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# THE SYLOW SUBGROUPS OF A FINITE REDUCTIVE GROUP

MICHEL ENGUEHARD AND JEAN MICHEL

*Dedicated to professor George Lusztig on the occasion of his 70th birthday*

ABSTRACT. We describe the structure of Sylow  $\ell$ -subgroups of a finite reductive group  $\mathbf{G}(\mathbb{F}_q)$  when  $q \not\equiv 0 \pmod{\ell}$  that we find governed by a complex reflection group attached to  $\mathbf{G}$  and  $\ell$ , which depends on  $\ell$  only through the set of cyclotomic factors of the generic order of  $\mathbf{G}(\mathbb{F}_q)$  whose value at  $q$  is divisible by  $\ell$ . We also tackle the more general case of groups  $\mathbf{G}^F$  where  $F$  is an isogeny some power of which is a Frobenius morphism.

## 1. INTRODUCTION

**Definition 1.1.** *Let  $\mathbf{G}$  be a connected reductive group over  $\overline{\mathbb{F}}_p$ , and  $F$  an isogeny such that some power of  $F$  is a Frobenius endomorphism; then  $\mathbf{G}^F$  is what we call a finite reductive group. To this situation we attach a positive real number  $q$  such that for some integer  $n$ , the isogeny  $F^n$  is the Frobenius endomorphism attached to a  $\mathbb{F}_{q^n}$ -structure.*

The goal of this note is to describe the Sylow  $\ell$ -subgroups of  $\mathbf{G}^F$  when  $\ell$  is a prime different from  $p$  and  $\mathbf{G}$  is semisimple. The structure of the Sylow  $\ell$ -subgroups of a Chevalley group was first described by [Gorenstein-Lyons] where they observed that they had a large normal abelian subgroup  $(\mathbb{Z}/n)_\ell^a$  where  $n$  is the  $\ell$ -part of  $\Phi_d(q)$ , where  $d$  is the multiplicative order of  $q \pmod{\ell}$ , and they computed  $a$  case by case.

In 1992 [Broué-Malle] exhibited subtori of  $\mathbf{G}^F$  attached to eigenspaces of elements of the Weyl reflection coset of  $(\mathbf{G}, F)$  whose  $F$ -stable points are the large abelian groups of [Gorenstein-Lyons]. To these eigenspaces are attached complex reflection groups by Springer's theory.

We show that the structure of the Sylow  $\ell$ -subgroups of  $\mathbf{G}^F$  is determined by these complex reflection groups. The results of this note in the case when  $F$  is a Frobenius were obtained by the first author in an unpublished note [Enguehard] of 1992; the second author has found a simpler (containing more casefree steps) proof which is an occasion to publish these results. Some of our results appeared also implicitly in [Malle].

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## 2. THE GENERIC SYLOW THEOREMS

Let  $\mathbf{G}$  be as in 1.1; an  $F$ -stable maximal torus  $\mathbf{T}$  of  $\mathbf{G}$  defines the Weyl group  $W = N_{\mathbf{G}}(\mathbf{T})/\mathbf{T}$ , that we may identify to a reflection subgroup of  $\mathrm{GL}(X(\mathbf{T}))$  where  $X(\mathbf{T}) := \mathrm{Hom}(\mathbf{T}, \mathbb{G}_m)$ , attached to the root system  $\Sigma \subset X(\mathbf{T})$  of  $\mathbf{G}$  with respect to  $\mathbf{T}$ . The isogeny  $F$  induces a  $p$ -morphism  $F^* \in \mathrm{End}(X(\mathbf{T}))$  by the formula  $F^*(x) = x \circ F$  for  $x \in X(\mathbf{T})$ , that is there is a permutation  $\sigma$  of  $\Sigma$  such that for  $\alpha \in \Sigma$  we have  $F^*(\alpha) = q_\alpha \sigma(\alpha)$  for some power  $q_\alpha$  of  $p$ ; in particular  $F^* \in N_{\mathrm{End}(X(\mathbf{T}))}(W)$ .

If  $q, n$  are as in 1.1 then  $F^{*n}$  is  $q^n$  times an element of  $\mathrm{GL}(X(\mathbf{T}))$  of finite order, thus over  $X(\mathbf{T}) \otimes \mathbb{Z}[q^{-1}]$  we have  $F^* = q\phi$  where  $\phi$  is an automorphism of finite order which normalizes  $W$ . We call  $W\phi$  the *reflection coset associated to*  $(\mathbf{G}, F)$ .

Our setting is more general than that of [Broué-Malle] who considered only the special cases where  $F$  is a Frobenius endomorphism, or where  $\mathbf{G}^F$  is a Ree or Suzuki group. The results of the next subsection allow to extend the definition of Sylow  $\Phi$ -subtori of [Broué-Malle] to any  $(\mathbf{G}, F)$  as in 1.1.

 **$F$ -indecomposable tori.**

**Definition 2.1.** For  $\mathbf{G}, F$  as in 1.1, a non-trivial subtorus of  $\mathbf{G}$  is called  $F$ -indecomposable if it is  $F$ -stable and contains no proper non-trivial  $F$ -stable subtorus.

We say that a group  $G$  is an almost direct product of subgroups  $G_1$  and  $G_2$  if they commute, generate  $G$  and have finite intersection, and we define similarly an almost direct product of  $k$  subgroups by induction on  $k$ .

**Proposition 2.2.** For  $\mathbf{G}, F$  as in 1.1, any  $F$ -stable subtorus  $\mathbf{T}$  of  $\mathbf{G}$  is an almost direct product of  $F$ -indecomposable tori  $\mathbf{S}_1, \dots, \mathbf{S}_k$  and  $|\mathbf{T}^F| = |\mathbf{S}_1^F| \dots |\mathbf{S}_k^F|$ .

*Proof.* An  $F$ -stable subtorus  $\mathbf{S}$  corresponds to a pure  $F$ -stable sublattice  $X' \subset X := X(\mathbf{T})$  (see for example [Borel, III, Proposition 8.12]). Let  $d$  be the smallest power of  $F$  which is a split Frobenius, thus on  $X(\mathbf{T})$  we have  $F^{*d} = q^d \mathrm{Id}$ . Let  $\pi \in \mathrm{End}(X \otimes \mathbb{Q})$  be a projector on  $X' \otimes \mathbb{Q}$ . Then in  $\mathrm{End}(X \otimes \mathbb{Q})$  we can define the  $F$ -invariant projector  $\pi' := d^{-1} \sum_{i=1}^d F^{*i} \pi F^{*-i}$  and  $\mathrm{Ker} \pi' \cap X$  is another  $F$ -stable pure sublattice which after tensoring by  $\mathbb{Q}$  becomes a complement to  $X' \otimes \mathbb{Q}$ . This corresponds to an  $F$ -stable subtorus  $\mathbf{S}'$  such that  $K := \mathbf{S} \cap \mathbf{S}'$  is finite and  $\mathbf{T} = \mathbf{S}\mathbf{S}'$ . Iterating, we get the first part of the proposition.

The second part of the proposition results from the next two lemmas.  $\square$

**Lemma 2.3.** For  $\mathbf{G}, F$  as in 1.1, and  $K$  an  $F$ -stable finite normal subgroup of  $\mathbf{G}$ , then  $|(\mathbf{G}/K)^F| = |\mathbf{G}^F|$ .

*Proof.* First, we notice that  $K$  is central, thus abelian, since conjugating by  $\mathbf{G}$  being continuous must be trivial on  $K$ .

Then, the Galois cohomology long exact sequence:  $1 \rightarrow K^F \rightarrow \mathbf{G}^F \rightarrow (\mathbf{G}/K)^F \rightarrow H^1(F, K) \rightarrow 1$  shows the result using that  $|K^F| = |H^1(F, K)|$ .  $\square$

**Lemma 2.4.** Let  $\mathbf{G}$  as 1.1 be an almost direct product of  $F$ -stable connected subgroups  $\mathbf{G} = \mathbf{G}_1 \dots \mathbf{G}_k$ . Then  $|\mathbf{G}^F| = |\mathbf{G}_1^F| \dots |\mathbf{G}_k^F|$ .

*Proof.* It is enough to consider the case  $k = 2$  and then iterate. Thus, we assume  $\mathbf{G} = \mathbf{G}_1 \mathbf{G}_2$  where  $K = \mathbf{G}_1 \cap \mathbf{G}_2$  is finite. We quotient by  $K$ , which makes the product direct, and apply Lemma 2.3 twice.  $\square$

**Lemma 2.5.** *Let  $\mathbf{S}$  be an  $F$ -indecomposable torus, let  $\eta$  be the smallest power such that  $q^\eta \in \mathbb{Z}$ , and let  $d$  be the smallest power such that  $F^{d\eta}$  is a split Frobenius on  $\mathbf{S}$ . Let  $F^* = q\phi$  on  $X(\mathbf{S})$ ; then the characteristic polynomial  $\Phi$  of  $\phi$  is a factor in  $\mathbb{Z}[x, q^{-1}]$  of  $\Phi_d(x^\eta)$ , where  $\Phi_d(x)$  denotes the  $d$ -th cyclotomic polynomial. Further  $q^{\deg \Phi} \Phi(x/q) \in \mathbb{Z}[x]$  is irreducible and  $|\mathbf{S}^F| = \Phi(q)$ .*

*Proof.* Since  $F^{*d\eta}$  acts as  $q^{d\eta}$  on  $X := X(\mathbf{S})$ , the minimal polynomial  $P$  of  $F^*$  divides  $x^{d\eta} - q^{d\eta}$ .

The polynomial  $P$  is irreducible over  $\mathbb{Z}$ , otherwise a proper nontrivial factor  $P_1$  defines an  $F^*$ -stable pure proper non-trivial sublattice  $\text{Ker}(P_1(F^*))$  of  $X$ , which contradicts  $F$ -indecomposability of  $\mathbf{S}$ .

It follows that  $X$  is a  $\mathbb{Z}[x]/P$ -module by making  $x$  act by  $F^*$ , and  $X \otimes \mathbb{Q}[x]/P$  is a one-dimensional  $\mathbb{Q}[x]/P$ -vector space, otherwise a proper nontrivial subspace would define an  $F^*$ -stable pure sublattice of  $X$ . It follows that  $\dim \mathbf{S} = \deg P = \dim X$  and thus  $P$  is also the characteristic polynomial of  $F^*$ .

We have in  $\mathbb{Z}[x]$  the equality  $x^{d\eta} - q^{d\eta} = \prod_{d' | d} (q^{\eta \deg \Phi_{d'}} \Phi_{d'}(x^\eta/q^\eta))$ . Since  $P$  is irreducible it divides one of the factors, and since  $d\eta$  is minimal such that  $F^{*d\eta} = q^{d\eta} \text{Id}$ , that is minimal such that  $P$  divides  $x^{d\eta} - q^{d\eta}$ , we have that  $P$  divides  $q^{\eta \deg \Phi_d} \Phi_d(x^\eta/q^\eta)$ ; equivalently  $\Phi = q^{-\deg P} P(qx)$  divides  $\Phi_d(x^\eta)$ .

We have  $|\mathbf{S}^F| = |\text{Irr}(\mathbf{S}^F)| = |X/(F^* - 1)X| = \det(F^* - 1) = (-1)^{\deg P} P(1) = (-q)^{\deg \Phi} \Phi(1/q)$  where the second equality reflects the well known group isomorphism  $\text{Irr}(\mathbf{S}^F) \simeq X/(F^* - 1)X$  and the third is a general property of lattices. Finally, since  $\Phi$  is real and divides  $\Phi_d(x^\eta)$ , its roots are stable under taking inverses, thus  $(-q)^{\deg \Phi} \Phi(1/q) = \Phi(q)$ .  $\square$

We call  $q$ -cyclotomic the polynomials  $\Phi$  of Lemma 2.5. In other terms

**Definition 2.6.** *For  $q$  as in 1.1, where  $q^\eta$  is the smallest power of  $q$  in  $\mathbb{Z}$ , we call  $q$ -cyclotomic the monic polynomials  $\Phi \in \mathbb{Z}[x, q^{-1}]$  such that  $q^{\deg \Phi} \Phi(x/q)$  is a  $\mathbb{Z}[x]$ -irreducible factor of some  $x^{d\eta} - q^{d\eta}$ .*

In the study of semisimple reductive groups we will need the  $q$ -cyclotomic polynomials of Lemma 2.7. Note that if  $d$  is minimal in Definition 2.6, then  $\Phi$  is a factor in  $\mathbb{Z}[x, q^{-1}]$  of  $\Phi_d(x^\eta)$ . We are interested in that number  $d$  rather than  $d\eta$ , and to emphasize this we write  $\Phi_{\eta, d}$  in the following examples.

**Lemma 2.7.** *When  $q \in \mathbb{Z}$ , the  $q$ -cyclotomic polynomials are the cyclotomic polynomials.*

*When  $q$  is an odd power of  $\sqrt{2}$ , the following polynomials are  $q$ -cyclotomic:  $\Phi_{2,1}(x) := \Phi_1(x^2)$ ,  $\Phi_{2,2}(x) := \Phi_2(x^2)$ ,  $\Phi_{2,6}(x) := \Phi_6(x^2)$ , the factors  $\Phi'_{2,4} := x^2 + \sqrt{2}x + 1$  and  $\Phi''_{2,4} := x^2 - \sqrt{2}x + 1$  of  $\Phi_4(x^2)$ , and the factors  $\Phi'_{2,12} := x^4 + x^3\sqrt{2} + x^2 + x\sqrt{2} + 1$  and  $\Phi''_{2,12} := x^4 - x^3\sqrt{2} + x^2 - x\sqrt{2} + 1$  of  $\Phi_{12}(x^2)$ .*

*When  $q$  is an odd power of  $\sqrt{3}$ , the following polynomials are  $q$ -cyclotomic:  $\Phi_{2,1}(x)$ ,  $\Phi_{2,2}(x)$  and the factors  $\Phi'_{2,6} := x^2 + x\sqrt{3} + 1$  and  $\Phi''_{2,6} := x^2 - x\sqrt{3} + 1$  of  $\Phi_6(x^2)$ .*

*Proof.* When  $q \in \mathbb{Z}$  the formula  $P \mapsto q^{-\deg P} P(qx)$  establishes a bijection between  $\mathbb{Z}[x]$ -irreducible factors of  $x^d - q^d$  and  $\mathbb{Z}[x]$ -irreducible factors of  $x^d - 1$ , that is cyclotomic polynomials, which gives the first case of the lemma.

For the other cases, we have to check for each given  $\Phi$  that  $q^{\deg \Phi} \Phi(x/q)$  is in  $\mathbb{Z}[x]$  and irreducible.  $\square$

**Proposition 2.8.** *Let  $\mathbf{S}, \eta, d, \Phi$  be as in 2.5 and let  $P = q^{\deg \Phi} \Phi(x^\eta/q^\eta)$  be the characteristic polynomial of  $F^*$ .*

- (1) *Assume that either  $q \in \mathbb{Z}$  or that  $\mathbb{Z}[x, q^{-\eta}]/P$  is integrally closed. Then  $\mathbf{S}^F \simeq \mathbb{Z}/\Phi(q)$ .*
- (2) *Let  $m$  be a divisor of  $\Phi(q)$ , and assume either that  $d \in \{1, 2\}$  and  $q \in \mathbb{Z}$  or that  $m$  is prime to  $d\eta$ ; then we have a natural isomorphism  $\text{Irr}(\mathbf{S}^F)/m \text{Irr}(\mathbf{S}^F) \simeq \text{Ker}(F^* - 1 \mid X(\mathbf{S})/mX(\mathbf{S}))$ .*

*Proof.* Proceeding as in the proof of Lemma 2.5 we set  $X = X(\mathbf{S})$  and  $\bar{X} = X/(F^* - 1)X \simeq \text{Irr}(\mathbf{S}^F)$ . Letting  $x$  act as  $F^*$  makes  $X$  into a  $\mathbb{Z}[x]/P$ -module, and  $\bar{X}$  a  $\mathbb{Z}[x]/(P, x - 1)$ -module. Since  $\mathbb{Z}[x]/(P, x - 1) = \mathbb{Z}/P(1) = \mathbb{Z}/\Phi(q)$  we find that the exponent of  $\bar{X}$  divides  $\Phi(q)$ .

Let  $A := \mathbb{Z}[x, q^{-\eta}]/P$ . The extension  $\mathbb{Z}[x]/P \hookrightarrow A/P$  is flat thus  $\bar{X} \otimes_{\mathbb{Z}[x]/P} A \simeq X'/(F^* - 1)X'$  where  $X' = X \otimes_{\mathbb{Z}[x]/P} A$ ; and since the exponent of  $\bar{X}$  divides  $\Phi(q)$  which is prime to  $q^\eta$ , we have  $\bar{X} \simeq \bar{X} \otimes_{\mathbb{Z}[x]/P} A$ . Under the assumptions of (1) the ring  $A$  is Dedekind: if  $\eta \neq 1$  then  $A$  is integrally closed thus Dedekind; if  $\eta = 1$  then  $A \simeq \mathbb{Z}[x, q^{-1}]/\Phi_d$  where the isomorphism is given by  $x \mapsto x/q$ , and is a localization of the Dedekind ring  $\mathbb{Z}[x]/\Phi_d$  by  $q$ . Thus  $X'$  identifies to a fractional ideal  $\mathfrak{J}$  of  $A$  and  $\bar{X} \simeq \mathfrak{J}/(x - 1)\mathfrak{J}$ . If  $e$  is the exponent of  $\bar{X}$  we have thus  $e\mathfrak{J} \subset (x - 1)\mathfrak{J}$ , which implies that  $x - 1$  divides  $e$  in  $A$ . This in turn implies that the norm  $(-1)^{\deg P} P(1) = \Phi(q)$  of  $(x - 1)$  divides  $e$  in  $\mathbb{Z}$ , thus  $e = \Phi(q)$  and  $\bar{X} \simeq \mathbb{Z}/\Phi(q)$  and the same isomorphism holds for the dual abelian group  $\mathbf{S}^F$ .

For (2), note that by construction  $\bar{X}/m\bar{X}$  is the biggest quotient of  $X$  on which both  $F^* - 1$  and the multiplication by  $m$  vanish. It is thus equal to the biggest quotient of  $X/mX$  on which  $F^* - 1$  vanishes. Thus the question is to see that  $\text{Ker}(F^* - 1)$  has a complement in  $X/mX$ .

If  $q \in \mathbb{Z}$  and  $d \in \{1, 2\}$  we have  $P = x \pm q$  so  $X \simeq \mathbb{Z}$  on which  $F^*$  acts by  $\mp q$  and  $\bar{X} = X/(q \pm 1)$  of which  $X/mX$  is a quotient, so  $F^* - 1$  vanishes on  $X/mX$  which is thus equal to  $\bar{X}/m\bar{X}$  and there is nothing to prove.

Assume now  $m$  prime to  $d\eta$ . There exists  $R \in \mathbb{Z}[x]$  such that in  $\mathbb{Z}[x]$  we have  $P = (x - 1)R + P(1)$ . Taking derivatives, we get  $P' = (x - 1)R' + R$ , whence  $R(1) = P'(1)$ . Let  $\delta$  be the discriminant of  $P$ ; we can find polynomials  $M, N \in \mathbb{Z}[x]$  such that  $MP + NP' = \delta$ , which evaluating at 1 gives  $M(1)P(1) + N(1)P'(1) = \delta$ . Since  $q$  is prime to  $P(1)$ , thus to  $m$ , and  $\delta$  is a divisor of the discriminant of  $X^{d\eta} - q^{d\eta}$ , equal to  $q^{d\eta(d\eta-1)}(d\eta)^{d\eta}$ , thus prime to  $m$ , we find that  $P'(1)$  is prime to  $m$ . In  $(\mathbb{Z}/m)[x]$  we have  $P = (x - 1)R$ , thus applied to  $F^*$  we get that on  $X/mX$  we have  $0 = P(F^*) = (F^* - 1)R(F^*)$ , whence  $\text{Ker}(F^* - 1) + \text{Ker}(R(F^*)) = X/mX$ . Since  $R(1)$  is prime to  $m$ , we can write  $1 \equiv Q(x - 1) + aR$  in  $(\mathbb{Z}/m)[x]$  for some  $Q \in (\mathbb{Z}/m)[x]$  and  $a$  the inverse (mod  $m$ ) of  $R(1)$ . This proves that  $\text{Ker}(F^* - 1) \cap \text{Ker}(R(F^*)) = 0$  thus  $X/mX$  is the direct sum of  $\text{Ker}(F^* - 1)$  and  $\text{Ker}(R(F^*))$  q.e.d.  $\square$

**Complex reflection cosets.** (1) to (3) below are classical results of Springer and Lehrer.

**Proposition 2.9.** *Let  $V$  be a finite dimensional vector space over a subfield  $k$  of  $\mathbb{C}$ , let  $W \subset \text{GL}(V)$  be a finite complex reflection group and let  $\phi \in N_{\text{GL}(V)}(W)$ , so that  $W\phi$  is a reflection coset; let  $(d_1, \varepsilon_1), \dots, (d_n, \varepsilon_n)$  be its generalized degrees (see for instance [Broué, 4.2]). For  $\zeta$  a root of unity define  $a(\zeta)$  as the multiset of the  $d_i$  such that  $\zeta^{d_i} = \varepsilon_i$ . Then:*

- (1) For any root of unity  $\zeta$ , the maximum dimension when  $w\phi$  runs over  $W\phi$  of a  $\zeta$ -eigenspace of  $w\phi$  on  $V \otimes_k k[\zeta]$  is  $|a(\zeta)|$ .
- (2) For  $w\phi \in W\phi$  denote  $V_{w,\zeta} \subset V \otimes_k k[\zeta]$  its  $\zeta$ -eigenspace. Assume  $\dim V_{w,\zeta} = |a(\zeta)|$  and let  $C = C_W(V_{w,\zeta})$  and  $N = N_W(V_{w,\zeta})$ . Then  $N/C$  is a complex reflection group acting on  $V_{w,\zeta}$ , with reflection degrees  $a(\zeta)$ .
- (3) Any two subspaces  $V_{w,\zeta}$  and  $V_{w',\zeta}$  of dimension  $|a(\zeta)|$  are  $W$ -conjugate.
- (4) For  $w\phi$  as in (2) the natural actions of  $w\phi$  on  $N$  and  $C$  induce the trivial action on  $N/C$ .
- (5) Let  $a \in \mathbb{Z}$  be such that  $(W\phi)^a = W\phi$  and such that  $\zeta$  and  $\zeta^a$  are conjugate by  $\text{Gal}(k[\zeta]/k)$ . Then for  $w\phi$  as in (2) there exists  $v \in N_W(N) \cap N_W(C)$  which conjugates  $w\phi C$  to  $(w\phi)^a C$ .

*Proof.* For (1) see for instance [Broué, 5.2], for (2) see [Broué, 5.6(3) and (4)] and for (3) see [Broué, 5.6 (1)]. (4) results from the observation that if  $n \in N$  and  $v \in V_{w,\zeta}$  then  $(n^{-1} \cdot {}^{w\phi}n)(v) = (n^{-1}w\phi n(w\phi)^{-1})(v) = (n^{-1}w\phi n)(\zeta^{-1}v) = (n^{-1}w\phi)(\zeta^{-1}n(v)) = (n^{-1})(n(v)) = v$  thus  $n^{-1} \cdot {}^{w\phi}n \in C$ .

For (5),  $\text{Gal}(k[\zeta]/k)$  acts naturally on  $V \otimes_k k[\zeta]$ , commuting with  $\text{GL}(V)$ , in particular with  $W$  and  $\phi$ . If  $\sigma \in \text{Gal}(k[\zeta]/k)$  is such that  $\sigma(\zeta) = \zeta^a$ , let  $\zeta^{a'} = \sigma^{-1}(\zeta)$ . Then  $\sigma^{-1}(V_{w,\zeta}) = V_{w,\zeta^{a'}}$ . It follows that  $N = N_W(V_{w,\zeta^{a'}})$  and  $C = C_W(V_{w,\zeta^{a'}})$ .

Now since  $a'$  is the inverse of  $a$  modulo the order of  $\zeta$  the space  $V_{w,\zeta^{a'}}$  is the  $\zeta$ -eigenspace of  $(w\phi)^a$ . By assumption we have  $(w\phi)^a \in W\phi$ . Since two maximal  $\zeta$ -eigenspaces of elements of  $W\phi$  are conjugate by (3) there exists  $v \in W$  which conjugates  $V_{w,\zeta}$  to  $V_{w,\zeta^{a'}}$ , and  $v \in N_W(N) \cap N_W(C)$  since  $N = N_W(V_{w,\zeta^{a'}})$  and  $C = C_W(V_{w,\zeta^{a'}})$ . The element  $v$  thus conjugates the set  $w\phi C$  of elements which have  $V_{w,\zeta}$  as  $\zeta$ -eigenspace to the set  $(w\phi)^a C$  of elements which have  $V_{w,\zeta^{a'}}$  as  $\zeta$ -eigenspace.  $\square$

**Generic Sylow subgroups.** We define the Sylow  $\Phi$ -subtori of  $(\mathbf{G}, F)$ , first in the case when  $\mathbf{G}$  is quasi-simple, then in the case of descent of scalars.

From now on we assume  $\mathbf{G}$  semisimple. Then, if  $(d_1, \varepsilon_1), \dots, (d_n, \varepsilon_n)$  are the generalized degrees of the reflection coset  $W\phi$ , we have (see [Steinberg, 11.16])

$$(2.10) \quad |\mathbf{G}^F| = q^{\sum_i (d_i - 1)} \prod_i (q^{d_i} - \varepsilon_i).$$

**Proposition 2.11.** *Let  $\mathbf{G}$  be as in 1.1 and quasi-simple. Then we can rewrite the order formula 2.10 for  $|\mathbf{G}^F|$  as*

$$(2.12) \quad |\mathbf{G}^F| = q^{\sum_i (d_i - 1)} \prod_{\Phi \in \mathcal{P}} \Phi(q)^{n_\Phi}$$

where  $\mathcal{P}$  is a set of  $q$ -cyclotomic polynomials, and where  $0 \neq n_\Phi = |a(\zeta)|$  (see 2.9) for any root  $\zeta$  of  $\Phi$ . For each  $\Phi \in \mathcal{P}$  there exists a non-trivial  $F$ -stable subtorus  $\mathbf{S}_\Phi$  of  $\mathbf{G}$  such that  $|\mathbf{S}_\Phi^F| = \Phi(q)^{n_\Phi}$ .

We note that if  $\mathbf{G}^F$  is a Ree or Suzuki group, the  $\eta$  of Definition 2.6 is 2. Otherwise  $\eta = 1$  and the  $q$ -cyclotomic polynomials are cyclotomic polynomials.

We call any  $F$ -stable torus  $\mathbf{S}$  such that  $|\mathbf{S}^F|$  is a power of  $\Phi(q)$  a  $\Phi$ -torus, and tori  $\mathbf{S}_\Phi$  as above are called *Sylow  $\Phi$ -subtori* of  $(\mathbf{G}, F)$  — we abuse notation and call

them Sylow  $\Phi$ -subtori of  $\mathbf{G}$  when  $F$  is clear from the context; they are the almost direct product of  $n_\Phi$   $F$ -indecomposable  $\Phi$ -tori.

*Proof.* Proposition 2.11 is essentially in [Broué-Malle] but let us reprove it.

First, we note that assuming  $|\mathbf{G}^F|$  has a decomposition of the form 2.12, the value of  $n_\Phi$  results from 2.10: let  $\zeta$  be any root of  $\Phi(x)$ . Then  $(x - \zeta)$  divides  $\Phi(x)$  with multiplicity one, and does not divide any another  $\Phi'(x)$  for  $\Phi' \in \mathcal{P}$  since the  $\Phi(x/q)$  are distinct irreducible polynomials in  $\mathbb{Q}[x]$ . Thus  $n_\Phi$  is the number of pairs  $(d_i, \varepsilon_i)$  such that  $x - \zeta$  divides  $x^{d_i} - \varepsilon_i$ .

There is a decomposition of the form 2.12: if  $\eta = 1$  we get such a decomposition of  $|\mathbf{G}^F|$  by decomposing each term of 2.10 into a product of cyclotomic polynomials. Otherwise  $\mathbf{G}^F$  is a Ree or Suzuki group,  $\eta = 2$  and  $q$  is an odd power of  $\sqrt{2}$  or  $\sqrt{3}$ , and the set  $\mathcal{P}$  and the decomposition of the form 2.12 is given by what follows:

$(\mathbf{G}, F)$	$ \mathbf{G}^F $	generalized degrees of $W\phi$
${}^2B_2(q^2)$	$q^4(\Phi_{2,1}\Phi'_{2,4}\Phi''_{2,4})(q)$	$\{(2, 1), (4, -1)\}$
${}^2F_4(q^2)$	$q^{24}(\Phi_{2,1}^2\Phi_{2,2}^2\Phi_{2,4}^2\Phi_{2,4}'^2\Phi_{2,6}\Phi_{2,6}'\Phi_{2,12}\Phi_{2,12}'^2)(q)$	$\{(2, 1), (6, -1), (8, 1), (12, -1)\}$
${}^2G_2(q^2)$	$q^6(\Phi_{2,1}\Phi_{2,2}\Phi_{2,6}\Phi_{2,6}')(q)$	$\{(2, 1), (6, -1)\}$

Note that for  $\eta = 2$  our “ $q$ -cyclotomic polynomials” are the “ $(tp)$ -cyclotomic polynomials” defined in [Broué-Malle, 3.14].

To construct the torus  $\mathbf{S}_\Phi$  for  $\Phi \in \mathcal{P}$ , let us choose  $\zeta$  a root of  $\Phi$  and  $w$  as in (2) of Proposition 2.9. Then if  $\mathbf{T}_w$  is a maximal torus of type  $w$  with respect to  $\mathbf{T}$ , so that  $(\mathbf{T}_w, F) \simeq (\mathbf{T}, wF)$ , the characteristic polynomial of  $w\phi$  on  $X(\mathbf{T})$  has  $\Phi(x)^{n_\Phi}$  as a factor; the kernel of  $\Phi(w\phi)$  on  $X(\mathbf{T})$  is a pure sublattice corresponding to a subtorus  $\mathbf{S}_\Phi$  of  $\mathbf{T}_w$  such that  $|\mathbf{S}_\Phi^F| = \Phi(q)^{n_\Phi}$ .  $\square$

**Proposition 2.13.** *Let  $(\mathbf{G}, F)$  be as in 1.1, semisimple and such that the Dynkin diagram of  $\mathbf{G}$  has  $n$  connected components permuted transitively by  $F$ . Then there exists a reductive group  $\mathbf{G}_1$  with isogeny  $F_1$  such that up to isomorphism  $\mathbf{G}$  is a “descent of scalars”  $\mathbf{G} = \mathbf{G}_1^n$  with  $F(g_1, \dots, g_n) = (g_2, \dots, g_n, F_1(g_1))$ .*

*Then  $\mathbf{G}^F \simeq \mathbf{G}_1^{F_1}$ , and if the scalar associated to  $(\mathbf{G}, F)$  is  $q$  that associated to  $(\mathbf{G}_1, F_1)$  is  $q_1 := q^n$ . Thus we have  $|\mathbf{G}^F| = q^n \sum_i (d_i - 1) \prod_{\Phi \in \mathcal{P}} \Phi(q^n)^{n_\Phi}$  where  $d_i, \mathcal{P}, n_\Phi$  are as given by 2.11 for  $(\mathbf{G}_1, F_1, q_1)$ .*

*Here again, for  $\Phi \in \mathcal{P}$  there exists a Sylow  $\Phi$ -subtorus of  $\mathbf{G}$ , that is an  $F$ -stable subtorus  $\mathbf{S}_\Phi$  such that  $|\mathbf{S}_\Phi^F| = \Phi(q^n)^{n_\Phi}$ .*

*Proof.* The proposition is obvious apart perhaps for the statement about the existence of  $\mathbf{S}_\Phi$ . This results in particular from the following lemma that we need for future reference.

**Lemma 2.14.** *In the situation of Proposition 2.13, let  $(\mathbf{T}, wF)$  where  $\mathbf{T} = \mathbf{T}_1^n$  be a maximal torus of type  $w = (1, \dots, 1, w_1)$  of  $\mathbf{G}$  and define  $\phi$  on  $V = X(\mathbf{T}) \otimes \mathbb{C}$  (resp.  $\phi_1$  on  $V_1 = X(\mathbf{T}_1) \otimes \mathbb{C}$ ) by  $F^* = q\phi$  (resp.  $F_1^* = q_1\phi_1$ ). Then if the characteristic polynomial of  $w_1\phi_1$  is  $P(x)$ , that of  $w\phi$  is  $P(x^n)$ . Let  $\Phi$  be a  $q_1$ -cyclotomic factor of  $P$  (corresponding to a  $\mathbb{Z}[x]$ -irreducible factor of the characteristic polynomial of  $w_1F_1^*$ ) and let  $\zeta$  be a root of  $\Phi(x^n)$ . Denote by  $V_\zeta$  the  $\zeta$ -eigenspace of  $w\phi$  (resp. by  $V_{1,\zeta^n}$  the  $\zeta^n$ -eigenspace of  $w_1\phi_1$ ).*

*Let  $\mathbf{S}_1$  be the Sylow  $\Phi$ -subtorus of  $(\mathbf{G}_1, F_1)$  determined by  $\text{Ker}(\Phi(w_1\phi_1))$ , and  $\mathbf{S}$  be the  $wF$ -stable subtorus of  $\mathbf{T}$  determined by  $\text{Ker}(\Phi((w\phi)^n))$ . Then  $\mathbf{S}$  is a Sylow*

$\Phi$ -subtorus of  $(\mathbf{G}, F)$  and

$$N_W(V_\zeta)/C_W(V_\zeta) \simeq N_{W_1}(V_{1,\zeta^n})/C_{W_1}(V_{1,\zeta^n}) \simeq N_{\mathbf{G}_1}(\mathbf{S}_1)/C_{\mathbf{G}_1}(\mathbf{S}_1) \simeq N_{\mathbf{G}}(\mathbf{S})/C_{\mathbf{G}}(\mathbf{S})$$

and we have an isomorphism  $\mathbf{S}^{wF} \simeq \mathbf{S}_1^{w_1F_1}$  compatible with the actions of  $N_{\mathbf{G}}(\mathbf{S})/C_{\mathbf{G}}(\mathbf{S})$  and  $N_{\mathbf{G}_1}(\mathbf{S}_1)/C_{\mathbf{G}_1}(\mathbf{S}_1)$  and the above isomorphism.

*Proof.* Let  $X = X(\mathbf{T})$ ,  $X_1 = X(\mathbf{T}_1)$ . On  $X \simeq X_1^n$  we have  $F^*(x_1, \dots, x_n) = (x_2, \dots, x_n, F_1^*(x_1))$ , thus  $\phi(x_1, \dots, x_n) = (q^{-1}x_2, \dots, q^{-1}x_n, q_1q^{-1}x_1)$ . It follows by an easy computation that  $V_\zeta$  is equal to the set of  $(x, (q\zeta)x, \dots, (q\zeta)^{n-1}x)$  where  $x \in V_{1,\zeta^n}$ , that  $C_W(V_\zeta) = \{(v_1, \dots, v_n) \mid v_i \in C_{W_1}(V_{1,\zeta^n})\}$  and that  $N_W(V_\zeta) = \{(vv_1, \dots, vv_n) \mid v \in N_{W_1}(V_{1,\zeta^n}), v_i \in C_{W_1}(V_{1,\zeta^n})\}$ . This shows that  $N_W(V_\zeta)/C_W(V_\zeta) \simeq N_{W_1}(V_{1,\zeta^n})/C_{W_1}(V_{1,\zeta^n})$ . Since when  $\zeta$  runs over the roots of  $\Phi(x^n)$  the  $q_1\zeta^n$  are roots of the same  $\mathbb{Z}[x]$ -irreducible polynomial  $q_1^{\deg \Phi} \Phi(x/q_1)$ , the  $\zeta^n$  are Galois conjugate thus  $C_{W_1}(V_{1,\zeta^n})$  (resp.  $N_{W_1}(V_{1,\zeta^n})$ ) centralizes (resp. normalizes) all the conjugate eigenspaces, whence our claim that  $N_{W_1}(V_{1,\zeta^n})/C_{W_1}(V_{1,\zeta^n}) \simeq N_{\mathbf{G}_1}(\mathbf{S}_1)/C_{\mathbf{G}_1}(\mathbf{S}_1)$ . Now  $\text{Ker}(\Phi((w\phi)^n))$  is the span of  $V_\zeta$  for all roots  $\zeta$  of  $\Phi(x^n)$  and by the analysis above  $C_W(V_\zeta)$  and  $N_W(V_\zeta)$  are independent of  $\zeta$ , thus isomorphic to  $C_W(\mathbf{S})$  and  $N_W(\mathbf{S})$ .

We have the following commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{wF^*-1} & X & \xrightarrow{\text{Res}} & \text{Irr}(\mathbf{T}^{wF}) & \longrightarrow & 1 \\ \downarrow \Sigma & & \downarrow \Sigma & & \downarrow \sim & & \\ X_1 & \xrightarrow{w_1F_1^*-1} & X_1 & \xrightarrow{\text{Res}} & \text{Irr}(\mathbf{T}_1^{w_1F_1}) & \longrightarrow & 1 \end{array}$$

where  $\Sigma$  is the map  $(x_1, \dots, x_n) \mapsto x_1 + \dots + x_n$ . Since we have  $\Sigma \circ (wF)^n = w_1F_1 \circ \Sigma$ , for any polynomial  $Q$  the morphism  $\Sigma$  induces a surjective morphism  $\text{Ker}(Q((wF^*)^n)) \rightarrow \text{Ker}(Q(w_1F_1^*))$  whence for  $Q = P$  a surjection  $\text{Irr}(\mathbf{S}^{wF}) \rightarrow \text{Irr}(\mathbf{S}_1^{w_1F_1})$ ; since  $|\mathbf{S}^{wF}|$  is prime to  $|\mathbf{T}^{wF}/\mathbf{S}^{wF}|$  this surjection must be an isomorphism. Extended to  $V = X \otimes \mathbb{C}$ , the map  $\Sigma$  sends  $V_\zeta$  to  $V_{1,\zeta^n}$  and sends the action of  $N_W(V_\zeta)/C_W(V_\zeta)$  to that of  $N_{W_1}(V_{1,\zeta^n})/C_{W_1}(V_{1,\zeta^n})$ , whence the last statement of the lemma.  $\square$

Note that any element of  $W\phi$  is conjugate to an element of the form  $(1, \dots, 1, w_1)\phi_1$  so the form of  $w$  in the statement of Lemma 2.14 covers all the types of maximal tori.  $\square$

**Remark 2.15.** If the generalized degrees of  $W_1\phi_1$  are  $(d_i, \varepsilon_i)_i$  those of  $W\phi$  are  $(d_i, \eta_{i,j})$  where  $\eta_{i,j}$  for  $j \in \{1, \dots, n\}$  runs over the  $n$ -th roots of  $\varepsilon_i$ . It follows that  $n_\Phi$  can be defined in terms of  $W\phi$  as it is also the number of  $(d_i, \eta_{i,j})$  such that  $\zeta^{d_i} = \eta_{i,j}$ , where  $\zeta$  is any root of  $\Phi(x^n)$ .

**Remark 2.16.** For  $\Phi \in \mathcal{P}(\mathbf{G})$ , a Sylow  $\Phi$ -subtorus of  $\mathbf{G}$  is a ‘‘power’’ of a subtorus  $\mathbf{S}_0$  such that  $|\mathbf{S}_0^F| = \Phi(q)$ . If  $\mathbf{G}$  is quasi-simple, such a subtorus  $\mathbf{S}_0$  is  $F$ -indecomposable (since then the polynomial  $\Phi$  is  $q$ -cyclotomic). But this is no longer true for a descent of scalars. First, a cyclotomic polynomial in  $x^n$  decomposes in several cyclotomic polynomials according to the formula  $\Phi_d(x^n) = \prod_{\{\mu \mid n, \frac{n}{\mu} \text{ prime to } d\}} \Phi_{\mu d}(x)$  (see [Broué-Malle, Appendice 2]). But there could be further decompositions: for instance, the characteristic polynomial of  $F^*$  on a Coxeter torus of a semisimple group  $\mathbf{G}$  of type  $B_2$  over  $\mathbb{F}_2$  is  $x^2 + 4$ , which is  $\mathbb{Z}$ -irreducible. But on a descent of scalars  $\mathbf{G} \times \mathbf{G}$ , the characteristic polynomial



of  $F^*$  on a lift of scalars of this torus is  $x^4 + 4$  which is no longer  $\mathbb{Z}$ -irreducible:  $x^4 + 4 = (x^2 + 2x + 2)(x^2 - 2x + 2)$ , so the torus seen inside the descent of scalars is no longer  $F$ -indecomposable.

We could have decomposed  $|\mathbf{G}^F|$  into a product of  $q$ -cyclotomic polynomials corresponding to  $F$ -indecomposable tori, but in the case of descent of scalars it was convenient to use larger tori.

**Remark 2.17.** An arbitrary semisimple reductive group is of the form  $\mathbf{G} = \mathbf{G}_1 \dots \mathbf{G}_k$ , an almost direct product of descents of scalars of quasi-simple groups  $\mathbf{G}_i$ , corresponding to the orbits of  $F$  on the connected components of the Dynkin diagram of  $\mathbf{G}$ . Then we have  $|\mathbf{G}^F| = |\mathbf{G}_1^F| \dots |\mathbf{G}_k^F|$  by Lemma 2.4, and similarly, if  $\mathbf{S}$  is an  $F$ -stable torus of  $\mathbf{G}$ , and  $\mathbf{S}_i = \mathbf{S} \cap \mathbf{G}_i$ , then  $|\mathbf{S}^F| = |\mathbf{S}_1^F| \dots |\mathbf{S}_k^F|$ . This can be used to give a global decomposition of  $|\mathbf{G}^F|$ , but the polynomials  $\mathcal{P}$  in one factor could divide those in another. For instance we could have  $\Phi'_{2,4}$  for a factor of  $\mathbf{G}$  of type  ${}^2B_2$  and  $\Phi_8$  for another factor of type  $B_2$ . Because of this it is cumbersome to give a global statement.

From now on we fix  $(\mathbf{G}, F)$  as in 2.13, which determines  $q, n$ , and  $\eta$  minimal such that  $q^{n\eta} \in \mathbb{Z}$ . This allows in the next definition to omit the mention of  $\mathbf{G}$  and  $F$  from the notation  $d(\ell)$ .

**Definition 2.18.** Let  $\ell$  be a prime number different from  $p$ . In the context of 2.13 we define  $d(\ell)$  as the order of  $q^{n\eta} \pmod{\ell}$  ( $\pmod{4}$  if  $\ell = 2$ ).

In particular  $\ell | \Phi_{d(\ell)}(q^{n\eta})$ .

The next proposition extends some of the Sylow theorems of [Broué-Malle], and introduces a complex reflection group  $W_\Phi$  attached to each  $\Phi$  in the set  $\mathcal{P}$  of 2.11.

**Proposition 2.19.** Under the assumptions of 2.13, let  $\mathbf{T}$  be an  $F$ -stable maximal torus of  $\mathbf{G}$  in an  $F$ -stable Borel subgroup, and let  $W\phi \subset \mathrm{GL}(X(\mathbf{T}))$  be the reflection coset associated to  $(\mathbf{G}, F)$ . Then for each  $\Phi \in \mathcal{P}$ :

- (1) If  $\zeta$  is a root of  $\Phi(x^n)$  and  $w$  is as in 2.9(2), a maximal torus of  $\mathbf{G}$  of type  $w$  with respect to  $\mathbf{T}$  contains a unique Sylow  $\Phi$ -subtorus  $\mathbf{S}$ .

For  $\zeta, w$  as in (1) let  $W_\Phi = N_W(V_\zeta)/C_W(V_\zeta)$  where  $V_\zeta$  is the  $\zeta$ -eigenspace of  $w\phi$  on  $V = X(\mathbf{T}) \otimes \mathbb{C}$ .

- (2) For  $\mathbf{S}$  as in (1) we have  $N_{\mathbf{G}^F}(\mathbf{S})/C_{\mathbf{G}^F}(\mathbf{S}) = N_{\mathbf{G}}(\mathbf{S})/C_{\mathbf{G}}(\mathbf{S}) \simeq W_\Phi$ , and  $W_\Phi$  can be identified to a subgroup of  $\mathrm{GL}(X(\mathbf{S}))$ .
- (3) The Sylow  $\Phi$ -tori of  $\mathbf{G}$  are  $\mathbf{G}^F$ -conjugate.
- (4) Let  $\ell \neq p$  be a prime number, and assume that  $\Phi$  divides  $\Phi_{d(\ell)}$  (see Definition 2.18). Then unless  $\ell = 2$  and  $(\mathbf{G}_1, F_1)$  is of type  ${}^2G_2$ , any Sylow  $\ell$ -subgroup of  $W_\Phi$  acts faithfully on the subgroup of  $\ell$ -elements  $\mathbf{S}_\ell^F$  of  $\mathbf{S}^F$ .

*Proof.* For (1) we consider a torus  $(\mathbf{T}, wF)$  of type  $w$ . Then a  $wF$ -stable subtorus corresponds to the span of a subset of eigenspaces of  $w\phi$  on  $V$ . Since the polynomials  $\Phi$  are prime to each other the polynomials  $\Phi(x^n)$  are also, thus  $q\zeta$  is root of no other factor of the characteristic polynomial of  $w\phi$  than  $\Phi(x^n)$ . Thus the  $\mathbf{S}$  defined in Lemma 2.14, which we will denote  $\mathbf{S}_0$ , is unique.

Let us show (2). Let  $(\mathbf{T}_w, F, \mathbf{S})$  be conjugate to  $(\mathbf{T}, wF, \mathbf{S}_0)$ . Let  $\mathbf{L} = C_{\mathbf{G}}(\mathbf{S})$ , which, as the centralizer of a torus, is a Levi subgroup. Then we note that  $N_{\mathbf{G}}(\mathbf{S}) \subset N_{\mathbf{G}}(\mathbf{L})$ . It follows that we can find representatives of  $N_{\mathbf{G}}(\mathbf{S})$  modulo  $\mathbf{L}$  in  $N_{\mathbf{G}}(\mathbf{T}_w)$  since for  $n \in N_{\mathbf{G}}(\mathbf{S})$  the torus  ${}^n\mathbf{T}_w$  is another maximal torus of  $\mathbf{L}$  which

is thus  $\mathbf{L}$ -conjugate to  $\mathbf{T}_w$ . We thus get that  $N_{\mathbf{G}}(\mathbf{S})/\mathbf{L} = N_{\mathbf{G}}(\mathbf{S}, \mathbf{T}_w)/(N_{\mathbf{G}}(\mathbf{T}_w) \cap \mathbf{L})$ ; transferring this to  $\mathbf{T}$  and then to  $W$  we get  $N_{\mathbf{G}}(\mathbf{S}, \mathbf{T}_w)/(N_{\mathbf{G}}(\mathbf{T}_w) \cap \mathbf{L}) \simeq N_W(\mathbf{S}_0)/C_W(\mathbf{S}_0)$  where  $\mathbf{S}_0$  is the subtorus of  $\mathbf{T}$  determined by  $\text{Ker}(P(wF^*))$  where  $P = \Phi(x^n/q^n)$ . The action of  $F$  is transferred to the action of  $w\phi$  on this quotient.

That  $N_W(\mathbf{S}_0) = N_W(V_{\zeta})$  and  $C_W(\mathbf{S}_0) = C_W(V_{\zeta})$  was given in 2.14.

By 2.9(4) we see that the action of  $w\phi$  on  $N_W(\mathbf{S}_0)/C_W(\mathbf{S}_0)$  is trivial, thus also that of  $F$  on  $N_{\mathbf{G}}(\mathbf{S})/C_{\mathbf{G}}(\mathbf{S})$ , thus  $N_{\mathbf{G}}(\mathbf{S})/C_{\mathbf{G}}(\mathbf{S}) = (N_{\mathbf{G}}(\mathbf{S})/C_{\mathbf{G}}(\mathbf{S}))^F = N_{\mathbf{G}}(\mathbf{S})^F/C_{\mathbf{G}}(\mathbf{S})^F = N_{\mathbf{G}^F}(\mathbf{S})/C_{\mathbf{G}^F}(\mathbf{S})$ , the second equality since  $\mathbf{L} = C_{\mathbf{G}}(\mathbf{S})$  is connected. Finally, the last part of (2) results from the fact that the representation of  $W_{\Phi}$  on  $X(\mathbf{S}_0)$ , extended to  $X(\mathbf{S}_0) \otimes \mathbb{C}$  has as summand the representation of  $W_{\Phi}$  on  $V_{\zeta}$ , which is the reflection representation, thus faithful.

(3) is a direct translation of 2.9(3): when brought to subtori of  $\mathbf{T}$  corresponding to eigenspaces of  $w\phi$  (resp.  $w'\phi$ ) the  $\mathbf{G}^F$ -conjugacy of two Sylow  $\Phi$ -subtori corresponds to the  $W$ -conjugacy of the corresponding eigenspaces.

For (4) we first remark that we can reduce to the case where  $\mathbf{G}$  is quasi-simple, using 2.14. Thus either  $q \in \mathbb{Z}$  or  $\mathbf{G}^F$  is a Ree or a Suzuki group. Let  $\delta$  be the order of the coset  $W\phi$ , that is the smallest integer such that  $(W\phi)^{\delta} = W$ . We have  $\delta \in \{1, 2, 3\}$ . We first show the

**Lemma 2.20.** *If  $\mathbf{G}$  is quasi-simple and we are in one of the cases:*

- (1)  $q \in \mathbb{Z}$  and  $\delta \in \{1, 2\}$ .
- (2)  $q \in \mathbb{Z}$ ,  $\delta = 3$  and  $d$  is prime to 3.
- (3)  $q$  is an odd power of  $\sqrt{2}$  and  $\ell = 3$ .

*then  $W_{\Phi}$  acts faithfully on  $\mathbf{S}_{\ell}^F$ .*

*Proof.* On  $X(\mathbf{T}) \otimes \mathbb{Q}(q^{-1})$  we have  $wF^* = qw\phi$ . The characteristic polynomial  $Q$  of  $wF^*$  on  $X(\mathbf{S})$  is  $q^{n_{\Phi} \deg \Phi} \Phi(x/q)^{n_{\Phi}}$ ; as  $wF^*$  is semisimple, the minimal polynomial of  $wF^*$  is  $P = q^{\deg \Phi} \Phi(x/q)$ . We can identify  $X(\mathbf{S})$  with  $\text{Ker}(P(qw\phi))$  on  $X(\mathbf{T})$ . As in the proof of Proposition 2.8, if  $X = X(\mathbf{S})$  we can make  $X' = X \otimes \mathbb{Z}[q^{-1}]$  an  $A$ -module where  $A = \mathbb{Z}[x, q^{-\eta}]/P$ . Under the assumptions of the lemma  $A$  is a Dedekind ring. This results from the proof of 2.8(1) when  $q \in \mathbb{Z}$ . In the remaining case (3) of Lemma 2.20,  $\eta = 2$  and the order of  $q^2 \pmod{3}$  is 2, thus  $\Phi = x^2 + 1$  and  $P = x^2 + q^2$ ; we have  $A = \mathbb{Z}[x, q^{-2}]/P \simeq \mathbb{Z}[1/2, \sqrt{-2}]$  which is integrally closed (thus Dedekind) since localized of  $\mathbb{Z}[\sqrt{-2}]$  which is integrally closed. As an  $A$ -module of rank  $n_{\Phi}$ , the module  $X'$  is a sum of projective rank 1 submodules thus  $\mathbf{S}$  is a product of  $n_{\Phi}$  copies of a  $wF$ -indecomposable torus. By Proposition 2.19(2) we can identify  $W_{\Phi}$  to a subgroup of  $\text{GL}(X)$ . With the notations of 2.8, since the assumption of 2.8(1) is satisfied,  $\bar{X} := X/(wF^* - 1)X \simeq \text{Irr}(\mathbf{S}^{wF})$  is isomorphic to  $(\mathbb{Z}/\Phi(q))^{n_{\Phi}}$ . The representation of  $W_{\Phi}$  on  $X$  reduces to  $\bar{X}$ . We will show it is faithful on  $\bar{X}/\ell\bar{X}$  (or  $\bar{X}/4\bar{X}$  when  $\ell = 2$ ).

If  $q \in \mathbb{Z}$  and  $\ell = 2$  then  $d \in \{1, 2\}$  and we can apply Proposition 2.8(2) taking  $m = 4$ . We get that  $\bar{X}/4\bar{X} \simeq \text{Ker}(wF^* - 1 | X/4X)$ . We have as observed in the proof of Proposition 2.8 that  $\text{Ker}(wF^* - 1) = X/4X$  and the representation of  $W_{\Phi}$  on  $\bar{X}/4\bar{X}$ , which is a quotient of  $\text{Irr}(\mathbf{S}_{\ell}^{wF})$ , is faithful by Lemma 4.3.

If  $q \in \mathbb{Z}$  and  $\ell \neq 2$  then  $d$  is prime to  $\ell$ ; and in case (3) of Lemma 2.20  $\eta = 2$ ,  $\ell = 3$  thus  $d = 2$  and  $\ell$  is prime to  $d\eta$ . In both cases we can apply Proposition 2.8(2) with  $m = \ell$  to get that  $\bar{X}/\ell\bar{X} \simeq \text{Ker}(wF^* - 1 | X/\ell X)$ . We know by Lemma 4.3 that the representation of  $W_{\Phi}$  on  $X/\ell X$  is faithful and we would like to conclude that it is faithful on the submodule  $\text{Ker}(wF^* - 1)$ . We use the element  $v$  given

by Proposition 2.9(5): it preserves the kernel of  $\Phi(w\phi)$  thus induces an element of  $\text{GL}(X)$  which defines an automorphism  $\sigma$  of  $W_\Phi$  which sends  $w\phi$  to  $(w\phi)^a$ , so it remains true after reduction (mod  $\ell$ ) that  $\sigma$  sends  $w\phi$  to  $(w\phi)^a$ , thus permutes the eigenspaces of  $wF^*$  on  $X/\ell X$ : since  $d$  is the order of  $q$  (mod  $\ell$ ), all the primitive  $d$ -th roots of unity live in  $\mathbb{F}_\ell$  and the eigenvalues of  $wF^*$  are the product of one primitive  $d$ -th root of unity, which is  $q$ , by the other primitive  $d$ -th roots of unity so are of the form  $q^{1-a}$  where  $a$  runs over  $(\mathbb{Z}/d)^\times$ . And under the assumption  $(W\phi)^a = W\phi$  of 2.9(5) we can find  $v$  thus  $\sigma$  which sends the  $q^{1-a}$ -eigenspace of  $wF^*$  to the  $q^{1-1} = 1$ -eigenspace.

If every  $a$  prime to  $d$  has a representative in  $1 + \delta\mathbb{Z}$  we can satisfy  $(W\phi)^a = W\phi$  for such  $a$  thus every eigenspace is isomorphic as a  $W_\Phi$ -module to  $\text{Ker}(wF^* - 1)$ . Then  $W_\Phi$  is faithful on the whole  $X/\ell X$  if and only if it is faithful on  $\text{Ker}(wF^* - 1)$ , thus we conclude. If  $a \equiv 1 \pmod{\text{gcd}(d, \delta)}$  then by Bezout's theorem there exist integers  $\alpha, \beta$  such that  $a = 1 + \alpha d + \beta \delta$ , and then  $a - \alpha d \in 1 + \delta\mathbb{Z}$  is a representative of  $a$ .

If  $\delta = 1$  or  $\delta = 2$  then every  $a$  prime to  $d$  is  $\equiv 1 \pmod{\text{gcd}(d, \delta)}$  and we conclude. We conclude similarly if  $\delta = 3$  and  $d$  is prime to 3, or in case (3) of Lemma 2.20 since in this case  $d = 2$ .  $\square$

When  $q \in \mathbb{Z}$  the only case not covered by the lemma is  ${}^3D_4$  and  $d$  divisible by 3, that is  $d \in \{3, 6, 12\}$ . But in this case  $\ell > 3$ , since  $d$  is the order of  $q$  (mod  $\ell$ ), thus  $|W|$  is prime to  $\ell$  and a fortiori the Sylow  $\ell$ -subgroup of  $W_\Phi$  is trivial.

For the Ree and Suzuki groups we do not have to consider  ${}^2B_2$  since  $W$  is a 2-group and  $\ell \neq p$ , and the groups  ${}^2G_2$  since only the prime  $\ell = 2$  divides  $|W|$  and is different from  $p$ , and this case is excluded in the proposition.

For the groups  ${}^2F_4$  the only prime  $\ell \neq p$  such that  $\ell || |W|$  is  $\ell = 3$  and we are in case (3) of the lemma.  $\square$

The Ree group  ${}^2G_2$  with  $\ell = 2$  is a genuine counterexample since the Sylow 2-subgroups of  ${}^2G_2(q)$  are isomorphic to  $(\mathbb{Z}/2)^3$ .

### 3. THE STRUCTURE OF THE SYLOW $\ell$ -SUBGROUPS

**Definition 3.1.** Let  $\mathbf{G}, F, \mathbf{G}_1, \mathcal{P}$  and  $n$  be as in 2.13 and let  $\ell \neq p$  be a prime number. We define  $D(\ell)$  as the set of integers  $d$  such that for some  $\Phi \in \mathcal{P}$  dividing  $\Phi_d(x^n)$  we have  $\ell | \Phi(q^n)$ , where  $\eta$  is as in Definition 2.18.

The following proposition is [Enguehard, Théorème 1] when  $\eta = 1$ ; we give here a shorter proof. Since [Enguehard] was written, Malle ([Malle, 5.14 and 5.19]) has published a proof of (2) below — thus implicitly of (1) also — when  $\eta = 1$  (giving more, see Theorem 3.3).

**Theorem 3.2.** Assume in the situation of 3.1 that  $D(\ell) \neq \emptyset$ , or equivalently that  $\ell || |\mathbf{G}^F|$ . Then

- (1)  $d(\ell) \in D(\ell)$ .
- (2) There exists a unique  $\Phi \in \mathcal{P}$  such that  $\ell | \Phi(q^n)$  and  $\Phi$  divides  $\Phi_{d(\ell)}(x^n)$ . If  $\mathbf{S}$  is a Sylow  $\Phi$ -torus then  $N_{\mathbf{G}}(\mathbf{S})$  contains a Sylow  $\ell$ -subgroup of  $\mathbf{G}^F$  which is an extension of  $(Z^0 C_{\mathbf{G}}(\mathbf{S}))_\ell^F$  by a Sylow  $\ell$ -subgroup of  $W_\Phi$ .
- (3) The Sylow  $\ell$ -subgroups of  $\mathbf{G}^F$  are abelian if and only if  $|D(\ell)| = 1$  (which is equivalent to  $W_\Phi$  being an  $\ell'$ -group), apart from the exception where

$(\mathbf{G}_1, F_1)$  is of type  ${}^2G_2$  and  $\ell = 2$  in which case  $|D(\ell)| = 2$  and  $|W_\Phi| = 6$  but the 2-Sylow is abelian, isomorphic to  $(\mathbb{Z}/2)^3$ .

Further, if  $\mathbf{S}$  is as in (2), then  $(Z^0 C_{\mathbf{G}}(\mathbf{S}))_\ell^F = \mathbf{S}_\ell^F$  except if:

- $\ell = 3$  and  $\mathbf{G}_1$  of type  ${}^3D_4$ .
- $\ell = 2, d = 1$  and for some odd degree  $\varepsilon_i = -1$ . Equivalently  $\mathbf{G}_1$  is non-split and has an odd reflection degree, that is, is one of  ${}^2A_n, {}^2D_{2n+1}$  or  ${}^2E_6$ .
- $\ell = 2, d = 2$  and for some odd degree  $\varepsilon_i = 1$ ; equivalently  $\mathbf{G}_1$  is split and has an odd reflection degree, that is, is one of  $A_n (n > 1), D_{2n+1}$  or  $E_6$ .

In the above exceptions,  $Z^0 C_{\mathbf{G}}(\mathbf{S}) = C_{\mathbf{G}}(\mathbf{S})$  is a maximal torus of  $\mathbf{G}$ .

*Proof.* Let us note that to prove (2) when we are not in an exception, that is the stronger statement that a Sylow  $\ell$ -subgroup is in an extension of  $\mathbf{S}^F$  by a Sylow  $\ell$ -subgroup of  $W_\Phi$ , it is enough to prove that

$$v_\ell(|\mathbf{G}^F|) = v_\ell(|\mathbf{S}^F|) + v_\ell(|W_\Phi|) \quad (*)$$

where  $v_\ell$  denotes the  $\ell$ -adic valuation, and in the exceptions, if we have proved that  $Z^0 C_{\mathbf{G}}(\mathbf{S}) = C_{\mathbf{G}}(\mathbf{S})$  it is enough to show

$$v_\ell(|\mathbf{G}^F|) = v_\ell(|C_{\mathbf{G}}(\mathbf{S})^F|) + v_\ell(|W_\Phi|) \quad (**)$$

Note also that by the definition of  $d(\ell)$  and  $D(\ell)$  in Proposition 2.13, assertion (1) as well as formulae (\*) and (\*\*) are equivalent in  $\mathbf{G}$  and  $\mathbf{G}_1$ , that is we may assume  $\mathbf{G}$  quasi-simple to prove them which we do now. Also, in view of (2) and Proposition 2.19(4), (3) reduces to proving:

(3')  $|D(\ell)| = 1$  is equivalent to  $W_\Phi$  being an  $\ell'$ -group.

We first look at the case of a Ree or Suzuki group, where  $\eta = 2$ .

Let us prove (1) first. By Lemma 4.2 if  $\ell$  divides  $|\mathbf{G}^F|$  then there is an element of  $D(\ell)$  of the form  $d(\ell)\ell^b$  with  $b \geq 0$ . By inspecting the order formula for  $|\mathbf{G}^F|$  given in the proof of 2.11 the elements of  $D(\ell)$  have all their prime factors in  $\{2, 3\}$ , so  $b > 0$  implies  $\ell \in \{2, 3\}$  thus  $d(\ell) \in \{1, 2\}$ ; inspecting again the formula, we see that then  $d(\ell)$  in  $D(\ell)$  and that  $|D(\ell)| = 1$  unless  $\ell \in \{2, 3\}$ .

To prove (2) for  $\ell \notin \{2, 3\}$ , we observe there is a single  $\Phi \in \mathcal{P}$  such that  $\ell | \Phi(q)$  since the two numbers  $\Phi'_{2,4}(q), \Phi''_{2,4}(q)$  are prime to each other, and the same observation applies to  $\Phi'_{2,6}(q), \Phi''_{2,6}(q)$  and  $\Phi'_{2,12}(q), \Phi''_{2,12}(q)$ . Thus for  $\ell \notin \{2, 3\}$  assertions (3') and (\*) are obvious since  $|\mathbf{G}^F|_\ell = |\mathbf{S}^F|_\ell$  and  $\ell \nmid |W|$ .

Let us prove (\*) for  $\ell \in \{2, 3\}$ ; since  $\ell \neq p$  and the elements of  $D(\ell)$  have only 2 as prime factor in the case  ${}^2B_2$ , we have just to consider:

- $\ell = 3$  for  ${}^2F_4$ : we have  $d(3) = 2$ ,  $W_{\Phi_{2,2}} = G_{12}$  of order 48; the only factor  $\Phi(q)$  with a value divisible by 3 apart from  $|\mathbf{S}^F| = \Phi_{2,2}(q)^2$  is  $\Phi_{2,6}(q)$  and  $v_3(\Phi_{2,6}(q)) = 1 = v_3(|G_{12}|)$  which proves this case.
- $\ell = 2$  for  ${}^2G_2$ : we have  $d(2) = 2$  and  $|W_{\Phi_{2,2}}| = 6$ ; the only factor  $\Phi(q)$  with an even value apart from  $|\mathbf{S}^F| = \Phi_{2,2}(q)$  is  $\Phi_{2,1}(q)$  and  $v_2(\Phi_{2,1}(q)) = 1 = v_2(|W_\Phi|)$  which proves this case.

We have seen (3') along the way.

Now we look at the other quasi-simple groups thus  $\eta = 1$ . We notice generally that, assuming we have proved (1) then if  $|D(\ell)| = 1$  assertion (2) is trivial since a Sylow  $\ell$ -subgroup is then in  $\mathbf{S}$ , and (3') reduces to checking that  $W_\Phi$  is an  $\ell'$ -group.

We consider separately  ${}^3D_4$  where  $|{}^3D_4(q)| = q^{12}(\Phi_1^2 \Phi_2^2 \Phi_3^2 \Phi_6^2 \Phi_{12})(q)$ . Again, since the only prime factors of elements of  $D(\ell)$  are  $\{2, 3\}$ , we see that  $d(\ell) \in D(\ell)$

except possibly if  $\ell \in \{2, 3\}$ ; but in that case  $d(\ell) \in \{1, 2\}$  and there is a factor  $\Phi_{d(\ell)}(q)$ , whence (1). Since  $|W| = 3 \cdot 2^6$  assertion (3') is proved when  $D(\ell) = 1$ . It remains to prove (2) when  $\ell \in \{2, 3\}$ . In both cases  $W_{\Phi_{d(\ell)}} = W(G_2)$  and by Lemma 4.2  $v_\ell(|\mathbf{G}^F|/|\mathbf{S}^F|) = 2$ . If  $\ell = 2$  then  $2 = v_\ell(|W(G_2)|)$  which proves (\*). If  $\ell = 3$  a Sylow  $\Phi$ -subtorus  $\mathbf{S}$  is in a torus  $\mathbf{T}_w = C_{\mathbf{G}}(\mathbf{S})$  where  $w = 1$  if  $d = 1$  (resp.  $w = w_0$  if  $d = 2$ ). We have  $|\mathbf{T}_1^F| = \Phi_1(q)^2 \Phi_3(q)$  (resp.  $|\mathbf{T}_{w_0}^F| = \Phi_2(q)^2 \Phi_6(q)$ ) which has same 3-valuation as  $|\mathbf{G}^F|/|W_\Phi|$  which proves (\*\*).

In the remaining cases  $\varepsilon_i = \pm 1$  for all  $i$ . Let us set  $\zeta_d = e^{2i\pi/d}$ . We have  $\Phi = \Phi_{d(\ell)}$  and  $v_\ell(|\mathbf{S}^F|) = |a(\zeta_{d(\ell)})|v_\ell(\Phi_{d(\ell)}(q))$ .

We first treat the case  $\ell$  odd. We have  $a(\zeta_d) = \{d_i \mid \zeta_d^{d_i} = \varepsilon_i\}$  and  $|W_\Phi| = \prod_{d_i \in a(\zeta_{d(\ell)})} d_i$ . By Lemma 4.2, a factor  $\Phi_e(q)$  of  $|\mathbf{G}^F|$  can contribute to the  $\ell$ -valuation only if  $e$  is of the form  $d(\ell)\ell^b$  for some  $b \geq 0$ . Further such a factor appears if and only if  $a(\zeta_e) \neq \emptyset$ , that is for some  $i$  we have  $\zeta_{d(\ell)\ell^b}^{d_i} = \varepsilon_i$ . Since  $\ell$  is odd raising this equality to the power  $\ell^b$  gives  $\zeta_{d(\ell)}^{d_i} = \varepsilon_i$  thus  $d_i \in a(\zeta_{d(\ell)})$  and in particular  $d(\ell) \in D(\ell)$ . And  $\zeta_{d(\ell)\ell^b}^{d_i} = \varepsilon_i$  implies that  $\ell^b$  divides  $d_i$ . Thus only the  $d_i$  in  $a(\zeta_{d(\ell)})$  contribute to  $v_\ell(|\mathbf{G}^F|)$  and each of them contributes  $v_\ell(\Phi_{d(\ell)}(q)) + v_\ell(\Phi_{d(\ell)\ell}(q)) + \dots + v_\ell(\Phi_{d(\ell)\ell^{v_\ell(d_i)}}(q))$ . By Lemma 4.2 this is  $v_\ell(\Phi_{d(\ell)}(q)) + v_\ell(d_i)$ . Summing over  $d_i \in a(\zeta_{d(\ell)})$  proves (\*).

It remains the case  $\ell = 2$  where we proceed similarly. We have  $d(2) \in \{1, 2\}$ . If  $d(2) = 1$  then  $a(1) = \{d_i \mid \varepsilon_i = 1\}$ . Thus the condition  $\zeta_{2^b}^{d_i} = \varepsilon_i$  is still equivalent to  $2^b \mid d_i$ ; but there could be some more solutions of this equation than elements of  $a(1)$  when  $b = 1$ : any odd  $d_i$  such that  $\varepsilon_i = -1$  brings an additional factor  $1 = v_2(\Phi_2(q))$ . If  $d(2) = 2$  then  $a(-1) = \{d_i \mid \varepsilon_i = (-1)^{d_i}\}$ . The contribution of the even  $d_i$  can be worked out as before; but this time the odd  $d_i$  where  $\varepsilon_i = 1$  bring additional factors  $v_2(\Phi_1(q))$ . In the exceptions in each case  $C_{\mathbf{G}}(\mathbf{S})$  is a maximal torus of type 1 or  $w_0$ ; looking at the orders of these tori, they contain enough extra  $\Phi_1$  or  $\Phi_2$  factors (which correspond to the eigenvalues 1 or  $-1$  of  $\phi$  or  $w_0\phi$ ) to compensate the discrepancy.

Let us show now (3'), which reduces to proving that  $|D(\ell)| > 1$  implies  $v_\ell(|W_\Phi|) > 0$ . Thus we assume  $|D(\ell)| > 1$ . We first do the case  $\ell = 2$ ; then  $d(\ell) \in \{1, 2\}$  from which it follows, since the 1 and  $-1$ -eigenspaces are defined over the reals, that  $W_\Phi$  is a Coxeter group, whose order is always even. We consider finally  $\ell$  odd; then  $D(\ell) \ni d(\ell)$  and  $d(\ell)\ell^a$  for some  $a > 0$ . But we have seen above that there exists a factor  $\Phi_{d(\ell)\ell^a}(q)$  only if  $\ell^a \mid d_i$  for some  $d_i$  in  $a(\zeta_{d(\ell)})$ .  $\square$

We remark that if  $\ell$  divides only one  $\Phi_d(q)$ , a Sylow  $\ell$ -subgroup  $S$  lies in a single Sylow  $\Phi$ -torus  $\mathbf{S}$  (the intersection of two tori has lower dimension so cannot have same order polynomial). It follows that  $N_{\mathbf{G}^F}(S) = N_{\mathbf{G}^F}(\mathbf{S})$  and  $C_{\mathbf{G}^F}(S) = C_{\mathbf{G}^F}(\mathbf{S})$ . This observation is a start for describing the  $\ell$ -Frobenius category of  $\mathbf{G}^F$  in terms of the category of  $\zeta_d$ -eigenspaces of  $W_{\Phi_d}$ .

In general, one can deduce the following unicity theorem from the work of Cabanes, Enguehard and Malle.

**Theorem 3.3.** *Consider  $\mathbf{G}, F, n, \mathbf{G}_1, q$  as in 2.13 with  $q^n \in \mathbb{Z}$  and let  $\Phi$  as defined in Theorem 3.2, (2). Assume that we are not in one of the following cases:*

- $\ell = 3$ ,  $\mathbf{G}_1$  simply connected of type  $A_2, {}^2A_2$  or  $G_2$ .
- $\ell = 2$ ,  $\mathbf{G}_1$  simply connected of type  $C_n, n \geq 1$ .

Let  $Q$  be a Sylow  $\ell$ -subgroup of  $\mathbf{G}^F$ . There is a unique Sylow  $\Phi$ -subtorus  $\mathbf{S}$  of  $\mathbf{G}$  such that  $Q \subseteq N_{\mathbf{G}}(\mathbf{S})$ .

*Proof.* In the context of Theorem 3.2(2), let  $Q$  be a Sylow  $\ell$ -subgroup of  $\mathbf{G}^F$  contained in  $N_{\mathbf{G}}(\mathbf{S})$ ; then according to [Cabanes],  $\mathbf{S}_{\ell}^F$  is often characteristic in  $Q$  (for example when  $l \geq 5$ ), thus in these cases  $N_{\mathbf{G}^F}(Q) \subseteq N_{\mathbf{G}}(\mathbf{S}_{\ell}^F)$ . Using inductively that property and inspecting small cases, G. Malle has proved the inclusion

$$(3.4) \quad N_{\mathbf{G}^F}(Q) \subseteq N_{\mathbf{G}}(\mathbf{S})$$

for all quasi-simple groups  $\mathbf{G}$  short of the cases excluded in Theorem 3.3, see [Malle, Theorems 5.14 and 5.19]. Here  $\mathbf{S}$  is a Sylow  $\Phi_{d(\ell)}$ -subtorus of  $(\mathbf{G}, F)$  as defined in Definition 2.18 with  $\eta = 1$  (note that  $N_{\mathbf{G}^F}(Q) \subseteq N_{\mathbf{G}}(\mathbf{S})$  implies  $Q \subseteq N_{\mathbf{G}}(\mathbf{S})$ ).

We first verify that the last inclusion holds more generally in a "descent of scalars". With hypotheses and notations of Proposition 2.13 and Lemma 2.14 assume  $q^n \in \mathbb{Z}$ . If  $e = d(\ell)$  is the order of  $q^n$  modulo  $\ell$ , take  $\Phi = \Phi_e \in \mathcal{P}$ , defining  $\mathbf{S} = \mathbf{S}_{\Phi}$  and  $\mathbf{S}_1$ . There is a morphism from  $\mathbf{G}$  onto  $\mathbf{G}_1$ , sending  $\mathbf{S}$  to  $\mathbf{S}_1$ , with restriction an isomorphism from  $\mathbf{G}^F$  to  $\mathbf{G}_1^F$ . Then a Sylow  $\ell$ -subgroup  $Q_1$  of  $\mathbf{G}_1^F$  contained in  $N_{\mathbf{G}_1}(\mathbf{S}_1)$  is the isomorphic image of a Sylow  $\ell$ -subgroup  $Q$  of  $\mathbf{G}^F$  contained in  $N_{\mathbf{G}}(\mathbf{S})$ . The inclusion 3.4 written with  $(\mathbf{G}_1, F_1, Q_1, \mathbf{S}_1)$  instead of  $(\mathbf{G}, F, Q, \mathbf{S})$  implies 3.4 in  $(\mathbf{G}, F)$ .

From 3.4 the unicity of  $\mathbf{S}$ , given  $Q$ , follows:

**Lemma 3.5.** *Let  $\Phi \in \mathcal{P}$ , let  $\mathbf{S}$  be a Sylow  $\Phi$ -subtorus of  $(\mathbf{G}, F)$  and  $Q$  a Sylow  $\ell$ -subgroup of  $\mathbf{G}^F$ . If  $N_{\mathbf{G}^F}(Q) \subseteq N_{\mathbf{G}}(\mathbf{S})$ , then  $\mathbf{S}$  is the unique Sylow  $\Phi$ -torus of  $(\mathbf{G}, F)$  such that  $Q \subseteq N_{\mathbf{G}}(\mathbf{S})$ .*

*Proof.* Assume  $Q \subseteq N_{\mathbf{G}}(\mathbf{S}')$  for some Sylow  $\Phi$ -torus  $\mathbf{S}'$  of  $(\mathbf{G}, F)$ . By Proposition 2.19 there exists  $g \in \mathbf{G}^F$  such that  $\mathbf{S} = (\mathbf{S}')^g$ , hence  $Q^g \subseteq N_{\mathbf{G}}(\mathbf{S})$ . By Sylow's theorem in  $N_{\mathbf{G}}(\mathbf{S})^F$ ,  $Q = Q^{gh}$  for some  $h \in N_{\mathbf{G}}(\mathbf{S})^F$  hence  $gh \in N_{\mathbf{G}}(\mathbf{S})$  by our hypothesis.  $\square$

#### 4. APPENDIX

We gather here arithmetical lemmas used above.

**Lemma 4.1.** *Let  $x, f, \ell \in \mathbb{N}$  where  $\ell$  is prime, and assume  $x \equiv 1 \pmod{\ell}$  (resp.  $\pmod{4}$ ) if  $\ell = 2$ ). Then  $v_{\ell}\left(\frac{x^f - 1}{x - 1}\right) = v_{\ell}(f)$ .*

*Proof.* From  $\frac{x^{f_1 f_2} - 1}{x - 1} = \frac{x^{f_1 f_2} - 1}{x^{f_2} - 1} \frac{x^{f_2} - 1}{x - 1}$  we see that it is enough to show the lemma when  $f$  is prime. We have  $\frac{x^f - 1}{x - 1} = f + \sum_{i=2}^{i=f} (x - 1)^{i-1} \binom{f}{i}$ . Let  $S$  be this last sum; we have  $S \equiv f \pmod{\ell}$ , since  $x - 1 \equiv 0 \pmod{\ell}$ , thus  $S$  is prime to  $\ell$  when  $f \neq \ell$  which shows the lemma in this case. When  $f = \ell$  then all the terms of  $S$  but the first one and possibly the last one are divisible by  $\ell^2$  since  $\binom{\ell}{i}$  is divisible by  $\ell$  when  $2 \leq i < \ell$ ; the last term is divisible by  $\ell^2$  when  $\ell - 1 \geq 2$  which fails only for  $f = \ell = 2$ ; but when  $\ell = 2$  we have arranged that  $v_{\ell}(x - 1) \geq 2$  and this time  $2(f - 1) \geq 1$ ; thus  $S \equiv f \pmod{\ell^2}$ , whence the lemma.  $\square$

The following lemma is in [Malle, 5.2]; a short elementary proof results immediately from Lemma 4.1.

**Lemma 4.2.** *Let  $q, \ell \in \mathbb{N}$  where  $\ell$  is prime. Let  $d$  be the order of  $q \pmod{\ell}$  (or  $\pmod{4}$  if  $\ell = 2$ ). Then  $\ell$  divides  $\Phi_e(q)$  if and only if  $e$  is of the form  $d\ell^b$  with  $b \in \mathbb{N}$  (or additionally  $b = -1$  when  $\ell = d = 2$ ), and  $v_\ell(\Phi_{d\ell^b}(q)) = 1$  if  $b \neq 0$ .*

The following lemma is in [Minkowski]; we give the proof since it is very short and the original German proof may be less accessible.

**Lemma 4.3.** *Let  $m \in \mathbb{N}, m > 2$ . Then the kernel of the reduction map  $\mathrm{GL}(\mathbb{Z}^n) \rightarrow \mathrm{GL}((\mathbb{Z}/m)^n)$  is torsion-free.*

Note that the bound  $m > 2$  is sharp since  $-\mathrm{Id} \equiv \mathrm{Id} \pmod{2}$ .

*Proof.* Let  $w \in \mathrm{GL}(\mathbb{Z}^n)$  be of finite order,  $w \neq \mathrm{Id}$  and assume its reduction  $v = \mathrm{Id}$ . We will derive a contradiction.

Possibly replacing  $w$  by a power, we may assume that  $w$  is of prime order  $p$ .

Also  $\mathrm{GL}(\mathbb{Z}^n/m) = \prod_i \mathrm{GL}(\mathbb{Z}^n/p_i)$  where  $m = \prod_i p_i$  is the decomposition of  $m$  into prime powers, thus we may assume that  $m$  is a prime power.

Since  $w$  is of order  $p$ , the polynomial  $\Phi_p(x)$  is a factor of the characteristic polynomial of  $w$ . The characteristic polynomial of  $v$  is the reduction  $\pmod{m}$  of that of  $w$ , thus we must have  $\Phi_p(x) \pmod{m} \equiv (x-1)^{p-1}$ ; in particular  $\binom{p-1}{1} \equiv -1 \pmod{m}$  thus  $m|p$  which implies  $m = p$ .

Write now  $w = \mathrm{Id} + xm^a$  where  $x \pmod{m} \not\equiv 0$  and  $a \in \mathbb{N}$ . Then the equation  $w^m = \mathrm{Id}$  gives  $\sum_{i=1}^m \binom{i}{m} x^i m^{ai} = 0$ , which after dividing by  $m^{a+1}$  becomes  $x = -\sum_{i=2}^m \binom{i}{m} x^i m^{a(i-1)-1}$  where all coefficients on the right-hand side are divisible by  $m$  (since  $m \geq 3$ ), which contradicts  $x \pmod{m} \not\equiv 0$ .  $\square$

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