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THE SYLOW SUBGROUPS OF A FINITE REDUCTIVE GROUP

MICHEL ENGUEHARD AND JEAN MICHEL

Dedicated to professor George Lusztig on the occasion of his 70th birthday

Abstract. We describe the structure of Sylow $\ell$-subgroups of a finite reductive group $G(F_q)$ when $q \not\equiv 0 \pmod{\ell}$ that we find governed by a complex reflection group attached to $G$ and $\ell$, which depends on $\ell$ only through the set of cyclotomic factors of the generic order of $G(F_q)$ whose value at $q$ is divisible by $\ell$. We also tackle the more general case of groups $G^F$ where $F$ is an isogeny some power of which is a Frobenius morphism.

1. Introduction

Definition 1.1. Let $G$ be a connected reductive group over $\mathbb{F}_p$, and $F$ an isogeny such that some power of $F$ is a Frobenius endomorphism; then $G^F$ is what we call a finite reductive group. To this situation we attach a positive real number $q$ such that for some integer $n$, the isogeny $F^n$ is the Frobenius endomorphism attached to a $\mathbb{F}_{q^n}$-structure.

The goal of this note is to describe the Sylow $\ell$-subgroups of $G^F$ when $\ell$ is a prime different from $p$ and $G$ is semisimple. The structure of the Sylow $\ell$-subgroups of a Chevalley group was first described by [Gorenstein-Lyons] where they observed that they had a large normal abelian subgroup $(\mathbb{Z}/n)^\ell$ where $n$ is the $\ell$-part of $\Phi_d(q)$, where $d$ is the multiplicative order of $q \pmod{\ell}$, and they computed a case by case.

In 1992 [Broué-Malle] exhibited subtori of $G^F$ attached to eigenspaces of elements of the Weyl reflection coset of $(G, F)$ whose $F$-stable points are the large abelian groups of [Gorenstein-Lyons]. To these eigenspaces are attached complex reflection groups by Springer’s theory.

We show that the structure of the Sylow $\ell$-subgroups of $G^F$ is determined by these complex reflection groups. The results of this note in the case when $F$ is a Frobenius were obtained by the first author in an unpublished note [Enguehard] of 1992; the second author has found a simpler (containing more casefree steps) proof which is an occasion to publish these results. Some of our results appeared also implicitly in [Malle].

The second author wishes to thank Carles Broto for a visit to Barcelona, which started him thinking about this topic.

We thank Raphaël Rouquier for discussions which helped with the proofs of Propositions 2.8 and 2.19(4).
2. The generic Sylow theorems

Let $G$ be as in 1.1; an $F$-stable maximal torus $T$ of $G$ defines the Weyl group $W = N_G(T)/T$, that we may identify to a reflection subgroup of $GL(X(T))$ where $X(T) := \text{Hom}(T, \mathbb{G}_m)$, attached to the root system $\Sigma \subset X(T)$ of $G$ with respect to $T$. The isogeny $F$ induces a $p$-morphism $F^* \in \text{End}(X(T))$ by the formula $F^*(x) = xqF$ for $x \in X(T)$, that is there is a permutation $\sigma$ of $\Sigma$ such that for $\alpha \in \Sigma$ we have $F^*(\alpha) = q_\alpha \sigma(\alpha)$ for some power $q_\alpha$ of $p$; in particular $F^* \in \text{N}_{\text{End}(X(T))}(W)$.

If $q, n$ are as in 1.1 then $F^{*n}$ is $q^n$ times an element of $GL(X(T))$ of finite order, thus over $X(T) \otimes \mathbb{Z}[q^{-1}]$ we have $F^* = q^\phi$ where $\phi$ is an automorphism of finite order which normalizes $W$. We call $W\phi$ the reflection coset associated to $(G, F)$.

Our setting is more general than that of [Broué-Malle] who considered only the special cases where $F$ is a Frobenius endomorphism, or where $G^F$ is a Ree or Suzuki group. The results of the next subsection allow to extend the definition of Sylow $\Phi$-subtori of [Broué-Malle] to any $(G, F)$ as in 1.1.

$F$-indecomposable tori.

**Definition 2.1.** For $G, F$ as in 1.1, a non-trivial subtorus of $G$ is called $F$-indecomposable if it is $F$-stable and contains no proper non-trivial $F$-stable subtorus.

We say that a group $G$ is an almost direct product of subgroups $G_1$ and $G_2$ if they commute, generate $G$ and have finite intersection, and we define similarly an almost direct product of $k$ subgroups by induction on $k$.

**Proposition 2.2.** For $G, F$ as in 1.1, any $F$-stable subtorus $T$ of $G$ is an almost direct product of $F$-indecomposable tori $S_1, \ldots, S_k$ and $|T^F| = |S_1^F| \cdots |S_k^F|$.

*Proof.* An $F$-stable subtorus $S$ corresponds to a pure $F$-stable sublattice $X' \subset X := X(T)$ (see for example [Borel, III, Proposition 8.12]). Let $d$ be the smallest power of $F$ which is a split Frobenius, thus on $X(T)$ we have $F^{*d} = q^d \text{Id}$. Let $\pi \in \text{End}(X \otimes \mathbb{Q})$ be a projector on $X' \otimes \mathbb{Q}$. Then in $\text{End}(X \otimes \mathbb{Q})$ we can define the $F$-invariant projector $\pi' := d^{-1} \sum_{i=1}^{d} F^{*i} \pi F^{*-i}$ and $\text{Ker } \pi' \cap X$ is another $F$-stable pure sublattice which after tensoring by $\mathbb{Q}$ becomes a complement to $X' \otimes \mathbb{Q}$. This corresponds to an $F$-stable subtorus $S'$ such that $K := S \cap S'$ is finite and $T = SS'$.

Iterating, we get the first part of the proposition.

The second part of the proposition results from the next two lemmas.

**Lemma 2.3.** For $G, F$ as in 1.1, and $K$ an $F$-stable finite normal subgroup of $G$, then $|G/K|^F = |G^F|$. 

*Proof.* First, we notice that $K$ is central, thus abelian, since conjugating by $G$ being continuous must be trivial on $K$.

Then, the Galois cohomology long exact sequence: $1 \to K^F \to G^F \to (G/K)^F \to H^1(F, K) \to 1$ shows the result using that $|K^F| = |H^1(F, K)|$. 

**Lemma 2.4.** Let $G$ as 1.1 be an almost direct product of $F$-stable connected subgroups $G = G_1 \ldots G_k$. Then $|G^F| = |G_1^F| \cdots |G_k^F|$. 

*Proof.* It is enough to consider the case $k = 2$ and then iterate. Thus, we assume $G = G_1G_2$ where $K = G_1 \cap G_2$ is finite. We quotient by $K$, which makes the product direct, and apply Lemma 2.3 twice. 

Lemma 2.5. Let $S$ be an $F$-indecomposable torus, let $\eta$ be the smallest power such that $q^\eta \in \mathbb{Z}$, and let $d$ be the smallest power such that $F^d \eta$ is a split Frobenius on $S$. Let $F^* = q^\eta$ on $X(S)$; then the characteristic polynomial $\Phi$ of $\Phi$ is a factor in $\mathbb{Z}[x, q^{-1}]$ of $\Phi_d(x^\eta)$, where $\Phi_d(x)$ denotes the $d$-th cyclotomic polynomial. Further $q^d \Phi(x/q) \in \mathbb{Z}[x]$ is irreducible and $|S^F| = \Phi(q)$.

Proof. Since $F^* \eta$ acts as $q^\eta$ on $X := X(S)$, the minimal polynomial $P$ of $F^*$ divides $x^{d\eta} - q^{d\eta}$.

The polynomial $P$ is irreducible over $\mathbb{Z}$, otherwise a proper nontrivial factor $P_1$ defines an $F^*$-stable pure nontrivial sublattice $\text{Ker}(P_1(F^*))$ of $X$, which contradicts $F$-indecomposability of $S$.

It follows that $X$ is a $\mathbb{Z}[x]/P$-module by making $x$ act by $F^*$, and $X \otimes \mathbb{Q}[x]/P$ is a one-dimensional $\mathbb{Q}[x]/P$-vector space, otherwise a proper nontrivial subspace would define an $F^*$-stable pure sublattice of $X$. It follows that $\dim S = \deg P = \dim X$ and thus $P$ is also the characteristic polynomial of $F^*$.

We have in $\mathbb{Z}[x]$ the equality $x^{d\eta} - q^{d\eta} = \prod_{d \mid d'}(q^{d\deg \Phi_{d'}} \Phi_{d'}(x^\eta/q^{d'}))$. Since $P$ is irreducible it divides one of the factors, and since $d\eta$ is minimal such that $F^{d\eta} = q^{d\eta} \text{Id}$, that is minimal such that $P$ divides $x^{d\eta} - q^{d\eta}$, we have that $P$ divides $q^{d\deg \Phi_{d'}} \Phi_{d'}(x^\eta/q^{d'})$; equivalently $\Phi = q^{-\deg P} P(qx)$ divides $\Phi_d(x^\eta)$.

We have $|S^F| = |\text{Irr}(S^F)| = |X/(F^*-1)X| = \det(F^*-1) = (-1)^{\deg P} P(1) = (-q)^{\deg \Phi(1/q)}$ where the second equality reflects the well known group isomorphism $\text{Irr}(S^F) \simeq X/(F^*-1)X$ and the third is a general property of lattices. Finally, since $\Phi$ is real and divides $\Phi_d(x^\eta)$, its roots are stable under taking inverses, thus $(-q)^{\deg \Phi(1/q)} = \Phi(q)$. 

We call $q$-cyclotomic the polynomials $\Phi$ of Lemma 2.5. In other terms

Definition 2.6. For $q$ as in 1.1, where $q^n$ is the smallest power of $q$ in $\mathbb{Z}$, we call $q$-cyclotomic the monic polynomials $\Phi \in \mathbb{Z}[x, q^{-1}]$ such that $q^{\deg \Phi} \Phi(x/q)$ is a $\mathbb{Z}[x]$-irreducible factor of some $x^{d\eta} - q^{d\eta}$.

In the study of semisimple reductive groups we will need the $q$-cyclotomic polynomials of Lemma 2.7. Note that if $d$ is minimal in Definition 2.6, then $\Phi$ is a factor in $\mathbb{Z}[x, q^{-1}]$ of $\Phi_d(x^\eta)$. We are interested in that number $d$ rather than $d\eta$, and to emphasize this we write $\Phi_{\eta, d}$ in the following examples.

Lemma 2.7. When $q \in \mathbb{Z}$, the $q$-cyclotomic polynomials are the cyclotomic polynomials.

When $q$ is an odd power of $\sqrt{2}$, the following polynomials are $q$-cyclotomic: $\Phi_{2,1}(x) := \Phi_1(x^2)$, $\Phi_{2,2}(x) := \Phi_2(x^2)$, $\Phi_{2,6}(x) := \Phi_6(x^2)$, the factors $\Phi_{2,4} := x^2 + \sqrt{2} + 1$ and $\Phi_{2,12}'' := x^2 - \sqrt{2} + 1$ of $\Phi_4(x^2)$, and the factors $\Phi_{2,12} := x^4 + x^3 \sqrt{2} + x^2 + x \sqrt{2} + 1$ and $\Phi_{2,12}'' := x^4 - x^3 \sqrt{2} + x^2 - x \sqrt{2} + 1$ of $\Phi_4(x^2)$.

When $q$ is an odd power of $\sqrt{3}$, the following polynomials are $q$-cyclotomic: $\Phi_{2,1}(x)$, $\Phi_{2,2}(x)$ and the factors $\Phi_{2,6} := x^2 + x \sqrt{3} + 1$ and $\Phi_{2,12}'' := x^2 - x \sqrt{3} + 1$ of $\Phi_6(x^2)$.

Proof. When $q \in \mathbb{Z}$ the formula $P \mapsto q^{-\deg P} P(qx)$ establishes a bijection between $\mathbb{Z}[x]$-irreducible factors of $x^{d} - q^{d}$ and $\mathbb{Z}[x]$-irreducible factors of $x^{d} - 1$, that is cyclotomic polynomials, which gives the first case of the lemma.

For the other cases, we have to check for each given factor that $q^{\deg \Phi} \Phi(x/q)$ is in $\mathbb{Z}[x]$ and irreducible. 

\qed
Proposition 2.8. Let $S, \eta, d, \Phi$ be as in 2.5 and let $P = q^{\deg \Phi} (x^n/q^n)$ be the characteristic polynomial of $F^*$. 

1. Assume that either $q \in \mathbb{Z}$ or that $\mathbb{Z}[x, q^{-1}]/P$ is integrally closed. Then $S^F \simeq \mathbb{Z}/\Phi(q)$.

2. Let $m$ be a divisor of $\Phi(q)$, and assume either that $d \in \{1, 2\}$ and $q \in \mathbb{Z}$ or that $m$ is prime to $dp$; then we have a natural isomorphism $\text{Irr}(S^F)/m \text{Irr}(S^F) \simeq \text{Ker}(F^* - 1 | X(S)/mX(S))$.

Proof. Proceeding as in the proof of Lemma 2.5 we set $X = X(S)$ and $X = X/(F^* - 1)X \simeq \text{Irr}(S^F)$. Letting $x$ act as $F^*$ makes $X$ into a $\mathbb{Z}[x]/P$-module, and $X$ a $\mathbb{Z}[x]/(P, x - 1)$-module. Since $\mathbb{Z}[x]/(P, x - 1) = \mathbb{Z}/P(1) = \mathbb{Z}/\Phi(q)$ we find that the exponent of $X$ divides $\Phi(q)$.

Let $A := \mathbb{Z}[x, q^{-1}]/P$. The extension $\mathbb{Z}[x]/P \hookrightarrow A/P$ is flat thus $\tilde{X} \otimes_{\mathbb{Z}[x]/P} A \simeq X'/((F^* - 1)X')$ where $X' = X \otimes_{\mathbb{Z}[x]/P} A$; and since the exponent of $X$ divides $\Phi(q)$ which is prime to $q^n$, we have $X \simeq X \otimes_{\mathbb{Z}[x]/P} A$. Under the assumptions of (1) the ring $A$ is Dedekind: if $\eta \neq 1$ then $A$ is integrally closed thus Dedekind; if $\eta = 1$ then $A \simeq \mathbb{Z}[x, q^{-1}]/\Phi_d$ where the isomorphism is given by $x \mapsto x/q$, and is a localization of the Dedekind ring $\mathbb{Z}[x]/\Phi_d$ by $q$. Thus $X'$ identifies to a fractional ideal $\mathfrak{I}$ of $A$ and $X \simeq X/(x - 1)\mathfrak{I}$. If $e$ is the exponent of $X$ we have thus $\mathfrak{c} \subset (x - 1)\mathfrak{I}$, which implies that $x - 1$ divides $e$ in $A$. This in turn implies that the norm $(-1)^{\deg F} P(1) = \Phi(q)$ of $(x - 1)$ divides $e$ in $\mathbb{Z}$, thus $e = \Phi(q)$ and $X \simeq \mathbb{Z}/\Phi(q)$ and the same isomorphism holds for the dual abelian group $S^F$.

For (2), note that by construction $X/mX$ is the biggest quotient of $X$ on which both $F^* - 1$ and the multiplication by $m$ vanish. It is thus equal to the biggest quotient of $X/mX$ on which $F^* - 1$ vanishes. Thus the question is that $\text{Ker}(F^* - 1)$ has a complement in $X/mX$.

If $q \in \mathbb{Z}$ and $d \in \{1, 2\}$ we have $P = x \pm q$ so $X \simeq \mathbb{Z}$ on which $F^*$ acts by $\mp q$ and $X = X/(q \pm 1)$ of which $X/mX$ is a quotient, so $F^* - 1$ vanishes on $X/mX$ which is thus equal to $X/mX$ and there is nothing to prove.

Assume now $m$ prime to $dp$. There exists $R \in \mathbb{Z}[x]$ such that in $\mathbb{Z}[x]$ we have $P = (x - 1)R + P(1)$. Taking derivatives, we get $P' = (x - 1)R' + R$, whence $R(1) = P'(1)$. Let $\delta$ be the discriminant of $P$; we can find polynomials $M, N \in \mathbb{Z}[x]$ such that $M P + N P' = \delta$, which evaluating at 1 gives $M(1) P(1) + N(1) P'(1) = \delta$. Since $q$ is prime to $P(1)$, thus to $m$, and $\delta$ is a divisor of the discriminant of $X^{dp^q} - q^{dp^q}$, equal to $q^{\deg(F^* - 1) (dp^q)}$, thus prime to $m$, we find that $P'(1)$ is prime to $m$. In $(\mathbb{Z}/m)[x]$ we have $P = (x - 1)R$, thus applied to $F^*$ we get that on $X/mX$ we have $0 = P(F^*) = (F^* - 1)R(F^*)$, whence $\text{Ker}(F^* - 1) \cap \text{Ker}(R(F^*)) = X/mX$. Since $R(1)$ is prime to $m$, we can write $1 \equiv Q(x - 1) + a R$ in $(\mathbb{Z}/m)[x]$ for some $Q \in (\mathbb{Z}/m)[x]$ and $a$ the inverse (mod $m$) of $R(1)$. This proves that $\text{Ker}(F^* - 1) \cap \text{Ker}(R(F^*)) = 0$ thus $X/mX$ is the direct sum of $\text{Ker}(F^* - 1)$ and $\text{Ker}(R(F^*))$ q.e.d.

Complex reflection cosets. (1) to (3) below are classical results of Springer and Lehrer.

Proposition 2.9. Let $V$ be a finite dimensional vector space over a subfield $k$ of $\mathbb{C}$, let $W \subset \text{GL}(V)$ be a finite complex reflection group and let $\phi \in N_{\text{GL}(V)}(W)$, so that $W \phi$ is a reflection coset, let $(d_1, \varepsilon_1), \ldots, (d_n, \varepsilon_n)$ be its generalized degrees (see for instance [Broué, 4.2]). For $\zeta$ a root of unity define $a(\zeta)$ as the multiset of the $d_i$ such that $\zeta^{d_i} = \varepsilon_i$. Then:
(1) For any root of unity $\zeta$, the maximum dimension when $w\phi$ runs over $W\phi$ of a $\zeta$-eigenspace of $w\phi$ on $V \otimes_k k[\zeta]$ is $|a(\zeta)|$.

(2) For $w\phi \in W\phi$ denote $V_{w,\zeta} \subset V \otimes_k k[\zeta]$ its $\zeta$-eigenspace. Assume $\dim V_{w,\zeta} = |a(\zeta)|$ and let $C = C_W(V_{w,\zeta})$ and $N = N_W(V_{w,\zeta})$. Then $N/C$ is a complex reflection group acting on $V_{w,\zeta}$, with reflection degrees $a(\zeta)$.

(3) Any two subspaces $V_{w,\zeta}$ and $V_{w',\zeta}$ of dimension $|a(\zeta)|$ are $W$-conjugate.

(4) For $w\phi$ as in (2) the natural actions of $w\phi$ on $N$ and $C$ induce the trivial action on $N/C$.

(5) Let $a \in \mathbb{Z}$ be such that $(W\phi)^a = W\phi$ and such that $\zeta$ and $\zeta^a$ are conjugate by $\Gal(k[\zeta]/k)$. Then for $w\phi$ as in (2) there exists $v \in N_W(N) \cap N_W(C)$ which conjugates $w\phi C$ to $(w\phi)^a C$.

Proof. For (1) see for instance [Broué, 5.2], for (2) see [Broué, 5.6(3) and (4)] and for (3) see [Broué, 5.6 (1)]. (4) results from the observation that if $n \in N$ and $v \in V_{w,\zeta}$ then $(n^{-1} w\phi n)(v) = (n^{-1} w\phi n)(\zeta^{-1} v) = (n^{-1} w\phi)(\zeta^{-1} n v) = (n^{-1}) (n(v)) = v$ thus $n^{-1} w\phi n \in C$.

For (5), $\Gal(k[\zeta]/k)$ acts naturally on $V \otimes_k k[\zeta]$, commuting with $GL(V)$, in particular with $W$ and $\phi$. If $\sigma \in \Gal(k[\zeta]/k)$ is such that $\sigma(\zeta) = \zeta^a$, let $\zeta' = \sigma^{-1}(\zeta)$. Then $\sigma^{-1}(V_{w,\zeta}) = V_{w,\zeta'}$. It follows that $N = N_W(V_{w,\zeta'})$ and $C = C_W(V_{w,\zeta'})$.

Now since $a'$ is the inverse of a modulo the order of $\zeta$ the space $V_{w,\zeta'}$ is the $\zeta$-eigenspace of $(w\phi)^a$. By assumption we have $(w\phi)^a \in W\phi$. Since two maximal $\zeta$-eigenspaces of elements of $W\phi$ are conjugate by (3) there exists $v \in W$ which conjugates $V_{w,\zeta}$ to $V_{w,\zeta'}$, and $v \in N_W(N) \cap N_W(C)$ since $N = N_W(V_{w,\zeta'})$ and $C = C_W(V_{w,\zeta'})$. The element $v$ thus conjugates the set $w\phi C$ of elements which have $V_{w,\zeta}$ as $\zeta$-eigenspace to the set $(w\phi)^a C$ of elements which have $V_{w,\zeta'}$ as $\zeta'$-eigenspace.

\section*{Generic Sylow subgroups.}

We define the Sylow $\Phi$-subtori of $(G, F)$, first in the case when $G$ is quasi-simple, then in the case of descent of scalars.

From now on we assume $G$ semisimple. Then, if $(d_1, \varepsilon_1), \ldots, (d_n, \varepsilon_n)$ are the generalized degrees of the reflection coset $W\phi$, we have (see [Steinberg, 11.16])

\begin{equation}
|G^F| = q^{\sum_i (d_i - 1)} \prod_i (q^{d_i} - \varepsilon_i).
\end{equation}

\begin{proposition}
Let $G$ be as in 1.1 and quasi-simple. Then we can rewrite the order formula 2.10 for $|G^F|$ as
\begin{equation}
|G^F| = q^{\sum_i (d_i - 1)} \prod_{\Phi \in \mathcal{P}} \Phi(q)^{n_\Phi}
\end{equation}
where $\mathcal{P}$ is a set of $q$-cyclotomic polynomials, and where $0 \neq n_\Phi = |a(\zeta)|$ (see 2.9) for any root $\zeta$ of $\Phi$. For each $\Phi \in \mathcal{P}$ there exists a non-trivial $F$-stable subtorus $S_\Phi$ of $G$ such that $|S_\Phi^F| = \Phi(q)^{n_\Phi}$.
\end{proposition}

We note that if $G^F$ is a Ree or Suzuki group, the $\eta$ of Definition 2.6 is 2. Otherwise $\eta = 1$ and the $q$-cyclotomic polynomials are cyclotomic polynomials.

We call any $F$-stable torus $S$ such that $|S^F|$ is a power of $\Phi(q)$ a $\Phi$-torus, and tori $S_\Phi$ as above are called Sylow $\Phi$-subtori of $(G, F)$ — we abuse notation and call
them Sylow $\Phi$-subtori of $G$ when $F$ is clear from the context; they are the most direct product of $n_\Phi$ $F$-indecomposable $\Phi$-tori.

**Proof.** Proposition 2.11 is essentially in [Broué-Malle] but let us reprove it.

First, we note that assuming $|G^F|$ has a decomposition of the form 2.12, the value of $n_\Phi$ results from 2.10: let $\zeta$ be any root of $\Phi(x)$. Then $(x - \zeta)$ divides $\Phi(x)$ with multiplicity one, and does not divide any another $\Psi(x)$ for $\Psi \in \mathcal{P}$ since the $\Phi(x/q)$ are distinct irreducible polynomials in $\mathbb{Q}[x]$. Thus $n_\Phi$ is the number of pairs $(d_i, \varepsilon_i)$ such that $x - \zeta$ divides $x^{d_i} - \varepsilon_i$.

There is a decomposition of the form 2.12: if $\eta = 1$ we get such a decomposition of $|G^F|$ by decomposing each term of 2.10 into a product of cyclotomic polynomials. Otherwise $G^F$ is a Ree or Suzuki group, $\eta = 2$ and $q$ is an odd power of $\sqrt{2}$ or $\sqrt{3}$, and the set $\mathcal{P}$ and the decomposition of the form 2.12 is given by what follows:

| $(G, F)$ | $|G^F|$ | Generalized degrees of $W_\phi$ |
|----------|--------|-----------------------------|
| $2B_2(q^r)$ | $q^4(\Phi_2, \Psi_4)(q)$ | $\{(2, 1), (4, 1)\}$ |
| $2F_4(q^2)$ | $q^{12}(\Phi_2, \Psi_4, \Psi_6, \Psi_{12})(q)$ | $\{(2, 1), (6, -1), (8, 1), (12, -1)\}$ |
| $2G_2(q^r)$ | $q^6(\Phi_2, \Psi_4, \Psi_6, \Psi_{12})(q)$ | $\{(2, 1), (6, -1)\}$ |

Note that for $\eta = 2$ our “$q$-cyclotomic polynomials” are the “$(tp)$-cyclotomic polynomials” defined in [Broué-Malle, 3.14].

To construct the torus $S_\Phi$ for $\Phi \in \mathcal{P}$, let us choose $\zeta$ a root of $\Phi$ and $w$ as in (2) of Proposition 2.9. Then if $T_w$ is a maximal torus of type $w$ with respect to $T$, so that $(T_w, F) \simeq (T, wF)$, the characteristic polynomial of $w\phi$ on $X(T)$ has $\Phi(x)^{n_\Phi}$ as a factor; the kernel of $\Phi(w\phi)$ on $X(T)$ is a pure sublattice corresponding to a subtorus $S_\Phi$ of $T_w$ such that $|S_\Phi| = \Phi(q)^{n_\Phi}$.

**Proposition 2.13.** Let $(G, F)$ be as in 1.1, semisimple and such that the Dynkin diagram of $G$ has $n$ connected components permuted transitively by $F$. Then there exists a reductive group $G_1$ with isogeny $F_1$ such that up to isomorphism $G$ is a “descent of scalars” $G = G_1^F$ with $F(g_1, \ldots, g_n) = (g_2, \ldots, g_n, F_1(g_1))$.

Then $G^F \simeq G_1^F$, and if the scalar associated to $(G, F)$ is $q$ that associated to $(G_1, F_1)$ is $q_1 := q^n$. Thus we have $|G^F| = q^n \sum_{1}^{d_1} \prod_{\phi \in \mathcal{P}} \Phi(q)^{n_\phi}$ where $d_i, \mathcal{P}, n_\phi$ are as given by 2.11 for $(G_1, F_1, q_1)$.

Here again, for $\Phi \in \mathcal{P}$ there exists a Sylow $\Phi$-subtorus of $G$, that is an $F$-stable subtorus $S_\Phi$ such that $|S_\Phi| = \Phi(q)^{n_\Phi}$.

**Proof.** The proposition is obvious apart perhaps for the statement about the existence of $S_\Phi$. This results in particular from the following lemma that we need for future reference.

**Lemma 2.14.** In the situation of Proposition 2.13, let $(T, wF)$ where $T = T_1^w$ be a maximal torus of type $w = (1, \ldots, 1, w_1)$ of $G$ and define $\phi$ on $V = X(T) \otimes \mathbb{C}$ (resp. $\phi_1$ on $V_1 = X(T_1) \otimes \mathbb{C}$) by $F^* = q\phi$ (resp. $F_1^* = q_1\phi_1$). Then if the characteristic polynomial of $w_1\phi_1$ is $P(x)$, that of $w\phi$ is $P(x^n)$. Let $\Phi$ be a $q_1$-cyclotomic factor of $P$ (corresponding to a $\mathbb{Z}[x]$-irreducible factor of the characteristic polynomial of $w_1\phi_1$) and let $\zeta$ be a root of $\Phi(x^n)$. Denote by $V_\zeta$ the $\zeta$-eigenspace of $w\phi$ (resp. by $V_1, \zeta^n$, the $\zeta^n$-eigenspace of $w_1\phi_1$).

Let $S_1$ be the Sylow $\Phi$-subtorus of $(G_1, F_1)$ determined by $\text{Ker}(\Phi(w_1\phi_1))$, and $S$ be the $wF$-stable subtorus of $T$ determined by $\text{Ker}(\Phi((w\phi)^n))$. Then $S$ is a Sylow
composes in several cyclotomic polynomials according to the formula $\Phi$
no longer true for a descent of scalars. First, a cyclotomic polynomial in
$\zeta$ and by the analysis above $C_{V_1} = \{(v_1, \ldots, v_n) \mid v_i \in C_{W_i(V_1, 1_{C_0})}\}$ and that $N_W(V_1) = \{(v_1, \ldots, v_n) \mid v_i \in C_{W_i(V_1, 1_{C_0})}\}$. This shows that $N_W(V_1)/C_{W_i(V_1)} \simeq N_W(V_1)/C_{W_i(V_1, 1_{C_0})}$. Since when $\zeta$ runs over the roots of $\Phi(x^n)$ the $q_i\zeta^n$ are roots of the same $\mathbb{Z}[x]$-irreducible polynomial $q_i^{d\Phi} \Phi(x/q_1)$, the $\zeta^n$ are Galois conjugate thus $C_{W_i(V_1, 1_{C_0})}$ (resp. $N_W(V_1, 1_{C_0})$) centralizes (resp. normalizes) all the conjugate eigenspaces, whence our claim that $N_W(V_1, 1_{C_0})/C_{W_i(V_1, 1_{C_0})} \simeq N_G(S_i)/C_{G_i(S_i)}$. Now $\ker(\Phi((w)\zeta^n))$ is the span of $V_0$ for all roots $\zeta$ of $\Phi(x^n)$ and by the analysis above $C_{W_i(V_1)}$ and $N_W(V_1)$ are independent of $\zeta$, thus isomorphic to $C_{W} G$ and $N_W(S)$.

We have the following commutative diagram

$$
\begin{array}{c}
X \xrightarrow{wF^{-1}} X \xrightarrow{\text{Res}} \text{Irr}(T^{wF}) \longrightarrow 1 \\
\downarrow \Sigma \downarrow \Sigma \downarrow \sim \\
X_1 \xrightarrow{w_1F_1^{-1}} X_1 \xrightarrow{\text{Res}} \text{Irr}(T^{w_1F_1}) \longrightarrow 1
\end{array}
$$

where $\Sigma$ is the map $(x_1, \ldots, x_n) \mapsto x_1 + \ldots + x_n$. Since we have $\Sigma \circ (wF)^n = w_1F_1 \circ \Sigma$, for any polynomial $Q$ the morphism $\Sigma$ induces a surjective morphism $\ker(Q(wF^n)) \to \ker(Q(w_1F_1^n))$ whence for $Q = P$ a surjection $\text{Irr}(S^{wF}) \to \text{Irr}(S_1^{w_1F_1})$; since $S^{wF}$ is prime to $T^{wF}/S^{wF}$ this surjection must be an isomorphism. Extended to $V = \mathbb{Z} \otimes \mathbb{C}$, the map $\Sigma$ sends $V_0$ to $V_1, 1_{C_0}$ and sends the action of $N_W(V_1, 1_{C_0})/C_{W_i(V_1, 1_{C_0})}$ to that of $N_W(V_1, 1_{C_0})/C_{W_i(V_1, 1_{C_0})}$, whence the last statement of the lemma.

Note that any element of $W \phi$ is conjugate to an element of the form $(1, \ldots, 1, w_1) \phi_1$ so the form of $w$ in the statement of Lemma 2.14 covers all the types of maximal tori.

Remark 2.15. If the generalized degrees of $W_1\phi_1$ are $(d_i, \varepsilon_i)$, those of $W_\phi$ are $(d_i, \eta_{i,j})$ where $\eta_{i,j}$ for $j \in \{1, \ldots, n\}$ runs over the $n$-th roots of $\varepsilon_i$. It follows that $n_q$ can be defined in terms of $W_\phi$ as it is also the number of $(d_i, \eta_{i,j})$ such that $\zeta^{d_i} = \eta_{i,j}$, where $\zeta$ is any root of $\Phi(x^n)$.

Remark 2.16. For $\Phi \in \mathcal{P}(G)$, a Sylow $\Phi$-subtorus of $G$ is a “power” of a subtorus $S_0$ such that $[S_0^\Phi] = \Phi(q)$. If $G$ is quasi-simple, such a subtorus $S_0$ is $F$-indecomposable (since then the polynomial $\Phi$ is $q$-cycloctom). But this is no longer true for a descent of scalars. First, a cyclotomic polynomial in $x^n$ decomposes in several cyclotomic polynomials according to the formula $\Phi_d(x^n) = \prod_{\mu \in \mathbb{P}(\mathbb{Z}, d \text{ prime to } d)} \Phi_{n_d}(x)$ (see [Broué-Malle, Appendice 2]). But there could be further decompositions: for instance, the characteristic polynomial of $F^*$ on a Coxeter torus of a semisimple group $G$ of type $B_2$ over $\mathbb{F}_2$ is $x^2 + 4$, which is $\mathbb{Z}$-irreducible. But on a descent of scalars $G \times G$, the characteristic polynomial
of $F^*$ on a lift of scalars of this torus is $x^4 + 4$ which is no longer $\mathbb{Z}$-irreducible: $x^4 + 4 = (x^2 + 2x + 2)(x^2 - 2x + 2)$, so the torus seen inside the descent of scalars is no longer $F$-indecomposable.

We could have decomposed $|G^F|$ into a product of $q$-cyclotomic polynomials corresponding to $F$-indecomposable tori, but in the case of descent of scalars it was convenient to use larger tori.

**Remark 2.17.** An arbitrary semisimple reductive group is of the form $G = G_1 \ldots G_k$, an almost direct product of descendents of quasi-simple groups $G_i$, corresponding to the orbits of $F$ on the connected components of the Dynkin diagram of $G$. Then we have $|G^F| = |G_1^F| \ldots |G_k^F|$ by Lemma 2.4, and similarly, if $S$ is an $F$-stable torus of $G$, and $S_0 = S \cap G_i$, then $|S^F| = |S_1^F| \ldots |S_k^F|$. This can be used to give a global decomposition of $|G^F|$, but the polynomials $P$ in one factor could divide those in another. For instance we could have $\Phi_{2,4}$ for a factor of $G$ of type $2B_2$ and $\Phi_8$ for another factor of type $B_2$. Because of this it is cumbersome to give a global statement.

From now on we fix $(G, F)$ as in 2.13, which determines $q, n$, and $\eta$ minimal such that $q^{\eta n} \in \mathbb{Z}$. This allows in the next definition to omit the mention of $G$ and $F$ from the notation $d(\ell)$.

**Definition 2.18.** Let $\ell$ be a prime number different from $p$. In the context of 2.13 we define $d(\ell)$ as the order of $q^{\eta n} \mod \ell$ (if $\ell = 2$).

In particular $\ell | \Phi_{d(\ell)}(q^{\eta n})$.

The next proposition extends some of the Sylow theorems of [Broué-Malle], and introduces a complex reflection group $W_\Phi$ attached to each $\Phi$ in the set $\mathcal{P}$ of 2.11.

**Proposition 2.19.** Under the assumptions of 2.13, let $T$ be an $F$-stable maximal torus of $G$ in an $F$-stable Borel subgroup, and let $W_\Phi \subset \text{GL}(X(T))$ be the reflection coset associated to $(G, F)$. Then for each $\Phi \in \mathcal{P}$:

1. If $\zeta$ is a root of $\Phi(x^n)$ and $w$ is as in 2.9(2), a maximal torus of $G$ of type $w$ with respect to $T$ contains a unique Sylow $\Phi$-subtorus $S$.

For $\zeta, w$ as in (1) let $W_\Phi = N_W(V_\zeta)/C_W(V_\zeta)$ where $V_\zeta$ is the $\zeta$-eigenspace of $w_\Phi$ on $V = X(T) \otimes \mathbb{C}$.

2. For $S$ as in (1) we have $N_{G^F}(S)/C_{G^F}(S) = N_G(S)/C_G(S) \simeq W_\Phi$, and $W_\Phi$ can be identified to a subgroup of $\text{GL}(X(S))$.

3. The Sylow $\Phi$-tori of $G$ are $G^F$-conjugate.

4. Let $\ell \neq p$ be a prime number, and assume that $\Phi$ divides $\Phi_{d(\ell)}$ (see Definition 2.18). Then unless $\ell = 2$ and $(G_1, F_1)$ is of type $2G_2$, any Sylow $\ell$-subgroup of $W_\Phi$ acts faithfully on the subgroup of $\ell$-elements $S^F_\ell$ of $S^F$.

**Proof.** For (1) we consider a torus $(T, wF)$ of type $w$. Then a $wF$-stable subtorus corresponds to the span of a subset of eigenspaces of $w_\Phi$ on $V$. Since the polynomials $\Phi$ are prime to each other the polynomials $\Phi(x^n)$ are also, thus $q^\zeta$ is root of no other factor of the characteristic polynomial of $w_\Phi$ than $\Phi(x^n)$. Thus the $S$ defined in Lemma 2.14, which we will denote $S_0$, is unique.

Let us show (2). Let $(T_w, F, S)$ be conjugate to $(T, wF, S_0)$. Let $L = C_G(S)$, which, as the centralizer of a torus, is a Levi subgroup. Then we note that $N_G(S) \subset N_G(L)$. It follows that we can find representatives of $N_G(S)$ modulo $L$ in $N_G(T_w)$ since for $n \in N_G(S)$ the torus $nT_w$ is another maximal torus of $L$ which...
is thus \( L \)-conjugate to \( T_w \). We thus get that \( N_G(S)/L = \{N_G(S, T_w)/(N_G(T_w) \cap L)\); transferring this to \( T \) and then to \( W \) we get \( N_G(S, T_w)/(N_G(T_w) \cap L) \simeq N_W(S_0)/C_W(S_0) \) where \( S_0 \) is the subtorus of \( T \) determined by \( \text{Ker}(P(wF^*)) \) where \( P = \Phi(x^n/q^n) \). The action of \( F \) is transferred to the action of \( wF \) on this quotient.

That \( N_W(S_0) = N_W(V_\zeta) \) and \( C_W(S_0) = C_W(V_\zeta) \) was given in 2.14.

By 2.9(4) we see that the action of \( wF \) on \( N_W(S_0)/C_W(S_0) \) is trivial, thus also that of \( F \) on \( N_G(S)/C_G(S) \), thus \( N_G(S)/C_G(S) = (N_G(S)/C_G(S))^F = N_G(S)^F/C_G(S)^F = N_{G^F}(S)/C_{G^F}(S) \), the second equality since \( L = C_G(S) \) is connected. Finally, the last part of (2) results from the fact that the representation of \( W_\Phi \) on \( X(S_0) \), extended to \( X(S_0) \otimes \mathbb{C} \) has as summand the representation of \( W_\Phi \) on \( V_\zeta \), which is the reflection representation, thus faithful.

(3) is a direct translation of 2.9(3): when brought to subtori of \( T \) corresponding to eigenspaces of \( wF \) (resp. \( wF \)) the \( G^F \)-conjugacy of two Sylow \( \Phi \)-subtori corresponds to the \( W \)-conjugacy of the corresponding eigenspaces.

For (4) we first remark that we can reduce to the case where \( G \) is quasi-simple, using 2.14. Thus either \( q \in \mathbb{Z} \) or \( G^F \) is a Ree or a Suzuki group. Let \( \delta \) be the order of the coset \( wF \), that is the smallest integer such that \( (W\Phi)^\delta = W \). We have \( \delta \in \{1, 2, 3\} \). We first show the

**Lemma 2.20.** If \( G \) is quasi-simple and we are in one of the cases:

1. \( q \in \mathbb{Z} \) and \( \delta \in \{1, 2\} \).
2. \( q \in \mathbb{Z}, \delta = 3 \) and \( d \) is prime to 3.
3. \( q \) is an odd power of \( \sqrt{2} \) and \( \ell = 3 \).

then \( W_\Phi \) acts faithfully on \( S_\ell^F \).

**Proof.** On \( X(T) \otimes \mathbb{Q}(q^{-1}) \) we have \( wF^* = qwF \). The characteristic polynomial \( Q \) of \( wF^* \) on \( X(S) \) is \( q^{n_\Phi} - \Phi(x/q^{n_\Phi}) \); as \( wF^* \) is semisimple, the minimal polynomial of \( wF^* \) is \( P = q^{k_\Phi} - \Phi(x/q) \). We can identify \( X(S) \) with \( \text{Ker}(P(qwF)) \) on \( X(T) \). As in the proof of Proposition 2.8, if \( X = X(S) \) we can make \( X' = X \otimes \mathbb{Z}[q^{-1}] \) an \( A \)-module where \( A = \mathbb{Z}[x, q^{-n_\Phi}]/P \). Under the assumptions of the lemma \( A \) is a Dedekind ring. This results from the proof of 2.8(1) when \( q \in \mathbb{Z} \). In the remaining case (3) of Lemma 2.20, \( \eta = 2 \) and the order of \( q^{k_\Phi} \) (mod 3) is 2, thus \( \Phi = x^2 + 1 \) and \( P = x^2 + q^2 \); we have \( A = \mathbb{Z}[x, x^{-2}]/P \simeq \mathbb{Z}[\sqrt{-2}] \) which is integrally closed (thus Dedekind) since localized at \( \mathbb{Z}[\sqrt{-2}] \) which is integrally closed. As an \( A \)-module of rank \( n_\Phi \), the module \( X' \) is a sum of projective rank 1 submodules thus \( S \) is a product of \( n_\Phi \) copies of a \( wF \)-indecomposable torus. By Proposition 2.19(2) we can identify \( W_\Phi \) to a subgroup of \( GL(X) \). With the notations of 2.8, since the assumption of 2.8(1) is satisfied, \( X := X/(wF^* - 1)X \simeq \text{Irr}(S_{wF}) \) is isomorphic to \( (Z/\Phi(q))^{n_\Phi} \). The representation of \( W_\Phi \) on \( X \) reduces to \( X \). We will show it is faithful on \( X/\ell X \) (or \( X/4X \) when \( \ell = 2 \)).

If \( q \in \mathbb{Z} \) and \( \ell = 2 \) then \( d \in \{1, 2\} \) and we can apply Proposition 2.8(2) taking \( m = 4 \). We get that \( X/4X \simeq \text{Ker}(wF^* - 1 | X/4X) \). We have as observed in the proof of Proposition 2.8 that \( \text{Ker}(wF^* - 1) = X/4X \) and the representation of \( W_\Phi \) on \( X/4X \), which is a quotient of \( \text{Irr}(S_{wF}) \), is faithful by Lemma 4.3.

If \( q \in \mathbb{Z} \) and \( \ell \neq 2 \) then \( d \) is prime to \( \ell \); and in case (3) of Lemma 2.20 \( \eta = 2, \ell = 3 \) thus \( d = 2 \) and \( \ell \) is prime to \( d_\Phi \). In both cases we can apply Proposition 2.8(2) with \( m = \ell \) to get that \( X/\ell X \simeq \text{Ker}(wF^* - 1 | X/\ell X) \). We know by Lemma 4.3 that the representation of \( W_\Phi \) on \( X/\ell X \) is faithful and we would like to conclude that it is faithful on the submodule \( \text{Ker}(wF^* - 1) \). We use the element \( v \) given
by Proposition 2.9(5): it preserves the kernel of $\Phi(w\phi)$ thus induces an element of $\text{GL}(X)$ which defines an automorphism $\sigma$ of $W_\Phi$ which sends $w\phi$ to $(w\phi)^a$, so it remains true after reduction (mod $\ell$) that $\sigma$ sends $w\phi$ to $(w\phi)^a$, thus permutes the eigenspaces of $wF^*$ on $X/\ell X$: since $d$ is the order of $q$ (mod $\ell$), all the primitive $d$-th roots of unity live in $\mathbb{F}_\ell$ and the eigenvalues of $wF^*$ are the product of one primitive $d$-th root of unity, which is $q$, by the other primitive $d$-th roots of unity so are of the form $q^{1-a}$ where $a$ runs over $(\mathbb{Z}/d)^\times$. And under the assumption $(W\phi)^a = W\phi$ of 2.9(5) we can find $v$ thus $\sigma$ which sends the $q^{1-a}$-eigenspace of $wF^*$ to the $q^{1-a}$ = 1-eigenspace.

If every $a$ prime to $d$ has a representative in $1 + \delta \mathbb{Z}$ we can satisfy $(W\phi)^a = W\phi$ for such $a$ thus every eigenspace is isomorphic as a $W_\Phi$-module to $\text{Ker}(wF^* - 1)$. Then $W_\Phi$ is faithful on the whole $X/\ell X$ if and only if it is faithful on $\text{Ker}(wF^* - 1)$, thus we conclude. If $a \equiv 1$ (mod $\gcd(d, \delta)$) then by Bezout’s theorem there exist integers $\alpha, \beta$ such that $a = 1 + ad + \beta\delta$, and then $a - ad \in 1 + \delta \mathbb{Z}$ is a representative of $a$.

If $\delta = 1$ or $\delta = 2$ then every $a$ prime to $d$ is $\equiv 1$ (mod $\gcd(d, \delta)$) and we conclude. We conclude similarly if $\delta = 3$ and $d$ is prime to 3, or in case (3) of Lemma 2.20 since in this case $d = 2$.

When $q \in \mathbb{Z}$ the only case not covered by the lemma is $^3D_4$ and $d$ divisible by 3, that is $d \in \{3, 6, 12\}$. But in this case $\ell > 3$, since $d$ is the order of $q$ (mod $\ell$), thus $|W|$ is prime to $\ell$ and a fortiori the Sylow $\ell$-subgroup of $W_\Phi$ is trivial.

For the Ree and Suzuki groups we do not have to consider $^2B_2$ since $W$ is a 2-group and $\ell \neq p$, and the groups $^2G_2$ since only the prime $\ell = 2$ divides $|W|$ and is different from $p$, and this case is excluded in the proposition.

For the groups $^2F_4$ the only prime $\ell \neq p$ such that $\ell||W|$ is $\ell = 3$ and we are in case (3) of the lemma.

The Ree group $^2G_2$ with $\ell = 2$ is a genuine counterexample since the Sylow 2-subgroups of $^2G_2(q)$ are isomorphic to $(\mathbb{Z}/2)^3$.

3. THE STRUCTURE OF THE SYLOW $\ell$-SUBGROUPS

Definition 3.1. Let $G, F, G_1, \mathcal{P}$ and $n$ be as in 2.12 and let $\ell \neq p$ be a prime number. We define $D(\ell)$ as the set of integers $d$ such that for some $\Phi \in \mathcal{P}$ dividing $\Phi_d(x^n)$ we have $\ell|\Phi(q^n)$, where $\eta$ is as in Definition 2.18.

The following proposition is [Enguehard, Théorème 1] when $\eta = 1$: we give here a shorter proof. Since [Enguehard] was written, Malle ([Malle, 5.14 and 5.19]) has published a proof of (2) below — thus implicitly of (1) also— when $\eta = 1$ (giving more, see Theorem 3.3).

Theorem 3.2. Assume in the situation of 3.1 that $D(\ell) \neq \emptyset$, or equivalently that $\ell|G^F$. Then

1. $d(\ell) \in D(\ell)$.
2. There exists a unique $\Phi \in \mathcal{P}$ such that $\ell|\Phi(q^n)$ and $\Phi$ divides $\Phi_d(x^n)$. If $S$ is a Sylow $\Phi$-torus then $N_G(S)$ contains a Sylow $\ell$-subgroup of $G^F$ which is an extension of $(Z^0C_G(S))^F$ by a Sylow $\ell$-subgroup of $W_\Phi$.
3. The Sylow $\ell$-subgroups of $G^F$ are abelian if and only if $|D(\ell)| = 1$ (which is equivalent to $W_\Phi$ being an $\ell$-group), apart from the exception where
is of type $^2G_2$ and $\ell = 2$ in which case $|D(\ell)| = 2$ and $|W_\Phi| = 6$ but the 2-Sylow is abelian, isomorphic to $(\mathbb{Z}/2)^3$.

Further, if $S$ is as in (2), then $(Z^0C_G(S))^F = S^F_\ell$ except if:

- $\ell = 3$ and $G_1$ of type $^3D_4$.
- $\ell = 2, d = 1$ and for some odd degree $\varepsilon_1 = -1$. Equivalently $G_1$ is non-split and has an odd reflection degree, that is, is one of $^2A_n, ^2D_{2n+1}$ or $^2E_6$.
- $\ell = 2, d = 2$ and for some odd degree $\varepsilon_1 = 1$; equivalently $G_1$ is split and has an odd reflection degree, that is, is one of $^2A_n(n > 1), ^2D_{2n+1}$ or $^2E_6$.

In the above exceptions, $Z^0C_G(S) = C_G(S)$ is a maximal torus of $G$.

Proof. Let us note that to prove (2) when we are not in an exception, that is the stronger statement that a Sylow $\ell$-subgroup is in an extension of $S^F$ by a Sylow $\ell$-subgroup of $W_\Phi$, it is enough to prove that

$$v_{\ell}(|G^F|) = v_{\ell}(|S^F|) + v_{\ell}(|W_\Phi|)$$

where $v_{\ell}$ denotes the $\ell$-adic valuation, and in the exceptions, if we have proved that $Z^0C_G(S) = C_G(S)$ it is enough to show

$$v_{\ell}(|G^F|) = v_{\ell}(|C_G(S)|) + v_{\ell}(|W_\Phi|)$$

Note also that by the definition of $d(\ell)$ and $D(\ell)$ in Proposition 2.13, assertion (1) as well as formulae (*) and (**) are equivalent in $G$ and $G_1$, that is we may assume $G$ quasi-simple to prove them which we do now. Also, in view of (2) and Proposition 2.19(4), (3) reduces to proving:

(3') $|D(\ell)| = 1$ is equivalent to $W_\Phi$ being an $\ell'$-group.

We first look at the cases of a Ree or Suzuki group, where $\eta = 2$.

Let us prove (1) first. By Lemma 4.2 if $\ell$ divides $|G^F|$ then there is an element of $D(\ell)$ of the form $d(\ell)^b$ with $b \geq 0$. By inspecting the order formula for $|G^F|$ given in the proof of 2.11 the elements of $D(\ell)$ have all their prime factors in $\{2, 3\}$, so $b > 0$ implies $\ell \in \{2, 3\}$ thus $d(\ell) \in \{1, 2\}$; inspecting again the formula, we see that then $d(\ell)$ is in $D(\ell)$ and that $|D(\ell)| = 1$ unless $\ell \in \{2, 3\}$.

To prove (2) for $\ell \notin \{2, 3\}$, we observe there is a single $\Phi \in P$ such that $\ell | \Phi(q)$ since the two numbers $\Phi_{2,4}(q), \Phi_{2,4}^+(q)$ are prime to each other, and the same observation applies to $\Phi_{4,6}(q), \Phi_{4,6}^+(q)$ and $\Phi_{12,12}(q), \Phi_{12,12}^+(q)$. Thus for $\ell \notin \{2, 3\}$ assertions (3') and (*) are obvious since $|G^F|_{\ell} = |S^F|_{\ell}$ and $\ell \not\mid |W|$.

Let us prove (*) for $\ell \in \{2, 3\}$; since $\ell \neq p$ and the elements of $D(\ell)$ have only 2 as prime factor in the case $^2B_2$, we have just to consider:

- $\ell = 3$ for $^2F_4$: we have $d(3) = 2, W_{\Phi_{2,2}} = G_{12}$ of order 48; the only factor $\Phi(q)$ with a value divisible by 3 apart from $|S^F| = |\Phi_{2,2}(q)|^2 \Phi_{2,6}(q)$ and $v_3(\Phi_{2,6}(q)) = 1 = v_3(|G_{12}|)$ which proves this case.
- $\ell = 2$ for $^2G_2$: we have $d(2) = 2$ and $|W_{\Phi_{2,2}}| = 6$; the only factor $\Phi(q)$ with an even value apart from $|S^F| = \Phi_{2,2}(q)$ is $\Phi_{2,1}(q)$ and $v_2(\Phi_{2,1}(q)) = 1 = v_2(|W_\Phi|)$ which proves this case.

We have seen (3') along the way.

Now we look at the other quasi-simple groups thus $\eta = 1$. We notice generally that, assuming we have proved (1) then if $|D(\ell)| = 1$ assertion (2) is trivial since a Sylow $\ell$-subgroup is then in $S$, and (3') reduces to checking that $W_\Phi$ is an $\ell'$-group.

We consider separately $^3D_4$ where $^3D_4(q) = q^{12}(\Phi_{1,1}^+, \Phi_{3,3}^+, \Phi_{12,12}^+)(q)$. Again, since the only prime factors of elements of $D(\ell)$ are $\{2, 3\}$, we see that $d(\ell) \in D(\ell)$
except possibly if \( \ell \in \{2, 3\} \); but in that case \( d(\ell) \in \{1, 2\} \) and there is a factor \( \Phi_d(q) \), whence (1). Since \( |W| = 3 \cdot 2^b \) assertion (3') is proved when \( d(\ell) = 1 \).

It remains to prove (2) when \( \ell \in \{2, 3\} \). In both cases \( W_{\Phi_d(q)} = W(G_2) \) and by Lemma 4.2 \( v_\ell(\mathbb{F}^F/[\mathbb{F}^S]) = 2 \). If \( \ell = 2 \) then \( 2 = v_\ell(|W(G_2)|) \) which proves (*). If \( \ell = 3 \) a Sylow \( \Phi \)-subtorus \( S \) is in a torus \( T_w = C_G(S) \) where \( w = 1 \) if \( d = 1 \) (resp. \( w = w_0 \) if \( d = 2 \)). We have \( |T_w^S| = \Phi_0(q) \Phi_2(q) \) (resp. \( |T_w^{\Phi_0}| = \Phi_2(q)^2 \Phi_6(q) \)) which has same 3-valuation as \( |G^F|/|W_\Phi| \) which proves (**).

In the remaining cases \( \varepsilon_i = \pm 1 \) for all \( i \). Let us set \( \zeta_d = e^{2i\pi/d} \). We have \( \Phi = \Phi_{d(\ell)} \) and \( v_\ell([S^F]) = |a(\zeta_{d(\ell)})|v_\ell(\Phi_{d(\ell)}(q)) \).

We first treat the case \( \ell \) odd. We have \( a(\zeta_d) = \{ d_i \ | \ z_{d_i} = \varepsilon_i \} \) and \( |W_\Phi| = \prod_{d_i \in a(\zeta_{d(\ell)})} d_i \). By Lemma 4.2, a factor \( \Phi_e(q) \) of \( |G^F| \) can contribute to the \( \ell \)-valuation only if \( e \) is of the form \( d(\ell)\ell^b \) for some \( b \geq 0 \). Further such a factor appears if and only if \( a(\zeta_e) \neq \emptyset \), that is for some \( i \) we have \( z_{d_i(\ell)} = \varepsilon_i \). Since \( \ell \) is odd raising this equality to the power \( \ell^b \) gives \( z_{d_i(\ell)} = \varepsilon_i \) thus \( d_i \in a(\zeta_{d(\ell)}) \) and in particular \( d(\ell) \in D(\ell) \). And \( z_{d_i(\ell)} = \varepsilon_i \) implies that \( \ell^b \) divides \( d_i \). Thus only the \( d_i \in a(\zeta_{d(\ell)}) \) contribute to \( v_\ell([S^F]) \) and each of them contributes \( v_\ell(\Phi_{d(\ell)}(q)) + v_\ell(\Phi_{d(\ell)}(q)) + \ldots + v_\ell(\Phi_{d(\ell)}(q)) \). By Lemma 4.2 this is \( v_\ell(\Phi_{d(\ell)}(q)) \). Summing over \( d_i \in a(\zeta_{d(\ell)}) \) proves (*).

It remains the case \( \ell = 2 \) where we proceed similarly. We have \( d(2) \in \{1, 2\} \). If \( d(2) = 1 \) then \( a(1) = \{ d_i \ | \ v_i = 1 \} \). Thus the condition \( z_{d_i} = \varepsilon_i \) is still equivalent to \( 2^b|d_i| \); but there could be some more solutions of this equation than elements of \( a(1) \) when \( b = 1 \): any odd \( d_i \) such that \( v_i = -1 \) brings an additional factor \( 1 = v_2(\Phi_2(q)) \). If \( d(2) = 2 \) then \( a(-1) = \{ d_i \ | \ v_i = -1 \} \). The contribution of the even \( d_i \) can be worked out as before; but this time the odd \( d_i \) where \( v_i = 1 \) bring additional factors \( v_2(\Phi_1(q)) \). In the exceptions in each case \( C_G(S) \) is a maximal torus of type 1 or \( w_0 \); looking at the orders of these tori, they contain enough extra \( \Phi_1 \) or \( \Phi_2 \) factors (which correspond to the eigenvalues 1 or \(-1 \) of \( \phi \) or \( w_0 \phi \)) to compensate the discrepancy.

Let us show now (3'), which reduces to proving that \( |D(\ell)| > 1 \) implies \( v_\ell(|W_\Phi|) > 0 \). Thus we assume \( |D(\ell)| > 1 \). We first do the case \( \ell = 2 \); then \( d(\ell) \in \{1, 2\} \) from which it follows, since the 1 and \(-1\)-eigenspaces are defined over the reals, that \( W_\Phi \) is a Coxeter group, whose order is always even. We consider finally \( \ell \)-odd; then it reduces to the case \( d(\ell) = (\ell^a)d_i \) for some \( a > 0 \). But we have seen above that there exists a factor \( \Phi_{d(\ell)^a}(q) \) only if \( \ell^a|d_i \) for some \( d_i \in a(\zeta_{d(\ell)}) \) \( \square \).

We remark that if \( \ell \) divides only one \( \Phi_d(q) \), a Sylow \( \ell \)-subgroup \( S \) lies in a single Sylow \( \Phi \)-torus \( S \) (the intersection of two tori has lower dimension so cannot have same order polynomial). It follows that \( N_{G^F}(S) = N_{G^F}(S) \) and \( C_{G^F}(S) = C_{G^F}(S) \). This observation is a start for describing the \( \ell \)-Frobenius category of \( G^F \) in terms of the category of \( \zeta \)-eigenspaces of \( W_\Phi \).

In general, one can deduce the following unicity theorem from the work of Cabanes, Enguehard and Malle.

**Theorem 3.3.** Consider \( G, F, n, G_1, q \) as in 2.13 with \( q^n \in \mathbb{Z} \) and let \( \Phi \) as defined in Theorem 3.2. (2). Assume that we are not in one of the following cases:

- \( \ell = 3 \), \( G_1 \) simply connected of type \( A_2 \), \( 2A_2 \) or \( G_2 \).
- \( \ell = 2 \), \( G_1 \) simply connected of type \( C_n, n \geq 1 \).


Let \( Q \) be a Sylow \( \ell \)-subgroup of \( G^F \). There is a unique Sylow \( \Phi \)-subtorus \( S \) of \( G \) such that \( Q \subseteq N_G(S) \).

**Proof.** In the context of Theorem 3.2(2), let \( Q \) be a Sylow \( \ell \)-subgroup of \( G^F \) contained in \( N_G(S) \); then according to [Cabanes], \( S_\ell^F \) is often characteristic in \( Q \) (for example when \( \ell \geq 5 \)), thus in these cases \( N_{G^F}(Q) \subseteq N_G(S_\ell^F) \). Using inductively that property and inspecting small cases, G. Malle has proved the inclusion

\[
N_{G^F}(Q) \subseteq N_G(S)
\]

for all quasi-simple groups \( G \) short of the cases excluded in Theorem 3.3, see [Malle, Theorems 5.14 and 5.19]. Here \( S \) is a Sylow \( \Phi_{d(\ell)} \)-subtorus of \( (G, F) \) as defined in Definition 2.18 with \( \eta = 1 \) (note that \( N_{G^F}(Q) \subseteq N_G(S) \) implies \( Q \subseteq N_G(S) \)).

We first verify that the last inclusion holds more generally in a "descent of scalars". With hypotheses and notations of Proposition 2.13 and Lemma 2.14 assume \( q^n \in \mathbb{Z} \). If \( e = d(\ell) \) is the order of \( q^n \) modulo \( \ell \), take \( \Phi = \Phi_e \in \mathcal{P} \), defining \( S = S_\Phi \) and \( S_1 \). There is a morphism from \( G \) onto \( G_1 \), sending \( S \) to \( S_1 \), with restriction an isomorphism from \( G^F \) to \( G_1^F \). Then a Sylow \( \ell \)-subgroup \( Q_1 \) of \( G_1^F \) contained in \( N_{G_1}(S_1) \) is the isomorphic image of a Sylow \( \ell \)-subgroup \( Q \) of \( G^F \) contained in \( N_G(S) \). The inclusion 3.4 written with (\( G_1, F_1, Q_1, S_1 \)) instead of (\( G, F, Q, S \)) implies 3.4 in \((G, F)\).

From 3.4 the unicity of \( S \), given \( Q \), follows:

**Lemma 3.5.** Let \( \Phi \in \mathcal{P} \), let \( S \) be a Sylow \( \Phi \)-subtorus of \( (G, F) \) and \( Q \) a Sylow \( \ell \)-subgroup of \( G^F \). If \( N_{G^F}(Q) \subseteq N_G(S) \), then \( S \) is the unique Sylow \( \Phi \)-torus of \((G, F)\) such that \( Q \subseteq N_G(S) \).

**Proof.** Assume \( Q \subseteq N_G(S') \) for some Sylow \( \Phi \)-torus \( S' \) of \((G, F)\). By Proposition 2.19 there exists \( g \in G^F \) such that \( S = (S')^g \), hence \( Q^g \subseteq N_G(S) \). By Sylow’s theorem in \( N_G(S)^F \), \( Q = Q^g \) for some \( h \in N_G(S)^F \) hence \( gh \in N_G(S) \) by our hypothesis. \( \square \)

4. Appendix

We gather here arithmetical lemmas used above.

**Lemma 4.1.** Let \( x, f, \ell \in \mathbb{N} \) where \( \ell \) is prime, and assume \( x \equiv 1 \pmod{\ell} \) (resp. \( \mod 4 \)) if \( \ell = 2 \). Then \( v_\ell\left(\frac{x^{f^\ell} - 1}{x - 1}\right) = v_\ell(f) \).

**Proof.** From \( \frac{x^{f^\ell f} - 1}{x - 1} = \frac{x^{f^{\ell f} - 1} - x^{f_1} - 1}{x - 1} \) we see that it is enough to show the lemma when \( f \) is prime. We have \( \frac{x^{f^\ell} - 1}{x - 1} = f + \sum_{i=2}^f (x - 1)^{i-1} \binom{f}{i} \). Let \( S \) be this last sum; we have \( S \equiv f \pmod{\ell} \), since \( x - 1 \equiv 0 \pmod{\ell} \), thus \( S \) is prime to \( \ell \) when \( f \neq \ell \) which shows the lemma in this case. When \( f = \ell \) then all the terms of \( S \) but the first one and possibly the last one are divisible by \( \ell^2 \) since \( \binom{f}{i} \) is divisible by \( \ell \) when \( 2 \leq i < \ell \); the last term is divisible by \( \ell^2 \) when \( \ell - 1 \geq 2 \) which fails only for \( f = \ell = 2 \); but when \( \ell = 2 \) we have arranged that \( v_2(x - 1) \geq 2 \) and this time \( 2(f - 1) \geq 1 \); thus \( S \equiv f \pmod{\ell^2} \), whence the lemma. \( \square \)

The following lemma is in [Malle, 5.2]; a short elementary proof results immediately from Lemma 4.1.
Lemma 4.2. Let \( q, \ell \in \mathbb{N} \) where \( \ell \) is prime. Let \( d \) be the order of \( q \mod \ell \) (or \( \mod 4 \) if \( \ell = 2 \)). Then \( \ell \) divides \( \Phi_d(q) \) if and only if \( e \) is of the form \( dtb \) with \( b \in \mathbb{N} \) (or additionally \( b = -1 \) when \( \ell = d = 2 \)), and \( \nu_{\ell}(\Phi_{dtb}(q)) = 1 \) if \( b \neq 0 \).

The following lemma is in [Minkowski]; we give the proof since it is very short and the original German proof may be less accessible.

Lemma 4.3. Let \( m \in \mathbb{N}, m > 2 \). Then the kernel of the reduction map \( \mathrm{GL}(\mathbb{Z}^n) \to \mathrm{GL}(\mathbb{Z}/m \mathbb{Z}^n) \) is torsion-free.

Note that the bound \( m > 2 \) is sharp since \(-\mathrm{Id} \equiv \mathrm{Id} \mod 2\).

Proof. Let \( w \in \mathrm{GL}(\mathbb{Z}^n) \) be of finite order, \( w \neq \mathrm{Id} \) and assume its reduction \( v = \mathrm{Id} \).

Possibly replacing \( w \) by a power, we may assume that \( w \) is of prime order \( p \).

Also \( \mathrm{GL}(\mathbb{Z}^n/m) = \prod_i \mathrm{GL}(\mathbb{Z}/p_i^n) \) where \( m = \prod_i p_i \) is the decomposition of \( m \) into prime powers, thus we may assume that \( m \) is a prime power.

Since \( w \) is of order \( p \), the polynomial \( \Phi_p(x) \) is a factor of the characteristic polynomial of \( w \). The characteristic polynomial of \( v \) is the reduction \( \mod m \) of that of \( w \), thus we must have \( \Phi_p(x) \mod m \equiv (x-1)^{p-1} \); in particular \( p-1 \equiv 1 \mod m \) thus \( m|p \) which implies \( m = p \).

Write now \( w = \mathrm{Id} + xmn^a \) where \( x \mod m \neq 0 \) and \( a \in \mathbb{N} \). Then the equation \( w^m = \mathrm{Id} \) gives \( \sum_{i=1}^m \binom{i}{m} x^i m^a i = 0 \), which after dividing by \( m^{a+1} \) becomes \( x = -\sum_{i=2}^m \binom{1}{m} x^i m^{a(i-1)-1} \) where all coefficients on the right-hand side are divisible by \( m \) (since \( m \geq 3 \), which contradicts \( x \mod m \neq 0 \)). \( \square \)

References


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