot x$ and the unique measure on X invariant under this action is called the *Haar measure* of X and is denoted by m_X .

Let τ_1, \dots, τ_d be commuting elements of G . For $i = 1, \dots, d$ let T_i be the translation $x \mapsto \tau_i \cdot x$ by τ_i on X . Then the system $(X, m_X, T_1, \dots, T_d)$ is called an *s-step nilsystem*. Nilsystems have been extensively studied and basic properties were established by Auslander [6], Parry [60, 61], Lesigne [58], Leibman [52, 53], and others.

Definition ([11]). If $X = G/\Gamma$ is an s -step nilmanifold, $\Psi \in C(X)$, and $\tau_1, \dots, \tau_d \in G$ are commuting elements, then the sequence $(\Psi(\tau_1^{n_1} \cdot \dots \cdot \tau_d^{n_d} \cdot e_X))_{n_1, \dots, n_d \in \mathbb{N}}$ is called an *s-step nilsequence in d variables*. Also, for notational convenience, we define a *0-step nilsequence* to be a constant sequence.

Remarks. • In [11] the notion “*basic s-step nilsequence*” is used for what we call here an “*s-step nilsequence*”.

• By [52, Paragraph 1.11] the nilmanifold X is isomorphic to a sub-nilmanifold of a nilmanifold $\tilde{X} = \tilde{G}/\tilde{\Gamma}$ where \tilde{G} is a connected and simply connected s -step nilpotent Lie group and all elements of G are represented in \tilde{G} . Hence, whenever needed, we can assume that the group G is connected and simply connected.

In recent years, nilsequences have played a key role in ergodic theory and additive combinatorics. They form the right substitute for linear exponential sequences needed to formalize certain inverse theorems which are used in the course of studying various multilinear expressions in analysis and number theory.

2.1.3. *Følner sequences and related averages.* We recall first some notions and introduce some notation.

Notation. We write $[N]$ for the interval $\{1, 2, \dots, N\}$ in \mathbb{N} .

Definition. A *Følner sequence in \mathbb{N}^d* is a sequence $\mathbf{I} = (I_j)_{j \in \mathbb{N}}$ of finite subsets of \mathbb{N}^d that satisfies

$$\lim_{j \rightarrow \infty} \frac{|(I_j + \mathbf{k}) \Delta I_j|}{|I_j|} = 0 \quad \text{for every } \mathbf{k} \in \mathbb{Z}^d,$$

where Δ denotes the symmetric difference and $I_j + \mathbf{k} := \{\mathbf{n} + \mathbf{k} : \mathbf{n} \in I_j\}$.

An example of a Følner sequence in \mathbb{N} is a sequence of intervals whose lengths tend to infinity. If $N_j \rightarrow +\infty$ and $(\mathbf{k}_j)_{j \in \mathbb{N}}$ is a sequence in \mathbb{N}^d , then $I_j := \mathbf{k}_j + [N_j]^d$, $j \in \mathbb{N}$, is a Følner sequence in \mathbb{N}^d . In all subsequent results and proofs we can replace the general Følner sequences by these particular examples.

If $a: \mathbb{N}^d \rightarrow \mathbb{C}$ is a sequence and $\mathbf{I} = (I_j)_{j \in \mathbb{N}}$ is a Følner sequence in \mathbb{N}^d , we let

$$\lim \text{Av}_{\mathbf{I}} a(\mathbf{n}) := \lim_{j \rightarrow +\infty} \frac{1}{|I_j|} \sum_{\mathbf{n} \in I_j} a(\mathbf{n}),$$

assuming of course that the previous limit exists. If the previous limit exists for every Følner sequence \mathbf{I} , then it is independent of \mathbf{I} ; we denote its common value with

$$\lim \text{Av } a(\mathbf{n})$$

and say that *the averages of a converge*. When it is unclear with respect to which variable we take the averages we use the notation

$$\lim \text{Av}_{\mathbf{n}, \mathbf{I}} a(\mathbf{n}), \quad \lim \text{Av}_{\mathbf{n}} a(\mathbf{n}).$$

Furthermore, we use the notation

$$\limsup |\text{Av}_{\mathbf{I}} a(\mathbf{n})| := \limsup_{j \rightarrow +\infty} \left| \frac{1}{|I_j|} \sum_{\mathbf{n} \in I_j} a(\mathbf{n}) \right|$$

and

$$\limsup |\text{Av } a(\mathbf{n})| := \sup_{\mathbf{I}} (\limsup |\text{Av}_{\mathbf{I}} a(\mathbf{n})|)$$

where the sup is taken over all Følner sequences $\mathbf{I} = (I_j)_{j \in \mathbb{N}}$ of subsets of \mathbb{N}^d .

We use similar notation for limits in $L^2(\mu)$ involving averages of functions $(f_{\mathbf{n}})_{\mathbf{n} \in \mathbb{N}^d}$ in $L^2(\mu)$ and write

$$\lim \text{Av}_{\mathbf{I}} f_{\mathbf{n}}, \quad \lim \text{Av} f_{\mathbf{n}}, \quad \limsup \|\text{Av}_{\mathbf{I}} f_{\mathbf{n}}\|_{L^2(\mu)}, \quad \limsup \|\text{Av} f_{\mathbf{n}}\|_{L^2(\mu)}$$

for the corresponding limits, where in the first two cases convergence takes place in $L^2(\mu)$ and in the last two cases we use $L^2(\mu)$ norms in place of the absolute values.

2.2. Convergence results for uniform averages. First we give convergence criteria for weighted ergodic averages which are defined using uniform averages, that is, averages over arbitrary Følner sequences in \mathbb{N}^d .

Definition. If $p_i: \mathbb{N}^d \rightarrow \mathbb{Z}$, $i = 1, \dots, \ell$, are polynomials, we call the map $\vec{p}: \mathbb{N}^d \rightarrow \mathbb{Z}^\ell$ defined by $\vec{p} := (p_1, \dots, p_\ell)$ a *polynomial mapping* from \mathbb{N}^d to \mathbb{Z}^ℓ . The *degree* $\deg(\vec{p})$ of \vec{p} is $\max_{i=1, \dots, \ell}(\deg(p_i))$.

We first state a strengthening of Theorem 1.1 that will be used frequently in this article. It is proved in Section 4.3.

Proposition 2.1. *For every $d, \ell, s \in \mathbb{N}$, polynomial mappings $\vec{p}_i: \mathbb{N}^d \rightarrow \mathbb{Z}^\ell$, $i = 1, \dots, s$, nilsequence $\psi: \mathbb{N}^d \rightarrow \mathbb{C}$, system $(X, \mu, T_1, \dots, T_\ell)$, and functions $f_1, \dots, f_s \in L^\infty(\mu)$, the limit*

$$\lim \text{Av} \psi(\mathbf{n}) \cdot T_{\vec{p}_1(\mathbf{n})} f_1 \cdot \dots \cdot T_{\vec{p}_s(\mathbf{n})} f_s$$

exists in $L^2(\mu)$.

Remark. For $d = 1$ this was proved in [30, Section 2.4].

Our main convergence criterion for uniform averages is the next result which is proved in Section 7.1.

Theorem 2.2. *Let $d, \ell, s, t \in \mathbb{N}$. Then there exists a positive integer $k = k(d, \ell, s, t)$, such that the following holds: If $w \in \ell^\infty(\mathbb{N}^d)$ is a sequence and for every k -step nilsequence $\psi: \mathbb{N}^d \rightarrow \mathbb{C}$ the limit*

$$(2.1) \quad \lim \text{Av} w(\mathbf{n}) \psi(\mathbf{n}) \quad \text{exists,}$$

then for every system $(X, \mu, T_1, \dots, T_\ell)$, functions $f_1, \dots, f_s \in L^\infty(\mu)$, and polynomial mappings $\vec{p}_i: \mathbb{N}^d \rightarrow \mathbb{Z}^\ell$, $i = 1, \dots, s$, of degree at most t , the limit

$$(2.2) \quad \lim \text{Av} w(\mathbf{n}) \cdot T_{\vec{p}_1(\mathbf{n})} f_1 \cdot \dots \cdot T_{\vec{p}_s(\mathbf{n})} f_s$$

exists in $L^2(\mu)$. Furthermore, if the limit in (2.1) is zero for every k -step nilsequence ψ in d variables, then the limit in (2.2) is always zero. Lastly, if the polynomial mappings $\vec{p}_i: \mathbb{N}^d \rightarrow \mathbb{Z}^\ell$, $i = 1, \dots, s$, are linear, then we can take $k = s$.

Remarks. • For $d = \ell = t = 1$ this result was proved in [48] and for $d = \ell = 1$ and $t \in \mathbb{N}$ arbitrary in [19].

• In Theorem 3.9 we give a characterization using “uniformity seminorms” of sequences satisfying the hypothesis of Theorem 2.2.

The next result shows that the hypothesis of Theorem 2.2 is necessary in order to have weak convergence of the averages in (2.2) for all linear polynomial mappings.

Proposition 2.3. *Let $d, s \in \mathbb{N}$ and $w \in \ell^\infty(\mathbb{N}^d)$ be a sequence. Suppose that for every system $(X, \mu, T_1, \dots, T_d)$, functions $f_0, \dots, f_s \in L^\infty(\mu)$, and linear forms $\vec{L}_i: \mathbb{N}^d \rightarrow \mathbb{Z}^d$, $i = 1, \dots, s$, the limit*

$$\lim \text{Av} w(\mathbf{n}) \int f_0 \cdot T_{\vec{L}_1(\mathbf{n})} f_1 \cdot \dots \cdot T_{\vec{L}_s(\mathbf{n})} f_s d\mu$$

exists. Then the limit

$$\lim Av w(\mathbf{n}) \psi(\mathbf{n})$$

exists for every s -step nilsequence ψ in d variables.

Remark. For $d = 1$ this was proved in [30].

Next for $d = 1$ we give some examples of sequences of weights in $\ell^\infty(\mathbb{N})$ for which Theorem 2.2 is applicable.

Examples. Let (Y, S) be a minimal uniquely ergodic system with invariant measure ν and for every $s \in \mathbb{N}$, let (Z_s, ν_s, S) be the “factor of order s ” defined in [46]. Suppose that for every $s \in \mathbb{N}$ the factor map $\pi_s: Y \rightarrow Z_s$ is continuous. Then for every $\Psi \in C(Y)$ and every $y_0 \in Y$, the sequence $w: \mathbb{N} \rightarrow \mathbb{C}$ defined by $w(n) := \Psi(S^n y_0)$, $n \in \mathbb{N}$, satisfies the hypothesis of Theorem 2.2 [48, Proposition 7.1]. Examples of this type include:

- (i) The Thue-Morse sequence which is the indicator function of those integers that have an odd sum of digits when expanded in base 2 (see [48, Proposition 2.21]).
- (ii) Bounded generalized polynomials (see [48, Corollary 2.23]).² These include sequences of the form $(\{p(n)\})_{n \in \mathbb{N}}$ or $(e(p(n)))_{n \in \mathbb{N}}$, where $p: \mathbb{N} \rightarrow \mathbb{Z}$ is an arbitrary generalized polynomial, $\{x\}$ denotes the fractional part of x , and $e(t) := e^{2\pi i t}$.

2.3. Convergence results for Cesàro averages. The assumptions of Theorem 2.2 are in many cases too strong to be of use (this is the case for the examples (i)-(v) below) and we would like to have a criterion that uses convergence assumptions of certain Cesàro averages instead of uniform averages. We obtain such a result by utilizing tools different from those used in the proof of Theorem 2.2 (Theorem 2.2 relies on Theorem 3.9 which necessitates the hypothesis (2.1)). The key new ingredients are decomposition results for multiple correlation sequences that are stated in Section 2.4 below.

Theorem 2.4. *Let $d, \ell, s, t \in \mathbb{N}$. Then there exists a positive integer $k = k(d, \ell, s, t)$, such that the following holds:*

(i) *If $w \in \ell^\infty(\mathbb{N}^d)$ and for every k -step nilsequence $\psi: \mathbb{N}^d \rightarrow \mathbb{C}$ the limit*

$$(2.3) \quad \lim_{N \rightarrow +\infty} \frac{1}{N^d} \sum_{\mathbf{n} \in [N]^d} w(\mathbf{n}) \psi(\mathbf{n}) \text{ exists,}$$

then for every system $(X, \mu, T_1, \dots, T_\ell)$, functions $f_0, \dots, f_s \in L^\infty(\mu)$, and polynomial mappings $\vec{p}_i: \mathbb{N}^d \rightarrow \mathbb{Z}^\ell$, $i = 1, \dots, s$, of degree at most t , the limit

$$(2.4) \quad \lim_{N \rightarrow +\infty} \frac{1}{N^d} \sum_{\mathbf{n} \in [N]^d} w(\mathbf{n}) \cdot \int f_0 \cdot T_{\vec{p}_1(\mathbf{n})} f_1 \cdot \dots \cdot T_{\vec{p}_s(\mathbf{n})} f_s d\mu$$

exists.

(ii) *If $w \in \ell^\infty(\mathbb{N}^d)$ and for every k -step nilsequence $\psi: \mathbb{N}^{2d} \rightarrow \mathbb{C}$ the limit*

$$(2.5) \quad \lim_{N, N' \rightarrow +\infty} \frac{1}{(NN')^d} \sum_{\mathbf{n} \in [N]^d, \mathbf{n}' \in [N']^d} w(\mathbf{n}) \overline{w(\mathbf{n}')} \psi(\mathbf{n}, \mathbf{n}')$$

exists, then for every system $(X, \mu, T_1, \dots, T_\ell)$, functions $f_1, \dots, f_s \in L^\infty(\mu)$, and polynomial mappings $\vec{p}_i: \mathbb{N}^d \rightarrow \mathbb{Z}^\ell$, $i = 1, \dots, s$, of degree at most t , the limit

$$(2.6) \quad \lim_{N \rightarrow +\infty} \frac{1}{N^d} \sum_{\mathbf{n} \in [N]^d} w(\mathbf{n}) \cdot T_{\vec{p}_1(\mathbf{n})} f_1 \cdot \dots \cdot T_{\vec{p}_s(\mathbf{n})} f_s$$

exists in $L^2(\mu)$.

²A *generalized polynomial* is a real valued function that is obtained from the identity function and real constants by using the operations of addition, multiplication, and taking the integer part.

Furthermore, if the limit in (2.3) (resp. (2.5)) is zero for every k -step nilsequence $\psi: \mathbb{N}^d \rightarrow \mathbb{C}$ (resp. $\psi: \mathbb{N}^{2d} \rightarrow \mathbb{C}$), then the limit in (2.4) (resp. (2.6)) is always zero.

Lastly, in (i) (resp. (ii)), if the polynomial mappings \vec{p}_i are linear, then we can take $k = s$ (resp. $k = 2s - 1$), and if in addition $\ell = s$ and $T_{\vec{p}_i(\mathbf{n})} = T_i^{L_i(\mathbf{n})}$ for $i = 1, \dots, s$, where L_1, \dots, L_s are linear forms spanning a subspace of dimension r , then we can take $k = s - r + 1$ (resp. $k = 2s - 2r + 1$).

Remarks. • For $d = 1$ the first part of this result was proved in [30].

• For single variable polynomials, in order to prove Part (ii) of this result, we rely on decomposition results of correlations sequences involving polynomials in two variables.

An analogue of Proposition 2.3, with Cesàro averages in place of uniform averages, holds with the same proof. This implies that the condition (2.3) is also necessary in order for the limit (2.4) to exist for all linear polynomial mappings.

Sequences $w \in \ell^\infty(\mathbb{N}^d)$ that satisfy the hypothesis (2.3) and (2.5) of Theorem 2.4 (but do not satisfy the condition (2.1) of Theorem 2.2 even for $\psi = 1$) include the following:

- (i) Any sequence of the form $(g(S_{\vec{n}}y))_{\vec{n} \in \mathbb{N}^d}$ where $(Y, \nu, S_1, \dots, S_d)$ is a system and $g \in L^\infty(\nu)$, for $y \in Y$ belonging to a set of full measure that depends only on the system and the function g . For $d = 1$ this was proved in [48, Theorem 2.22] for hypothesis (2.3) but a similar argument also gives hypothesis (2.5) and works (using Theorem 3.1) for general $d \in \mathbb{N}$.
- (ii) Any “good” multiplicative function $\phi: \mathbb{N}^d \rightarrow \mathbb{C}$ (see Section 2.6 and Theorem 8.1). For $d = 1$ an alternate proof which depends on [31], and thus on deep results from [40] and [42], was given in [32].
- (iii) The indicator function of all vectors of \mathbb{N}^d whose coordinates have an even (or an odd) number of distinct prime factors; or more generally, the indicator function of any set S defined as in Theorem 2.12 below (this follows from Theorem 8.1 and the argument in Section 8.2).
- (iv) Any sequence of the form $(e(\sum_{i=1}^d n_i^{a_i}))_{n_1, \dots, n_d \in \mathbb{N}}$, where a_1, \dots, a_d are positive non-integers. In this case the limits in (2.3) and (2.5) are always 0 (see Theorem 9.1 and Proposition 9.3). Moreover, for $d = 1$, we give necessary and sufficient conditions for a sequence of the form $(e(f(n)))_{n \in \mathbb{N}}$, where f is a Hardy field function of at most polynomial growth, to be a good universal weight for mean convergence of the averages (2.6) (see Corollary 9.2).
- (v) The indicator function of any set of the form

$$S := \{n_1, \dots, n_d \in \mathbb{N} : \|f_1(n_1)\| \in [a_1, b_1], \dots, \|f_d(n_d)\| \in [a_d, b_d]\},$$

where $0 \leq a_i < b_i \leq 1/2$, $\|x\| := d(x, \mathbb{Z})$, and f_i are Hardy field functions of at most polynomial growth that stay away from polynomials (this follows from Proposition 9.3 and an approximation argument that uses the estimate (9.1)). Furthermore, the set S is good for multiple recurrence and mean convergence and the $L^2(\mu)$ limit

$$\lim_{N \rightarrow +\infty} \frac{1}{|S \cap [N]^d|} \sum_{\mathbf{n} \in S \cap [N]^d} T_{\vec{p}_1(\mathbf{n})} f_1 \cdots T_{\vec{p}_s(\mathbf{n})} f_s$$

is equal to the limit obtained when S is replaced by \mathbb{N}^d (see Theorem 9.5).

2.4. Multiple correlations in ergodic theory. Multiple correlation sequences are a well studied object in ergodic theory and form an indispensable tool in the study of various multiple ergodic averages. For single variable sequences, structural results have been obtained in [11, 30, 55, 56, 59]; we extend some of these results to sequences in several variables. These extensions turn out to be key for the proof of the convergence criterion given in Theorem 2.4. Our argument follows closely the method used in [30] to

obtain similar results for single variable sequences; but also some refinements obtained (for example Theorem 2.6) require new methodology.

Notation. For a bounded sequence $a: \mathbb{N}^d \rightarrow \mathbb{C}$ we let

$$(2.7) \quad \|a\|_2 := (\limsup \text{Av} |a(\mathbf{n})|^2)^{\frac{1}{2}}.$$

Definition. A bounded sequence $a: \mathbb{N}^d \rightarrow \mathbb{C}$ is an *approximate s -step nilsequence in d variables* if for every $\varepsilon > 0$ it admits a decomposition as $a = a_{\text{st}} + a_{\text{er}}$, where

- (i) $a_{\text{st}}: \mathbb{N}^d \rightarrow \mathbb{C}$ is an s -step nilsequence in d variables with $\|a_{\text{st}}\|_\infty \leq \|a\|_\infty$;
- (ii) $\|a_{\text{er}}\|_2 \leq \varepsilon$.

The subscripts “st” and “err” are used to indicate “structured” and “error” respectively. In Section 6.2 we show:

Theorem 2.5. *Let $d, \ell, s, t \in \mathbb{N}$. Then there exists a positive integer $k = k(d, \ell, s, t)$, such that for every system $(X, \mu, T_1, \dots, T_\ell)$, functions $f_0, \dots, f_s \in L^\infty(\mu)$, and polynomial mappings $\vec{p}_i: \mathbb{N}^d \rightarrow \mathbb{Z}^\ell$, $i = 1, \dots, s$, of degree at most t , the sequence $a: \mathbb{N}^d \rightarrow \mathbb{C}$ given by*

$$(2.8) \quad a(\mathbf{n}) := \int f_0 \cdot T_{\vec{p}_1(\mathbf{n})} f_1 \cdots \cdots T_{\vec{p}_s(\mathbf{n})} f_s d\mu, \quad \mathbf{n} \in \mathbb{N}^d,$$

is an approximate k -step nilsequence in d variables.

Furthermore, if the polynomial mappings $\vec{p}_1, \dots, \vec{p}_s$ are linear, then we can take $k = s$.

Remark. For $d = 1$ this result was proved in [30].

The degree of nilpotency provided in the last part of Theorem 2.5 is not always optimal. In Section 6.3 we establish the following improvement for particular correlation sequences:

Theorem 2.6. *For $d, \ell \in \mathbb{N}$ let $(X, \mu, T_1, \dots, T_\ell)$ be a system, $f_0, \dots, f_\ell \in L^\infty(\mu)$ be functions, and $L_1, \dots, L_\ell: \mathbb{N}^d \rightarrow \mathbb{Z}$ be linear forms spanning a space of dimension r . Then the sequence $a: \mathbb{N}^d \rightarrow \mathbb{C}$ given by*

$$a(\mathbf{n}) := \int f_0 \cdot T_1^{L_1(\mathbf{n})} f_1 \cdots \cdots T_\ell^{L_\ell(\mathbf{n})} f_\ell d\mu, \quad \mathbf{n} \in \mathbb{N}^d,$$

is an approximate $(\ell - r + 1)$ -step nilsequence in d variables.

Remark. Examples of sequences for which this theorem applies and gives the optimal degree of nilpotency are the three sequences in (1.4).

A crucial ingredient in the proof of the previous two decomposition results is Theorem 3.10 which gives a characterization involving uniformity seminorms of approximate nilsequences. Furthermore, the proof of Theorem 2.6 uses a structural result for the generalized Kronecker factor of a not necessarily ergodic system that is of independent interest (see Theorem 5.2).

Lastly, we give an interesting corollary of Theorem 2.5. For $d \in \mathbb{N}$ we consider various subsets of $\ell^\infty(\mathbb{N}^d)$. The first is the set

$$\mathcal{N}_d := \{(\psi(\mathbf{n}))_{\mathbf{n} \in \mathbb{N}^d} : \psi \text{ is a nilsequence in } d \text{ variables}\}.$$

With $\mathcal{MC}_{d, \text{pol}}$ we denote the set that contains all sequences of the form

$$\left(\int f_0 \cdot T_{\vec{p}_1(\mathbf{n})} f_1 \cdots \cdots T_{\vec{p}_s(\mathbf{n})} f_s d\mu \right)_{\mathbf{n} \in \mathbb{N}^d}$$

for arbitrary systems $(X, \mu, T_1, \dots, T_\ell)$, functions $f_0, \dots, f_s \in L^\infty(\mu)$, polynomial mappings $\vec{p}_1, \dots, \vec{p}_s: \mathbb{N}^d \rightarrow \mathbb{Z}^\ell$, and $\ell, s \in \mathbb{N}$.

We also define as $\mathcal{MC}_{d, \text{lin}}$ the set of multiple correlation sequences defined as above using linear polynomial mappings only.

Theorem 2.7. *For every $d \in \mathbb{N}$ the sets \mathcal{N}_d , $\mathcal{MC}_{d,lin}$, $\mathcal{MC}_{d,pol}$ are subspaces of $\ell^\infty(\mathbb{N}^d)$ and we have*

$$\overline{\mathcal{N}_d} = \overline{\mathcal{MC}_{d,lin}} = \overline{\mathcal{MC}_{d,pol}}$$

where the closure is taken with respect to the seminorm $\|\cdot\|_2$ defined in (2.7).

2.5. Multiple correlations for sequences in \mathbb{N}^d and \mathbb{Z}_N^d . We use the decomposition results of the previous subsection in order to deduce similar results for multiple correlations of bounded sequences in \mathbb{N}^d .

Definition. Let \mathcal{A} be a finite collection of bounded complex valued sequences in ℓ variables and $\mathbf{I} = (I_j)_{j \in \mathbb{N}}$ be a Følner sequence in \mathbb{N}^ℓ . We say that the collection \mathcal{A} admits correlations along \mathbf{I} if for every $s \in \mathbb{N}$ and all $\mathbf{h}_1, \dots, \mathbf{h}_s \in \mathbb{N}^\ell$, the limit

$$\lim \text{Av}_{\mathbf{k}, \mathbf{I}} \prod_{j=1}^s b_j(\mathbf{k} + \mathbf{h}_j)$$

exists, where for $j = 1, \dots, s$ the sequence b_j or the sequence $\overline{b_j}$ belongs to \mathcal{A} .

Combining Theorem 2.5 with the correspondence principle stated in Proposition 6.4 below, we deduce the following statement:

Theorem 2.8. *Let $d, \ell, s, t \in \mathbb{N}$. Then there exists a positive integer $k = k(d, \ell, s, t)$ such that the following holds: If $a_1, \dots, a_s: \mathbb{Z}^\ell \rightarrow \mathbb{C}$ are bounded sequences admitting correlations along a Følner sequence \mathbf{I} in \mathbb{N}^ℓ and $\vec{p}_i: \mathbb{N}^d \rightarrow \mathbb{Z}^\ell$, $i = 1, \dots, s$, are polynomial mappings of degree at most t , then the sequence $b: \mathbb{N}^d \rightarrow \mathbb{C}$ defined by*

$$b(\mathbf{n}) := \lim \text{Av}_{\mathbf{k}, \mathbf{I}} \prod_{i=1}^s a_i(\mathbf{k} + \vec{p}_i(\mathbf{n})), \quad \mathbf{n} \in \mathbb{N}^d,$$

is an approximate k -step nilsequence in d variables.

Moreover, if the polynomial mappings are linear, then we can take $k = s - 1$.

If A is a finite set we let $\mathbb{E}_{n \in A} := \frac{1}{|A|} \sum_{n \in A}$. Decomposition results of similar nature also hold in the finite world, for example, the following is true:

Theorem 2.9. *Let $d, \ell, s, t \in \mathbb{N}$. Then there exists a positive integer $k = k(d, \ell, s, t)$ such that the following holds: For every $\varepsilon > 0$ there exists a k -step nilmanifold $X = X(d, \ell, s, t, \varepsilon)$ such that for every $N \in \mathbb{N}$, finite sequences $a_1, \dots, a_s: \mathbb{Z}_N^\ell \rightarrow \mathbb{C}$ of modulus at most 1, and polynomial mappings $\vec{p}_i: \mathbb{N}^d \rightarrow \mathbb{Z}^\ell$, $i = 1, \dots, s$, having integer coefficients and degree at most t , the sequence $b: \mathbb{N}^d \rightarrow \mathbb{C}$ defined by*

$$(2.9) \quad b(\mathbf{n}) := \mathbb{E}_{\mathbf{k} \in \mathbb{Z}_N^\ell} \prod_{i=1}^s a_i(\mathbf{k} + \vec{p}_i(\mathbf{n})), \quad \mathbf{n} \in \mathbb{Z}_N^d,$$

admits a decomposition of the form $b = b_{st} + b_{er}$ where

- (i) $b_{st}: \mathbb{N}^d \rightarrow \mathbb{C}$ is a convex combination of k -step nilsequences defined by functions on X with Lipschitz norm at most 1;
- (ii) $\mathbb{E}_{\mathbf{n} \in \mathbb{Z}_N^d} |b_{er}(\mathbf{n})| \leq \varepsilon$.

Furthermore, if the polynomial mappings are linear, then we can take $k = s - 1$.

Remark. It is important that the nilmanifold X and the Lipschitz norm of the function defining the nilsequence are independent of $N \in \mathbb{N}$.

2.6. Applications to arithmetic. Next we give some applications of number theoretic and combinatorial flavor.

Definition. A function $\phi: \mathbb{N} \rightarrow \mathbb{C}$ is called *multiplicative* if

$$\phi(mn) = \phi(m)\phi(n) \quad \text{whenever } (m, n) = 1.$$

It is called *completely multiplicative* if this relation holds for all $m, n \in \mathbb{N}$.

We say that a multiplicative function $\phi: \mathbb{N} \rightarrow \mathbb{C}$ that is bounded by 1 is *good* if the limit

$$(2.10) \quad \lim_{N \rightarrow +\infty} \frac{1}{N} \sum_{n=1}^N \phi(an + b)$$

exists for all $a \in \mathbb{N}, b \in \mathbb{Z}_+$. It is called *aperiodic* if all these limits are equal to 0.

For $d \in \mathbb{N}$ a function $\phi: \mathbb{N}^d \rightarrow \mathbb{C}$ is called *multiplicative* if it is of the form

$$\phi(n_1, \dots, n_d) = \phi_1(n_1) \cdots \phi_d(n_d), \quad n_1, \dots, n_d \in \mathbb{N},$$

for some multiplicative functions $\phi_i: \mathbb{N} \rightarrow \mathbb{C}, i = 1, \dots, d$, which we call the *components* of ϕ . We call a multiplicative function $\phi: \mathbb{N}^d \rightarrow \mathbb{C}$ *good* if all its component functions are good and *aperiodic* if at least one of its component functions is aperiodic.

By a classical result of Wirsing [67], every real valued multiplicative function that is bounded by 1 is good. A result of Halász [44] allows to characterize good and aperiodic multiplicative functions. Let \mathbb{P} be the set of primes. A Dirichlet character is a periodic completely multiplicative function which takes the value 1 at 1.

Notation ([37]). If $\phi_1, \phi_2: \mathbb{N} \rightarrow \mathbb{C}$ are multiplicative functions, bounded by 1, we define $\mathbb{D}(\phi_1, \phi_2) \in [0, +\infty]$ by

$$\mathbb{D}(\phi_1, \phi_2)^2 := \sum_{p \in \mathbb{P}} \frac{1}{p} (1 - \Re(\phi_1(p)\overline{\phi_2(p)})).$$

Remark. Note that if $|\phi_1| = |\phi_2| = 1$, then $\mathbb{D}(\phi_1, \phi_2)^2 = \sum_{p \in \mathbb{P}} \frac{1}{2p} |\phi_1(p) - \phi_2(p)|^2$.

The next result can be deduced from [28, Theorem 6.3].

Theorem 2.10. *Let $\phi: \mathbb{N} \rightarrow \mathbb{C}$ be a multiplicative function that is bounded by 1. Then ϕ is good if and only if for every Dirichlet character χ we either have*

- (i) $\mathbb{D}(\phi\chi, n^{it}) = +\infty$ for every $t \in \mathbb{R}$, or
 - (ii) for some $t \in \mathbb{R}$ we have $\mathbb{D}(\phi\chi, n^{it}) < \infty$ and $\chi(2)^k \phi(2^k) = -2^{ikt}$ for all $k \in \mathbb{N}$,
- or
- (iii) $\sum_{p \in \mathbb{P}} \frac{1}{p} (1 - \phi(p)\chi(p))$ converges.

Moreover, ϕ is aperiodic if and only if either the condition (i) or (ii) is satisfied for every Dirichlet character χ .

For a more complete discussion of these notions, see [32, Section 2.5]. The next result is proved in Section 8.1.

Theorem 2.11. *Let $d \in \mathbb{N}$ and $\phi: \mathbb{N}^d \rightarrow \mathbb{C}$ be a good multiplicative function. Then for every $\ell, s \in \mathbb{N}$, system $(X, \mu, T_1, \dots, T_\ell)$, functions $f_1, \dots, f_s \in L^\infty(\mu)$, and polynomial mappings $\vec{p}_i: \mathbb{N}^d \rightarrow \mathbb{Z}^\ell, i = 1, \dots, s$, the limit*

$$(2.11) \quad \lim_{N \rightarrow +\infty} \frac{1}{N^d} \sum_{\mathbf{n} \in [N]^d} \phi(\mathbf{n}) \cdot T_{\vec{p}_1(\mathbf{n})} f_1 \cdots T_{\vec{p}_s(\mathbf{n})} f_s \quad \text{exists in } L^2(\mu).$$

Furthermore, if the multiplicative function ϕ is aperiodic, then the limit is equal to 0.

Remarks. • This result and its consequences below were proved in [32] for $d = 1$ using a deep structural result for multiplicative functions from [31]. The current argument relies on the convergent criterion of Theorem 2.4 and uses much softer number theoretic input (we only use Theorem 8.1).

• Conversely, if for $d = \ell = s = 1$ the averages in (2.11) converge weakly, then examples of periodic systems show that ϕ has to be good, and if the averages in (2.11) converge weakly to 0, then ϕ has to be aperiodic.

• Similar statements, with similar proofs, hold if in (2.11) we use averages of the form $\frac{1}{N_1 \cdots N_d} \sum_{\mathbf{n} \in [N_1] \times \cdots \times [N_d]}$ and take the limit as $N_1, \dots, N_d \rightarrow +\infty$. A similar comment applies for the next two results.

Definition. For $a \in \mathbb{Z}_+$ and $b \in \mathbb{N}$ we let $S_{a,b}$ consist of those $n \in \mathbb{N}$ whose number of distinct prime factors is congruent to $a \pmod{b}$.

We can also define $S_{a,b}$ by counting prime factors with multiplicity; then all results stated below continue to hold with similar proofs. The next result is proved in Section 8.2.

Theorem 2.12. Let $d \in \mathbb{N}$, $a_i, c_i \in \mathbb{Z}_+$, $b_i \in \mathbb{N}$, $i = 1, \dots, d$, and let

$$(2.12) \quad S := (S_{a_1, b_1} + c_1) \times \cdots \times (S_{a_d, b_d} + c_d).$$

Then for all $\ell, s \in \mathbb{N}$, polynomial mappings $\vec{p}_1, \dots, \vec{p}_s: \mathbb{N}^d \rightarrow \mathbb{Z}^\ell$, system $(X, \mu, T_1, \dots, T_\ell)$, and functions $f_1, \dots, f_s \in L^\infty(\mu)$, the limit

$$\frac{1}{|S \cap [N]^d|} \sum_{\mathbf{n} \in S \cap [N]^d} T_{\vec{p}_1(\mathbf{n})} f_1 \cdots T_{\vec{p}_s(\mathbf{n})} f_s$$

exists in $L^2(\mu)$ and is equal to the limit obtained when one replaces S with \mathbb{N}^d .

Remark. It follows from our argument that $\lim_{N \rightarrow +\infty} |S \cap [N]^d|/N^d = (\prod_{i=1}^d b_i)^{-1}$.

In Section 8.2 we deduce from this result the following multiple recurrence statement:

Theorem 2.13. We use the notation of Theorem 2.12 and assume in addition that $\vec{p}_i(\mathbf{0}) = \vec{0}$ for $i = 1, \dots, s$. Then for S as in (2.12), for every $A \in \mathcal{X}$ with $\mu(A) > 0$ we have

$$\lim_{N \rightarrow +\infty} \frac{1}{|S \cap [N]^d|} \sum_{\mathbf{n} \in S \cap [N]^d} \mu(A \cap T_{-\vec{p}_1(\mathbf{n})} A \cap \cdots \cap T_{-\vec{p}_s(\mathbf{n})} A) > 0.$$

Lastly, we give some combinatorial implications of the previous multiple recurrence result. We define the *upper Banach density* $d^*(E)$ of a set $E \subset \mathbb{Z}^\ell$ by $d^*(E) := \limsup_{|I| \rightarrow +\infty} \frac{|E \cap I|}{|I|}$, where the lim sup is taken over all parallelepipeds $I \subset \mathbb{Z}^\ell$ whose side lengths tend to infinity and the *lower natural density* as $\liminf_{N \rightarrow \infty} \frac{|E \cap [-N, N]^\ell|}{(2N+1)^\ell}$. Using a modification of the correspondence principle of H. Furstenberg (proved as in [12]) we deduce from Theorem 2.13 the following result:

Theorem 2.14. We use the notation of Theorem 2.12 and assume in addition that $\vec{p}_i(\mathbf{0}) = \vec{0}$ for $i = 1, \dots, s$. Then for S as in (2.12), for every set $E \subset \mathbb{Z}^\ell$ with $d^*(E) > 0$, the set

$$\{\mathbf{n} \in S: d^*(E \cap (E - \vec{p}_1(\mathbf{n})) \cap \cdots \cap (E - \vec{p}_s(\mathbf{n}))) > 0\}$$

has positive lower natural density.

Applying this for $d = 2$, $a_i = 2$, $b_i = 0$ or 1 , $c_i = 0$, and $\vec{p}_i(n_1, n_2) = i(n_1^2 + n_2^2)$ for $i = 1, \dots, s$, we obtain the refinement of Szemerédi's theorem mentioned towards the end of the introduction.

2.7. Open problems. When all the maps T_1, \dots, T_ℓ are powers of the same transformation and $d = 1$, a strengthening of Theorem 2.5 holds which shows that the error term can be taken to converge to 0 in uniform density (see [11, 55, 56, 57]). It is not clear whether a similar result holds for arbitrary commuting transformations, even when $d = 1$, $\ell = 2$, and the polynomial mappings are linear.

Problem 1. *Let (X, μ, T_1, T_2) be a system and $f_0, f_1, f_2 \in L^\infty(\mu)$. Is it true that the sequence $a: \mathbb{N} \rightarrow \mathbb{C}$ defined by*

$$a(n) := \int f_0 \cdot T_1^n f_1 \cdot T_2^n f_2 d\mu, \quad n \in \mathbb{N},$$

can be decomposed as $a = a_{\text{st}} + a_{\text{er}}$ where a_{st} is a uniform limit of 2-step nilsequences and $\|a_{\text{er}}\|_2 = 0$?

When $T_2 = T_1^2$ this is shown to be the case in [11, 56, 57].

Theorem 2.4 shows that condition (2.3) is sufficient for weak convergence of the averages in (2.6), but we needed the stronger hypothesis (2.5) in order to guarantee mean convergence. This is probably an artifact of our proof.

Problem 2. *Show that condition (2.3) is sufficient for mean convergence of the averages in (2.6).*

When $d = 1$ and all the maps T_1, \dots, T_ℓ are powers of the same transformation this is shown to be the case in [19, Theorem 1.3].

In Theorem 2.5, even in seemingly simple cases, it is not clear what the optimal dependence of k on d, ℓ, s, t is, even when the polynomial mappings are linear. It is expected (but we are unable to verify this) that this optimal dependence can already be inferred from the case where all the T_1, \dots, T_ℓ are powers of the same transformation (a case which is much more tractable using the theory of characteristic factors). We record here a relevant open problem.

Problem 3. *Let (X, μ, T_1, T_2) be a system and $f, g, h \in L^\infty(\mu)$. Show that the sequence $a: \mathbb{N}^2 \rightarrow \mathbb{C}$ defined by*

$$a(m, n) := \int f \cdot T_1^m T_2^n g \cdot T_1^n T_2^m h d\mu, \quad m, n \in \mathbb{N},$$

is an approximate 1-step nilsequence in 2 variables.

When T_1, T_2 are powers of the same transformation this can be verified by an argument similar to the one used in the proof of Theorem 2.6. Note also that Theorem 2.5 gives that the sequence a is an approximate 2-step nilsequence in 2 variables.

2.8. Notation and conventions. For readers convenience, we gather here some notation used throughout the article.

We denote by \mathbb{N} the set of positive integers, by \mathbb{Z}_+ the set of non-negative integers, and by \mathbb{R}_+ the set of non-negative real numbers.

For $N \in \mathbb{N}$ we denote by $[N]$ the set $\{1, \dots, N\}$.

With $\ell^\infty(\mathbb{N}^d)$ we denote the space of all bounded sequences $a: \mathbb{N}^d \rightarrow \mathbb{C}$.

If A is a finite set we let $\mathbb{E}_{n \in A} := \frac{1}{|A|} \sum_{n \in A}$.

We write $\mathcal{C}: \mathbb{C} \rightarrow \mathbb{C}$ for the complex conjugation.

If x is a real, $e(x)$ denotes the number $e^{2\pi i x}$ and $\|x\|$ denotes the distance between x and the nearest integer.

Given $d \in \mathbb{N}$ we write $\mathbf{n} = (n_1, \dots, n_d)$ for a point of \mathbb{Z}^d .

We typically use the letter ψ to denote nilsequences.

3. UNIFORMITY SEMINORMS AND DECOMPOSITION RESULTS

In this section we extend to sequences in $\ell^\infty(\mathbb{N}^d)$ some results established in [48] and [30] for sequences in $\ell^\infty(\mathbb{N})$. The statements and the proofs are completely similar and we only give the necessary definitions and sketch the main steps of the proofs.

Some definitions and notation. We write $\mathcal{C}: \mathbb{C} \rightarrow \mathbb{C}$ for the complex conjugation; then $\mathcal{C}^k z = z$ if k is even and $\mathcal{C}^k z = \bar{z}$ if k is odd. We let $\llbracket k \rrbracket := \{0, 1\}^k$ and $\llbracket k^* \rrbracket := \llbracket k \rrbracket \setminus \{\mathbf{0}\}$. Elements of $\llbracket k \rrbracket$ are written as $\underline{\epsilon} = (\epsilon_1, \dots, \epsilon_k)$. We let $|\underline{\epsilon}| := \epsilon_1 + \dots + \epsilon_k$. Elements of $(\mathbb{N}^d)^k$ are written as $\underline{\mathbf{h}} = (\mathbf{h}_1, \dots, \mathbf{h}_k)$ where $\mathbf{h}_i \in \mathbb{N}^d$ for $i = 1, \dots, k$. For $\underline{\mathbf{h}} \in (\mathbb{N}^d)^k$ and $\underline{\epsilon} \in \llbracket k \rrbracket$ we let $\underline{\epsilon} \cdot \underline{\mathbf{h}} := \epsilon_1 \mathbf{h}_1 + \dots + \epsilon_k \mathbf{h}_k \in \mathbb{N}^d$.

3.1. The definition of the seminorms. We follow Section 2 of [48].

We say that a finite or countable family \mathcal{F} of bounded sequences in $\ell^\infty(\mathbb{N}^d)$ admits correlations along a Følner sequence $\mathbf{I} = (I_j)_{j \in \mathbb{N}}$ in \mathbb{N}^d if the limit

$$\lim \text{Av}_{\mathbf{n}, \mathbf{I}} \left(\prod_{i=1}^m b_i(\mathbf{n} + \mathbf{h}_i) \right)$$

exists for every $m \in \mathbb{N}$, all $\mathbf{h}_1, \dots, \mathbf{h}_m \in \mathbb{N}^d$, and all sequences $b_1, \dots, b_m \in \ell^\infty(\mathbb{N}^d)$ such that either b_i or $\overline{b_i}$ belongs to \mathcal{F} for $i = 1, \dots, m$. We remark that from every Følner sequence \mathbf{I} we can extract a subsequence \mathbf{I}' so that a given family of sequences admits correlations along \mathbf{I}' .

Suppose that the sequence $a \in \ell^\infty(\mathbb{N}^d)$ admits correlations along \mathbf{I} . Then for $k \in \mathbb{N}$ and $\underline{\mathbf{h}} = (\mathbf{h}_1, \dots, \mathbf{h}_k) \in (\mathbb{N}^d)^k$ we write

$$\text{Corr}_{\mathbf{I}}(a; \underline{\mathbf{h}}) := \lim \text{Av}_{\mathbf{n}, \mathbf{I}} \left(\prod_{\underline{\epsilon} \in \llbracket k \rrbracket} \mathcal{C}^{|\underline{\epsilon}|} a(\mathbf{n} + \underline{\epsilon} \cdot \underline{\mathbf{h}}) \right)$$

and

$$\|a\|_{\mathbf{I}, k} := \left(\lim_{H \rightarrow +\infty} \frac{1}{H^{dk}} \sum_{\mathbf{h}_1, \dots, \mathbf{h}_k \in [H]^d} \text{Corr}_{\mathbf{I}}(a; \underline{\mathbf{h}}) \right)^{1/2^k}.$$

In [48, Proposition 2.4] it is shown that for $d = 1$ the previous limit exists and is non-negative; the proof is similar for general $d \in \mathbb{N}$. Furthermore, the map $a \mapsto \|a\|_{\mathbf{I}, k}$ is subadditive ([48, Proposition 2.5] for $d = 1$), that is, if the sequences a, b , and $a + b$ admit correlations along \mathbf{I} , then $\|a + b\|_{\mathbf{I}, k} \leq \|a\|_{\mathbf{I}, k} + \|b\|_{\mathbf{I}, k}$.

For $a \in \ell^\infty(\mathbb{N}^d)$ we define

$$\|a\|_{U^k(\mathbb{N}^d)} := \sup_{\mathbf{I}} \|a\|_{\mathbf{I}, k}$$

where the supremum is taken over all Følner sequences \mathbf{I} in \mathbb{N}^d for which the sequence a admits correlations. Then the map $a \mapsto \|a\|_{U^k(\mathbb{N}^d)}$ is a seminorm on $\ell^\infty(\mathbb{N}^d)$, we call it the *uniformity seminorm of order k* of a .

Interpretation. Next, we interpret the previous definitions and results in dynamical terms. We use a variant of Furstenbergs correspondence principle that enables to transfer results from ergodic theory to results about bounded sequences of complex numbers. We follow the method used in [48, Section 6.1] when $d = 1$; similar arguments work for general $d \in \mathbb{N}$ and we summarize them here. For notational convenience we restrict to the case where the family \mathcal{F} contains only a single sequence $a: \mathbb{N}^d \rightarrow \mathbb{C}$, the general case being completely similar. Let D be the closed disk in \mathbb{C} of radius $\|a\|_\infty$ and let $D^{\mathbb{Z}^d}$ be endowed with the product topology and with the natural shifts T_1, \dots, T_d given by

$$(T_i x)(n_1, \dots, n_d) = x(n_1, \dots, n_{i-1}, n_i + 1, n_{i+1}, \dots, n_d)$$

for $i = 1, \dots, d$, where we use the notation $\mathbf{n} = (n_1, \dots, n_d) \in \mathbb{Z}^d$ and $x = (x(\mathbf{n}))_{\mathbf{n} \in \mathbb{Z}^d} \in D^{\mathbb{Z}^d}$. We define the continuous function $f: D^{\mathbb{Z}^d} \rightarrow \mathbb{C}$ by $f(x) := x(\mathbf{0})$. Furthermore, we

define the point ω in $D^{\mathbb{Z}^d}$ by $\omega(\mathbf{n}) := a(\mathbf{n})$ for $\mathbf{n} \in \mathbb{N}^d$ and $\omega(\mathbf{n}) = 0$ otherwise. Then we have

$$f(T_{\mathbf{n}}\omega) = a(\mathbf{n}), \quad \mathbf{n} \in \mathbb{N}^d.$$

Let X be the closed orbit of ω under T_1, \dots, T_d . Then (X, T_1, \dots, T_d) is a topological dynamical system and ω is a transitive point of this system, meaning it has a dense orbit in X .

Let $\mathbf{I} = (I_j)_{j \in \mathbb{N}}$ be a Følner sequence in \mathbb{N}^d for which the sequence a admits correlations. Let μ be a w^* -limit point for the sequence of measures

$$\mu_j := \frac{1}{|I_j|} \sum_{\mathbf{n} \in I_j} \delta_{T_{\mathbf{n}}\omega}, \quad j \in \mathbb{N}.$$

Then μ is a probability measure on X , invariant under T_1, \dots, T_d , and by construction, for every $m \in \mathbb{N}$, all $\eta_1, \dots, \eta_m \in \{0, 1\}$, and all $\mathbf{h}_1, \dots, \mathbf{h}_m \in \mathbb{Z}^d$, we have

$$\lim \text{Av}_{\mathbf{n}, \mathbf{I}} \left(\prod_{i=1}^m \mathcal{C}^{\eta_i} a(\mathbf{n} + \mathbf{h}_i) \right) = \lim \text{Av}_{\mathbf{n}, \mathbf{I}} \left(\prod_{i=1}^m \mathcal{C}^{\eta_i} T_{\mathbf{h}_i} f(T_{\mathbf{n}}\omega) \right) = \int \prod_{i=1}^m \mathcal{C}^{\eta_i} T_{\mathbf{h}_i} f d\mu.$$

In particular, for $\underline{\mathbf{h}} = (\mathbf{h}_1, \dots, \mathbf{h}_k) \in (\mathbb{N}^d)^k$, we have

$$\text{Corr}_{\mathbf{I}}(a; \underline{\mathbf{h}}) = \int \prod_{\epsilon \in [k]} \mathcal{C}^{|\epsilon|} T_{\epsilon, \underline{\mathbf{h}}} f d\mu$$

and thus

$$\|a\|_{\mathbf{I}, k} = \|f\|_{\mu, k}$$

where $\|\cdot\|_{\mu, k}$ is the seminorm on $L^\infty(\mu)$ defined in [46] in the ergodic case and in [20] in the general case.³ We recall the definition and some properties of these seminorms in Appendix A. Note also that if $\psi: \mathbb{N}^d \rightarrow \mathbb{C}$ is a nilsequence of the form $(\Phi(T_{\mathbf{n}}x))_{\mathbf{n} \in \mathbb{N}^d}$, then ψ admits correlations along every Følner sequence \mathbf{I} and

$$(3.1) \quad \|\psi\|_{\mathbf{I}, k} = \|\psi\|_{U^k(\mathbb{N}^d)} = \|\Phi\|_{\mu, k}$$

where the last seminorm is defined with respect to the action of $T_{\mathbf{n}}$, $\mathbf{n} \in \mathbb{N}^d$, on X . From the properties (A.1) and (A.4) we deduce

$$(3.2) \quad \limsup |\text{Av}_{\mathbf{I}} a(\mathbf{n})| \leq \|a\|_{\mathbf{I}, 1};$$

$$(3.3) \quad \|a\|_{\mathbf{I}, k+1}^{2^{k+1}} = \lim_{H \rightarrow +\infty} \frac{1}{H^d} \sum_{\mathbf{h} \in [H]^d} \|\sigma_{\mathbf{h}} a \cdot \bar{a}\|_{\mathbf{I}, k}^{2^k}, \quad k \in \mathbb{N},$$

where $\sigma_{\mathbf{h}} a(\mathbf{n}) := a(\mathbf{n} + \mathbf{h})$ for $\mathbf{h}, \mathbf{n} \in \mathbb{N}^d$.

Then we have (see [48, Proposition 4.5] for $d = 1$)

$$(3.4) \quad \begin{aligned} \|a\|_{U^k(\mathbb{N}^d)} &= \sup_{\mu \text{ invariant probability measure on } X} \|f\|_{\mu, k} \\ &= \sup_{\mu \text{ invariant ergodic probability measure on } X} \|f\|_{\mu, k} \end{aligned}$$

where the last equality follows by using the ergodic decomposition of the measure μ .

³The seminorms were defined for a single transformation but the properties we use extend immediately to the case of several commuting transformations.

3.2. Tools. For an ergodic system (X, μ, T) the structure theorem of [46] links the seminorms $\|\cdot\|_{\mu, k}$ with the factors of the system (X, μ, T) that are $(k-1)$ -step nilsystems. This result was generalized to \mathbb{Z}^d -actions by Griesmer [43, Lemma 4.4.3 and Theorem 4.10.1] and an alternate proof that is based on finitestic inverse theorems was recently given by Tao [65, Remark 4]. We record an immediate corollary of this result that is more convenient for our purposes.

Theorem 3.1 ([46] for $d = 1$, [43, 65] for general d). *Let $d, k \in \mathbb{N}$, $(X, \mu, T_1, \dots, T_d)$ be an ergodic system, $f \in L^\infty(\mu)$, and $\varepsilon > 0$. Then there exists a $(k-1)$ -step nilsystem $(Y, \nu, T_1, \dots, T_d)$, a factor map $\pi: X \rightarrow Y$, and a continuous function Φ on Y such that*

$$\|\Phi\|_\infty \leq \|f\|_\infty \quad \text{and} \quad \|f - \Phi \circ \pi\|_{\mu, k} \leq \varepsilon.$$

The next result can be considered as a strengthening of the correspondence principle of Furstenberg. We recall that a topological dynamical system (Y, T_1, \dots, T_d) is *distal* if for all $y \neq y' \in Y$ we have $\inf_{\mathbf{n} \in \mathbb{Z}^d} d_Y(T_{\mathbf{n}}y, T_{\mathbf{n}}y') > 0$, where d_Y is the distance on Y defining its topology. It is known that every nilsystem is distal [6].

Proposition 3.2 ([48, Proposition 6.1] for $d = 1$). *Let $d \in \mathbb{N}$, (X, T_1, \dots, T_d) be a topological dynamical system, $\omega \in X$ be a transitive point, and μ be an invariant ergodic measure on X . Moreover, let (Y, T_1, \dots, T_d) be a distal topological dynamical system, ν be an invariant measure on Y , and $\pi: X \rightarrow Y$ be a measure theoretic factor map. Then there exists a point $y_0 \in Y$ and a Følner sequence $\mathbf{I} = (I_j)_{j \in \mathbb{N}}$ on \mathbb{N}^d such that*

$$\lim_{j \rightarrow +\infty} \frac{1}{|I_j|} \sum_{\mathbf{n} \in I_j} f(T_{\mathbf{n}}\omega) g(T_{\mathbf{n}}y_0) = \int_X f \cdot g \circ \pi d\mu$$

for every $f \in C(X)$ and every $g \in C(Y)$.

The important point in this statement is that we do not assume that the map π is continuous. The proof is exactly the same as in the case of a single transformation. The classical property of distal systems that we use is:

Theorem 3.3 ([5, Chapter 5]). *Let $d \in \mathbb{N}$ and (Y, T_1, \dots, T_d) be a distal system. Then for every $y_1 \in Y$ and every sequence $(\mathbf{m}_i)_{i \in \mathbb{N}}$ with values in \mathbb{N}^d , there exists $y_0 \in Y$ and a subsequence $(\mathbf{m}'_i)_{i \in \mathbb{N}}$ of $(\mathbf{m}_i)_{i \in \mathbb{N}}$ such that $T_{\mathbf{m}'_i}y_0$ converges to y_1 .*

From Theorem 3.1 and the discussion of Section 3.1 we deduce:

Proposition 3.4. [48, Proposition 6.2 for $d = 1$] *Let $d, k \in \mathbb{N}$, $a \in \ell^\infty(\mathbb{N}^d)$ be a sequence, and $\varepsilon > 0$. Then there exists a Følner sequence $\mathbf{I} = (I_j)_{j \in \mathbb{N}}$ and a $(k-1)$ -step nilsequence ψ_1 in d variables such that the sequences a and $a - \psi_1$ admit correlations along \mathbf{I} and we have*

$$\|a\|_{\mathbf{I}, k} \geq \|a\|_{U^k(\mathbb{N}^d)} - \varepsilon, \quad \|\psi_1\|_\infty \leq \|a\|_\infty, \quad \text{and} \quad \|a - \psi_1\|_{\mathbf{I}, k} \leq \varepsilon.$$

Proof. As explained above, there exists a system (X, T_1, \dots, T_d) , a transitive point $\omega \in X$, and a continuous function f on X such that $a(\mathbf{n}) = f(T_{\mathbf{n}}\omega)$ for every $\mathbf{n} \in \mathbb{N}^d$. Note that then $\|a\|_\infty = \|f\|_\infty$. Moreover, by (3.4) there exists an invariant ergodic probability measure μ on X with $\|f\|_{\mu, k} \geq \|a\|_{U^k(\mathbb{N}^d)} - \varepsilon$.

Let the nilsystem $(Y, \nu, T_1, \dots, T_d)$, the factor map π , and the function Φ be defined as in Theorem 3.1. Let σ be the measure on $X \times Y$ which is the image of μ under the map $\text{id} \times \pi$. This measure is ergodic under the product action and thus admits a generic point (x_1, y_1) . Since ω is a transitive point of X , there exists a sequence $(\mathbf{m}_j)_{j \in \mathbb{N}}$ in \mathbb{N}^d such that $T_{\mathbf{m}_j}\omega \rightarrow x_1$. By Theorem 3.3, substituting a subsequence for the sequence $(\mathbf{m}_j)_{j \in \mathbb{N}}$, we can assume that there exists a point $y_0 \in Y$ such that $T_{\mathbf{m}_j}y_0 \rightarrow y_1$ and thus $(T \times T)_{\mathbf{m}_j}(\omega, x_0) \rightarrow (x_1, y_1)$. Since the point (x_1, y_1) is generic, substituting again

a subsequence for $(\mathbf{m}_j)_{j \in \mathbb{N}}$ and defining the Følner sequence \mathbf{I} by $I_j = \mathbf{m}_j + [j]^d$, $j \in \mathbb{N}$, we obtain that the sequence of probability measures

$$\frac{1}{|I_j|} \sum_{\mathbf{n} \in I_j} \delta_{T_{\mathbf{n}}\omega} \times \delta_{T_{\mathbf{n}}y_0}, \quad j \in \mathbb{N},$$

on $X \times Y$ converges weak* to a probability measure σ . Let the nilsequence ψ_1 be defined by $\psi_1(\mathbf{n}) := \Phi(T_{\mathbf{n}}y_0)$, $\mathbf{n} \in \mathbb{N}^d$. We have $\|\psi_1\|_\infty \leq \|\Phi\|_\infty \leq \|f\|_\infty = \|a\|_\infty$. By applying the preceding discussion to the product system on $X \times Y$, for the point (ω, y_0) , the Følner sequence \mathbf{I} , and the function given by $F(x, y) = f(x)$, we obtain $\|a\|_{\mathbf{I}, k} = \|F\|_{\sigma, k}$. Letting $G(x, y) = f(x) - \Phi(y)$ we have $\|a - \psi_1\|_{\mathbf{I}, k} = \|G\|_{\sigma, k}$. By the definition of σ and of the seminorms $\|\cdot\|_k$ we have $\|F\|_{\sigma, k} = \|f\|_{\mu, k}$ and by the definition of Φ we have $\|G\|_{\sigma, k} = \|f - \Phi \circ \pi\|_{\mu, k} \leq \varepsilon$. This completes the proof. \square

3.3. Anti-uniformity. We introduce certain classes of sequences that are asymptotically approximately orthogonal to k -uniform sequences; in Theorem 3.1 we give a characterization of such sequences in terms of $(k-1)$ -step nilsequences.

Definition. Let $a \in \ell^\infty(\mathbb{N}^d)$. We say that

- The sequence a is *strongly k -anti-uniform* if there exists a constant $C \geq 0$ such that for every $b \in \ell^\infty(\mathbb{N}^d)$ we have

$$(3.5) \quad \limsup |Av a(\mathbf{n}) b(\mathbf{n})| \leq C \|b\|_{U^k(\mathbb{N}^d)}.$$

In this case, we write $\|a\|_{U^k(\mathbb{N}^d)}^*$ for the smallest constant C such that (3.5) holds.

- The sequence a is *k -anti-uniform* if for every $\varepsilon > 0$ there exists $C = C(\varepsilon) \geq 0$ such that for every $b \in \ell^\infty(\mathbb{N}^d)$ we have

$$\limsup |Av a(\mathbf{n}) b(\mathbf{n})| \leq C \|b\|_{U^k(\mathbb{N}^d)} + \varepsilon \|b\|_\infty.$$

Proposition 3.5 ([48, Section 5] for $d = 1$). *Let $d, k \in \mathbb{N}$, (X, T_1, \dots, T_d) be an ergodic $(k-1)$ -step nilsystem, and $f_\varepsilon \in C(X)$ for $\varepsilon \in \llbracket k^* \rrbracket$. Then the limit*

$$(3.6) \quad \Phi(x) := \lim_{H \rightarrow +\infty} \frac{1}{H^{dk}} \sum_{\mathbf{h}_1, \dots, \mathbf{h}_k \in [0, H]^d} \prod_{\varepsilon \in \llbracket k^* \rrbracket} f_\varepsilon(T_{\varepsilon \mathbf{h}} x)$$

exists for every $x \in X$ and the convergence is uniform in $x \in X$ (hence, $\Phi \in C(X)$). If $x_0 \in X$ and $a \in \ell^\infty(\mathbb{N}^d)$ is the sequence defined by

$$a(\mathbf{n}) := \Phi(T_{\mathbf{n}}x_0), \quad \mathbf{n} \in \mathbb{N}^d,$$

then a is strongly k -anti-uniform and

$$\|a\|_{U^k(\mathbb{N}^d)}^* \leq \prod_{\varepsilon \in \llbracket k^* \rrbracket} \|f_\varepsilon\|_k \leq \prod_{\varepsilon \in \llbracket k^* \rrbracket} \|f_\varepsilon\|_\infty.$$

Sketch of the proof. The first part of the result is proved in [48, Corollary 5.2].

To prove the second part, let $b \in \ell^\infty(\mathbb{N}^d)$ and $\mathbf{I} = (I_j)_{j \in \mathbb{N}}$ be a Følner sequence such that b admits correlations along \mathbf{I} . We use (3.6) for $x := T_{\mathbf{n}}x_0$, take the averages for $\mathbf{n} \in I_j$ and exchange the limits in j and in H ; this can be achieved because of the uniform convergence in (3.6) (see [48, Theorem 5.4]). By an iterated use of the Cauchy-Schwarz inequality (this is estimate (12) in [48]) we obtain that

$$(3.7) \quad \limsup |Av_{\mathbf{I}} a(\mathbf{n}) b(\mathbf{n})| \leq \|b\|_{\mathbf{I}, k} \cdot \prod_{\varepsilon \in \llbracket k^* \rrbracket} \|f_\varepsilon\|_k.$$

Taking the supremum over all Følner sequences \mathbf{I} in the left hand side, we obtain the announced bound. For $d = 1$ the details can be found in [48, Section 5.4], the proof for general $d \in \mathbb{N}$ is similar. \square

Let (X, T_1, \dots, T_d) be a $(k-1)$ -step nilsystem defining the nilsequence $\psi: \mathbb{N}^d \rightarrow \mathbb{C}$. By [48, Proposition 5.6] (see also [20, Proposition 3.2]), the linear span of the functions defined as in (3.6) is dense in $C(X)$ with the uniform norm. By Proposition 3.5, the sequence ψ is a uniform limit of strongly k -anti-uniform sequences. We deduce:

Corollary 3.6. *Every $(k-1)$ -step nilsequence is k -anti-uniform.*

Remark. Alternatively, this follows by combining Proposition 4.2 and Proposition 6.1.

For $d = 1$, the first statement of the next Proposition is [48, Theorem 2.16] and the second statement is [30, Theorem 2.1].

Proposition 3.7. *Let $d, k \in \mathbb{N}$, $a \in \ell^\infty(\mathbb{N}^d)$ be a sequence, and $\delta > 0$. Then there exists a Følner sequence $\mathbf{I} = (I_j)_{j \in \mathbb{N}}$ such that the following holds:*

(i) *There exists a $(k-1)$ -step nilsequence $\psi_2: \mathbb{N}^d \rightarrow \mathbb{C}$ such that*

$$\|\psi_2\|_{U^k(\mathbb{N}^d)}^* \leq 1 \quad \text{and} \quad \limsup |\text{Av}_{\mathbf{I}} a(\mathbf{n}) \psi_2(\mathbf{n})| \geq \|a\|_{U^k(\mathbb{N}^d)} - \delta.$$

Moreover, if $b \in \ell^\infty(\mathbb{N}^d)$ is a sequence that admits correlations along a Følner sequence \mathbf{J} , then we have

$$(3.8) \quad \limsup |\text{Av}_{\mathbf{J}} \psi_2(\mathbf{n}) b(\mathbf{n})| \leq \|b\|_{\mathbf{J}, k}.$$

(ii) *If $\|a\|_\infty \leq 1$, then there exists a $(k-1)$ -step nilsequence $\psi_3: \mathbb{N}^d \rightarrow \mathbb{C}$ such that*

$$\|\psi_3\|_\infty \leq 1 \quad \text{and} \quad \limsup |\text{Av}_{\mathbf{I}} a(\mathbf{n}) \psi_3(\mathbf{n})| \geq \|a\|_{U^k(\mathbb{N}^d)}^{2^k} - \delta.$$

Proof. Let the $(k-1)$ -step nilsequence $\psi_1: \mathbb{N}^d \rightarrow \mathbb{C}$ and the Følner sequence \mathbf{I} be given by Proposition 3.4 for an $\varepsilon > 0$ that will be specified later. Then $a - \psi_1$ admits correlations along \mathbf{I} and

$$(3.9) \quad \|a - \psi_1\|_{\mathbf{I}, k} \leq \varepsilon, \quad \|\psi_1\|_{\mathbf{I}, k} \geq \|a\|_{U^k(\mathbb{N}^d)} - 2\varepsilon, \quad \text{and} \quad \|\psi_1\|_\infty \leq \|a\|_\infty.$$

The sequence ψ_1 has the form $\psi_1(\mathbf{n}) = \Phi_1(T_{\mathbf{n}}x_0)$, $\mathbf{n} \in \mathbb{N}^d$, for some $(k-1)$ -step ergodic nilsystem $(X, \mu, T_1, \dots, T_d)$, point $x_0 \in X$, and function $\Phi_1 \in C(X)$ with $\|\Phi_1\|_\infty \leq \|a\|_\infty$. For $x \in X$, we define the function Φ on X as in Proposition 3.5 taking $f_\varepsilon := \mathcal{C}^{|\varepsilon|} \Phi_1$ for $\varepsilon \in \llbracket k^* \rrbracket$. Then $\Phi \in C(X)$ and $\int \Phi \Phi_1 d\mu = \|\Phi_1\|_{\mu, k}^{2^k}$. Let $\psi_3: \mathbb{N}^d \rightarrow \mathbb{C}$ be the nilsequence defined by

$$\psi_3(\mathbf{n}) := \Phi(T_{\mathbf{n}}x_0), \quad \mathbf{n} \in \mathbb{N}^d.$$

If we assume that $\|a\|_\infty \leq 1$, then we have $\|\Phi_1\|_\infty \leq 1$, thus $\|\Phi\|_\infty \leq 1$ and $\|\psi_3\|_\infty \leq 1$. By unique ergodicity of $(X, \mu, T_1, \dots, T_d)$ we have

$$\begin{aligned} \lim \text{Av}_{\mathbf{I}} \psi_1(\mathbf{n}) \psi_3(\mathbf{n}) &= \int \Phi_1 \Phi d\mu = \|\Phi_1\|_{\mu, k}^{2^k} = \|\psi_1\|_{\mathbf{I}, k}^{2^k} \quad \text{by (3.1)} \\ &\geq (\|a\|_{U^k(\mathbb{N}^d)} - 2\varepsilon)^{2^k} \geq \|a\|_{U^k(\mathbb{N}^d)}^{2^k} - \delta/2 \end{aligned}$$

by (3.9) if ε is chosen sufficiently small. Moreover, since $a - \psi_1$ admits correlations along \mathbf{I} , by (3.7) we have

$$\limsup |\text{Av}_{\mathbf{I}} \psi_3(\mathbf{n}) (a - \psi_1)(\mathbf{n})| \leq \|a - \psi_1\|_{\mathbf{I}, k} \|\Phi_1\|_\infty^{2^k - 1} \leq \varepsilon \leq \delta/2$$

by (3.9) if $\varepsilon \leq \delta/2$. Combining the last two estimates we get Part (ii).

We move now to the proof of Part (i). Notice first that using Proposition 3.5 and (3.1) we get that

$$\|\psi_3\|_{U^k(\mathbb{N}^d)}^* \leq \|\Phi_1\|_{\mu, k}^{2^k - 1} = \|\psi_1\|_{U^k(\mathbb{N}^d)}^{2^k - 1}.$$

Moreover, if the sequence $b \in \ell^\infty(\mathbb{N}^d)$ admits correlations along a Følner sequence \mathbf{J} , then (3.7) and (3.1) give

$$\limsup |\text{Av}_{\mathbf{J}} \psi_3(\mathbf{n}) b(\mathbf{n})| \leq \|b\|_{\mathbf{J}, k} \|\Phi_1\|_{\mu, k}^{2^k - 1} = \|b\|_{\mathbf{J}, k} \|\psi_1\|_{U^k(\mathbb{N}^d)}^{2^k - 1}.$$

Defining

$$\psi_2 := (\|\psi_1\|_{U^k(\mathbb{N}^d)}^{2^k-1})^{-1} \cdot \psi_3$$

we deduce that $\|\psi_2\|_{U^k(\mathbb{N}^d)}^* \leq 1$ and estimate (3.8) is satisfied. Furthermore, as before we get

$$\lim \text{Av}_{\mathbf{I}} \psi_1(\mathbf{n}) \psi_2(\mathbf{n}) = \|\psi_1\|_{U^k(\mathbb{N}^d)}^{2^k} / \|\psi_1\|_{U^k(\mathbb{N}^d)}^{2^k-1} = \|\psi_1\|_{U^k(\mathbb{N}^d)} \geq \|a\|_{U^k(\mathbb{N}^d)} - \delta/2$$

by (3.9) if $\varepsilon \leq \delta/4$. Using (3.7) and (3.1) we get

$$\limsup |\text{Av}_{\mathbf{I}} \psi_2(\mathbf{n}) (a - \psi_1)(\mathbf{n})| \leq \|a - \psi_1\|_{\mathbf{I},k} \frac{\|\Phi_1\|_{\mu,k}^{2^k-1}}{\|\psi_1\|_{U^k(\mathbb{N}^d)}^{2^k-1}} = \|a - \psi_1\|_{\mathbf{I},k} \leq \varepsilon \leq \delta/2$$

by (3.9) if $\varepsilon \leq \delta/2$. Combining the last two estimates finishes the proof of Part (i). \square

Corollary 3.8. *Let $d, k \in \mathbb{N}$ and $a \in \ell^\infty(\mathbb{N}^d)$ be such that the averages of $a(\mathbf{n}) \psi(\mathbf{n})$ converge to 0 for every $(k-1)$ -step nilsequence $\psi: \mathbb{N}^d \rightarrow \mathbb{C}$. Then $\|a\|_{U^k(\mathbb{N}^d)} = 0$.*

3.4. Regular sequences and their structure. Next we introduce certain classes of sequences for which we are able to prove the two decomposition results of this section.

Definition. Let $d \in \mathbb{N}$, $k \in \mathbb{Z}_+$. We say that a sequence $a \in \ell^\infty(\mathbb{N}^d)$ is k -regular if the limit

$$\lim \text{Av} a(\mathbf{n}) \psi(\mathbf{n})$$

exists for every k -step nilsequence $\psi: \mathbb{N}^d \rightarrow \mathbb{C}$.

Remarks. • Every nilsequence is k -regular for every $k \in \mathbb{N}$.

- The product of two k -regular sequences may not be k -regular.⁴

Theorem 3.9 ([48, Theorem 2.19] for $d = 1$). *Let $d, k \in \mathbb{N}$ and $a \in \ell^\infty(\mathbb{N}^d)$ be a sequence. Then the following are equivalent:*

- a is $(k-1)$ -regular.
- For every $\delta > 0$, the sequence a can be written as $a = \psi + u$, where ψ is a $(k-1)$ -step nilsequence with $\|\psi\|_\infty \leq \|a\|_\infty$, and $\|u\|_{U^k(\mathbb{N}^d)} \leq \delta$.

Proof. The implication (ii) \implies (i) is a simple consequence of Corollary 3.6 and the fact that the product of two nilsequences is a nilsequence and thus has convergent averages.

We establish now the converse implication. For $\delta > 0$ let $\psi := \psi_1$ and \mathbf{I} be as in Proposition 3.4 with $\varepsilon := \delta/2$. Then we have $\|a - \psi\|_{\mathbf{I},k} \leq \delta/2$. Since $a = \psi + (a - \psi)$, it remains to show that $\|a - \psi\|_{U^k(\mathbb{N}^d)} \leq \delta$. Suppose that this is not the case. Then Part (i) of Proposition 3.7 provides a Følner sequence \mathbf{J} and a $(k-1)$ -step nilsequence ψ_2 with

$$\limsup |\text{Av}_{\mathbf{J}} (a(\mathbf{n}) - \psi(\mathbf{n})) \psi_2(\mathbf{n})| \geq \|a - \psi\|_{U^k(\mathbb{N}^d)} - \delta/3 \geq 2\delta/3.$$

Since a is a $(k-1)$ -regular sequence and $\psi\psi_2$ is a $(k-1)$ -step nilsequence, the sequence $(a - \psi)\psi_2$ has convergent averages. We deduce that

$$|\lim \text{Av}_{\mathbf{I}} (a(\mathbf{n}) - \psi(\mathbf{n})) \psi_2(\mathbf{n})| = |\lim \text{Av}_{\mathbf{J}} (a(\mathbf{n}) - \psi(\mathbf{n})) \psi_2(\mathbf{n})| \geq 2\delta/3.$$

On the other hand, we have

$$|\lim \text{Av}_{\mathbf{I}} (a(\mathbf{n}) - \psi(\mathbf{n})) \psi_2(\mathbf{n})| \leq \|a - \psi\|_{\mathbf{I},k} \leq \delta/2$$

where the second estimate follows from our data and the first estimate follows from the estimate (3.8) since the sequence $a - \psi$ admits correlations along \mathbf{I} . Combining the above we get a contradiction. \square

⁴Let $a(n) := e(n^{k+1}\alpha)$, $n \in \mathbb{N}$, where $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and $b := \bar{a} \cdot \sum_{k=0}^{\infty} \mathbf{1}_{[2^{2k}, 2^{2k+1})}$. The sequences a, b are k -regular for every $k \in \mathbb{N}$, but the sequence $a \cdot b$ is not even 0-regular since the averages $\frac{1}{N} \sum_{n=1}^N a(n) b(n)$ do not converge as $N \rightarrow +\infty$.

3.5. The structure of regular anti-uniform sequences. The next result gives a characterization of regular anti-uniform sequences in $\ell^\infty(\mathbb{N}^d)$. Assuming the multiparameter inverse theorem of Proposition 3.7, its proof is identical to the proof of Theorem 1.2 in [30] where the case $d = 1$ was treated; we only sketch the main idea of the proof below.

Theorem 3.10 ([30, Theorem 1.2] for $d = 1$). *Let $d, k \in \mathbb{N}$ and $a \in \ell^\infty(\mathbb{N}^d)$ be a sequence. Then the following properties are equivalent:*

- (i) *a is $(k - 1)$ -regular and k -anti-uniform.*
- (ii) *a is an approximate $(k - 1)$ -step nilsequence.*

Idea of the proof. The implication (ii) \implies (i) follows from Corollary 3.6 and the fact that a product of two nilsequences is a nilsequence, hence a regular sequence.

We explain now the main idea of the converse implication. Let $a \in \ell^\infty(\mathbb{N}^d)$ be $(k - 1)$ -regular and k -anti-uniform. Let \mathcal{N} be the linear space of $(k - 1)$ -step nilsequences in d variables and let \mathcal{H} be the linear span of \mathcal{N} and a . Then it follows from our $(k - 1)$ -regularity assumption and Theorem 3.9 that for all $c, c' \in \mathcal{H}$ the averages of $c(\mathbf{n}) \overline{c'(\mathbf{n})}$ converge and we write $\langle c, c' \rangle$ for this limit. We remark that $\|c\|_2^2 = \langle c, c \rangle$.

If the space \mathcal{H} endowed with the “scalar product” $\langle \cdot, \cdot \rangle$ was a Hilbert space and \mathcal{N} was a closed subspace, then we could define ψ to be the orthogonal projection of a on \mathcal{N} . Then $\langle a - \psi, \psi' \rangle = 0$ for all $\psi' \in \mathcal{N}$ and Corollary 3.8 would imply that $\|a - \psi\|_{U^k(\mathbb{N}^d)} = 0$. Since, by assumption, a is k -anti-uniform, we deduce that $\|a - \psi\|_2 = 0$, that is, a is a $(k - 1)$ -step nilsequence plus a sequence that converges to 0 in uniform density. In our present setup there is lack of completeness, so we choose ψ to be an “approximate orthogonal projection”, in the sense that $\|a - \psi\|_2$ is sufficiently close to the distance of a to \mathcal{N} , and we obtain using Part (i) of Proposition 3.7 the announced decomposition. For the details see the proof of Theorem [30, Theorem 1.2] which contains a proof for $d = 1$ under the assumption of strong k -anti-uniformity (it is called k -anti-uniformity there); the same argument works without change for general $d \in \mathbb{N}$ under the weaker assumption of k -anti-uniformity. \square

4. CORRELATIONS ARE REGULAR SEQUENCES

The goal of this section is to show that modulo small ℓ^∞ -errors, nilsequences can be represented as multiple correlation sequences, and then use the mean convergence result of Walsh (Theorem 1.1) in order to show that multiple correlation sequences are regular sequences.

4.1. Producing nilsequences as correlations. The argument we use below is analogous to the one used in [30] to handle single variable nilsequences.

Lemma 4.1 ([38, Lemma 14.2]). *Let $d, k \in \mathbb{N}$ and $X = G/\Gamma$ be a $(k - 1)$ -step nilmanifold. Then there exists a continuous map $P: X^k \rightarrow X$ such that*

$$(4.1) \quad P(hg \cdot e_X, h^2g \cdot e_X, \dots, h^k g \cdot e_X) = g \cdot e_X \quad \text{for all } g, h \in G.$$

Remark. The result in [38, Lemma 14.2] gives $P(g\Gamma, hg\Gamma, h^2g\Gamma, \dots, h^{\ell-1}g\Gamma) = h^\ell g\Gamma$. Inserting $h^{-\ell}g$ in place of g , then h^{-1} in place of h , and rearranging coordinates, we get (4.1).

Proposition 4.2. *Let $d, k \in \mathbb{N}$ and $\psi: \mathbb{N}^d \rightarrow \mathbb{C}$ be a $(k - 1)$ -step nilsequence. Then for every $\varepsilon > 0$ there exists a system $(Y, \nu, S_1, \dots, S_d)$ and functions $F_1, \dots, F_k \in L^\infty(\nu)$, such that the sequence $b: \mathbb{N}^d \rightarrow \mathbb{C}$ defined by*

$$(4.2) \quad b(\mathbf{n}) := \int \prod_{j=1}^k \left(\prod_{i=1}^d S_i^{\ell_j n_i} \right) F_j d\nu, \quad \mathbf{n} = (n_1, \dots, n_d) \in \mathbb{N}^d,$$

where $\ell_j = k!/j$ for $j = 1, \dots, k$, satisfies

$$\|\psi - b\|_{\ell^\infty(\mathbb{N}^d)} \leq \varepsilon.$$

Proof. The sequence $\psi: \mathbb{N}^d \rightarrow \mathbb{C}$ has the form

$$\psi(\mathbf{n}) = \Psi\left(\prod_{i=1}^d \tau_i^{n_i} \cdot e_X\right), \quad \mathbf{n} = (n_1, \dots, n_d) \in \mathbb{N}^d,$$

for some $(k-1)$ -step nilmanifold $X = G/\Gamma$, commuting elements τ_1, \dots, τ_d of G , and function $\Psi \in C(X)$. Let $\varepsilon > 0$. As remarked after the definition of the nilsequence in Section 2.1.2, we can assume that the group G is connected.

For $i = 1, \dots, d$, let $g_i \in G$ be such that $g_i^{k!} = \tau_i$ (such elements exist since G is connected, hence divisible). Let $\mathbf{m} = (m_1, \dots, m_d) \in \mathbb{N}^d$. Using Lemma 4.1 with $g := \prod_{i=1}^d g_i^{k!n_i}$ and $h := \prod_{i=1}^d g_i^{m_i}$ and writing $H = \Psi \circ P$, we have $H \in C(X^k)$ and we obtain

$$\psi(\mathbf{n}) = H\left(\left(\prod_{i=1}^d g_i^{m_i+k!n_i}\right), \left(\prod_{i=1}^d g_i^{2m_i+k!n_i}\right), \dots, \left(\prod_{i=1}^d g_i^{km_i+k!n_i}\right)\right) \cdot e_{X^k}$$

for all \mathbf{m} and $\mathbf{n} \in \mathbb{N}^d$. Letting

$$\tilde{\alpha}_i := (g_i, g_i^2, \dots, g_i^k) \in G^k, \quad i = 1, \dots, d,$$

averaging with respect to $\mathbf{m} \in \mathbb{N}^d$, and using the equidistribution results for sequences on nilmanifolds (see [58] or [52]), we get that for every $\mathbf{n} \in \mathbb{N}^d$ we have

$$\begin{aligned} \psi(\mathbf{n}) &= \lim_{M \rightarrow +\infty} \frac{1}{M^d} \sum_{\mathbf{m} \in [M]^d} H\left(\left(\prod_{i=1}^d g_i^{k!n_i}\right), \left(\prod_{i=1}^d g_i^{k!n_i}\right), \dots, \left(\prod_{i=1}^d g_i^{k!n_i}\right)\right) \cdot \tilde{\alpha}_1^{m_1} \cdot \dots \cdot \tilde{\alpha}_d^{m_d} \cdot e_{X^k} \\ &= \int_{\tilde{Y}} H\left(\left(\prod_{i=1}^d g_i^{k!n_i}\right) \cdot x_1, \left(\prod_{i=1}^d g_i^{k!n_i}\right) \cdot x_2, \dots, \left(\prod_{i=1}^d g_i^{k!n_i}\right) \cdot x_k\right) dm_{\tilde{Y}}(x_1, \dots, x_k) \end{aligned}$$

where \tilde{Y} is the closure of the sequence $\{\tilde{\alpha}_1^{m_1} \cdot \dots \cdot \tilde{\alpha}_d^{m_d} \cdot e_{X^k} : m_1, \dots, m_d \in \mathbb{N}\}$ in X^k and $m_{\tilde{Y}}$ is the Haar measure of this sub-nilmanifold of X^k . For $i = 1, \dots, d$, let $\tilde{S}_i: \tilde{Y} \rightarrow \tilde{Y}$ be the translation by $\tilde{\alpha}_i$. Note that $(\tilde{Y}, m_{\tilde{Y}}, \tilde{S}_1, \dots, \tilde{S}_d)$ is a nilsystem.

The continuous function H on X^k can be approximated uniformly by linear combinations of functions of the form $f_1 \otimes \dots \otimes f_k$ where $f_j \in C(X)$ for $j = 1, \dots, k$. Since finite linear combinations of sequences of the form (4.2) have the same form (see the proof of Theorem 2.7 in Section 6.6 below), it remains to show that any sequence $\psi': \mathbb{N}^d \rightarrow \mathbb{C}$, given by

$$\psi'(\mathbf{n}) := \int_{\tilde{Y}} \prod_{j=1}^k f_j\left(\left(\prod_{i=1}^d g_i^{k!n_i}\right) \cdot x_j\right) dm_{\tilde{Y}}(x_1, \dots, x_k), \quad \mathbf{n} = (n_1, \dots, n_d) \in \mathbb{N}^d,$$

has the form (4.2). To this end, for $j = 1, \dots, k$, let $F_j \in C(X^k)$ be given by $F_j(\tilde{x}) := f_j(x_j)$ for $\tilde{x} = (x_1, \dots, x_k) \in X^k$. Recall that $\ell_j = k!/j$ for $j = 1, \dots, k$. Since $\tilde{S}_i \tilde{x} = \tilde{\alpha}_i \tilde{x} = (g_i, g_i^2, \dots, g_i^k) \tilde{x}$, $i = 1, \dots, d$, the j^{th} coordinate of the element $(\prod_{i=1}^d \tilde{S}_i^{\ell_j n_i}) \tilde{x}$ is $\prod_{i=1}^d g_i^{j \ell_j n_i} \cdot x_j = \prod_{i=1}^d g_i^{k!n_i} \cdot x_j$ and thus for $j = 1, \dots, k$ we have

$$F_j\left(\left(\prod_{i=1}^d \tilde{S}_i^{\ell_j n_i}\right) \tilde{x}\right) = f_j\left(\prod_{i=1}^d g_i^{k!n_i} \cdot x_j\right), \quad n_1, \dots, n_d \in \mathbb{N}.$$

Therefore, we have

$$\psi'(\mathbf{n}) = \int_{\tilde{Y}} \prod_{j=1}^k F_j \left(\left(\prod_{i=1}^d \tilde{S}_i^{\ell_j n_i} \right) \tilde{x} \right) dm_{\tilde{Y}}(\tilde{x}), \quad \mathbf{n} = (n_1, \dots, n_d) \in \mathbb{N}^d.$$

This completes the proof. \square

4.2. Proof of Proposition 2.3. Let $d, s \in \mathbb{N}$ and $w \in \ell^\infty(\mathbb{N}^d)$ satisfy the hypothesis of Proposition 2.3. Let ψ be an s -step nilsequence in d variables.

We set $k := s + 1$. Let $\varepsilon > 0$ and let the system $(Y, \nu, S_1, \dots, S_d)$ and the functions $F_1, \dots, F_k \in L^\infty(\nu)$ be as in Proposition 4.2. Letting $h_j := F_{j+1}$, for $j = 0, \dots, s$, the sequence $b: \mathbb{N}^d \rightarrow \mathbb{C}$ defined by (4.2) can be rewritten as

$$b(\mathbf{n}) = \int h_0 \cdot \prod_{j=0}^s S_{(\ell_{j+1} - \ell_1) \cdot \mathbf{n}} h_j d\nu, \quad \mathbf{n} \in \mathbb{N}^d.$$

By hypothesis the averages of $w(\mathbf{n})b(\mathbf{n})$ converge. Since $|b(\mathbf{n}) - \psi(\mathbf{n})|$ is uniformly bounded by ε , the oscillations of the averages of $w(\mathbf{n})\psi(\mathbf{n})$ are uniformly bounded by 2ε . Since this holds for every $\varepsilon > 0$, the averages of $w(\mathbf{n})\psi(\mathbf{n})$ converge, completing the proof. \square

4.3. Proof of Proposition 2.1. By Proposition 4.2 it suffices to prove that the limit

$$\lim \text{Av}_{\mathbf{n}} b(\mathbf{n}) T_{p_1(\mathbf{n})}^{\rightarrow} f_1 \cdot \dots \cdot T_{p_s(\mathbf{n})}^{\rightarrow} f_s$$

exists in $L^2(\mu)$ for every sequence $(b(\mathbf{n}))_{\mathbf{n} \in \mathbb{N}^d}$ defined as in (4.2).

By Theorem 1.1 the limit

$$\lim \text{Av}_{\mathbf{n}} \left(\prod_{j=1}^k F_j \left(\prod_{i=1}^d S_i^{\ell_j n_i} y \right) \cdot \prod_{m=1}^s f_m \left(T_{p_m(\mathbf{n})}^{\rightarrow} x \right) \right)$$

exists in $L^2(\nu \times \mu)$. Taking the integral over Y with respect to ν we obtain the announced result. \square

5. THE STRUCTURE OF SYSTEMS OF ORDER 1

In this section we prove a structural result which is going to be used in the proof of Theorem 2.6 in the next section. Here we work only with systems (X, μ, T) with a single transformation. We denote by $\mathcal{I}(T)$ the σ -algebra of T -invariant subsets of X and write the ergodic decomposition of μ under T as

$$\mu = \int \mu_x d\mu(x),$$

where for μ -a.e. $x \in X$ the measure μ_x is invariant and ergodic under T and the map $x \mapsto \mu_x$ is invariant under T . Let $f \in L^1(\mu)$. Then for μ -a.e. $x \in X$ the function f is defined μ_x -a.e. and belongs to $L^1(\mu_x)$; the map $x \mapsto \int f d\mu_x$ is measurable with respect to $\mathcal{I}(T)$, and we have

$$\mathbb{E}_\mu(f | \mathcal{I}(T))(x) = \int f d\mu_x \quad \text{for } \mu\text{-a.e. } x \in X,$$

meaning that for every set $A \in \mathcal{I}(T)$ we have

$$\int_A f d\mu = \int_A \left(\int f d\mu_x \right) d\mu(x).$$

The factor \mathcal{Z}_1 is defined in Appendix A.2 and reduces to the Kronecker factor for ergodic systems.

Definition. We say that a system (X, μ, T) has *order 1* if $\mathcal{Z}_1 = \mathcal{X}$.

Throughout this section we work only with systems of order 1. Generalizing the construction of a Fourier basis of a compact Abelian group, our goal is to construct a “relative orthonormal” basis for systems of order 1 consisting of “relative” eigenfunctions. This is the context of Theorem 5.2 below.

5.1. Relative orthonormal basis.

Definition. A *relative orthonormal system* (with respect to the T -invariant σ -algebra $\mathcal{I}(T)$) is a countable family $(\phi_j)_{j \in \mathbb{N}}$ of functions belonging to $L^2(\mu)$ such that

- (i) $\mathbb{E}_\mu(|\phi_j|^2 | \mathcal{I}(T))$ has value 0 or 1 μ -a.e. for every $j \in \mathbb{N}$;
- (ii) $\mathbb{E}_\mu(\phi_j \overline{\phi_k} | \mathcal{I}(T)) = 0$ μ -a.e. for all $j, k \in \mathbb{N}$ with $j \neq k$.

The family $(\phi_j)_{j \in \mathbb{N}}$ is a *relative orthonormal basis* if it also satisfies

- (iii) the linear space spanned by all functions of the form $\phi_j \psi$, where $j \in \mathbb{N}$ and $\psi \in L^\infty(\mu)$ varies over all T -invariant functions, is dense in $L^2(\mu)$.

We do not make the apparently natural assumption that $\mathbb{E}_\mu(|\phi_j|^2 | \mathcal{I}(T)) = 1$ μ -a.e., as there does not exist in general a relative orthonormal basis satisfying this additional condition (consider a system with ergodic components given by rotations on cyclic groups of different order). This creates a few minor complications in the statements and the proofs below. We remark that the definition allows some of the elements of the base to be identically 0. This explains why we can assume without loss of generality that all relative orthonormal systems are countably infinite, and thus indexed by \mathbb{N} .

Definition. Given a relative orthonormal system $(\phi_j)_{j \in \mathbb{N}}$ and $f \in L^2(\mu)$ we let

$$f_j := \mathbb{E}_\mu(f \overline{\phi_j} | \mathcal{I}(T)), \quad j \in \mathbb{N}.$$

If $(\phi_j)_{j \in \mathbb{N}}$ is a relative orthonormal basis, we say that the T -invariant functions $f_j, j \in \mathbb{N}$, are the *coordinates* of f in this basis.

Example 5.1. On \mathbb{T}^2 with the Haar measure $m_{\mathbb{T}^2}$ let $T: \mathbb{T}^2 \rightarrow \mathbb{T}^2$ be given by $T(x, y) = (x, y + x)$. Then $(e(jy))_{j \in \mathbb{Z}}$ is a relative orthonormal basis for $L^2(m_{\mathbb{T}^2})$. The coordinates $(f_j)_{j \in \mathbb{Z}}$ of a function $f \in L^2(m_{\mathbb{T}^2})$ are given by $f_j(x) := \int f(x, y) e(-jy) dy, j \in \mathbb{Z}$.

We remark that if $\|f\|_{L^\infty(\mu)} \leq 1$, then $\|f_j\|_{L^\infty(\mu)} \leq 1$ for every $j \in \mathbb{N}$. We will use repeatedly that condition (i) implies the identity

$$(5.1) \quad f_j = f_j \cdot \mathbb{E}_\mu(|\phi_j|^2 | \mathcal{I}(T)) \quad \mu\text{-a.e.}, \quad j \in \mathbb{N};$$

in particular, $f_j(x) = 0$ μ -a.e. on the set where $\mathbb{E}(|\phi_j|^2 | \mathcal{I}(T))(x) = 0$. We remark also that for μ -a.e. $x \in X$ for every $j \in \mathbb{N}$ we have the identities

$$f_j(x) = \int f \overline{\phi_j} d\mu_x \quad \text{and} \quad \mathbb{E}(|f|^2 | \mathcal{I}(T))(x) = \|f\|_{L^2(\mu_x)}^2.$$

Next we establish a relative version of the Parseval and Fourier identity and give necessary and sufficient conditions for a relative orthonormal system to be a relative orthonormal basis.

Proposition 5.1. *Let (X, μ, T) be a system with ergodic decomposition $\mu = \int \mu_x d\mu(x)$ and $(\phi_j)_{j \in \mathbb{N}}$ be a relative orthonormal system for $L^2(\mu)$. Then*

- (i) *For every $f \in L^2(\mu)$ and for μ -a.e. $x \in X$ the series*

$$(5.2) \quad Pf := \sum_{j \in \mathbb{N}} f_j \phi_j$$

converges in $L^2(\mu_x)$ and we have

$$(5.3) \quad \|Pf\|_{L^2(\mu_x)}^2 = \sum_{j \in \mathbb{N}} |f_j(x)|^2 \quad \text{and} \quad \|f\|_{L^2(\mu_x)}^2 = \|Pf\|_{L^2(\mu_x)}^2 + \|f - Pf\|_{L^2(\mu_x)}^2.$$

(ii) $(\phi_j)_{j \in \mathbb{N}}$ is a relative orthonormal basis if and only if for every $f \in L^2(\mu)$ and for μ -a.e. $x \in X$ the series (5.2) converges to f in $L^2(\mu_x)$. In this case, we have

$$(5.4) \quad \|f\|_{L^2(\mu_x)}^2 = \sum_{j \in \mathbb{N}} |f_j(x)|^2 \quad \mu\text{-a.e.}$$

(iii) $(\phi_j)_{j \in \mathbb{N}}$ is a relative orthonormal basis if and only if for every $f \in L^2(\mu)$ we have

$$(5.5) \quad \|f\|_{L^2(\mu)}^2 = \sum_{j \in \mathbb{N}} \|f_j\|_{L^2(\mu)}^2.$$

Proof. We prove (i). Let $f \in L^2(\mu)$; then $f \in L^2(\mu_x)$ for μ -a.e. $x \in X$. Recall that $\int |\phi_j|^2 d\mu_x = \mathbb{E}_\mu(|\phi_j|^2 \mid \mathcal{I}(T))(x) \in \{0, 1\}$ for μ -a.e. $x \in X$. Let $E_x = \{j \in \mathbb{N} : \int |\phi_j|^2 d\mu_x = 1\}$. Then for μ -a.e. $x \in X$ the functions $(\phi_j)_{j \in E_x}$ form an orthonormal system in the Hilbert space $L^2(\mu_x)$ and viewing f as an element of $L^2(\mu_x)$, the functions $(f_j)_{j \in E_x}$ are the coordinates of f in this system. Note also that by (5.1), we have $f_j = 0$ if $j \notin E_x$. This establishes (i).

We prove (ii). Suppose first that for every $f \in L^2(\mu)$ for μ -a.e. $x \in X$ we have $Pf = f$ μ_x -a.e.. It follows that the series in (5.2) converges to f in $L^2(\mu)$. Therefore, f belongs to the closed linear subspace of $L^2(\mu)$ spanned by all functions of the form $\phi_j \psi$ where $j \in \mathbb{N}$ and $\psi \in L^\infty(\mu)$ varies over all T -invariant functions. By definition, $(\phi_j)_{j \in \mathbb{N}}$ is a relative orthonormal system.

We establish now the converse implication. Suppose that $(\phi_j)_{j \in \mathbb{N}}$ is a relative orthonormal basis and let $f \in L^2(\mu)$. For μ -a.e. $x \in X$, the function \overline{Pf} defined by (5.2) satisfies $\int \overline{Pf} \phi_j d\mu_x = f_j$ for every $j \in \mathbb{N}$, hence $\mathbb{E}_\mu((f - Pf) \overline{\phi_j} \mid \mathcal{I}(T)) = 0$ and $\int (f - Pf) \overline{\phi_j} \psi d\mu = 0$ for every T -invariant function $\psi \in L^\infty(\mu)$. Therefore, the function $f - Pf$ is orthogonal in $L^2(\mu)$ to the linear space spanned by all functions of the form $\phi_j \psi$ where $j \in \mathbb{N}$ and $\psi \in L^\infty(\mu)$ varies over all T -invariant functions. Thus, $f - Pf = 0$ by hypothesis. Inserting f in place of Pf in (5.3) gives identity (5.4).

We prove (iii). If $(\phi_j)_{j \in \mathbb{N}}$ is a relative orthonormal basis, then (5.5) follows by integrating the identity in (5.4) over X with respect to μ . To prove the converse implication, let $f \in L^2(\mu)$. Using (5.5), inserting the first identity of (5.3) in the second, and integrating over X with respect to μ , we deduce that $f = Pf$ for μ -a.e. $x \in X$. Hence, Property (iii) of the definition of a relative orthonormal basis is satisfied. \square

5.2. Relative orthonormal basis of eigenfunctions.

Definition. Let $\lambda \in L^\infty(\mu)$ be a T -invariant function and $\phi \in L^\infty(\mu)$. We say that ϕ is an *eigenfunction* with eigenvalue λ if

- (i) $|\phi(x)|$ has value 0 or 1 for μ -a.e. $x \in X$;
- (ii) $\lambda(x) = 0$ for μ -a.e. $x \in X$ such that $\phi(x) = 0$;
- (iii) $\phi \circ T = \lambda \cdot \phi$ μ -a.e..

The role of Property (ii) is to avoid ambiguities. Note also that Property (iii) does not imply anything about the value of $\lambda(x)$ at the points $x \in X$ where $\phi(x) = 0$.

Property (iii) gives that for μ -a.e. $x \in X$ such that $\phi(x) \neq 0$ we have $\phi(T^{-1}x) \neq 0$ and thus $|\phi(x)| = |\phi(T^{-1}x)| = 1$ by Property (i) above and $|\lambda(x)| = 1$ by Property (iii). On the other hand, for μ -a.e. $x \in X$ such that $\phi(x) = 0$ we have $\phi(T^{-1}x) = 0$ by Property (iii) and $\lambda(x) = 0$ by Property (ii). Therefore, the function $|\phi|$ is T -invariant and

$$(5.6) \quad |\phi| = |\lambda|.$$

Next, we state the main result of this section and we prove it in Section 5.4.

Theorem 5.2. *Let (X, μ, T) be a system of order 1. Then $L^2(\mu)$ admits a relative orthonormal basis of eigenfunctions.*

Remarks. • It is true and not difficult to prove that if a system has a relative orthonormal basis of eigenfunctions, then it has order 1.

• For ergodic systems, Theorem 5.2 is well known, but it is not easy to deduce the general case from the ergodic one. The reason is that although the ergodic components of a system of order 1 are ergodic rotations (Proposition A.1), we cannot simply “glue” together their eigenfunctions, because of measurability issues.

Example 5.2. For the system described in the Example 5.1 we have that $(e(jy))_{j \in \mathbb{Z}}$ is a relative orthonormal basis of eigenfunctions with eigenvalues $(e(jx))_{j \in \mathbb{Z}}$.

The next proposition will be used in the proof of Theorem 2.6.

Proposition 5.3. *Let (X, μ, T) be a system of order 1 with ergodic decomposition $\mu = \int \mu_x d\mu(x)$. Suppose that $(\phi_j)_{j \in \mathbb{N}}$ is an orthonormal basis of eigenfunctions, $f \in L^\infty(\mu)$, and let $(f_j)_{j \in \mathbb{N}}$ be the coordinates of f in this base. Then for μ -a.e. $x \in X$ we have*

$$(5.7) \quad \|f\|_{T, \mu_x, 2}^4 = \sum_{j \in \mathbb{N}} |f_j(x)|^4 \quad \mu\text{-a.e.}$$

Remark. After integrating (5.7) over X with respect to μ it follows from (A.3) that $\|f\|_{T, \mu, 2}^4 = \sum_{j \in \mathbb{N}} \|f_j\|_{L^4(\mu)}^4$.

In the proof of Proposition 5.3 we will use the following basic fact:

Lemma 5.4. *Let (X, μ, T) be a system, $(\phi_j)_{j \in \mathbb{N}}$ be a relative orthonormal system of eigenfunctions, and $(\lambda_j)_{j \in \mathbb{N}}$ be the corresponding eigenvalues. Then for μ -a.e. $x \in X$ we have $\lambda_j(x) \overline{\lambda_k(x)} \neq 1$ for all $j, k \in \mathbb{N}$ with $j \neq k$.*

Proof of Lemma 5.4. Note that the set $A = \{x: \lambda_j(x) \overline{\lambda_k(x)} = 1\}$ is T -invariant and the function $\mathbf{1}_A \phi_j \overline{\phi_k}$ is T -invariant by Part (iii) of the definition of an eigenfunction, and thus equal to 0 by Part (ii) of the definition of a relative orthonormal system. On the other hand, for μ -a.e. $x \in A$ we have $|\phi_j(x) \overline{\phi_k(x)}| = 1$ by (5.6). Thus, $\mu(A) = 0$ and the claim follows. \square

Proof of Proposition 5.3. By Part (ii) of Proposition 5.1 we have for μ -a.e. $x \in X$ that

$$f = \sum_{j \in \mathbb{N}} f_j \cdot \phi_j$$

where $f_j = \int f \overline{\phi_j} d\mu_x$ and the convergence takes place in $L^2(\mu_x)$. It follows that for μ -a.e. $x \in X$ for every $n \in \mathbb{N}$ we have

$$T^n f \cdot \overline{f} = \sum_{j, k \in \mathbb{N}} \lambda_j^n f_j \overline{f_k} \phi_j \overline{\phi_k}$$

where convergence takes place in $L^1(\mu_x)$. Using this, identity (5.1), and that $\int \phi_j \overline{\phi_k} d\mu_x = 0$ for $j, k \in \mathbb{N}$ with $j \neq k$, we deduce that for μ -a.e. $x \in X$ we have

$$\int T^n f \cdot \overline{f} d\mu_x = \sum_{j \in \mathbb{N}} \lambda_j^n |f_j|^2.$$

Note that by (5.4) the above series converges absolutely for μ -a.e. $x \in X$. Hence, for μ -a.e. $x \in X$ we have

$$\left| \int T^n f \cdot \overline{f} d\mu_x \right|^2 = \sum_{j, k \in \mathbb{N}} (\lambda_j \overline{\lambda_k})^n |f_j|^2 |f_k|^2.$$

Averaging in $n \in \mathbb{N}$ and using that by (A.2) we have

$$\|f\|_{T, \mu_x, 2}^4 = \lim_{N \rightarrow +\infty} \frac{1}{N} \sum_{n=1}^N \left| \int T^n f \cdot \bar{f} d\mu_x \right|^2,$$

we obtain for μ -a.e. $x \in X$ that

$$\|f\|_{T, \mu_x, 2}^4 = \sum_{j, k \in \mathbb{N}} \left(\lim_{N \rightarrow +\infty} \frac{1}{N} \sum_{n=1}^N (\lambda_j \bar{\lambda}_k)^n \right) \cdot |f_j|^2 |f_k|^2$$

where the interchange of limits is justified because $\sum_{j, k \in \mathbb{N}} |f_j|^2 |f_k|^2$ converges μ -a.e. by (5.4).

If $j \neq k$, since $|\lambda_j \bar{\lambda}_k| \leq 1$ and $\lambda_j \bar{\lambda}_k \neq 1$ by Lemma 5.4, we have for μ -a.e. $x \in X$ that

$$\lim_{N \rightarrow +\infty} \frac{1}{N} \sum_{n=1}^N (\lambda_j \bar{\lambda}_k)^n = 0.$$

Suppose now that $j = k$. For μ -a.e. $x \in X$ the following holds: If $\lambda_j(x) = 0$, then $\phi_j(x) = 0$ by (5.6) and $f_j(x) = 0$ by (5.1). If $\lambda_j(x) \neq 0$, then $|\lambda_j(x)| = 1$ by (5.6) and Part (i) of the definition of an eigenfunction. In both cases we have $|\lambda_j(x)|^{2n} |f_j(x)|^4 = |f_j(x)|^4$. Combining the above we get (5.7). \square

5.3. A Borel selection result. The proof of Theorem 5.2 needs some technical preliminaries. We will need the following selection theorem of Lusin-Novikov.

Theorem 5.5 (see for example [51, Theorem 18.10]). *Let X, Y be Polish (i.e. complete separable metric) spaces and $P \subset X \times Y$ be a Borel set such that every vertical section $P_x = \{y : (x, y) \in P\}$ is a countable set. Then the vertical projection A of P on X is Borel and there exists a Borel function $f : A \rightarrow Y$ such that $f(x) \in P_x$ for every $x \in A$.*

Proposition 5.6. *Let X, K be Polish spaces and $F : X \times K \rightarrow \mathbb{R}_+$ be a bounded Borel function. Suppose that for every $x \in X$ the set*

$$P_x := \{y \in K : F(x, y) > 0\}$$

is countable and

$$(5.8) \quad \Sigma(x) := \sum_{y \in P_x} F(x, y) < +\infty.$$

Let $N(x) := |P_x| \in [0, +\infty]$. Then there exists a sequence $(t_j)_{j \in \mathbb{N}}$ of Borel maps $X \rightarrow K$ such that for every $x \in X$ the values $t_j(x)$, $1 \leq j < 1 + N(x)$, are pairwise distinct and $P_x = \{t_j(x) : 1 \leq j \leq N(x)\}$.

Remarks. • The condition $j < 1 + N(x)$ means $j \leq N(x)$ if $N(x) < +\infty$ and j is arbitrary if $N(x) = +\infty$.

• Note that if $P_x = \emptyset$, then the values of $t_j(x)$ are not determined by the statement.

Proposition 5.6 is a consequence of the next lemma.

Lemma 5.7. *Let F, K, X be as in Proposition 5.6 and for $x \in X$ let*

$$S(x) := \sup_{y \in K} F(x, y).$$

Then there exists a Borel map $t : X \rightarrow K$ such that $F(x, t(x)) = S(x)$ for every $x \in X$.

Proof of Lemma 5.7. Note that (5.8) implies that this supremum $S(x)$ is attained. Let \mathcal{X} and \mathcal{K} be the Borel σ -algebras of the spaces X and K respectively.

We first claim that the function S is Borel. Indeed, let $s \geq 0$. The set $\{(x, y) \in X \times K : F(x, y) > s\}$ belongs to $\mathcal{X} \otimes \mathcal{K}$ and has countable fibers. By Theorem 5.5 its

projection on X belongs to \mathcal{X} . Since this projection is the set $\{x \in X : S(x) > s\}$, the map S is Borel.

For $m \in \mathbb{N}$, let

$$A_m := \{(x, y) \in X \times K : F(x, y) \geq (1 - 2^{-m})S(x)\}.$$

Then A_m belongs to $\mathcal{X} \otimes \mathcal{K}$, the projection of A_m on X is onto, and this projection is countable to one. By Theorem 5.5, for every $m \in \mathbb{N}$ there exists a Borel map $t_m : X \rightarrow K$ such that $(x, t_m(x)) \in A_m$ for every $x \in X$, that is,

$$F(x, t_m(x)) \geq (1 - 2^{-m})S(x) \quad \text{for every } x \in X.$$

If x is such that $S(x) = 0$, then $t_m(x) = 0$ for every $m \in \mathbb{N}$. If not, then $F(x, t_m(x)) \geq S(x)/2$ for every $m \in \mathbb{N}$ and thus the set $\{t_m(x) : m \geq 1\}$ contains at most $2\Sigma(x)S(x)^{-1}$ distinct elements. It follows that for every $x \in X$ the sequence $(t_m(x))_{m \in \mathbb{N}}$ is eventually constant. The limit value $t(x)$ of this sequence is therefore well defined, it is a Borel map from X to K , and satisfies $F(x, t(x)) = S(x)$. This completes the proof. \square

Proof of Proposition 5.6. We build by induction the family of Borel maps $t_j : X \rightarrow K, j \in \mathbb{N}$. Let $t_1 : X \rightarrow K$ be given by Lemma 5.7. We have that $F(x, t_1(x)) = 0$ if and only if P_x is empty, that is, if $N(x) = 0$. We define

$$F'(x, t) := \begin{cases} 0 & \text{if } t = t_1(x); \\ F(x, t) & \text{otherwise,} \end{cases}$$

$$P'_x := \{t : F'(x, t) > 0\} \quad \text{for } x \in X.$$

The function F' is Borel and for every $x \in X$ for which P_x is non-empty, this set is the disjoint union of P'_x and $\{t_1(x)\}$. We replace the function F with F' , and Lemma 5.7 provides a map $t_2 : X \rightarrow K$. Iterating, we obtain a sequence $(t_j)_{j \in \mathbb{N}}$ of Borel maps $X \rightarrow K$ that satisfy

$$F(x, t_1(x)) \geq F(x, t_2(x)) \geq \dots \geq F(x, t_j(x));$$

$$F(x, t_j(x)) = 0 \text{ if and only if } j > N(x);$$

$$\text{if } N(x) \geq j \text{ then } t_{j+1}(x) \notin \{t_1(x), \dots, t_j(x)\};$$

$$F(x, t_{j+1}(x)) = \sup\{F(x, t) : t \notin \{t_1(x), \dots, t_j(x)\}\}.$$

We have $\{t_j(x) : 1 \leq j \leq N(x)\} \subset P_x$ and we claim that equality holds. Suppose that this is not the case and let $t \in P_x \setminus \{t_j(x) : 1 \leq j \leq N(x)\}$. Then, by construction, $t_j(x) \geq t$ for $j \leq N(x)$ and thus $\sigma(x) \geq N(x)t$. It follows that $N(x)$ is finite. By construction, $|\{t_j(x) : 1 \leq j \leq N(x)\}| = N(x) = |P_x|$ which is a contradiction. This completes the proof. \square

5.4. Proof of Theorem 5.2. In this subsection we use a different presentation of the ergodic decomposition of a system that is more convenient for our purposes.⁵ Recall that we assume that (X, \mathcal{X}, μ) is a Lebesgue space. It is well known (see for example [36, Theorems 8.7 and A.7]) that there exists a Lebesgue space (Y, \mathcal{Y}, ν) , a measure preserving map $\pi : X \rightarrow Y$ that satisfies $\pi \circ T = \pi$, $\mathcal{I}(T) = \pi^{-1}(\mathcal{Y})$ up to μ -null sets, and for $y \in Y$ a probability measure μ_y on X such that

- (i) The map $y \mapsto \mu_y$ is Borel, meaning that for every bounded Borel function on X , the function $y \mapsto \int f d\mu_y$ is Borel.
- (ii) For every bounded Borel function on X ,

$$\mathbb{E}_\mu(f \mid (\mathcal{I}(T)))(x) = \int f d\mu_{\pi(x)} \quad \text{for } \mu\text{-a.e. } x \in X.$$

⁵The measures μ_x of the top of Section 5 are written as $\mu_{\pi(x)}$ here.

- (iii) For ν -a.e. $y \in Y$ the measure μ_y on X is concentrated on $\pi^{-1}(\{y\})$ and is invariant and ergodic under T .

Taking the integral in (ii), we obtain

$$\mu = \int_Y \mu_y d\nu(y).$$

By density,

- (iv) For $f \in L^1(\mu)$, we have $f \in L^1(\mu_y)$ for ν -a.e. $y \in Y$, $\|f\|_{L^1(\mu)} = \int \|f\|_{L^1(\mu_y)} d\nu(y)$, and the equality in (ii) remains valid.

Lastly, since our standing assumption in this section is that the system (X, μ, T) has order 1, by Proposition A.1 we have:

- (v) For ν -a.e. $y \in Y$ the system (X, μ_y, T) is an ergodic rotation.

We first prove the following intermediate result:

Lemma 5.8. *Let (X, μ, T) be a system of order 1 and $f \in L^2(\mu)$. Then there exists a relative orthonormal system of eigenfunctions $(\phi_j)_{j \in \mathbb{N}}$ such that*

$$(5.9) \quad f = \sum_{j \in \mathbb{N}} \mathbb{E}_\mu(f \overline{\phi_j} \mid \mathcal{I}(T)) \cdot \phi_j$$

where convergence takes place in $L^2(\mu)$.

Moreover, for every $j \in \mathbb{N}$ the eigenfunction ϕ_j belongs to the smallest closed T -invariant subspace of $L^2(\mu)$ containing the set $\{f\phi: \phi \in L^\infty(\mu) \text{ is } T\text{-invariant}\}$.

Remark. For notational convenience, the orthonormal system we build below is indexed by \mathbb{Z}_+ instead of \mathbb{N} .

Proof. We can assume that f is a Borel function defined everywhere. The set Y_0 of $y \in Y$ such that μ_y is invariant under T is Borel and has full measure. Since the map $y \mapsto \|f\|_{L^2(\mu_y)}$ is Borel, and since $\int \|f\|_{L^2(\mu_y)}^2 d\nu(y) = \|f\|_{L^2(\mu)}^2 < +\infty$, the set Y_1 of points $y \in Y_0$ such that $f \in L^2(\mu_y)$ is Borel and has full measure. Substituting Y_1 for Y , we are reduced to the case where the measure μ_y is T -invariant and $f \in L^2(\mu_y)$ for every $y \in Y$. Since (Y, \mathcal{Y}, ν) is a Lebesgue space, we can assume that Y is a Polish space and \mathcal{Y} is its Borel σ -algebra.

For $y \in Y$, we write σ_y for the spectral measure of f with respect to the system (X, μ_y, T) ; it is the finite positive measure on \mathbb{T} defined by

$$\widehat{\sigma}_y(n) := \int T^n f \cdot \overline{f} d\mu_y, \quad n \in \mathbb{Z}.$$

For ν -a.e. $y \in Y$ this measure is atomic because (X, μ_y, T) is a rotation. For every $t \in \mathbb{T}$ and every $y \in Y$ the limit

$$F(y, t) := \lim_{N \rightarrow +\infty} \frac{1}{N} \sum_{n=1}^N \widehat{\sigma}_y(n) e(-nt)$$

exists and we have

$$F(y, t) = \sigma_y(\{t\}).$$

Since for every $n \in \mathbb{N}$ the function $T^n f \cdot \overline{f}$ is Borel, we get that the map $y \mapsto \widehat{\sigma}_y(n)$ is Borel on Y . Thus, the function F satisfies the hypothesis of Proposition 5.6 for the Polish space $Y \times \mathbb{T}$. Henceforth, we use the notation of this proposition and let $t_j: Y \rightarrow \mathbb{T}$, $j \in \mathbb{N}$, be the Borel maps obtained. For $j \in \mathbb{Z}_+$, let

$$A_j := \{y \in Y: N(y) > j\} = \{y \in Y: \sigma_y(\{t_j(y)\}) > 0\};$$

$$\lambda_j(y) := \mathbf{1}_{A_j}(y) e(t_j(y)) \text{ for } y \in Y.$$

For $y \in A_j, j \in \mathbb{Z}_+$, we have $\lambda_j(y) \in P_y$, that is, $\sigma_y(\{t_j(y)\}) > 0$. By the Wiener-Wintner Theorem, the limit

$$\psi_j(x) := \lim_{N \rightarrow +\infty} \frac{1}{N} \sum_{n=1}^N f(T^n x) \overline{\lambda_j(\pi(x))}^n$$

exists in $L^2(\mu)$ (and for μ -a.e. $x \in X$). We remark that if $\nu(A_j) = 0$, then the function ψ_j is equal to 0 μ -a.e., and the same holds for the functions θ_j and ϕ_j defined below. For ν -a.e. $y \in A_j$ we have

$$(5.10) \quad \int f \overline{\psi_j} d\mu_y = \lim_{N \rightarrow +\infty} \frac{1}{N} \sum_{n=1}^N \int f \cdot T^n \overline{f} d\mu_y \cdot \lambda_j(y)^n \\ = \lim_{N \rightarrow +\infty} \frac{1}{N} \sum_{n=1}^N \widehat{\sigma}_y(-n) \lambda_j(y)^n = \sigma_y(\{t_j(y)\}).$$

Furthermore, for $j \in \mathbb{Z}_+$ the function ψ_j satisfies

$$(5.11) \quad \psi_j(Tx) = \psi_j(x) \lambda_j(\pi(x)).$$

It follows that $|\psi_j|$ is T -invariant. Expressing $|\psi_j|^2$ as

$$(5.12) \quad |\psi_j|^2 = \theta_j \circ \pi, \quad \text{for some } \theta_j \in L^\infty(Y, \nu),$$

a similar computation gives

$$(5.13) \quad \theta_j(y) = \int |\psi_j|^2 d\mu_y = \lim_{N \rightarrow +\infty} \frac{1}{N^2} \sum_{m, n \in [N]} \int T^n f \cdot T^m \overline{f} d\mu_y \cdot \lambda_j(y)^m \overline{\lambda_j(y)}^n \\ = \sigma_y(\{t_j(y)\}) > 0 \text{ for } \nu\text{-a.e. } y \in A_j.$$

We let

$$\phi_j(x) := \begin{cases} |\psi_j(x)|^{-1} \psi_j(x) & \text{for } x \in \pi^{-1}(A_j); \\ 0 & \text{for } x \notin \pi^{-1}(A_j). \end{cases}$$

It is immediate to check that ϕ_j is an eigenfunction for the eigenvalue λ_j . By construction, for $i \neq j$ we have $\lambda_i(x) \neq \lambda_j(x)$ for μ -a.e. $x \in X$ except when $\lambda_i(x) = \lambda_j(x) = 0$; thus $\lambda_i(x) \overline{\lambda_j(x)} \neq 1$ μ -a.e.. On the other hand, $\mathbb{E}_\mu(\phi_i \overline{\phi_j} | \mathcal{I}(T)) = \lambda_i \overline{\lambda_j} \mathbb{E}_\mu(\phi_i \overline{\phi_j} | \mathcal{I}(T))$. Therefore,

$$\mathbb{E}_\mu(\phi_i \cdot \overline{\phi_j} | \mathcal{I}(T)) = 0 \quad \text{for all } i \neq j, i, j \in \mathbb{Z}_+.$$

Furthermore, for every $j \in \mathbb{Z}_+$ by the definition of the ϕ_j we have $\mathbb{E}_\mu(|\phi_j|^2 | \mathcal{I}(T))$ takes the values 0 or 1. Hence, $(\phi_j)_{j \in \mathbb{Z}_+}$ is a relative orthonormal system.

Next we establish identity (5.9). Arguing as in Part (iii) of Proposition 5.1 it suffices to show that

$$(5.14) \quad \|f\|_{L^2(\mu)}^2 = \sum_{j \in \mathbb{Z}_+} \|\mathbb{E}_\mu(f \cdot \overline{\phi_j} | \mathcal{I}(T))\|_{L^2(\mu)}^2.$$

Using the definition of the function ϕ_j and then (5.12) combined with (5.13), we get

$$\sum_{j \in \mathbb{Z}_+} \left| \int f \cdot \overline{\phi_j} d\mu_y \right|^2 = \sum_{j \in \mathbb{Z}_+} \mathbf{1}_{A_j}(y) \frac{1}{|\psi_j|^2} \left| \int f \cdot \overline{\psi_j} d\mu_y \right|^2 = \sum_{j \in \mathbb{Z}_+} \mathbf{1}_{A_j}(y) \frac{1}{\sigma_y(\{t_j(y)\})} \left| \int f \cdot \overline{\psi_j} d\mu_y \right|^2.$$

By (5.10), for ν -a.e. $y \in Y$ the last sum is equal to

$$\sum_{j \in \mathbb{Z}_+} \mathbf{1}_{A_j}(y) \sigma_y(\{t_j(y)\}) = \sum_{j \in \mathbb{Z}_+} \sigma_y(\{t_j(y)\}) = \sigma_y(\mathbb{T}) = \widehat{\sigma}_y(0) = \int |f|^2 d\mu_y,$$

where we used the definition of the set A_j to get the first identity, and that for ν -a.e. $y \in Y$ the measure σ_y is atomic and the defining property of the maps $t_j, j \in \mathbb{Z}_+$, to

get the second identity. Integrating the established identity over Y with respect to ν we obtain (5.14).

Lastly, the last claim of the lemma follows by the construction of the functions ψ_j and ϕ_j for $j \in \mathbb{Z}_+$. This completes the proof. \square

We are now ready to complete the proof of Theorem 5.2.

End of proof of Theorem 5.2. Let $(f_k)_{k \in \mathbb{N}}$ be a dense sequence in $L^2(\mu)$. For every $k \in \mathbb{N}$, we build by induction a countable family \mathcal{F}_k of functions in $L^2(\mu)$ such that for every $k \in \mathbb{N}$ we have

- (i) $\mathcal{F}_1 \cup \dots \cup \mathcal{F}_k$ is a relative orthonormal system of eigenfunctions;
- (ii) the function f_k belongs to the closed subspace \mathcal{H}_k of $L^2(\mu)$ spanned by all functions of the form ϕw where $\phi \in \mathcal{F}_1 \cup \dots \cup \mathcal{F}_k$ and $w \in L^\infty(\mu)$ varies over all T -invariant functions.

For $k = 1$, the result is given by Lemma 5.8 with $f = f_1$. Suppose that the result holds for $k \in \mathbb{N}$, we shall show that it holds for $k + 1$. We can decompose f_{k+1} as

$$f_{k+1} = g + f \quad \text{where } g \in \mathcal{H}_k \text{ and } f \perp \mathcal{H}_k.$$

The space \mathcal{H}_k is invariant under multiplication by bounded T -invariant functions and thus $\mathbb{E}_\mu(f \cdot \bar{h} \mid \mathcal{I}(T)) = 0$ for every $h \in \mathcal{H}_k$. On the other hand, every $\phi \in \mathcal{F}_1 \cup \dots \cup \mathcal{F}_k$ is an eigenfunction, and thus, by definition, $T\phi$ belongs to the space \mathcal{H}_k . It follows that \mathcal{H}_k is invariant under T and thus

$$(5.15) \quad \mathbb{E}_\mu(T^n f \cdot \bar{h} \mid \mathcal{I}(T)) = 0 \quad \text{for every } n \in \mathbb{N} \text{ and every } h \in \mathcal{H}_k.$$

Applying Lemma 5.8 to the function f , we obtain a relative orthonormal system $\mathcal{F}_{k+1} = (\phi_j)_{j \in \mathbb{N}}$ of eigenfunctions such that f belongs to the closed linear span of all functions of the form $\phi_j w$ where $j \in \mathbb{N}$ is arbitrary and $w \in L^\infty(\mu)$ varies over all T -invariant functions. Moreover, for every $j \in \mathbb{N}$, ϕ_j belongs to the smallest T -invariant subspace of $L^2(\mu)$ containing all functions of the form $f w$ where w varies over all T -invariant functions in $L^\infty(\mu)$. Hence, by (5.15), for every $j \in \mathbb{N}$ and every $h \in \mathcal{H}_k$ we have $\mathbb{E}_\mu(\phi_j \cdot \bar{h} \mid \mathcal{I}(T)) = 0$; this holds, in particular, for all functions h belonging to $\mathcal{F}_1 \cup \dots \cup \mathcal{F}_k$. Therefore, $\mathcal{F}_1 \cup \dots \cup \mathcal{F}_k \cup \mathcal{F}_{k+1}$ is a relative orthonormal system.

The closed subspace of $L^2(\mu)$ spanned by functions of the form ϕw , where $\phi \in \mathcal{F}_1 \cup \dots \cup \mathcal{F}_k \cup \mathcal{F}_{k+1}$ and $w \in L^\infty(\mu)$ varies over all T -invariant functions, contains f and g , and thus f_{k+1} . This completes the induction.

We choose an enumeration of the countable set $\mathcal{F} := \bigcup_{k=1}^\infty \mathcal{F}_k$ and write it as $(\phi'_j)_{j \in \mathbb{N}}$; then $(\phi'_j)_{j \in \mathbb{N}}$ is a relative orthonormal system. The closed subspace of $L^2(\mu)$ spanned by all functions of the form $\phi'_j w$, where $j \in \mathbb{N}$ and $w \in L^\infty(\mu)$ varies over all T -invariant functions, contains the functions f_k for every $k \in \mathbb{N}$, and thus is equal to $L^2(\mu)$. Hence, $(\phi'_j)_{j \in \mathbb{N}}$ is a relative orthonormal basis of eigenfunctions, completing the proof. \square

6. DECOMPOSITION OF CORRELATION SEQUENCES

In this section we prove the decomposition results stated in Sections 2.4 and 2.5.

6.1. Anti-uniformity in norm. We start with some preparatory results. The main tool in verifying anti-uniformity properties is the following inner product space variant of a well known estimate of van der Corput (for a proof see [10]):

van der Corput Lemma. *Let $d \in \mathbb{N}$, \mathcal{H} be an inner product space, $\xi: \mathbb{N}^d \rightarrow \mathcal{H}$ be a bounded sequence, and let \mathbf{I} be a Følner sequence in \mathbb{N}^d . Then*

$$\limsup \|\text{Av}_{\mathbf{I}} \xi_{\mathbf{n}}\|^2 \leq 4 \limsup_{H \rightarrow +\infty} \frac{1}{H^d} \sum_{\mathbf{h} \in [H]^d} \limsup |\text{Av}_{\mathbf{n}, \mathbf{I}} \langle \xi_{\mathbf{n}+\mathbf{h}}, \xi_{\mathbf{n}} \rangle|.$$

Proposition 6.1. *Let $d, \ell, s, t \in \mathbb{N}$. Then there exists a positive integer $k = k(d, \ell, s, t)$ such that for every system $(X, \mu, T_1, \dots, T_\ell)$, functions $f_1, \dots, f_s \in L^\infty(\mu)$ bounded by 1, polynomial mappings $\vec{p}_1, \dots, \vec{p}_s: \mathbb{N}^d \rightarrow \mathbb{Z}^\ell$ of degree at most t , and sequence $w \in \ell^\infty(\mathbb{N}^d)$, we have*

$$\limsup \left\| \text{Av } w(\mathbf{n}) T_{\vec{p}_1(\mathbf{n})} f_1 \cdots T_{\vec{p}_s(\mathbf{n})} f_s \right\|_{L^2(\mu)} \leq 4 \|w\|_{U^{k+1}(\mathbb{N}^d)}.$$

Furthermore, if the polynomial mappings are linear, then we can take $k = s$.

Sketch of the proof. Let $\mathbf{I} = (I_i)_{j \in \mathbb{N}}$ be a Følner in \mathbb{N}^d . After passing to a subsequence, we can assume that the sequence w admits correlations along \mathbf{I} . It suffices to show that

$$\limsup \left\| \text{Av}_{\mathbf{I}} w(\mathbf{n}) \cdot T_{\vec{p}_1(\mathbf{n})} f_1 \cdots T_{\vec{p}_s(\mathbf{n})} f_s \right\|_{L^2(\mu)} \leq 4 \|w\|_{\mathbf{I}, k+1}$$

for some $k = k(d, \ell, s, t)$. To verify this, one applies an inductive argument, often called PET induction, introduced by V. Bergelson in [10]. Each step uses the van der Corput Lemma in $L^2(\mu)$, invariance of the measure under some of the transformations, and the Cauchy-Schwarz inequality. The details are similar to several other arguments in the literature (see for example the proof of [33, Lemma 3.5]) and so we do not give the proof.

In the case of linear polynomials, it can be shown by induction on $s \in \mathbb{Z}_+$ that one can take $k = s$; for $s = 0$ the statement is trivial and the inductive step can be carried out as the inductive step in the proof of Theorem 2.6 below. \square

6.2. Proof of Theorem 2.5. By Theorem 3.10 it suffices to show that the sequence $a: \mathbb{N}^d \rightarrow \mathbb{C}$ given by

$$a(\mathbf{n}) := \int f_0 \cdot T_{\vec{p}_1(\mathbf{n})} f_1 \cdots T_{\vec{p}_s(\mathbf{n})} f_s d\mu, \quad \mathbf{n} \in \mathbb{N}^d,$$

is k -regular and $(k+1)$ -anti-uniform for some $k \in \mathbb{N}$ that depends only on the integers d, ℓ, s, t . The regularity is given by Proposition 2.1 and the anti-uniformity by Proposition 6.1. \square

6.3. Proof of Theorem 2.6. We start with some preparatory results.

Lemma 6.2. *Let $d, \ell \in \mathbb{N}$ and $L_1, \dots, L_\ell: \mathbb{N}^d \rightarrow \mathbb{Z}$ be linearly independent linear forms. Then there exists a constant $C := C(d, L_1, \dots, L_\ell)$ such that the following holds: If $(X, \mu, T_1, \dots, T_\ell)$ is a system and $f_0, \dots, f_\ell \in L^\infty(\mu)$ are functions bounded by 1, then*

$$(6.1) \quad \lim \text{Av} \left| \int f_0 \cdot T_1^{L_1(\mathbf{n})} f_1 \cdots T_\ell^{L_\ell(\mathbf{n})} f_\ell d\mu \right|^2 \leq C \min_{1 \leq i \leq \ell} \|f_i\|_{T_i, \mu, 2}^2.$$

Proof. We first note that the limit on the left hand side of (6.1) can be rewritten as

$$\lim \text{Av} \left(\int (f_0 \otimes \overline{f_0}) \cdot \prod_{i=1}^{\ell} (T_i \times T_i)^{L_i(\mathbf{n})} (f_i \otimes \overline{f_i}) d(\mu \times \mu) \right)$$

and thus exists by Theorem 1.1. Therefore, in (6.1) we can restrict to averages taken on the cubes $[N]^d$, $N \in \mathbb{N}$, that is, it suffices to obtain bounds for the following limit

$$(6.2) \quad \lim_{N \rightarrow +\infty} \frac{1}{N^d} \sum_{\mathbf{n} \in [N]^d} \left| \int f_0 \cdot T_1^{L_1(\mathbf{n})} f_1 \cdots T_\ell^{L_\ell(\mathbf{n})} f_\ell d\mu \right|^2.$$

Next, we claim that it suffices to consider the case where $d = \ell$. Indeed, since the linear forms are linearly independent, we have $d \geq \ell$, and if $d > \ell$, then there exist linear forms $L_{\ell+1}, \dots, L_d: \mathbb{N}^d \rightarrow \mathbb{Z}$ such that the linear forms L_1, \dots, L_d are linearly independent. Then applying the $d = \ell$ case of the result for this set of linear forms and the functions f'_0, \dots, f'_d defined by $f'_i := f_i$ for $i = 0, \dots, \ell$ and $f'_i := 1$ for $i = \ell+1, \dots, d$, we get the asserted estimate. Henceforth, we assume that $d = \ell$.

Let $\vec{L}: \mathbb{N}^d \rightarrow \mathbb{N}^d$ be defined by $\vec{L}(\mathbf{n}) := (L_1(\mathbf{n}), \dots, L_d(\mathbf{n}))$, $\mathbf{n} \in \mathbb{N}^d$. Since the linear forms L_1, \dots, L_d are linearly independent, the linear map \vec{L} is injective. Furthermore,

there exists a positive integer $M = M(L_1, \dots, L_d)$ such that $\vec{L}([N]^d) \subset [-MN, MN]^d$ for every $N \in \mathbb{N}$. This easily implies that the limit in (6.2) is bounded by

$$\begin{aligned} & (3M)^d \lim_{N \rightarrow +\infty} \frac{1}{(2N+1)^d} \sum_{\mathbf{n} \in [-N, N]^d} \left| \int f_0 \cdot T_1^{n_1} f_1 \cdots \cdots T_d^{n_d} f_d d\mu \right|^2 \\ &= (3M)^d \lim_{N \rightarrow +\infty} \frac{1}{(2N+1)^d} \sum_{\mathbf{n} \in [-N, N]^d} \int (f_0 \otimes \overline{f_0}) \cdot \prod_{i=1}^d (T_i \times T_i)^{n_i} (f_i \otimes \overline{f_i}) d(\mu \times \mu). \end{aligned}$$

By the ergodic theorem, the last limit is equal to

$$\int (f_0 \otimes \overline{f_0}) \cdot \prod_{i=1}^d \mathbb{E}_\mu(f_i \otimes \overline{f_i} \mid \mathcal{I}(T_i \times T_i)) d(\mu \times \mu).$$

For $i = 1, \dots, d$ this quantity is bounded by

$$\|\mathbb{E}_\mu(f_i \otimes \overline{f_i} \mid \mathcal{I}(T_i \times T_i))\|_{L^2(\mu)} = \|f_i \otimes \overline{f_i}\|_{T_i \times T_i, \mu \times \mu, 1} \leq \|f_i\|_{T_i, \mu, 2}^2$$

where we used that all the functions are bounded by 1 and the estimate (A.5). This completes the proof. \square

Proposition 6.3. *Let $d, \ell \in \mathbb{N}$ and $L_1, \dots, L_\ell: \mathbb{N}^d \rightarrow \mathbb{Z}$ be linearly independent linear forms. Then for every $\varepsilon > 0$ there exists a constant $C := C(d, L_1, \dots, L_\ell, \varepsilon) > 0$ such that the following holds: If $(X, \mu, T_1, \dots, T_\ell)$ is a system and $f_0, \dots, f_\ell \in L^\infty(\mu)$ are functions bounded by 1, then for every sequence $w \in \ell^\infty(\mathbb{N}^d)$ we have*

$$(6.3) \quad \limsup \left| \text{Av} \left(w(\mathbf{n}) \cdot \int f_0 \cdot T_1^{L_1(\mathbf{n})} f_1 \cdots \cdots T_\ell^{L_\ell(\mathbf{n})} f_\ell d\mu \right) \right| \leq C \|w\|_{U^2(\mathbb{N}^d)} + \varepsilon \|w\|_\infty.$$

Remark. This proves that the correlation sequence defined by the integral is 2-anti-uniform with constants C that do not depend on the functions f_0, \dots, f_ℓ as long as they are bounded by 1. This condition is essential in the proof of Theorem 2.6 and Theorem 5.2 is key in establishing the condition.

Proof. Let $\varepsilon \in (0, 1)$, $w \in \ell^\infty(\mathbb{N}^d)$, and $\mathbf{I} = (I_j)_{j \in \mathbb{N}}$ be a Følner sequence in \mathbb{N}^d . By passing to a Følner subsequence we can assume that w admits correlations along \mathbf{I} .

By the defining property of the factor \mathcal{Z}_1 (see Section A.2) we have that $\|f_i - \mathbb{E}_\mu(f_i \mid \mathcal{Z}_1(X, \mu, T_i))\|_{\mu, T_i, 2} = 0$ for $i = 1, \dots, \ell$. Hence, by Lemma 6.2, the limsup in (6.3) remains unchanged if we replace each function f_i with $\mathbb{E}_\mu(f_i \mid \mathcal{Z}_1(X, \mu, T_i))$. Therefore, we can and will assume that for $i = 1, \dots, \ell$, the function f_i is measurable with respect to $\mathcal{Z}_1(X, \mu, T_i)$. For $i = 1, \dots, \ell$ let $\mu = \int \mu_{i,x} d\mu(x)$ be the ergodic decomposition of the system (X, μ, T_i) .

By Theorem 5.2, for $i = 1, \dots, \ell$ the space $L^2(\mathcal{Z}_1(X, \mu, T_i), \mu)$ admits a relative orthonormal basis $(\phi_{i,j})_{j \in \mathbb{N}}$ such that $\phi_{i,j}$ is an eigenfunction of (X, μ, T_i) with eigenvalue $\lambda_{i,j}$ for $j \in \mathbb{N}$. We write $(f_{i,j})_{j \in \mathbb{N}}$ for the coordinates of f_i in this base. We recall that $f_{i,j}$ is invariant under T_i and that

$$f_{i,j} = \mathbb{E}_\mu(f_i \overline{\phi_{i,j}} \mid \mathcal{I}(T_i)).$$

Then $\|f_{i,j}\|_{L^\infty(\mu)} \leq 1$ for all $i, j \in \mathbb{N}$. Moreover, by Part (ii) of Proposition 5.1 we have

$$(6.4) \quad \mathbb{E}_\mu(|f_i|^2 \mid \mathcal{I}(T_i)) = \sum_{j \in \mathbb{N}} |f_{i,j}|^2, \quad \mu\text{-a.e.};$$

$$(6.5) \quad f_i = \sum_{j \in \mathbb{N}} f_{i,j} \phi_{i,j},$$

where convergence in (6.4) is pointwise and in (6.5) is in $L^2(\mu)$ and in $L^2(\mu_{i,x})$ μ -a.e..

For $i = 1, \dots, \ell$ we separate the series (6.5) in two parts. For $x \in X$ we let

$$E_i(x) := \{j \in \mathbb{N} : |f_{i,j}(x)|^2 \geq \varepsilon^{10^i}\};$$

$$g_i(x) := \sum_{j \in E_i(x)} f_{i,j}(x) \phi_{i,j}(x) \quad \text{and} \quad h_i := f_i - g_i.$$

By (6.4) and since all functions are bounded by 1, we have

$$(6.6) \quad |E_i(x)| \leq \varepsilon^{-10^i} \quad \mu\text{-a.e.}, \quad \|g_i\|_{L^\infty(\mu)} \leq \varepsilon^{-10^i}, \quad i = 1, \dots, \ell.$$

Furthermore, since $f_{i,j}$ are T_i -invariant we have $E_i(T_i x) = E_i(x)$ μ -a.e. and the set $A_{i,j} = \{x \in X : j \notin E_i(x)\}$ is invariant under T_i . We have

$$h_i = \sum_{j \in \mathbb{N}} \mathbf{1}_{A_{i,j}} f_{i,j} \phi_{i,j}$$

and thus the coordinates of the function h_i in the base $(\phi_{i,j})_{j \in \mathbb{N}}$ are the functions $\mathbf{1}_{A_{i,j}} f_{i,j}$. By Proposition 5.3, we obtain

$$(6.7) \quad \|h_i\|_{T_i, \mu_{i,x,2}}^4 = \sum_{j \in \mathbb{N}} |\mathbf{1}_{A_{i,j}}(x) f_{i,j}(x)|^4 = \sum_{j \notin E_i(x)} |f_{i,j}(x)|^4 \quad \mu\text{-a.e.}, \quad i = 1, \dots, \ell.$$

We therefore have

$$\|h_i\|_{T_i, \mu, 2}^4 = \int \|h_i\|_{T_i, \mu_{i,x,2}}^4 d\mu = \int \sum_{j \notin E_i(x)} |f_{i,j}(x)|^4 d\mu \leq \varepsilon^{10^i} \int \sum_{j \in \mathbb{N}} |f_{i,j}(x)|^2 d\mu \leq \varepsilon^{10^i},$$

where we used (A.3) in the appendix to get the first identity, (6.7) to get the second identity, the definition of the sets $E_i(x)$ to get the first estimate, and (6.4) combined with the fact that the functions f_i are bounded by 1 to get the last estimate.

Let C be the constant defined in Lemma 6.2. Combining this lemma with the preceding estimates we deduce for $m = 1, \dots, \ell$ that

$$\begin{aligned} \limsup \text{Av}_{\mathbf{I}} \left| \int f_0 \cdot \left(\prod_{i=1}^{m-1} T_i^{L_i(\mathbf{n})} g_i \right) \cdot T_m^{L_m(\mathbf{n})} h_m \cdot \left(\prod_{i=m+1}^{\ell} T_i^{L_i(\mathbf{n})} f_i \right) d\mu \right|^2 \\ \leq C \prod_{i=1}^{m-1} \|g_i\|_{L^\infty(\mu)}^2 \cdot \|h_m\|_{T_m, \mu, 2}^2 \leq C \varepsilon^{-2 \sum_{i=1}^{m-1} 10^i} \varepsilon^{5 \cdot 10^{m-1}} \leq C \varepsilon^2. \end{aligned}$$

Using the Cauchy-Schwarz inequality and telescoping, we obtain

$$\limsup \left| \text{Av}_{\mathbf{I}} \left(w(\mathbf{n}) \int f_0 \cdot \prod_{i=1}^{\ell} T_i^{L_i(\mathbf{n})} f_i d\mu - w(\mathbf{n}) \int f_0 \cdot \prod_{i=1}^{\ell} T_i^{L_i(\mathbf{n})} g_i d\mu \right) \right| \leq \ell C^{1/2} \varepsilon \|w\|_\infty.$$

On the other hand, using the definition of the functions g_1, \dots, g_ℓ and recalling that for $i = 1, \dots, \ell$ and $j \in \mathbb{N}$ the function $\phi_{i,j}$ is a T_i -eigenfunction with eigenvalue $\lambda_{i,j}$, we get

$$\begin{aligned} \int f_0 \cdot \prod_{i=1}^{\ell} T_i^{L_i(\mathbf{n})} g_i d\mu &= \int \sum_{j_1 \in E_1, \dots, j_\ell \in E_\ell} f_0 \cdot \prod_{i=1}^{\ell} f_{i,j_i} \cdot T_i^{L_i(\mathbf{n})} \phi_{i,j_i} d\mu \\ &= \int \sum_{j_1 \in E_1, \dots, j_\ell \in E_\ell} g_{j_1, \dots, j_\ell} \cdot \prod_{i=1}^{\ell} \lambda_{i,j_i}^{L_i(\mathbf{n})} d\mu \end{aligned}$$

where

$$g_{j_1, \dots, j_\ell} := f_0 \cdot \prod_{i=1}^{\ell} f_{i,j_i} \cdot \phi_{i,j_i}.$$

Since for μ -a.e. $x \in X$ we have by (6.6) that $|E_i(x)| \leq \varepsilon^{-10^i}$ for $i = 1, \dots, \ell$, we deduce that for μ -a.e. $x \in X$ the sum contains at most $\varepsilon^{-10^{\ell+1}}$ terms. Moreover, since L_1, \dots, L_ℓ are linear, using the van der Corput Lemma on \mathbb{C} and the fact that the functions g_{j_1, \dots, j_ℓ} are bounded by 1, we have the pointwise estimate

$$\begin{aligned} \limsup \left| \text{Av}_{\mathbf{I}} w(\mathbf{n}) \cdot \prod_{i=1}^{\ell} \lambda_{i, j_i}^{L_i(\mathbf{n})} \right|^2 &\leq 4 \limsup_{H \rightarrow +\infty} \frac{1}{H^d} \sum_{\mathbf{h} \in [H]^d} \left| \lim \text{Av}_{\mathbf{n}, \mathbf{I}} w(\mathbf{n} + \mathbf{h}) \overline{w(\mathbf{n})} \right| \\ &\leq 4 \lim_{H \rightarrow +\infty} \frac{1}{H^d} \sum_{\mathbf{h} \in [H]^d} \|\sigma_{\mathbf{h}} w \cdot \overline{w}\|_{\mathbf{I}, 1} \leq 4 \|w\|_{\mathbf{I}, 2}^2 \leq 4 \|w\|_{U^2(\mathbb{N}^d)}^2 \end{aligned}$$

where we used (3.2) and (3.3). Combining the above estimates, we deduce that the left hand side of (6.3) is bounded by $2\varepsilon^{-10^{\ell+1}} \|w\|_{U^2(\mathbb{N}^d)} + \ell C^{1/2} \varepsilon \|w\|_{\infty}$. This completes the proof. \square

End of proof of Theorem 2.6. We can assume that the functions f_0, \dots, f_ℓ are bounded by 1. Furthermore, we can extract a linearly independent subset of r elements of $\{L_1, \dots, L_\ell\}$; hence, after reordering the linear forms we can assume that the first r ones are linearly independent.

Let $k := \ell - r + 1$. In order to prove that the sequence $a: \mathbb{N}^d \rightarrow \mathbb{C}$ defined by

$$a(\mathbf{n}) := \int f_0 \cdot T_1^{L_1(\mathbf{n})} f_1 \cdot \dots \cdot T_\ell^{L_\ell(\mathbf{n})} f_\ell d\mu, \quad \mathbf{n} \in \mathbb{N}^d,$$

admits a decomposition of the announced type, it suffices by Theorem 3.10 to show that it is k -regular and $(k+1)$ -anti-uniform.

The regularity follows from Proposition 2.1.

Next we verify $(k+1)$ -anti-uniformity. Let $C = C(d, L_1, \dots, L_r, \varepsilon)$ be the constant defined by Proposition 6.3. We can assume that $C > 1$. For fixed $d \in \mathbb{N}$ we will prove by induction on $k \in \mathbb{N}$ that if the functions f_0, \dots, f_ℓ are bounded by 1, then for every $\varepsilon > 0$ and every $w \in \ell^\infty(\mathbb{N}^d)$ with $\|w\|_{\infty} = 1$ we have

$$\limsup \left| \text{Av} a(\mathbf{n}) w(\mathbf{n}) \right| \leq 4C \|b\|_{U^{k+1}(\mathbb{N}^d)} + 4\varepsilon^{\frac{1}{2^{k-1}}}.$$

It will then follow that the sequence a is $(k+1)$ -anti-uniform with anti-uniformity constant $C' := 4C(d, L_1, \dots, L_r, \frac{1}{4}\varepsilon^{2^{k-1}})$.

For $k = 1$ the statement follows from Proposition 6.3.

Suppose that $k \geq 2$ and that the statement holds for $k-1$. Let $\varepsilon > 0$ and $w \in \ell^\infty(\mathbb{N}^d)$. Let also $\mathbf{I} = (I_j)_{j \in \mathbb{N}}$ be a Følner sequence in \mathbb{N}^d . By passing to a Følner subsequence we can assume that w admits correlations along \mathbf{I} .

Composing with $T_\ell^{-L_\ell(\mathbf{n})}$ we rewrite the sequence $a: \mathbb{N}^d \rightarrow \mathbb{C}$ as⁶

$$a(\mathbf{n}) = \int T_\ell^{-L_\ell(\mathbf{n})} f_0 \cdot T_\ell^{-L_\ell(\mathbf{n})} T_1^{L_1(\mathbf{n})} f_1 \cdot \dots \cdot T_\ell^{-L_\ell(\mathbf{n})} T_{\ell-1}^{L_{\ell-1}(\mathbf{n})} f_{\ell-1} \cdot f_\ell d\mu, \quad \mathbf{n} \in \mathbb{N}^d.$$

Using the Cauchy-Schwarz inequality and that $\|f_\ell\|_{L^\infty(\mu)} \leq 1$, we get that

$$(6.8) \quad \limsup \left| \text{Av}_{\mathbf{I}} a(\mathbf{n}) w(\mathbf{n}) \right|^2 \leq \limsup \left\| \text{Av}_{\mathbf{I}} \xi_{\mathbf{n}} \right\|_{L^2(\mu)}^2$$

where $\xi: \mathbb{N}^d \rightarrow L^2(\mu)$ is defined by

$$\xi_{\mathbf{n}} := w(\mathbf{n}) \cdot T_\ell^{-L_\ell(\mathbf{n})} f_0 \cdot T_\ell^{-L_\ell(\mathbf{n})} T_1^{L_1(\mathbf{n})} f_1 \cdot \dots \cdot T_\ell^{-L_\ell(\mathbf{n})} T_{\ell-1}^{-L_{\ell-1}(\mathbf{n})} f_{\ell-1}, \quad \mathbf{n} \in \mathbb{N}^d.$$

⁶This maneuver is necessary; a direct application of the van der Corput Lemma produces a weaker estimate involving a seminorm of order $k+2$.

Using van der Corput's Lemma in $L^2(\mu)$ for the sequence ξ , we obtain that the right hand side of (6.8) is bounded by

$$(6.9) \quad 4 \limsup_{H \rightarrow +\infty} \frac{1}{H^d} \sum_{\mathbf{h} \in [H]^d} \limsup |Av_{\mathbf{n}, \mathbf{I}} \langle \xi_{\mathbf{n}+\mathbf{h}}, \xi_{\mathbf{n}} \rangle|.$$

Recall that $\sigma_{\mathbf{h}} w(\mathbf{n}) = w(\mathbf{n} + \mathbf{h})$ for all $\mathbf{h}, \mathbf{n} \in \mathbb{N}^d$. A simple computation gives that for every $\mathbf{h} \in \mathbb{N}^d$ we have

$$(6.10) \quad \frac{1}{|I_j|} \sum_{\mathbf{n} \in I_j} \langle \xi_{\mathbf{n}+\mathbf{h}}, \xi_{\mathbf{n}} \rangle = \frac{1}{|I_j|} \sum_{\mathbf{n} \in I_j} \sigma_{\mathbf{h}} w(\mathbf{n}) \cdot \overline{w(\mathbf{n})} \int \tilde{f}_{0,\mathbf{h}} \cdot T_1^{L_1(\mathbf{n})} \tilde{f}_{1,\mathbf{h}} \cdot \dots \cdot T_{\ell-1}^{L_{\ell-1}(\mathbf{n})} \tilde{f}_{\ell-1} d\mu$$

where $\tilde{f}_{j,\mathbf{h}} := T_{\ell}^{-L_{\ell}(\mathbf{h})} T_j^{L_j(\mathbf{h})} f_j \cdot \overline{f_j}$ for $j = 0, \dots, \ell - 1$ and $L_0 := 0$.

Note for every $\mathbf{h} \in \mathbb{N}^d$ the sequence $(w(\mathbf{n} + \mathbf{h}) \overline{w(\mathbf{n})})_{\mathbf{n} \in \mathbb{N}^d}$ admits correlations along \mathbf{I} and that $\|\tilde{f}_{j,\mathbf{h}}\|_{L^\infty(\mu)} \leq 1$ for $j = 0, \dots, \ell - 1$. The expression on the right hand side of (6.10) is thus of the type studied, with $(\ell - 1)$ in place of ℓ . Hence, the induction hypothesis applies and gives that for every $\mathbf{h} \in \mathbb{N}^d$ we have

$$\limsup |Av_{\mathbf{n}, \mathbf{I}} \langle \xi_{\mathbf{n}+\mathbf{h}}, \xi_{\mathbf{n}} \rangle| \leq 4C \|\sigma_{\mathbf{h}} w \cdot w\|_{\mathbf{I},k} + 4\varepsilon^{\frac{1}{2^{k-2}}}.$$

It is important for the last part of the argument that the constant C is independent of the parameter \mathbf{h} . Combining the above, we deduce that the expression (6.9) is bounded by

$$\begin{aligned} & 16C \limsup_{H \rightarrow +\infty} \frac{1}{H^d} \sum_{\mathbf{h} \in [H]^d} \|\sigma_{\mathbf{h}} w \cdot \overline{w}\|_{\mathbf{I},k} + 16\varepsilon^{\frac{1}{2^{k-2}}} \leq \\ & 16C \limsup_{H \rightarrow +\infty} \left(\frac{1}{H^d} \sum_{\mathbf{h} \in [H]^d} \|\sigma_{\mathbf{h}} w \cdot \overline{w}\|_{\mathbf{I},k}^{2^k} \right)^{\frac{1}{2^k}} + 16\varepsilon^{\frac{1}{2^{k-2}}} = 16C \|w\|_{\mathbf{I},k+1}^2 + 16\varepsilon^{\frac{1}{2^{k-2}}} \end{aligned}$$

where the last identity follows from the inductive property (3.3) of the uniformity seminorms. Putting together the previous estimates and taking square roots we get the announced bound. This completes the induction and the proof. \square

6.4. Proof of Theorem 2.8. We use the variant of Furstenberg's correspondence principle already used in Section 3.1 for a single sequence. In the case of several sequences it is proved in a similar fashion and gives the following:

Proposition 6.4. *Let $\ell, s \in \mathbb{N}$ and $a_1, \dots, a_s: \mathbb{Z}^\ell \rightarrow \mathbb{C}$ be bounded sequences such that the family $\mathcal{F} = \{a_1, \dots, a_s\}$ admits correlations along the Følner sequence \mathbf{I} in \mathbb{N}^ℓ . Then there exists a topological dynamical system (X, T_1, \dots, T_ℓ) , where T_1, \dots, T_ℓ are commuting homeomorphisms, functions $f_1, \dots, f_s \in C(X)$, and a Borel probability measure μ on X that is T_i -invariant for $i = 1, \dots, \ell$, such that*

$$\int T_{\vec{n}_1} f_1 \cdot \dots \cdot T_{\vec{n}_s} f_s d\mu = \lim Av_{\mathbf{k}, \mathbf{I}} \left(\prod_{i=1}^s a_i(\mathbf{k} + \vec{n}_i) \right)$$

for every $\vec{n}_1, \dots, \vec{n}_s \in \mathbb{Z}^\ell$.

Combining this with Theorem 2.5 we immediately deduce Theorem 2.8. \square

6.5. Proof of Theorem 2.9. We start by recalling the definition of the Gowers norms in \mathbb{Z}_N^d .

Definition. Let $d, N \in \mathbb{N}$ and $f: \mathbb{Z}_N^d \rightarrow \mathbb{C}$ be a function. For every $\mathbf{h} \in \mathbb{Z}_N^d$ we write $f_{\mathbf{h}}(\mathbf{n}) := f(\mathbf{n} + \mathbf{h})$, $\mathbf{n} \in \mathbb{Z}_N^d$. For $s \in \mathbb{N}$ we denote by $\|f\|_{U^s(\mathbb{Z}_N^d)}$ the Gowers $U^s(\mathbb{Z}_N^d)$ -norm of f that is defined inductively as follows: We let

$$\|f\|_{U^1(\mathbb{Z}_N^d)} := |\mathbb{E}_{\mathbf{n} \in \mathbb{Z}_N^d} f(\mathbf{n})| = \left(\mathbb{E}_{\mathbf{n}, \mathbf{h} \in \mathbb{Z}_N^d} f(\mathbf{n}) \overline{f(\mathbf{n} + \mathbf{h})} \right)^{1/2},$$

and for every $s \geq 1$ we let

$$\|f\|_{U^{s+1}(\mathbb{Z}_N^d)} := \left(\mathbb{E}_{\mathbf{h} \in \mathbb{Z}_N^d} \|f \cdot \overline{f_{\mathbf{h}}}\|_{U^s(\mathbb{Z}_N^d)}^{2^s} \right)^{1/2^{s+1}}.$$

For $d = 1$ the next result is deduced in [49, Section 3.1] from the inverse theorem for the Gowers norms in \mathbb{Z}_N [42]. A multidimensional extension of this inverse theorem was established in [18, 63] (for an alternate proof see [65]) and the argument in [49, Section 3.1] allows us to deduce the following result:

Proposition 6.5 ([49, Section 3.1] for $d = 1$). *Let $d, k \in \mathbb{N}$, $C > 0$, and $\varepsilon > 0$. Then there exists a k -step nilmanifold X such that the following holds: If $N \in \mathbb{N}$ and $a: \mathbb{Z}_N^d \rightarrow \mathbb{C}$ satisfies*

$$(6.11) \quad \left| \mathbb{E}_{\mathbf{n} \in \mathbb{Z}_N^d} a(\mathbf{n}) b(\mathbf{n}) \right| \leq C \|b\|_{U^{k+1}(\mathbb{Z}_N^d)} \quad \text{for every } b: \mathbb{Z}_N^d \rightarrow \mathbb{C},$$

then we have the decomposition $a = a_{\text{st}} + a_{\text{er}}$ where

- (i) a_{st} is a convex combination of k -step nilsequences defined by functions on X with Lipschitz norm at most 1;
- (ii) $\mathbb{E}_{\mathbf{n} \in \mathbb{Z}_N^d} |a_{\text{er}}(\mathbf{n})| \leq \varepsilon$.

It follows from the previous result that in order to complete the proof of Theorem 2.9 it suffices to show that the sequence $b: \mathbb{Z}_N^d \rightarrow \mathbb{C}$ defined by (2.9) satisfies the property (6.11) for some $k \in \mathbb{N}$ and $C > 0$ that depend only on the integers d, ℓ, s, t . This can be verified directly with $C = 1$ by using a PET-induction argument (as in [33, Lemma 3.5]) and the estimate $|\mathbb{E}_{\mathbf{n} \in \mathbb{Z}_N^d} a(\mathbf{n})|^2 \leq \mathbb{E}_{\mathbf{h} \in \mathbb{Z}_N^d} |\mathbb{E}_{\mathbf{n} \in \mathbb{Z}_N^d} a(\mathbf{n} + \mathbf{h}) \overline{a(\mathbf{n})}|$, but it turns out to be simpler to deduce (6.11) directly from Proposition 6.1 as follows:

For fixed $N \in \mathbb{N}$ we interpret $b: \mathbb{Z}_N^d \rightarrow \mathbb{C}$ as a periodic sequence in \mathbb{N}^d , and note that b can be represented as

$$b(\mathbf{n}) = \int T_{\vec{p}_1(\mathbf{n})} f_1 \cdots \cdots T_{\vec{p}_s(\mathbf{n})} f_s d\mu, \quad \mathbf{n} \in \mathbb{N}^d,$$

where $X := \mathbb{Z}_N^\ell$, μ is the Haar measure on X , and for $i = 1, \dots, s$, $f_i := a_i$ and T_i is the measure preserving transformation on \mathbb{Z}_N^ℓ (with the Haar measure) that shifts the i -th coordinate of an element of \mathbb{Z}_N^ℓ by 1 and leaves the other coordinates unchanged. Notice also that if the sequence $c: \mathbb{N}^d \rightarrow \mathbb{N}$ is N -periodic on every coordinate direction, then

$$\lim \text{Av } c(\mathbf{n}) = \mathbb{E}_{\mathbf{n} \in \mathbb{Z}_N^d} c(\mathbf{n}) \quad \text{and} \quad \|c\|_{U^k(\mathbb{Z}^d)} = \|c\|_{U^k(\mathbb{Z}_N^d)}$$

for every $k \in \mathbb{N}$. Keeping all these in mind and using Proposition 6.1, we deduce that the estimate (6.11) holds with $C = 4$ for some $k = k(d, \ell, s, t)$ and we can take $k = s$ if the polynomial mappings are linear. This completes the proof of Theorem 2.9. \square

6.6. Proof of Theorem 2.7. First we check that \mathcal{N}_d is a linear subspace of $\ell^\infty(\mathbb{N}^d)$. For $i = 1, 2$, let $(\psi_i(\mathbf{n}))_{\mathbf{n} \in \mathbb{N}^d}$ be a k_i -step nilsequence given by $\psi_i(\mathbf{n}) = \Psi_i(\prod_{j=1}^d \tau_{i,j}^{n_j} \cdot e_{X_i})$ where $X_i = G_i/\Gamma_i$ is a k_i -step nilmanifold and $\tau_i \in G_i$. Then for $k := \max(k_1, k_2)$ their sum is the k -step nilsequence $(\Psi(\prod_{i=1}^d \tau_i^{n_i} \cdot e_X))_{n_1, \dots, n_d \in \mathbb{N}}$ where $X := X_1 \times X_2 = (G_1 \times G_2)/(\Gamma_1 \times \Gamma_2)$, $\Psi(x_1, x_2) := \Psi_1(x_1) + \Psi_2(x_2)$, $e_X := (e_{X_1}, e_{X_2})$, and $\tau_j := (\tau_{1,j}, \tau_{2,j})$ for $j = 1, \dots, d$.

Next we show that the space $\mathcal{MC}_{d,\text{pol}}$ is linear, a similar argument works for the space $\mathcal{MC}_{d,\text{lin}}$. Let $a: \mathbb{N}^d \rightarrow \mathbb{C}$ be given by

$$a(\mathbf{n}) = \int f_0 \cdot T_{\vec{p}_1(\mathbf{n})} f_1 \cdots \cdots T_{\vec{p}_s(\mathbf{n})} f_s d\mu, \quad \mathbf{n} \in \mathbb{N}^d,$$

for some system $(X, \mu, T_1, \dots, T_\ell)$, functions $f_0, \dots, f_s \in L^\infty(\mu)$, and polynomial mappings $\vec{p}_1, \dots, \vec{p}_s: \mathbb{N}^d \rightarrow \mathbb{Z}^\ell$. Let also $a': \mathbb{N}^d \rightarrow \mathbb{C}$ be defined by a similar formula, with ℓ' in place of ℓ , $(X', \mu', T'_1, \dots, T'_{\ell'})$ in place of $(X, \mu, T_1, \dots, T_\ell)$, s' in place of s , $\vec{p}'_1, \dots, \vec{p}'_{s'}: \mathbb{Z}^d \rightarrow \mathbb{Z}^{\ell'}$ in place of $\vec{p}_1, \dots, \vec{p}_s$, and $f'_0, \dots, f'_{s'} \in L^\infty(\mu')$ in place of f_0, \dots, f_s . We define a system $(Y, \nu, S_1, \dots, S_{\ell+\ell'})$ by letting Y to be the disjoint union $X \uplus X'$ of X and X' , $\nu = \frac{1}{2}(\mu + \mu')$, and defining the transformations $S_1, \dots, S_{\ell+\ell'}$ of Y by

$$\begin{aligned} S_j|_X &:= T_j \quad \text{and} \quad S_j|_{X'} := \text{Id}, & \text{for } 1 \leq j \leq \ell; \\ S_j|_X &:= \text{Id} \quad \text{and} \quad S_j|_{X'} := T'_j, & \text{for } \ell < j \leq \ell + \ell'. \end{aligned}$$

We also define the polynomial mappings $\vec{q}_1, \dots, \vec{q}_s: \mathbb{N}^d \rightarrow \mathbb{Z}^{\ell+\ell'}$ and $\vec{q}'_1, \dots, \vec{q}'_{s'}: \mathbb{N}^d \rightarrow \mathbb{Z}^{\ell+\ell'}$ as follows:

$$\begin{aligned} \vec{q}_i(\mathbf{n}) &:= (p_{i,1}(\mathbf{n}), \dots, p_{i,\ell}(\mathbf{n}), 0, \dots, 0), & \text{for } 1 \leq i \leq s; \\ \vec{q}'_i(\mathbf{n}) &:= (0, \dots, 0, p'_{i,1}(\mathbf{n}), \dots, p'_{i,\ell'}(\mathbf{n})), & \text{for } 1 \leq i \leq s'. \end{aligned}$$

We also let $\vec{q}_0 = \vec{q}'_0 := 0$. Finally, we define the functions $g_i \in L^\infty(\nu)$, $0 \leq i \leq s$, and $g'_i \in L^\infty(\nu)$, $0 \leq i \leq s'$, by

$$g_i := \mathbf{1}_X f_i + \mathbf{1}_{X'} \quad \text{and} \quad g'_i := \mathbf{1}_X + \mathbf{1}_{X'} f'_i.$$

Then we have

$$\int_Y \prod_{i=0}^s S_{\vec{q}_i(\mathbf{n})} g_i \cdot \prod_{i=0}^{s'} S_{\vec{q}'_i(\mathbf{n})} g'_i d\nu = \frac{1}{2}(a(\mathbf{n}) + a'(\mathbf{n})), \quad \mathbf{n} \in \mathbb{N}^d.$$

This completes the proof of the linearity of the space $\mathcal{MC}_{d,\text{pol}}$.

The inclusion $\overline{\mathcal{N}}_d \subset \overline{\mathcal{MC}}_{d,\text{lin}}$ follows from Proposition 4.2. The inclusion $\overline{\mathcal{MC}}_{d,\text{lin}} \subset \overline{\mathcal{MC}}_{d,\text{pol}}$ is obvious. The inclusion $\overline{\mathcal{MC}}_{d,\text{pol}} \subset \overline{\mathcal{N}}_d$ follows from Theorem 2.5. This completes the proof of Theorem 2.7. \square

7. CONVERGENCE CRITERIA FOR WEIGHTED AVERAGES

In this short section we use the machinery developed in the previous sections in order to prove the convergence criteria stated in Sections 2.2 and 2.3.

7.1. Proof of Theorem 2.2. Let $d, \ell, s, t \in \mathbb{N}$, $(X, \mu, T_1, \dots, T_\ell)$ be a system, f_1, \dots, f_s be functions in $L^\infty(\mu)$, and $\vec{p}_i: \mathbb{N}^d \rightarrow \mathbb{Z}^\ell$, $i = 1, \dots, s$, be polynomial mappings of degree at most t . We can assume that $\|f_i\|_{L^\infty(\mu)} \leq 1$ for $i = 1, \dots, s$.

Let $\delta > 0$ and $k = k(d, \ell, s, t)$ be given by Proposition 6.1 and suppose that the bounded sequence $w: \mathbb{N}^d \rightarrow \mathbb{C}$ is k -regular. As remarked there, if all the polynomials are linear, then we can take $k = s$. By Theorem 3.9, the sequence w can be written as $w = \psi + u$ where ψ is a k -step nilsequence in d variables and $u \in \ell^\infty(\mathbb{N}^d)$ satisfies $\|u\|_{U^{k+1}(\mathbb{N}^d)} < \delta$. By Proposition 2.1, the limit

$$\lim \text{Av} \psi(\mathbf{n}) \cdot T_{\vec{p}_1(\mathbf{n})} f_1 \cdots \cdots T_{\vec{p}_s(\mathbf{n})} f_s$$

exists in $L^2(\mu)$ and by Proposition 6.1 for every Følner sequence $\mathbf{I} = (I_j)_{j \in \mathbb{N}}$ we have

$$\lim \left\| \text{Av}_{\mathbf{I}} u(\mathbf{n}) \cdot T_{\vec{p}_1(\mathbf{n})} f_1 \cdots \cdots T_{\vec{p}_s(\mathbf{n})} f_s \right\|_{L^2(\mu)} \leq 4\delta.$$

It follows that for all sufficiently large $j, j' \in \mathbb{N}$, the difference between

$$(7.1) \quad \frac{1}{|I_j|} \sum_{\mathbf{n} \in I_j} u(\mathbf{n}) T_{\vec{p}_1} f_1 \cdots T_{\vec{p}_s} f_s$$

and the similar average on $I_{j'}$ has a norm in $L^2(\mu)$ bounded by 8δ . Therefore, the averages (7.1) form a Cauchy sequence and thus converge in $L^2(\mu)$.

Furthermore, if we assume that the averages of $w(\mathbf{n})\psi(\mathbf{n})$ converge to 0 for every k -step nilsequence ψ in d variables, then by Corollary 3.8 we have $\|w\|_{U^k(\mathbb{N}^d)} = 0$ and the averages (7.1) converge to 0 in $L^2(\mu)$ by Proposition 6.1. This completes the proof. \square

7.2. Proof of Theorem 2.4. Let $k = k(d, \ell, s, t)$ be as in Theorem 2.5 and the sequence $a: \mathbb{N}^d \rightarrow \mathbb{C}$ be defined by

$$a(\mathbf{n}) := \int f_0 \cdot T_{\vec{p}_1} f_1 \cdots T_{\vec{p}_s} f_s d\mu, \quad \mathbf{n} \in \mathbb{N}^d.$$

By Theorem 2.5 a is an approximate k -step nilsequence in d variables and by hypothesis (2.3) the averages (2.4) converge. This proves Part (i) of Theorem 2.4. Furthermore, we deduce that if the averages (2.3) converge to 0 for every nilsequence ψ in d variables, then the averages (2.4) converge to 0.

Next we prove Part (ii). Let $k = k(2d, \ell, 2s - 1, t)$ be as in Theorem 2.5 and let the sequence $b: \mathbb{N}^d \times \mathbb{N}^d \rightarrow \mathbb{C}$ be defined by

$$b(\mathbf{n}, \mathbf{n}') := \int T_{\vec{p}_1} f_1 \cdots T_{\vec{p}_s} f_s \cdot T_{\vec{p}_1} \bar{f}_1 \cdots T_{\vec{p}_s} \bar{f}_s d\mu, \quad \mathbf{n}, \mathbf{n}' \in \mathbb{N}^d.$$

By Theorem 2.5 the sequence b is an approximate k -step nilsequence in $2d$ variables. Hence, by hypothesis (2.5), the averages

$$\frac{1}{N^d N'^d} \sum_{\mathbf{n} \in [N]^d} \sum_{\mathbf{n}' \in [N']^d} w(\mathbf{n}) w(\mathbf{n}') b(\mathbf{n}, \mathbf{n}')$$

converge to some limit L when N and N' tend to $+\infty$. Let N_0 be such that the difference between this average and L is bounded in absolute value by ε for all $N, N' > N_0$. Expanding the square

$$\left\| \frac{1}{N^d} \sum_{\mathbf{n} \in [N]^d} w(\mathbf{n}) \cdot T_{\vec{p}_1} f_1 \cdots T_{\vec{p}_s} f_s - \frac{1}{N'^d} \sum_{\mathbf{n}' \in [N']^d} w(\mathbf{n}') \cdot T_{\vec{p}_1} f_1 \cdots T_{\vec{p}_s} f_s \right\|_{L^2(\mu)}^2$$

we obtain 4 terms of this form with alternate signs, and thus this square is bounded by 4ε . Hence, the averages in (2.6) form a Cauchy sequence in $L^2(\mu)$ and thus converge. This proves Part (ii) of Theorem 2.4. As above, if the limit (2.5) is zero for all k -step nilsequences ψ in $2d$ variables, then the limit (2.6) is equal to 0.

In order to get the last part of Theorem 2.4 we argue as before and use the last parts of Theorems 2.5 and 2.6. \square

8. APPLICATIONS TO ARITHMETIC WEIGHTS

In this section we prove the main results stated in Section 2.6.

8.1. Proof of Theorem 2.11. Theorem 2.11 follows immediately by the next result which shows that hypothesis (ii) of Theorem 2.4 is satisfied.

Theorem 8.1. *Let $d \in \mathbb{N}$ and $\phi: \mathbb{N}^d \rightarrow \mathbb{C}$ be a good multiplicative function. Then for every nilsequence $\psi: \mathbb{N}^d \rightarrow \mathbb{C}$ the limit*

$$(8.1) \quad \lim_{N_1, \dots, N_d \rightarrow +\infty} \frac{1}{N_1 \cdots N_d} \sum_{\mathbf{n} \in [N_1] \times \cdots \times [N_d]} \phi(\mathbf{n}) \psi(\mathbf{n})$$

exists. Moreover, if the multiplicative function ϕ is aperiodic, then the limit is equal to 0 for all nilsequences ψ in d variables.

Remarks. • One can also deduce variants of the previous result that deal with polynomial sequences on nilmanifolds; indeed, any such sequence can be represented as a linear sequence on a different nilmanifold [53, Section 2.11].

• The case where $d = 1$ can be deduced from Theorem 6.1 in [31], but due to the finitary nature of the statement in [31] the argument used there is vastly more complicated.

We prove Theorem 8.1 in Section 8.3.

8.2. Proof of Theorems 2.12 and 2.13. For $b \in \mathbb{N}$ we let ζ be a root of unity of order b and let f_b be the multiplicative function defined by $f_b(p^k) = \zeta$ for all primes p and all $k \in \mathbb{N}$.⁷ Note that

$$(8.2) \quad \mathbf{1}_{S_{a,b}}(n) = \frac{1}{b} \sum_{j=0}^{b-1} \zeta^{-aj} (f_b(n))^j, \quad n \in \mathbb{N}.$$

It can be seen using Theorem 2.10 (see [32, Proposition 2.10]) that for $j = 1, \dots, b-1$ the multiplicative function f_b^j is aperiodic.

Let

$$V_{\mathbf{n}} := T_{\vec{p}_1(\mathbf{n})} f_1 \cdots T_{\vec{p}_s(\mathbf{n})} f_s, \quad \mathbf{n} \in \mathbb{N}^d.$$

Recall that $S := (S_{a_1, b_1} + c_1) \times \cdots \times (S_{a_d, b_d} + c_d)$. Using (8.2) we get that the averages

$$\frac{1}{N^d} \sum_{\mathbf{n} \in S \cap [N]^d} V_{\mathbf{n}}$$

are asymptotically equal (meaning, the relevant difference converges to 0 in $L^2(\mu)$) to the averages

$$(8.3) \quad \frac{1}{N^d} \sum_{\mathbf{n} \in [N]^d} \prod_{i=1}^d \frac{1}{b_i} \sum_{j=0}^{b_i-1} \zeta^{-a_i j} (f_{b_i}(n_i))^j \cdot V_{\mathbf{n}+\mathbf{c}}$$

where $\mathbf{c} := (c_1, \dots, c_d)$ and as usual n_1, \dots, n_d denote the coordinates of the vector $\mathbf{n} \in \mathbb{N}^d$. Since for $i = 1, \dots, d$ and $j = 1, \dots, b_i - 1$ the multiplicative function $f_{b_i}^j$ is aperiodic, we get that for $j_i \in \{0, \dots, b_i - 1\}$, $i = 1, \dots, d$, the multiplicative function $(n_1, \dots, n_d) \mapsto \prod_{i=1}^d (f_{b_i}(n_i))^{j_i}$ is aperiodic unless $j_1 = \cdots = j_d = 0$. Keeping this in mind, expanding the product in (8.3), and using the second part of Theorem 2.11, we deduce that the averages (8.3) are asymptotically equal in $L^2(\mu)$ to the averages

$$\prod_{i=1}^d \frac{1}{b_i} \cdot \frac{1}{N^d} \sum_{\mathbf{n} \in [N]^d} V_{\mathbf{n}}.$$

Furthermore, the previous argument, applied for $V_{\mathbf{n}} := 1$, $\mathbf{n} \in [N]^d$, gives that

$$\lim_{N \rightarrow +\infty} \frac{|S \cap [N]^d|}{N^d} = \prod_{i=1}^d \frac{1}{b_i}.$$

Combining the above, we deduce that the difference

$$\frac{1}{|S \cap [N]^d|} \sum_{\mathbf{n} \in S \cap [N]^d} V_{\mathbf{n}} - \frac{1}{N^d} \sum_{\mathbf{n} \in [N]^d} V_{\mathbf{n}}$$

converges to 0 in $L^2(\mu)$. Using this, in conjunction with Theorem 1.1, completes the proof of Theorem 2.12.

⁷In order to get results when the set $S_{a,b}$ is defined by counting prime factors with multiplicity, we define the completely multiplicative function f_b by $f_b(p^k) = \zeta^k$ for all primes p and all $k \in \mathbb{N}$.

Using the previously established identity for $f_i := \mathbf{1}_A$, $i = 1, \dots, d$, where $A \in \mathcal{X}$ is a set of positive measure, we deduce that

$$\begin{aligned} \lim_{N \rightarrow +\infty} \frac{1}{|S \cap [N]^d|} \sum_{\mathbf{n} \in S \cap [N]^d} \mu(A \cap T_{-\vec{p}\mathbf{1}(\mathbf{n})} A \cap \dots \cap T_{-\vec{p}\vec{s}(\mathbf{n})} A) = \\ \lim_{N \rightarrow +\infty} \frac{1}{N^d} \sum_{\mathbf{n} \in [N]^d} \mu(A \cap T_{-\vec{p}\mathbf{1}(\mathbf{n})} A \cap \dots \cap T_{-\vec{p}\vec{s}(\mathbf{n})} A) > 0, \end{aligned}$$

where the positiveness of the last limit follows from the multiparameter polynomial Szemerédi theorem [14, Theorem 0.9]. This proves Theorem 2.13. \square

8.3. Correlations of multiplicative functions with nilsequences. In this subsection we prove Theorem 8.1 for $d = 2$; the general case is identical to this one modulo changes in notation. We begin with some preliminaries.

8.3.1. *Some classical facts about commutators.*

Lemma 8.2. *Let G be a group and $i, j \in \mathbb{N}$. Then the commutator map $(g, h) \rightarrow [g, h]$ maps $G_i \times G_j$ to G_{i+j} . Moreover, it induces a bi-homomorphism from $(G_i/G_{i+1}) \times (G_j/G_{j+1})$ to G_{i+j}/G_{i+j+1} .*

Sketch of the proof. The first statement follows by induction from the classical three subgroups lemma:

Let $H, K, L \subset G$ and N be a normal subgroup of G . If $[[H, K], L] \subset N$ and $[[L, H], K] \subset N$, then $[[K, L], H] \subset N$.

The second statement follows from the identity:

$$(8.4) \quad [xy, z] = x[y, z]x^{-1} \cdot [x, z] = [x, [y, z]] \cdot [y, z] \cdot [x, z] \quad \text{for all } x, y, z \in G.$$

\square

Lemma 8.3. *Let G be a group, $H^{(1)}, \dots, H^{(k)}$ and $Q^{(1)}, \dots, Q^{(\ell)}$ be normal subgroups of G , $H = H^{(1)} \cdot \dots \cdot H^{(k)}$ and $Q = Q^{(1)} \cdot \dots \cdot Q^{(\ell)}$. Then the commutator group $[H, Q]$ is the product of the groups $[H^{(i)}, Q^{(j)}]$ for $i = 1, \dots, k$ and $j = 1, \dots, \ell$.*

Proof. All the groups $[H^{(i)}, Q^{(j)}]$ are normal and included in $[H, Q]$, thus it suffices to prove that $[H, Q]$ is contained in the product of these groups. If $h_i \in H^{(i)}$ for $i = 1, 2$ and $q \in Q^{(1)}$, it follows from (8.4) and from the normality of $[H^{(1)}, Q^{(1)}]$ that $[h_1 h_2, q] \in [H^{(1)}, Q^{(1)}] \cdot [H^{(2)}, Q^{(1)}]$. This proves the result in the case $k = 2, \ell = 1$. The result for k arbitrary and $\ell = 1$ follows by induction on k . The result for k, ℓ arbitrary follows by exchanging the roles of H and Q . \square

8.3.2. *Some reductions.* We prove Theorem 8.1 for $d = 2$. Recall that ψ is an s -step nilsequence in 2 variables defined by

$$\psi(n_1, n_2) := \Psi(\tau_1^{n_1} \tau_2^{n_2} \cdot e_X), \quad n_1, n_2 \in \mathbb{N},$$

where $X = G/\Gamma$ is a nilmanifold, $\Psi \in C(X)$, and τ_1, τ_2 are two commuting elements of G . It is known [53, 58] that the closure X' of $\{\tau_1^{n_1} \tau_2^{n_2} \cdot e_X : n_1, n_2 \in \mathbb{N}\}$ in X is a subnilmanifold of X . Substituting X' for X we can assume that $\{\tau_1^{n_1} \tau_2^{n_2} \cdot e_X : n_1, n_2 \in \mathbb{N}\}$ is dense in X ; it then follows by [53, Theorem 1.4] that it is equidistributed in X .

We can assume without loss of generality that G is spanned by the connected component of the unit element and the elements τ_1 and τ_2 . This implies that all commutator subgroups G_i , $i \geq 2$, are connected (see for example [11, Theorem 4.1]).

Suppose that $s \geq 2$ and that G is s -step but not $(s-1)$ -step nilpotent. Then the group $K_s := G_s/(G_s \cap \Gamma)$ is a finite dimensional torus, sometimes called the *vertical torus* [40], and acts freely on X . We denote this action by $(u, x) \mapsto u \cdot x$ for $u \in K_s$ and $x \in X$. Let \widehat{K}_s be the dual group of K_s , that is, the group of continuous group homomorphisms

from K_s to the circle group \mathcal{S}^1 . If for some $\chi \in \widehat{K_s}$ the function $\Psi \in C(X)$ satisfies $\Psi(u \cdot x) = \chi(u)\Psi(x)$ for every $u \in K_s$ and $x \in X$, then it is called a *nilcharacter* of X with *vertical frequency* χ . The linear span of nilcharacters is dense in $C(X)$ for the uniform norm. Therefore, it suffices to prove (8.1) when the function Ψ defining the nilsequence ψ is a nilcharacter.

If the vertical frequency χ of Ψ is the trivial character of K_s , then the function Ψ factorizes through the quotient of X under the action of this group. This quotient is the $(s-1)$ -step nilmanifold $X' = G/(G_s\Gamma) = (G/G_s)/((\Gamma \cap G_s)/G_s)$. Writing τ'_1, τ'_2 for the images of τ_1, τ_2 in G/G_s , we have that $\psi(n_1, n_2) = \Psi'(\tau_1^{n_1} \tau_2^{n_2} \cdot e_{X'})$ for some $\Psi' \in C(X')$.

Iterating this procedure, we reduce matters to considering the following two cases: (i) X is a 1-step nilmanifold, and (ii) X is an s -step nilmanifold for some $s \geq 2$ and Ψ is a nilcharacter with vertical frequency $\chi \neq 1$.

If (i) holds, then X is a compact Abelian group and we can further reduce matters to the case where Ψ is a character of X . Then the average in (8.1) can be rewritten as

$$(8.5) \quad \left(\frac{1}{N_1} \sum_{n_1=1}^{N_1} \phi_1(n_1) e(n_1 t_1) \right) \cdot \left(\frac{1}{N_2} \sum_{n_2=1}^{N_2} \phi_2(n_2) e(n_2 t_2) \right)$$

for some $t_1, t_2 \in \mathbb{R}$ and good multiplicative functions $\phi_1, \phi_2: \mathbb{N} \rightarrow \mathbb{C}$. If t_1 or t_2 is irrational, then the limit is equal to 0 by a Theorem of Daboussi [23] (see also [24, 25]). If both t_1 and t_2 are rational, then the limit exists since by hypothesis the multiplicative functions ϕ_1 and ϕ_2 are good. Furthermore, this limit is equal to 0 if either ϕ_1 or ϕ_2 is aperiodic, that is, if ϕ is aperiodic.

Hence, it suffices to consider case (ii), that is, we can assume that X is an s -step nilmanifold for some $s \geq 2$ and Ψ is a nilcharacter with vertical frequency $\chi \neq 1$. Replacing X with its quotient by the kernel of χ , we are reduced to the case where $G_s = \mathcal{S}^1$ and Ψ has vertical frequency 1, that is,

$$(8.6) \quad \Psi(u \cdot x) = u \Psi(x) \quad \text{for every } x \in X \text{ and } u \in G_s = \mathcal{S}^1.$$

It therefore suffices to prove:

Proposition 8.4 (Theorem 8.1-Nilcharacter form). *Let $X = G/\Gamma$ be an s -step nilmanifold for some $s \geq 2$. Suppose that $G_s = \mathcal{S}^1$, $\Psi \in C(X)$ satisfies (8.6), and τ_1, τ_2 are two commuting elements of G such that the sequence $(\tau_1^{n_1} \tau_2^{n_2} \cdot e_X)_{n_1, n_2 \in \mathbb{N}}$ is equidistributed in X . Then for all multiplicative functions $\phi_1, \phi_2: \mathbb{N} \rightarrow \mathbb{C}$ that are bounded by 1 we have*

$$(8.7) \quad \lim_{N_1, N_2 \rightarrow +\infty} \frac{1}{N_1 N_2} \sum_{n_1 \in [N_1], n_2 \in [N_2]} \phi_1(n_1) \phi_2(n_2) \Psi(\tau_1^{n_1} \tau_2^{n_2} \cdot e_X) = 0.$$

We proceed now to establish Proposition 8.4.

8.3.3. A variant of Katái's Lemma. We use a two dimensional variant of a result of Katái ([50], see also [23]); we omit its proof since it is identical with its one dimensional version modulo changes in notation.

Proposition 8.5. *Let $P_0 \in \mathbb{N}$, $\phi_1, \phi_2: \mathbb{N} \rightarrow \mathbb{C}$ be multiplicative functions, bounded by 1, and $a \in \ell^\infty(\mathbb{N}^2)$. Suppose that*

$$\lim_{N_1, N_2 \rightarrow +\infty} \frac{1}{N_1 N_2} \sum_{n_1 \in [N_1], n_2 \in [N_2]} a(p_1 n_1, p_2 n_2) \overline{a(p'_1 n_1, p'_2 n_2)} = 0$$

for all distinct primes $p_1, p_2, p'_1, p'_2 \geq P_0$. Then

$$\lim_{N_1, N_2 \rightarrow +\infty} \sup_{\phi_1, \phi_2} \left| \frac{1}{N_1 N_2} \sum_{n_1 \in [N_1], n_2 \in [N_2]} \phi_1(n_1) \phi_2(n_2) a(n_1, n_2) \right| = 0$$

where the sup is taken over all $\phi_1, \phi_2: \mathbb{N} \rightarrow \mathbb{C}$ that are multiplicative and bounded by 1.

8.3.4. *The nilmanifold Y .* In the sequel, p_1, p_2, p'_1, p'_2 are fixed distinct primes. By Proposition 8.5, in order to prove Proposition 8.4, it suffices to show that

$$(8.8) \quad \lim_{N_1, N_2 \rightarrow +\infty} \frac{1}{N_1 N_2} \sum_{n_1 \in [N_1], n_2 \in [N_2]} \Psi(\tau_1^{p_1 n_1} \tau_2^{p_2 n_2} \cdot e_X) \overline{\Psi(\tau_1^{p'_1 n_1} \tau_2^{p'_2 n_2} \cdot e_X)} = 0$$

where Ψ satisfies (8.6). We let

$$Y := \overline{\{(\tau_1^{p_1 n_1} \tau_2^{p_2 n_2} \cdot e_X, \tau_1^{p'_1 n_1} \tau_2^{p'_2 n_2} \cdot e_X) : n_1, n_2 \in \mathbb{N}\}} \subset X \times X.$$

Then Y is a sub-nilmanifold of $X \times X$ and by [53, Theorem 1.4] the 2-variable sequence above is equidistributed in Y . Writing m_Y for the Haar measure of Y , the limit in (8.8) is equal to

$$(8.9) \quad \int_Y \Psi(x) \overline{\Psi(x')} dm_Y(x, x').$$

Hence, it remains to show that this integral is zero.

Let H be the smallest closed subgroup of G containing $\Gamma \times \Gamma$ and the shift elements $(\tau_1^{p_1}, \tau_1^{p'_1})$ and $(\tau_2^{p_2}, \tau_2^{p'_2})$. We claim that

$$Y = H/(\Gamma \times \Gamma).$$

Indeed, by the definition of H and Y , we have $H \cdot (e_X, e_X) \supset Y$. Furthermore, by the remark following [52, Theorem 2.21], we have $Y = H_1 \cdot (e_X, e_X)$ for some closed subgroup H_1 of $G \times G$ containing the shift elements $(\tau_1^{p_1}, \tau_1^{p'_1})$ and $(\tau_2^{p_2}, \tau_2^{p'_2})$. Since Y is compact, $H_1 \cap (\Gamma \times \Gamma)$ is cocompact in H_1 and thus $H_2 := H_1 \cdot (\Gamma \times \Gamma)$ is a closed subgroup of G . Since H_2 is a closed subgroup that contains the shift elements and $\Gamma \times \Gamma$, we have $H_2 \supset H$, hence $H \cdot (e_X, e_X) \subset H_2 \cdot (e_X, e_X) = H_1 \cdot (e_X, e_X) = Y$. Therefore, $H \cdot (e_X, e_X) = Y$, which implies that $Y = H/(\Gamma \times \Gamma)$. This proves the claim.

8.3.5. *Projection on the Kronecker factor.* We denote by Z the compact Abelian group $G/(G_2\Gamma)$ and let $\pi: X \rightarrow Z$ and $p: G \rightarrow Z$ be the natural projections. For $i = 1, 2$ let $\alpha_i := p(\tau_i)$ and Z_i be the closed subgroup of Z spanned by α_i . Since $\{\tau_1^{n_1} \tau_2^{n_2} \cdot e_X : n_1, n_2 \in \mathbb{N}\}$ is dense in X , we have that $\{\alpha_1^{n_1} \alpha_2^{n_2} \cdot e_X : n_1, n_2 \in \mathbb{N}\}$ is dense in Z and $Z = Z_1 Z_2$. For $i = 1, 2$ we let

$$G^{(i)} := p^{-1}(Z_i).$$

Then $G^{(i)}$ is a closed subgroup of G containing Γ and G_2 , hence normal in G , and

$$G = G^{(1)} G^{(2)}.$$

Let

$$W := (p \times p)(H) = (\pi \times \pi)(Y).$$

By the definition of Y , we have that W is the closure in $Z \times Z$ of

$$\{(\alpha_1^{p_1 n_1} \alpha_2^{p_2 n_2} \cdot e_X, \alpha_1^{p'_1 n_1} \alpha_2^{p'_2 n_2} \cdot e_X) : n_1, n_2 \in \mathbb{N}\}$$

and thus

$$(8.10) \quad W = \{(z_1^{p_1} z_2^{p_2}, z_1^{p'_1} z_2^{p'_2}) : z_1 \in Z_1, z_2 \in Z_2\}.$$

8.3.6. *Starting the induction.* For $i = 1, 2$ let $g_i \in G^{(i)}$. Then by (8.10) we have

$$(p \times p)(g_1^{p_1}, g_1^{p_1'}) \in W \quad \text{and} \quad (p \times p)(g_2^{p_2}, g_2^{p_2'}) \in W.$$

We have $(p \times p)(H) = W$, the kernel of $p \times p$ is $(G_2 \times G_2)(\Gamma \times \Gamma)$, and $\Gamma \times \Gamma \subset H$, thus

$$(g_1^{p_1}, g_1^{p_1'}) \in H(G_2 \times G_2) \quad \text{and} \quad (g_2^{p_2}, g_2^{p_2'}) \in H(G_2 \times G_2).$$

For $i = 1, 2$, let $h_i \in G^{(i)}$. By (8.4) we have

$$(H(G_2 \times G_2))_2 \subset H_2(G_3 \times G_3)$$

and thus

$$[(g_1^{p_1}, g_1^{p_1'}), (h_1^{p_1}, h_1^{p_1'})], [(g_1^{p_1}, g_1^{p_1'}), (h_2^{p_2}, h_2^{p_2'})], \text{ and } [(g_2^{p_2}, g_2^{p_2'}), (h_2^{p_2}, h_2^{p_2'})]$$

belong to $H_2(G_3 \times G_3)$. By Lemma 8.2, these elements are equal modulo $G_3 \times G_3$ to

$$([g_1, h_1]^{p_1^2}, [g_1, h_1]^{p_1'^2}), ([g_1, h_2]^{p_1 p_2}, [g_1, h_2]^{p_1' p_2'}), \text{ and } ([g_2, h_2]^{p_2^2}, [g_2, h_2]^{p_2'^2})$$

respectively.

Henceforth, for $i = 1, 2$ we denote $(G^{(i)})_2$ by $G_2^{(i)}$. For $u, v \in G_2$ we have $(uv)^{p_1^2} = u^{p_1^2} v^{p_1^2} \pmod{G_3}$ and $(uv)^{p_1'^2} = u^{p_1'^2} v^{p_1'^2} \pmod{G_3}$ and thus the set

$$L := \{u \in G_2^{(1)} : (u^{p_1^2}, u^{p_1'^2}) \in H_2(G_3 \times G_3)\}$$

is a subgroup of $G_2^{(1)}$. By the previous discussion, for $g_1, h_1 \in G^{(1)}$ the set L contains $[g_1, h_1]$ and thus it is equal to $G_2^{(1)}$. Hence,

$$(u^{p_1^2}, u^{p_1'^2}) \in H_2(G_3 \times G_3) \quad \text{for every } u \in G_2^{(1)}.$$

In the same way, we have that

$$(u^{p_2^2}, u^{p_2'^2}) \in H_2(G_3 \times G_3) \quad \text{for every } u \in G_2^{(2)};$$

$$(u^{p_1 p_2}, u^{p_1' p_2'}) \in H_2(G_3 \times G_3) \quad \text{for every } u \in [G^{(1)}, G^{(2)}].$$

8.3.7. *The induction.* By induction on r we show:

Lemma 8.6. *For $2 \leq r \leq s$ and $i_1, \dots, i_r \in \{1, 2\}$ let*

$$s_i := |\{j : 1 \leq j \leq r, i_j = i\}| \quad \text{for } i = 1, 2;$$

$$G^{(i_1, \dots, i_r)} := [G^{(i_1)}, [G^{(i_2)}, \dots, [G^{(i_{r-1})}, G^{(i_r)}] \dots]] \subset G_r.$$

Then, for every $u \in G^{(i_1, \dots, i_r)}$ we have $(u^{p_1^{s_1} p_2^{s_2}}, u^{p_1'^{s_1} p_2'^{s_2}}) \in H_r(G_{r+1} \times G_{r+1})$.

Proof. For $r = 2$ this was proved in the preceding subsection and the inductive step is proved by the same method. \square

In the sequel we only use this lemma for $r = s$.

8.3.8. *Conclusion of the proof.* We argue by contradiction. Suppose that

$$I := \int \Psi(x) \overline{\Psi}(x') dm_Y(x, x') \neq 0.$$

Recall that $G_s = \mathcal{S}^1$ and that Ψ satisfies (8.6).

Let $i_1, \dots, i_s \in \{1, 2\}$ and $s_i = |\{j : 1 \leq j \leq s, i_j = i\}|$ for $i = 1, 2$. Since G is s -step nilpotent, the subgroup G_{s+1} is trivial. Hence, by Lemma 8.6, for every $u \in G^{(i_1, \dots, i_s)}$, we have $(u^{p_1^{s_1} p_2^{s_2}}, u^{p_1'^{s_1} p_2'^{s_2}}) \in H_s$. Therefore, the measure m_Y is invariant under translation by $(u^{p_1^{s_1} p_2^{s_2}}, u^{p_1'^{s_1} p_2'^{s_2}})$. Hence,

$$I = \int \Psi(u^{p_1^{s_1} p_2^{s_2}} \cdot x) \overline{\Psi}(u^{p_1'^{s_1} p_2'^{s_2}} \cdot x') dm_Y(x, x') = u^{p_1^{s_1} p_2^{s_2} - p_1'^{s_1} p_2'^{s_2}} \cdot I.$$

Thus $u^{p_1^{s_1} p_2^{s_2} - p_1'^{s_1} p_2'^{s_2}} = 1$ for every $g \in G^{(i_1, \dots, i_s)}$. Since $G^{(i_1, \dots, i_s)}$ is a subgroup of the torus G_s and p_1, p_1', p_2, p_2' are distinct primes, it follows that the group $G^{(i_1, \dots, i_s)}$ is finite.

Furthermore, since $G = G^{(1)}G^{(2)}$, using Lemma 8.3 and induction, we have that the group G_s is the product of the groups $G^{(i_1, \dots, i_s)}$ for $i_1, \dots, i_s \in \{1, 2\}$ and thus is finite, a contradiction since $G_s = \mathcal{S}^1$. This completes the proof of Proposition 8.4 and hence of Theorem 8.1. \square

9. APPLICATIONS TO HARDY FIELD WEIGHTS

9.1. Hardy field functions. Let B be the collection of equivalence classes of real valued functions defined on some half line $[c, +\infty)$, where we identify two functions if they agree on some half line. A *Hardy field* H is a subfield of the ring $(B, +, \cdot)$ that is closed under differentiation. A *Hardy field function* is a function that belongs to some Hardy field. An example of a Hardy field consists of all *logarithmic-exponential functions*, meaning, all functions defined on some half line $[c, +\infty)$ by a finite combination of the symbols $+, -, \times, :, \log, \exp$ operating on the real variable t and on real constants. Examples include functions of the form $t^a(\log t)^b$ for every $a, b \in \mathbb{R}$. An important property of Hardy field functions is that we can relate their growth rates with the growth rates of their derivatives. The reader can find further discussion about Hardy fields in [15, 16, 17, 29] and the references therein.

Let f be a Hardy field function. We say that it

- (i) has *at most polynomial growth* if $f(t)/t^m \rightarrow 0$ for some $m \in \mathbb{N}$;
- (ii) *stays away from polynomials* if $|f(t) - p(t)|/\log t \rightarrow +\infty$ for every $p \in \mathbb{R}[t]$;
- (iii) is *asymptotically polynomial* if $f(t) - p(t) \rightarrow 0$ for some $p \in \mathbb{R}[t]$.

For our purposes, the key property of Hardy field functions that stay away from polynomials is that they satisfy Lemma 9.6 below. Examples of such functions include:

- t^a where a is a positive non-integer;
- $t^a(\log t)^b$, where $a > 0$ and $b \in \mathbb{R} \setminus \{0\}$;
- $t^a + (\log t)^b$, where $a \in \mathbb{R}$ and $b > 1$.

9.2. Convergence and recurrence results. The main result of this section is the following:

Theorem 9.1. *Let $d \in \mathbb{N}$ and f_1, \dots, f_d be Hardy field functions with at most polynomial growth that stay away from polynomials. We define the sequence $w: \mathbb{N}^d \rightarrow \mathbb{C}$ by*

$$w(\mathbf{n}) := e\left(\sum_{i=1}^d f_i(n_i)\right), \quad \mathbf{n} = (n_1, \dots, n_d) \in \mathbb{N}^d.$$

Then for every system $(X, \mu, T_1, \dots, T_\ell)$, functions $F_1, \dots, F_s \in L^\infty(\mu)$, and polynomial mappings $\vec{p}_i: \mathbb{N}^d \rightarrow \mathbb{Z}^\ell$, $i = 1, \dots, s$, we have

$$\lim_{N \rightarrow +\infty} \frac{1}{N^d} \sum_{\mathbf{n} \in [N]^d} w(\mathbf{n}) \cdot T_{\vec{p}_1(\mathbf{n})} F_1 \cdot \dots \cdot T_{\vec{p}_s(\mathbf{n})} F_s = 0$$

where the limit is taken in $L^2(\mu)$.

Remark. Related work for pointwise convergence when $d = \ell = s = 1$ appears in [27].

Using the $d = 1$ case of the previous result we deduce in Section 9.3 the following:

Corollary 9.2. *Let f be a Hardy field function with at most polynomial growth. Then the sequence $w: \mathbb{N} \rightarrow \mathbb{C}$ defined by $w(n) := e(f(n))$, $n \in \mathbb{N}$, is a good universal weight for mean convergence of the averages (2.6) if and only if either f is asymptotically polynomial or f stays away from polynomials.*

Theorem 9.1 follows immediately from Part (ii) of Theorem 2.4 and the next result.

Proposition 9.3. *Let $d \in \mathbb{N}$ and f_1, \dots, f_d be Hardy field functions with at most polynomial growth that stay away from polynomials. Then for every nilsequence $\psi: \mathbb{N}^d \rightarrow \mathbb{C}$ we have*

$$\lim_{N_1, \dots, N_d \rightarrow +\infty} \frac{1}{N_1 \cdots N_d} \sum_{\mathbf{n} \in [N_1] \times \cdots \times [N_d]} e\left(\sum_{i=1}^d f_i(n_i)\right) \psi(\mathbf{n}) = 0.$$

We prove Proposition 9.3 in Section 9.3.

Next, we give some applications. Note that for $0 < a < b < 1/2$ and $t \in [0, 1)$ we have

$$\mathbf{1}_{[a,b]}(\|t\|) = \mathbf{1}_{[a,b]}(t) + \mathbf{1}_{[1-b, 1-a]}(t).$$

Since $\mathbf{1}_{[c,d]}(t)$ is Riemann integrable, for all $c, d \in \mathbb{R}$ with $0 \leq c < d < 1$, we have that for every $\varepsilon > 0$ there exist (1-periodic) trigonometric polynomials P_1, P_2 with zero constant terms such that

$$(9.1) \quad P_1(t) - \varepsilon \leq \mathbf{1}_{[c,d]}(t) - (d - c) \leq P_2(t) + \varepsilon, \quad t \in [0, 1].$$

Using Proposition 9.3 with $2\pi k_i f_i$ in place of f_i for $k_i \in \mathbb{Z}$ not all of them zero, for $i = 1, \dots, d$, we deduce using the estimate (9.1) the following:

Corollary 9.4. *Let $d \in \mathbb{N}$ and f_1, \dots, f_d be Hardy field functions with at most polynomial growth that stay away from polynomials. Let $a_i, b_i \in \mathbb{R}$ with $0 \leq a_i < b_i \leq 1/2$, $i = 1, \dots, d$, and*

$$S := \{(n_1, \dots, n_d) \in \mathbb{N}^d : \|f_1(n_1)\| \in [a_1, b_1], \dots, \|f_d(n_d)\| \in [a_d, b_d]\}.$$

Then for every nilsequence $\psi: \mathbb{N}^d \rightarrow \mathbb{C}$ we have

$$\lim_{N \rightarrow +\infty} \frac{1}{|S \cap [N]^d|} \sum_{\mathbf{n} \in S \cap [N]^d} \psi(\mathbf{n}) = \lim_{N \rightarrow +\infty} \frac{1}{N^d} \sum_{\mathbf{n} \in [N]^d} \psi(\mathbf{n}).$$

We also deduce the following mean convergence and multiple recurrence result:

Theorem 9.5. *Let $S \subset \mathbb{N}^d$ be as in Corollary 9.4. Then the density $d(S)$ of S is $(\prod_{i=1}^d 2(b_i - a_i))^{-1}$ and*

(i) *The sequence $w := \mathbf{1}_S$ is a good universal weight for mean convergence of the averages (2.6) and the limit of these averages is equal to the limit obtained when $w := d(S)$.*

(ii) *For every $d, \ell, s \in \mathbb{N}$, polynomial mappings $\vec{p}_1, \dots, \vec{p}_s: \mathbb{N}^d \rightarrow \mathbb{Z}^\ell$ with zero constant term, system $(X, \mu, T_1, \dots, T_\ell)$, and set $A \in \mathcal{X}$ with $\mu(A) > 0$, we have*

$$(9.2) \quad \lim_{N \rightarrow +\infty} \frac{1}{N^d} \sum_{\mathbf{n} \in [N]^d} \mathbf{1}_S(\mathbf{n}) \mu(A \cap T_{-\vec{p}_1(\mathbf{n})} A \cap \cdots \cap T_{-\vec{p}_s(\mathbf{n})} A) > 0.$$

Proof. If f is a Hardy field function of at most polynomial growth that stays away from polynomials, then the sequence $(f(n))_{n \in \mathbb{N}}$ is uniformly distributed mod 1 (see [16]). The statement about the density of S follows from this fact.

Using Theorem 9.1 with $2\pi k_i f_i$ in place of f_i for $k_i \in \mathbb{Z}$ not all of them zero, for $i = 1, \dots, d$, and the estimate (9.1), we deduce Part (i).

To prove Part (ii) we use Part (i) for $f_1 = \cdots = f_s = \mathbf{1}_A$, multiply by $\mathbf{1}_A$, and integrate with respect to μ . We deduce that the limit in (9.2) is the same as the one obtained when the constant sequence $d(S)$ takes the place of $\mathbf{1}_S$. The asserted positiveness then follows from the multiparameter polynomial Szemerédi theorem [14, Theorem 0.9]. \square

9.3. Proof of Proposition 9.3 and Corollary 9.2. We start with some preliminary facts. Our assumptions on the functions f_1, \dots, f_d are used via the following lemma:

Lemma 9.6. *Let f be a Hardy field function that stays away from polynomials and satisfies $f(t)/t^m \rightarrow 0$ as $t \rightarrow +\infty$ for some $m \in \mathbb{N}$. Let $k \geq m$ be an integer. Then there exist real numbers α_N with $\alpha_N \rightarrow 0$, polynomials $q_N \in \mathbb{R}[t]$ with $\deg(q_N) < k$, positive integers L_N with $L_N/N \rightarrow 0$ and $L_N^k |\alpha_N| \rightarrow +\infty$, such that*

$$f(N+n) = n^k \alpha_N + q_N(n) + o_{N \rightarrow +\infty}(1), \quad n \in [L_N].$$

Proof. This follows by noticing that the proof of [29, Lemma 3.5] applies to all $k \in \mathbb{N}$ with $k \geq m$ and then following the argument in the proof of [29, Lemma 3.4]. \square

In the proof of Proposition 9.3 we use some quantitative equidistribution results from [40]. We record here some relevant notions:

- If G is a nilpotent group, then $g: \mathbb{N}^d \rightarrow G$ is a *polynomial sequence* if it has the form $g(\mathbf{n}) = \prod_{i=1}^s \tau_i^{p_i(\mathbf{n})}$, where for $i = 1, \dots, s$ we have $\tau_i \in G$ and $p_i: \mathbb{N}^d \rightarrow \mathbb{Z}$ are polynomials. The *degree* of the polynomial sequence (with a given representation) is the maximum of the degrees of the polynomials p_1, \dots, p_s .
- For $N_1, \dots, N_d \in \mathbb{N}$ we say that the finite sequence $(g(\mathbf{n}) \cdot e_X)_{\mathbf{n} \in [N_1] \times \dots \times [N_d]}$ is δ -*equidistributed* in the nilmanifold X , if for every Lipschitz function $\Psi: X \rightarrow \mathbb{C}$ with $\|\Psi\|_{\text{Lip}(X)} \leq 1$ and $\int_X \Psi dm_X = 0$, we have

$$\left| \frac{1}{N_1 \cdots N_d} \sum_{\mathbf{n} \in [N_1] \times \dots \times [N_d]} \Psi(g(\mathbf{n}) \cdot e_X) \right| \leq \delta.$$

- An infinite sequence $(g(\mathbf{n}) \cdot e_X)_{\mathbf{n} \in \mathbb{N}^d}$ is *equidistributed in X* if for all $\Psi \in C(X)$ with $\int_X \Psi dm_X = 0$ we have (note that the averages below are uniform)

$$\lim \text{Av} \Psi(g(\mathbf{n}) \cdot e_X) = 0.$$

It is *totally equidistributed in X* if the sequence $(\mathbf{1}_{P_1 \times \dots \times P_d}(\mathbf{n}) \cdot g(\mathbf{n}) \cdot e_X)_{\mathbf{n} \in \mathbb{N}^d}$ is equidistributed in X for all infinite arithmetic progressions $P_1, \dots, P_d \subset \mathbb{N}$.

- The *horizontal torus* of the nilmanifold $X = G/\Gamma$ is the compact Abelian group $Z := G/(G_2\Gamma)$. If G is connected, it is a finite dimensional torus. A *horizontal character* is a continuous group homomorphism $G \rightarrow \mathbb{T}$. It factors through the horizontal torus and induces a character $\eta: Z \rightarrow \mathbb{T}$; when G is connected, it is of the form $\mathbf{t} \mapsto \mathbf{k} \cdot \mathbf{t}$, where $\mathbf{k} \in \mathbb{Z}^s$, $\mathbf{t} \in \mathbb{T}^s$, $s := \dim(Z)$. In this case, we define $\|\eta\| := \|\mathbf{k}\|_1$, that is, the sum of the absolute values of the coordinates of \mathbf{k} .
- If $p: \mathbb{N}^d \rightarrow \mathbb{T}$ has the form $p(n_1, \dots, n_d) = \sum_{j_1, \dots, j_d} \alpha_{j_1, \dots, j_d} n_1^{j_1} \cdots n_d^{j_d}$, we define

$$\|p\|_{C^\infty [N_1] \times \dots \times [N_d]} := \max_{(j_1, \dots, j_d) \neq (0, \dots, 0)} N_1^{j_1} \cdots N_d^{j_d} \|\alpha_{j_1, \dots, j_d}\|.$$

- If $X = G/\Gamma$ is a nilmanifold, then γ is a *rational element of G* if $\gamma^k \in \Gamma$ for some $k \in \mathbb{N}$.

We will use the following quantitative equidistribution result:

Theorem 9.7 ([40, Theorem 8.6] and [41]). *Let $X := G/\Gamma$ be a nilmanifold with G connected and simply connected, $d, t \in \mathbb{N}$, and $\varepsilon > 0$. There exists $M := M(X, d, t, \varepsilon) > 0$ such that the following holds: For all $N_1, \dots, N_d \in \mathbb{N}$, greater than M , if $g: \mathbb{N}^d \rightarrow G$ is a polynomial sequence of degree t and $(g(\mathbf{n}) \cdot e_X)_{\mathbf{n} \in [N_1] \times \dots \times [N_d]}$ is not ε -equidistributed in X , then there exists a non-trivial horizontal character η such that*

$$0 < \|\eta\| \leq M \quad \text{and} \quad \|\eta \circ g\|_{C^\infty [N_1] \times \dots \times [N_d]} \leq M.$$

Remark. For every horizontal character η , the sequence $\eta \circ g$ is a polynomial sequence in \mathbb{T} of degree at most t .

We will use the following elementary result which is a two dimensional variant of [29, Lemma 3.3].

Lemma 9.8. *Let $a \in \ell^\infty(\mathbb{N}^2)$ be such that*

$$(9.3) \quad \lim_{N, N' \rightarrow +\infty} \frac{1}{L_N L_{N'}} \sum_{\mathbf{n} \in (N+[L_N]) \times (N'+[L'_{N'}])} a(\mathbf{n}) = 0$$

for some sequences of positive integers $(L_N)_{N \in \mathbb{N}}$, $(L'_{N'})_{N' \in \mathbb{N}}$ that satisfy $L_N/N \rightarrow 0$ and $L'_{N'}/N' \rightarrow 0$ as $N \rightarrow +\infty$. Then

$$\lim_{N, N' \rightarrow +\infty} \frac{1}{NN'} \sum_{\mathbf{n} \in [N] \times [N']} a(\mathbf{n}) = 0.$$

Proof. Let the sequence of positive integers $(k_i)_{i \in \mathbb{N}}$ be defined by $k_1 := 1$, $k_{i+1} := k_i + L_{k_i}$, $i \in \mathbb{N}$, and similarly let the sequence $(k'_i)_{i \in \mathbb{N}}$ be defined by $k'_1 := 1$, $k'_{i+1} := k'_i + L'_{k'_i}$, $i \in \mathbb{N}$. For $N \in \mathbb{N}$ let $i_N := \max\{i \in \mathbb{N} : k_i \leq N\}$ and $i'_{N'} := \max\{i \in \mathbb{N} : k'_i \leq N'\}$. Then the rectangles $(k_i, k_{i+1}] \times (k'_i, k'_{i+1}]$, where $i \in [i_N - 1]$ and $i' \in [i'_{N'} - 1]$, have the form $(k, k + L_k] \times (k', k' + L'_{k'})$, and together with a leftover set $E_{N, N'}$ form a partition of the rectangle $[N] \times [N']$. The set $E_{N, N'}$ is contained in the union of the rectangles $[N] \times (N' - L'_{k'_{i_N}}, N']$ and $(N - L_{k_{i_N}}, N] \times [N']$, and since $k_{i_N} \leq N$ and $k'_{i'_N} \leq N'$, we have

$$|E_{N, N'}| \leq N \max_{k \leq N'}(L'_k) + N' \max_{k \leq N}(L_k).$$

Since $L_N/N, L'_{N'}/N' \rightarrow 0$ as $N \rightarrow +\infty$, we get that $|E_{N, N'}|/(NN') \rightarrow 0$ as $N, N' \rightarrow +\infty$. Using this, the fact that $a: \mathbb{N}^2 \rightarrow \mathbb{C}$ is bounded, and our assumption (9.3), we deduce that

$$\lim_{N, N' \rightarrow +\infty} \frac{1}{NN'} \sum_{\mathbf{n} \in [N] \times [N']} a(\mathbf{n}) = \lim_{k, k' \rightarrow +\infty} \frac{1}{L_k L'_{k'}} \sum_{\mathbf{n} \in (k, k+L_k] \times (k', k'+L'_{k'})} a(\mathbf{n}) = 0.$$

This completes the proof. \square

Proof of Proposition 9.3. We give the proof for $d = 2$, the proof in the general case is completely similar.

Suppose that the nilsequence ψ has the form

$$\psi(n, n') = \Psi(\tau^n \tau'^{n'} e_X), \quad n, n' \in \mathbb{N},$$

for some nilmanifold $X = G/\Gamma$, commuting elements $\tau, \tau' \in G$, and function $\Psi \in C(X)$. By a remark made in Section 2.1.2, we can assume that the group G is connected and simply connected. Moreover, we can assume that Ψ is a Lipschitz function with $\|\Psi\|_{\text{Lip}(X)} \leq 1$.

By the infinitary factorization theorem [40, Corollary 1.12] (the same argument works for sequences in several variables) the sequence $g: \mathbb{N}^2 \rightarrow G$ defined by $g(n, n') := \tau^n \tau'^{n'}$, $n, n' \in \mathbb{N}$, can be factorized as follows

$$g(n, n') = g'(n, n') \gamma(n, n'), \quad n, n' \in \mathbb{N},$$

where

- $g': \mathbb{N}^2 \rightarrow G'$ is a polynomial sequence on a closed, connected and simply connected subgroup G' of G such that $X' := G'/(G' \cap \Gamma)$ is a nilmanifold;
- the sequence $(g'(n, n') \cdot e_{X'})_{n, n' \in \mathbb{N}}$ is totally equidistributed on X' ;
- the sequence $(\gamma(n, n') \cdot e_X)_{n, n' \in \mathbb{N}}$ is periodic and $\gamma(n, n')$ is a rational element of G for every $n, n' \in \mathbb{N}$.

Then for some $r \in \mathbb{N}$ and all $(i_1, i_2) \in \{0, \dots, r-1\}^2$ the sequence $(\gamma(n, n') \cdot e_X)_{n, n' \in \mathbb{N}}$ is constant in the set $r\mathbb{N}^2 + (i_1, i_2)$; say that it is equal to $\gamma_{i_1, i_2} \cdot e_X$ for some rational element γ_{i_1, i_2} in G . After partitioning \mathbb{N}^2 as a union of such sets, we are reduced to showing that

$$\lim_{N, N' \rightarrow +\infty} \frac{1}{NN'} \sum_{n \in [N], n' \in [N']} e(f(rn + i_1) + f'(rn' + i_2)) \Psi(g'(rn + i_1, rn' + i_2) \gamma_{i_1, i_2} \cdot e_X) = 0$$

for all $(i_1, i_2) \in \{0, \dots, r-1\}^2$. Notice that if h is a Hardy field function of at most polynomial growth that stays away from polynomials, then also $t \mapsto h(kt + l)$ has the same property for all $k \in \mathbb{N}$ and $l \in \mathbb{Z}$. Hence, it suffices to show that if f, f' satisfy the assumptions of Proposition 9.3, then

$$(9.4) \quad \lim_{N, N' \rightarrow +\infty} \frac{1}{NN'} \sum_{n \in [N], n' \in [N']} e(f(n) + f'(n')) \Psi(g'(n, n') \gamma \cdot e_X) = 0,$$

where $g': \mathbb{N}^2 \rightarrow G'$ is such that the infinite polynomial sequence $(g'(n, n') \cdot e_X)_{n, n' \in \mathbb{N}}$ is equidistributed on X' and γ is a rational element of G .

Let $(L_N)_{N \in \mathbb{N}}, (L'_N)_{N \in \mathbb{N}}$ be sequences of positive integers that will be specified later; for the moment we only assume that $L_N, L'_N \rightarrow +\infty$ and $L_N/N, L'_N/N \rightarrow 0$ (we will also impose condition (9.8) later). Using Lemma 9.8 we see that it suffices to show that

$$(9.5) \quad \lim_{N, N' \rightarrow +\infty} \frac{1}{L_N L_{N'}} \sum_{n \in [L_N], n' \in [L_{N'}]} e(f(N+n) + f(N'+n')) \Psi_\gamma(g'_\gamma(N+n, N'+n') \cdot e_X) = 0,$$

where $g'_\gamma := \gamma^{-1} g' \gamma$ is a polynomial sequence on $G'_\gamma := \gamma^{-1} G \gamma$ and $\Psi_\gamma \in \text{Lip}(X)$ is defined by $\Psi_\gamma(g \cdot e_X) := \Psi(\gamma g \cdot e_X)$ for $g \in G$.

In order to prove (9.5) we need to first gather some data. First, we claim that for $N, N' \in \mathbb{N}$ the finite sequence

$$(g'(N+n, N'+n') \cdot e_X)_{n \in [L_N], n' \in [L_{N'}]}$$

is $\delta_{N, N'}$ -equidistributed on X' for some $\delta_{N, N'} > 0$ that satisfy $\delta_{N, N'} \rightarrow 0$ as $N, N' \rightarrow +\infty$. Indeed, if this is not the case, then there exists $\delta > 0$, $N_m, N'_m \rightarrow +\infty$, and $\Psi_m \in \text{Lip}(X')$ with $\|\Psi_m\|_{\text{Lip}(X')} \leq 1$ and $\int_{X'} \Psi_m dm_{X'} = 0$, such that

$$(9.6) \quad \left| \frac{1}{L_{N_m} L_{N'_m}} \sum_{(n, n') \in (N_m + [L_{N_m}]) \times (N'_m + [L_{N'_m}])} \Psi_m(g(n, n') \cdot e_{X'}) \right| \geq \delta, \quad \text{for all } m \in \mathbb{N}.$$

By the Arzelá-Ascoli theorem, a subsequence of Ψ_m converges uniformly to some $\Psi_0 \in \text{Lip}(X')$ with $\|\Psi_0\|_{\text{Lip}(X')} \leq 1$ and $\int_{X'} \Psi_0 dm_{X'} = 0$. Then (9.6) is satisfied for infinitely many $m \in \mathbb{N}$ with Ψ_0 in place of Ψ_m and $\delta/2$ in place of δ . Since $L_{N_m}, L_{N'_m} \rightarrow +\infty$, $((N_m + [L_{N_m}]) \times (N'_m + [L_{N'_m}]))_{m \in \mathbb{N}}$ is a Følner sequence in \mathbb{N}^2 , and we deduce that

$$\lim \text{Av} \Psi_0(g'(n, n') \cdot e_X) \neq 0.$$

This contradicts our assumption that $(g'(n, n') \cdot e_X)_{n, n' \in \mathbb{N}}$ is equidistributed in X' .

From the aforementioned equidistribution property, we deduce using [31, Corollary 5.5] that the finite sequence

$$(g'_\gamma(N+n, N'+n') \cdot e_X)_{n \in [L_N], n' \in [L_{N'}]}$$

is $\delta'_{N, N'}$ -equidistributed, where $\delta'_{N, N'} \rightarrow 0$ as $N, N' \rightarrow +\infty$, on the nilmanifold $X'_\gamma := G'_\gamma / \Gamma'_\gamma$, where $\Gamma'_\gamma := \Gamma \cap G'_\gamma$.

We move now to the proof of (9.5). We apply Lemma 9.6 for the functions f, f' and we get sequences $(L_N)_{N \in \mathbb{N}}, (L'_N)_{N \in \mathbb{N}}$ satisfying the conditions in the lemma for some $k, k' \in \mathbb{N}$ such that $k, k' > \deg(g'_\gamma)$. After ignoring negligible errors, we deduce that in

(9.5) we can replace the finite sequences $(f(N+n))_{n \in [L_N]}$ and $(f'(N'+n'))_{n' \in [L'_{N'}]}$ by finite polynomial sequences $(p_N(n))_{n \in [L_N]}$ and $(p'_{N'}(n'))_{n' \in [L'_{N'}]}$, where

$$(9.7) \quad p_N(n) = n^k \alpha_N + q_N(n), \quad p'_{N'}(n') = n'^{k'} \alpha'_{N'} + q'_{N'}(n'), \quad n, n', N, N' \in \mathbb{N},$$

for some real numbers $\alpha_N, \alpha'_{N'}$ satisfying $\alpha_N \rightarrow 0$ and $\alpha'_{N'} \rightarrow 0$, and polynomials $q_N, q'_{N'} \in \mathbb{R}[t]$, $N \in \mathbb{N}$, that satisfy

$$(9.8) \quad \deg(q_N) < k, \quad \deg(q'_{N'}) < k', \quad \text{and} \quad L_N^k |\alpha_N|, L_{N'}^{k'} |\alpha'_{N'}| \rightarrow +\infty.$$

We claim that the finite polynomial sequence

$$(9.9) \quad (p_N(n), p'_{N'}(n'), g'_\gamma(N+n, N'+n') \cdot e_X)_{n \in [L_N], n' \in [L'_{N'}]}$$

is $\delta''_{N, N'}$ -equidistributed in the nilmanifold $\mathbb{T}^2 \times X'_\gamma$ where $\delta''_{N, N'} \rightarrow 0$ as $N, N' \rightarrow +\infty$. Arguing by contradiction, suppose that this is not true. Then there exists $\delta > 0$ such that

$$(9.10) \quad \text{the sequence (9.9) is not } \delta\text{-equidistributed in } \mathbb{T}^2 \times X'_\gamma \text{ for some } N_m, N'_m \rightarrow +\infty.$$

The horizontal torus of the nilmanifold X'_γ has the form \mathbb{T}^s for some $s \in \mathbb{N}$. Then the horizontal torus of the nilmanifold $\mathbb{T}^2 \times X'_\gamma$ is $\mathbb{T}^2 \times \mathbb{T}^s$. Let $\pi: G \rightarrow \mathbb{T}^s$ be the natural projection on the horizontal torus of X'_γ and $r_{N, N'}: \mathbb{N}^2 \rightarrow \mathbb{T}^s$ be the polynomial sequence defined by $r_{N, N'}(n, n') := \pi(g'_\gamma(N+n, N'+n'))$. Then

$$(9.11) \quad \deg(r_{N, N'}) < \min(k, k') \quad \text{for every } N, N' \in \mathbb{N}.$$

By Theorem 9.7, we deduce that there exists $M > 0$ and $k_m, k'_m \in \mathbb{Z}, k''_m \in \mathbb{Z}^s$ such that for those N_m, N'_m for which (9.10) holds and are greater than M we have

$$(9.12) \quad 0 < |k_m| + |k'_m| + \|k''_m\|_1 \leq M$$

and

$$(9.13) \quad \|k_m p_{N_m}(n) + k'_m p'_{N'_m}(n') + k''_m \cdot r_{N_m, N'_m}(n, n')\|_{C^\infty([L_{N_m}] \times [L'_{N'_m}])} \leq M.$$

If $k_m = k'_m = 0$ for infinitely many $m \in \mathbb{N}$, then k''_m is non-zero for infinitely many $m \in \mathbb{N}$, and using [31, Lemma 5.3] we get a contradiction from the fact that the sequence $(g'_\gamma(N+n, N'+n') \cdot e_X)_{n \in [L_N], n' \in [L'_{N'}]}$ is $\delta'_{N, N'}$ -equidistributed on the nilmanifold X'_γ where $\delta'_{N, N'} \rightarrow 0$ as $N, N' \rightarrow +\infty$.

Suppose next that $k_m \neq 0$ for infinitely many $m \in \mathbb{N}$. Using (9.13) (note that the polynomials p_N and $p_{N'}$ depend on different variables) in conjunction with (9.7), (9.8), (9.11), we obtain that

$$L_{N_m}^k \|k_m \alpha_{N_m}\| \leq M, \quad \text{for infinitely many } m \in \mathbb{N}.$$

Since $\alpha_N \rightarrow 0$ and $1 \leq |k_m| \leq M$, we get that $\|k_m \alpha_{N_m}\| = |k_m \alpha_{N_m}| \geq |\alpha_{N_m}|$ for infinitely many $m \in \mathbb{N}$. We deduce that

$$L_{N_m}^k |\alpha_{N_m}| \leq M, \quad \text{for infinitely many } m \in \mathbb{N}.$$

This contradicts (9.8). The argument is similar if $k'_m \neq 0$ for infinitely many $m \in \mathbb{N}$.

Hence, the finite polynomial sequence (9.9) is $\delta''_{N, N'}$ -equidistributed in the nilmanifold $\mathbb{T}^2 \times X'_\gamma$ where $\delta''_{N, N'} \rightarrow 0$ as $N, N' \rightarrow +\infty$. We deduce that the limit in the left hand side of (9.5) is equal to

$$\int e(t) \cdot e(t') \cdot F_\gamma(x) dm_{\mathbb{T}^2 \times X'_\gamma} = 0$$

where the last identity follows since $m_{\mathbb{T}^2 \times X'_\gamma} = m_{\mathbb{T}^2} \times m_{X'_\gamma}$. This completes the proof. \square

Proof of Corollary 9.2. Let f be a Hardy field function of polynomial growth. We consider the following three cases:

If f stays away from polynomials, then the conclusion follows from the $d = 1$ case of Theorem 9.1 and the corresponding averages converge to 0 in $L^2(\mu)$.

Suppose next that f is asymptotically polynomial, that is, $f(t) - p(t) \rightarrow 0$ for some $p \in \mathbb{R}[t]$. In this case, the mean convergence of the averages (2.6) follows from Proposition 2.1 and the well known fact that sequences of the form $n \mapsto e(p(n))$ are nilsequences.

Lastly, suppose that $f = p + g$ for some polynomial $p \in \mathbb{R}[t]$ and Hardy field function g that satisfies $|g(t)| \rightarrow +\infty$ and $|g(t)| \leq C \log t$ for some $C > 0$ and all sufficiently large $t \in \mathbb{R}_+$. Let $p(t) = \sum_{i=0}^{\ell} \alpha_i t^i$, $t \in \mathbb{R}$, for some $\ell \in \mathbb{N}$ and $\alpha_1, \dots, \alpha_\ell \in \mathbb{R}$. For $i = 1, \dots, \ell$, we consider the commuting transformations $T_i t := t + \alpha_i$, $t \in \mathbb{T}$, acting on \mathbb{T} with the Haar measure $m_{\mathbb{T}}$, and the function $h \in L^\infty(m_{\mathbb{T}})$ defined by $h(t) := e(-t)$, $t \in \mathbb{T}$. Then

$$\frac{1}{N} \sum_{n=1}^N e(f(n)) h\left(\prod_{i=1}^{\ell} T_i^{n^i} t\right) = e(-t + \alpha_0) \frac{1}{N} \sum_{n=1}^N e(g(n)), \quad \text{for every } N \in \mathbb{N}, t \in \mathbb{T}.$$

By [29, Proof of Theorem 3.1] (see also [17, Proof of Theorem 3.3]) we get that the last averages do not converge as $N \rightarrow +\infty$. Hence, the sequence $n \mapsto e(f(n))$ is not a good universal weight for weak convergence of averages of the form (2.6) even when $s = 1$.

If f is any Hardy field function and $p \in \mathbb{R}[t]$ is any polynomial, then it is known that the limit $\lim_{t \rightarrow +\infty} (f(t) - p(t)) / \log t$ either exists or else is $\pm\infty$. Hence, every Hardy field function with at most polynomial growth is covered in one of the previous three cases and the proof is complete. \square

APPENDIX A. SEMINORMS ON $L^\infty(\mu)$ AND RELATED FACTORS

Let $(X, \mu, T_1, \dots, T_\ell)$ be a system. We recall here the definition and some properties of the seminorms $\|\cdot\|_k$ on $L^\infty(\mu)$ and of the factors \mathcal{Z}_k defined in [46] for the ergodic case and in [20] for the general case. These two papers deal only with the case of a single transformation, the generalization to the case of several commuting transformations is completely similar and is given below.

A.1. The seminorms $\|\cdot\|_k$. We write $\mathcal{I}(\vec{T})$ for the σ -algebra of sets invariant under all transformations T_1, \dots, T_ℓ . For $f \in L^\infty(\mu)$, we define

$$(A.1) \quad \|f\|_1 := \|\mathbb{E}_\mu(f \mid \mathcal{I}(\vec{T}))\|_{L^2(\mu)}$$

and for $k \in \mathbb{N}$ we let

$$(A.2) \quad \|f\|_{k+1} := \left(\text{Lim Av}_{\vec{n}} \|f \cdot T_{\vec{n}} \bar{f}\|_k^{2^k} \right)^{1/2^{k+1}}$$

where, as usual, we use the notation $T_{\vec{n}} = \prod_{i=1}^{\ell} T_i^{n_i}$ for $\vec{n} = (n_1, \dots, n_\ell)$. By induction, we have

$$\|f\|_k \leq \|f\|_{L^\infty(\mu)} \quad \text{for every } k \in \mathbb{N}.$$

In case of ambiguity, we write $\|f\|_{\mu, k}$ or $\|f\|_{\vec{T}, \mu, k}$. If $\mu = \int \mu_x d\mu(x)$ is the ergodic decomposition of μ under \vec{T} , then for every $f \in L^\infty(\mu)$ and every $k \in \mathbb{N}$ we have

$$(A.3) \quad \|f\|_{\mu, k}^{2^k} = \int \|f\|_{\mu_x, k}^{2^k} d\mu(x).$$

For $f \in L^\infty(\mu)$, by (A.1) we have

$$(A.4) \quad \left| \int f d\mu \right| \leq \|f\|_1.$$

Writing $\vec{T} \times \vec{T}$ for the \mathbb{Z}^ℓ -action on $X \times X$ induced by $T_1 \times T_1, \dots, T_\ell \times T_\ell$, we have

$$\begin{aligned} \|f \otimes \bar{f}\|_{\vec{T} \times \vec{T}, \mu \times \mu, 1}^2 &= \|\mathbb{E}_{\mu \times \mu}(f \otimes \bar{f} | \mathcal{I}(\vec{T} \times \vec{T}))\|_{L^2(\mu \times \mu)}^2 \\ &= \lim \text{Av}_{\vec{n}} \left| \int f \cdot T_{\vec{n}} \bar{f} d\mu \right|^2 \text{ by the ergodic theorem} \\ &\leq \limsup \text{Av}_{\vec{n}} \|\mathbb{E}_\mu(f \cdot T_{\vec{n}} \bar{f} | \mathcal{I}(\vec{T}))\|_{L^2(\mu)}^2 \\ &= \limsup \text{Av}_{\vec{n}} \|f \cdot T_{\vec{n}} \bar{f}\|_{\vec{T}, \mu, 1}^2 = \|f\|_{\vec{T}, \mu, 2}^4 \text{ by (A.1) and (A.2)}. \end{aligned}$$

By induction, using the relation (A.2) for the measures $\mu \times \mu$ and μ , we deduce that for every $k \in \mathbb{N}$ we have

$$(A.5) \quad \|f \otimes \bar{f}\|_{\vec{T} \times \vec{T}, \mu \times \mu, k} \leq \|f\|_{\vec{T}, \mu, k+1}^2.$$

A.2. The factors \mathcal{Z}_k . For $k \in \mathbb{Z}_+$, the factor \mathcal{Z}_k of X is characterized by the following property

$$\text{for } f \in L^\infty(\mu), \mathbb{E}_\mu(f | \mathcal{Z}_k) = 0 \text{ if and only if } \|f\|_{k+1} = 0.$$

Equivalently, one has

$$L^\infty(\mathcal{Z}_k, \mu) = \left\{ f \in L^\infty(\mu) : \int f \cdot g d\mu = 0 \text{ for every } g \in L^\infty(\mu) \text{ with } \|g\|_{k+1} = 0 \right\}.$$

In case of ambiguity, we write $\mathcal{Z}_k(X, \mu, \vec{T})$.

We say that (X, μ, \vec{T}) is a *system of order k* if the σ -algebra \mathcal{Z}_k coincides with the σ -algebra \mathcal{X} . Equivalently, if $\|\cdot\|_{k+1}$ is a norm on $L^\infty(\mu)$.

Proposition A.1. *Let (X, μ, \vec{T}) be a system of order 1 and $\mu = \int \mu_x d\mu(x)$ be the ergodic decomposition of μ under \vec{T} . Then for μ -a.e. $x \in X$ the system (X, μ_x, \vec{T}) is isomorphic to an ergodic rotation on a compact Abelian group.*

Remark. It is not hard to show that the converse also holds.

Proof. Since (X, \mathcal{X}, μ) is a Lebesgue space, there exists a countable sequence $(f_n)_{n \in \mathbb{N}}$ of bounded Borel functions (defined everywhere) that is dense in $L^1(\mu)$ and in $L^1(\mu_x)$ for every $x \in X$. By [20, Corollary 3.3], there exists a Borel set $X_1 \subset X$ with $\mu(X_1) = 1$, such that for every $x \in X_1$ and every $n \in \mathbb{N}$ the function f_n belongs to $L^\infty(\mathcal{Z}_1(X, \mu_x, \vec{T}))$. For $x \in X_1$ it follows by density that every $f \in L^1(\mu_x)$ belongs to $L^1(X, \mathcal{Z}_1(X, \mu_x, \vec{T}))$. The σ -algebras \mathcal{X} and $\mathcal{Z}_1(X, \mu_x, \vec{T})$ coincide up to μ_x -null sets and (X, μ_x, \vec{T}) is a system of order 1 for μ -a.e. $x \in X$. It is well known that an ergodic system of order 1 is isomorphic to an ergodic rotation on a compact Abelian group and the proof is complete. \square

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