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ON ESTIMATES FOR THE $\bar{\partial}$ EQUATION IN STEIN MANIFOLDS.

ERIC AMAR

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ABSTRACT. We generalize to intersection of strictly c -convex domains in Stein manifold, $L^r - L^s$ and Lipschitz estimates for the solutions of the $\bar{\partial}$ equation obtained by Ma and Vassiliadou for domains in \mathbb{C}^n . For this we use a Docquier-Grauert holomorphic retraction plus the raising steps method I introduce earlier. This gives results in the case of intersection of domains with low regularity, \mathcal{C}^3 , for their boundary.

1. INTRODUCTION.

The solutions with L^r and Lipschitz estimates of the equation $\bar{\partial}u = \omega$, $\bar{\partial}\omega = 0$ are known to be very important in complex analysis and geometry.

The first results of this kind were obtained by the use of solving kernels: Grauert-Lieb [8], Henkin [11], Ovreid [20], Skoda [23], Krantz [15], in the case of strictly pseudo-convex domains with \mathcal{C}^∞ smooth boundary in \mathbb{C}^n , with the exception of Kerzman [14] in the case of $(0, 1)$ forms in strictly pseudo-convex domains with \mathcal{C}^4 smooth boundary in Stein manifolds.

Here we shall be interested in strictly c -convex, s.c.c. for short, domain D in a complex manifold. Such a domain is defined by a function ρ of class \mathcal{C}^3 in a neighbourhood U of \bar{D} and such that $i\partial\bar{\partial}\rho$ has at least $n - c + 1$ strictly positive eigenvalues in U .

These domains in \mathbb{C}^n have been studied in the case of smooth \mathcal{C}^∞ boundary by Fisher and Lieb [7].

Ma and Vassiliadou [18] obtained very nice estimates even in the case of intersections of s.c.c. domains with \mathcal{C}^3 boundary. I shall use their results here.

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Quite recently C. Laurent-Thiébaud [16] got this kind of result for s.c.c. domains with smooth \mathcal{C}^∞ boundary in complex manifold by use of the Grauert's method of "bumps".

Concerning the study of transverse intersection of domains, one can cite the works of Henkin and Leiterer [10], Menini [19] for strictly pseudo convex domains and G. Schmalz [22] and Ma and Vassiliadou [18] for c -convex domains. C. Laurent-Thiébaud and J. Leiterer [17] solved the $\bar{\partial}$ equation in a case of intersection of s.c.c. domains more general than the one considered by Ma and Vassiliadou [18] but for bounded forms and they got solutions in Lipschitz spaces. It seems that the L^p case is still open for their situation.

Let us state our first result which is completely analogous to the one Ma and Vassiliadou [18] obtained for domains in \mathbb{C}^n .

Theorem 1.1. *Let Ω be a Stein manifold of dimension n . Let D be a strictly c -convex (s.c.c.) domain relatively compact with smooth \mathcal{C}^3 boundary in Ω . Let ω be a (p, q) form in $L_{p,q}^r(D)$, $\bar{\partial}\omega = 0$, with $1 < r < 2n+2$, $c \leq q \leq n$. Then there is a $(p, q-1)$ form u in $L_{p,q-1}^s(D)$, with $\frac{1}{s} = \frac{1}{r} - \frac{1}{2n+2}$, such that $\bar{\partial}u = \omega$.*

If ω is in $L_{p,q}^r(D)$, $\bar{\partial}\omega = 0$ with $r \geq 2n+2$, $c \leq q \leq n$, then there is a $(p, q-1)$ form u in $\Lambda_{(p,q-1)}^\epsilon(\bar{D})$ such that $\bar{\partial}u = \omega$ with $\epsilon = \frac{1}{2} - \frac{n+1}{r}$.

The spaces $\Lambda_{(p,q-1)}^\epsilon(\bar{D})$ are the (isotropic) Lipschitz spaces of order ϵ and we set $\Lambda_{(p,q-1)}^0(\bar{D}) := L_{(p,q-1)}^\infty(D)$.

It has to be noticed that the boundary regularity is just \mathcal{C}^3 , so it seems that this is a new result in a Stein manifold for such a low regularity.

In the case of a \mathcal{C}^∞ boundary regularity then this result is contained in C. Laurent-Thiébaud [16] corollary 2.11, but the proof here is completely different and, in some sense, "lighter" because it never uses Beals, Greiner and Stanton [5] heavy technology. We use for the analytic part kernels methods plus essentially geometric ones. Nevertheless we can recover the Sobolev estimates by a direct use of Beals, Greiner and Stanton [5] in the case of a \mathcal{C}^∞ boundary regularity by theorem 6.2 here. This avoid the use of the "bumps method" but this is valid only in Stein manifolds although C. Laurent-Thiébaud [16] results are valid in any complex manifold.

To state our next result, we need the definition of a \mathcal{C}^3 c convex intersection, still taken from [18].

Definition 1.2. *A relatively compact domain D in a Stein manifold Ω shall be called a \mathcal{C}^3 c -convex intersection if there exists a relatively compact neighbourhood W in Ω of \bar{D} and a finite number of real \mathcal{C}^3 functions ρ_1, \dots, ρ_N where $n \geq N+3$ defined on W such that $D = \{z \in W :: \rho_1(z) < 0, \dots, \rho_N(z) < 0\}$ and the following are true:*

i) For $1 \leq i_1 < \dots < i_l \leq N$ the 1-forms $d\rho_{i_1}, \dots, d\rho_{i_l}$ are \mathbb{R} -linearly independent on $\bigcap_{j=1}^l \{\rho_{i_j} \leq 0\}$.

ii) For $1 \leq i_1 < \dots < i_l \leq N$, for every $z \in \bigcap_{j=1}^l \{\rho_{i_j} \leq 0\}$, if we set $I := (i_1, \dots, i_l)$, there exists a

linear subspace T_z^I of Ω of complex dimension at least $n - c + 1$ such that for $i \in I$ the Levi forms $L\rho_i$ restricted on T_z^I are positive definite.

We notice that, in \mathbb{C}^n , Ma and Vassiliadou need $N \leq n - 2$ and here we need $N \leq n - 3$. Now we can state:

Theorem 1.3. *Let Ω be a Stein manifold of dimension n and a \mathcal{C}^3 c -convex intersection D such that D is relatively compact in Ω . There exists a $\nu \in \mathbb{N}^+$ (which depends on the maximal number of non empty intersections of $\{\rho_j = 0\}$) such that :*

if ω is a (p, q) form in $L_{p,q}^r(D)$, $\bar{\partial}\omega = 0$ with $q \geq c$, $1 < r < 2n + 2$, then there is a $(p, q - 1)$ form u in $L_{p,q-1}^s(D)$ such that $\bar{\partial}u = \omega$ with $\frac{1}{s} = \frac{1}{r} + \frac{1}{\lambda} - 1$, where $1 \leq \lambda < \frac{2n + 2\nu}{2n - 1 + 2\nu}$.

More precisely,

i) For any $1 < r < 2n + 2\nu$, there exists $c_r(D)$, a positive constant such that

$$\|u\|_{L_{(p,q-1)}^s(D)} \leq c_r(D) \|\omega\|_{L_{(p,q)}^r(D)}$$

with $\frac{1}{s} = \frac{1}{r} - \frac{1}{2n + 2\nu}$.

ii) For $r \geq 2n + 2\nu$, we have $\|u\|_{L_{(p,q-1)}^\infty(D)} \leq a_r(D) \|\omega\|_{L_{(p,q)}^r(D)}$ for some positive constant $a_r(D)$.

This also seems to be new in case \mathbb{C}^n is replaced by a Stein manifold.

The results of Ma and Vassiliadou [18] give good estimates in case of domains in \mathbb{C}^n . The first point here was to pass from \mathbb{C}^n to a submanifold of \mathbb{C}^n . To do this I was inspired by a nice paper of H. Rossi [21] on Docquier Grauert holomorphic retraction. The first result is based on it and is the following non optimal theorem.

Theorem 1.4. *Let M be a closed submanifold of dimension d of a Stein domain U_0 in \mathbb{C}^n . Let D be a s.c.c. domain relatively compact in M ($\bar{D} \subset M$) with \mathcal{C}^3 boundary. Then, with $r \geq 2n + 2$, we can solve in D the equation $\bar{\partial}u = \omega$ when $\bar{\partial}\omega = 0$ and with $u \in \Lambda_{(p,q-1)}^\epsilon(\bar{D})$ if $\omega \in L_{(p,q)}^r(D)$, $c \leq q \leq n$, with $\epsilon = \frac{1}{2} - \frac{n+1}{r}$.*

Then we use the raising steps method [2] (see also [4] for more general operators than $\bar{\partial}$ and [3] in the non compact case). Let me recall it in this specific situation.

Theorem 1.5. *Let M be a closed complex manifold and D a relatively compact domain in M . Suppose there is $\delta > 0$ and a finite covering $\{U_j\}_{j=1,\dots,N}$ of \bar{D} such that :*

(i) $\forall r > 1$, $\forall \omega \in L_{(p,q)}^r(D)$, $\bar{\partial}\omega = 0$, $\exists u_j \in L_{(p,q-1)}^t(D \cap U_j) :: \bar{\partial}u_j = \omega$ in $D \cap U_j$

and $\|u_j\|_{L^t(D \cap U_j)} \lesssim \|\omega\|_{L^r(D \cap U_j)}$, with $\frac{1}{t} = \frac{1}{r} - \delta$.

(ii) $\exists s > 1$, $\forall \omega \in L_{(p,q)}^s(\Omega)$, $\bar{\partial}\omega = 0$, $\exists w \in L_{(p,q-1)}^s(\Omega) :: \bar{\partial}w = \omega$ and $\|w\|_{L^s(\Omega)} \lesssim \|\omega\|_{L^s(\Omega)}$.

Then there is a constant $c > 0$ such that, for $r \leq s$, if $\omega \in L_{(p,q)}^r(D)$, $\bar{\partial}\omega = 0$, it exists $u \in L_{(p,q-1)}^t(D)$ with $\lambda := \min(\delta, \frac{1}{r} - \frac{1}{s})$ and $\frac{1}{t} = \frac{1}{r} - \lambda$, such that $\bar{\partial}u = \omega$, $u \in L_{(p,q-1)}^t(D)$, $\|u\|_{L^t(D)} \leq c \|\omega\|_{L^r(D)}$.

The local estimates (i) are given by "localizing s.c.c. domain", proposition 7.2, plus the results of Ma and Vassiliadou, theorem 2.1 here. The global estimate (ii), the threshold, is given by the $L^r - \Lambda^\epsilon$ estimates done in theorem 1.4. We get the same optimal results as for domains in \mathbb{C}^n .

Theorem 1.6. *Let M be a complex submanifold of dimension d in \mathbb{C}^n and a s.c.c. domain D such that D is relatively compact with smooth boundary of class \mathcal{C}^3 in M . Let ω be a (p, q) form in $L_{p,q}^r(D)$, $\bar{\partial}\omega = 0$, $c \leq q \leq n$, with $1 < r < 2d + 2$. Then there is a $(p, q - 1)$ form u in $L_{p,q-1}^s(D)$ with $\frac{1}{s} = \frac{1}{r} - \frac{1}{2d + 2}$, such that $\bar{\partial}u = \omega$.*

If $r \geq 2n + 2$ then there is a $(p, q - 1)$ form u in $\Lambda_{(p,q-1)}^\epsilon(\bar{D})$ such that $\bar{\partial}u = \omega$ with $\epsilon = \frac{1}{2} - \frac{d+1}{r}$.

We follow exactly the same path to work with \mathcal{C}^3 c -convex intersection with again local estimates given by "localizing s.c.c. intersection", proposition 7.3 plus the results of Ma and Vassiliadou, theorem 2.2. The global estimate (ii), the threshold, is given by the $L^r - L^\infty$ estimates done in theorem 2.2 plus the generalization of a theorem of Rossi done in theorem 7.6.

To pass to Stein manifold, we use an embedding theorem of Bishop and Narashiman (see theorem 5.3.9. of Hörmander [13]) to see an abstract Stein manifold of dimension d as a submanifold of \mathbb{C}^{2d+1} . So we get our main results.

This work will be presented in the following way.

- First we recall the estimates in the case of strictly c -convex domains in \mathbb{C}^n done by Ma and Vassiliadou [18].
- We recall the Docquier Grauert holomorphic retraction on a complex submanifold M of \mathbb{C}^n .
- We extend a form ω from a domain D s.c.c. in M to a domain E s.c.c. in \mathbb{C}^n by use of a generalization of a theorem of H. Rossi [21]. We then solve the form in E by the known estimates in \mathbb{C}^n . Then we show that the solution in E can be restricted to D to get a solution in D with good enough estimates, for $r \geq 2n + 2$. This gives theorem 1.4.
- We use the raising steps theorem with the threshold given by theorem 1.4. So we have theorem 1.6 for the case of a submanifold of \mathbb{C}^n .
- Then by the same way, using ad-hoc modifications of propositions done in the appendix, we get theorem 1.3 for s.c.c. intersections in the case of a submanifold of \mathbb{C}^n .
- By use of a theorem of Bishop and Narashiman, i.e. the proper embedding of a Stein manifold of dimension d in \mathbb{C}^{2d+1} , we get our main theorems 1.1, 1.3 for any Stein manifold.
- Finally we prove technical results we need in the appendix.

I am indebted to C. Laurent-Thiébaud who pointed to me the precise link between the work of Beals, Greiner and Stanton [5] and the existence of actual solutions for the $\bar{\partial}$ Neuman problem.

Moreover I thank the referee for his nice suggestions : in particular the proof of theorem 1.4 comes from a slight modification of one of them and it simplifies substantially my original proof.

2. STRICTLY c -CONVEX DOMAIN IN \mathbb{C}^n .

We shall use the nice estimates for a smoothly \mathcal{C}^3 bounded c convex domains in \mathbb{C}^n obtained by Ma and Vassiliadou [18] , lemma 5.3. in their paper.

Theorem 2.1. *Let D be a bounded s.c.c. domain in \mathbb{C}^n with a \mathcal{C}^3 defining function. Then*

$$\forall \omega \in L_{(p,q)}^r, \bar{\partial}\omega = 0, \quad c \leq q \leq n,$$

there exists $u \in L_{(p,q-1)}^s(D)$, $\frac{1}{s} = \frac{1}{r} - \frac{1}{2n+2}$, with the following properties:

i) if $1 \leq r < 2n + 2$, $\bar{\partial}u = \omega$ in the sense of currents in D .

ii) if $r = 1$, $u \in L_{(p,q-1)}^{\frac{2n+2}{2n+1}-\eta}$ for any $\eta > 0$.

iii) if $2n + 2 \leq r \leq \infty$, $u \in \Lambda_{(p,q-1)}^\epsilon(\bar{D})$ with $\epsilon = \frac{1}{2} - \frac{n+1}{r}$.

They also prove results in the case of intersections.

Theorem 2.2. *Let a \mathcal{C}^3 c -convex intersection domain D be such that D is relatively compact in \mathbb{C}^n . Then there exists a $\nu \in \mathbb{N}^+$ (which depends on the maximal number of non empty intersections of $\{\rho_j = 0\}$) such that :*

if ω a (p, q) form in $L^r_{p,q}(D)$, $\bar{\partial}\omega = 0$ with $q \geq c$, $1 < r < 2n + 2\nu$, there is a $(p, q - 1)$ form u in $L^s(D)$, such that $\bar{\partial}u = \omega$ with $\frac{1}{s} = \frac{1}{r} + \frac{1}{\lambda} - 1$, where $1 \leq \lambda < \frac{2n + 2\nu}{2n - 1 + 2\nu}$.

More precisely

i) For any $1 < r < 2n + 2\nu$, there exists $c_r(D)$ positive constant such that

$$\|u\|_{L^s_{(p,q-1)}(D)} \leq c_r(D) \|\omega\|_{L^r_{(p,q)}(D)}$$

with $\frac{1}{s} = \frac{1}{r} - \frac{1}{2n + 2\nu}$.

ii) For $r \geq 2n + 2\nu$, we have $\|u\|_{L^\infty_{(p,q-1)}(D)} \leq a_r(D) \|\omega\|_{L^r_{(p,q)}(D)}$ for some positive constant $a_r(D)$.

3. THE DOCQUIER - GRAUERT HOLOMORPHIC RETRACTION.

We have the Docquier-Grauert lemma [6] :

Lemma 3.1. *Let K be a compact subset of a closed complex submanifold M of dimension d in \mathbb{C}^n . There is a neighbourhood U of K and a holomorphic map $\pi : U \rightarrow U \cap M$ such that $\pi(\zeta) = \zeta$ for $\zeta \in U \cap M$.*

In fact we have more (Rossi [21], p 172) from the argument of Docquier-Grauert we have that the fibers $\pi^{-1}\pi\zeta$ of π intersect M transversely at all points of M and are of dimension $n - d$.

Let M be a complex submanifold of dimension d in \mathbb{C}^n and D a relatively compact domain strictly c -convex in M . We have the following lemma.

Lemma 3.2. *Let $\zeta \in \bar{D}$, there is a neighborhood U of ζ in \mathbb{C}^n and a bi-holomorphic mapping (U, φ) , $\varphi : U \rightarrow T := \varphi(U)$, such that, with $z = (z_1, \dots, z_n)$ the coordinates in T , we have: $\varphi(D) \cap T = \{z_{d+1} = \dots = z_n = 0\}$ and the retraction $\tilde{\pi} := \varphi \circ \pi \circ \varphi^{-1}$ read in the mapping φ is given by $\tilde{\pi}(z) = (z_1, \dots, z_d, 0, \dots, 0)$ i.e. this is the orthogonal projection onto the subspace of $z' := (z_1, \dots, z_d)$. Moreover one can choose for T a tube around $\varphi(M)$ of width $\delta > 0$.*

Proof.

The manifold M is given, by use of the retraction π , by the functions $f_k(\zeta) := \zeta_k - \pi_k(\zeta)$, $k = 1, \dots, n$. We have if $\zeta \in M$, $\zeta - \pi(\zeta) = 0$; if $\zeta \notin M$, $\zeta - \pi(\zeta) \neq 0$, because $\pi(\zeta) \in M$. The transversality of the fibers with respect to M at all points of \bar{D} insures that the Jacobian of the map $f = (f_1, \dots, f_n)$ has rank $n - d$, which is the complex co-dimension of M . Take a point $\zeta^0 \in \bar{D}$, there are $n - d$ functions f_j which are independent in a neighborhood U of ζ^0 . Re-numerating the functions f_j and the variables ζ_k , we may suppose that the determinant $(\frac{\partial f_j}{\partial \zeta_k})_{j,k=d+1,\dots,n}$ is different from zero.

Now we shall make the change of variables $z = \varphi(\zeta)$ with $z_j = \zeta_j$, $j = 1, \dots, d$; $z_j = f_j(\zeta)$, $j = d + 1, \dots, n$. This is actually a change of variables because the Jacobian of φ is different from zero in the open set U . We have that the mapping φ is a bi-holomorphism from the open set U onto the open set $T := \varphi(U)$.

Let $z' = (z_1, \dots, z_d)$ and $z'' = (z_{d+1}, \dots, z_n)$; we have in T that:

$$N := \varphi(M) = \{z = (z', z'') \in T :: z'' = 0\}.$$

Now take a tube around N , $T = \{z = (z', z'') :: z'' \in B((z', 0), \delta)\}$, we call it again T , and we still denote by U the set $\varphi^{-1}(T)$.

We cover \bar{D} by a finite number of these charts (U_j, φ_j) . We note N_j the manifold $N_j := \varphi_j(M \cap U_j) \subset T_j := \varphi_j(U_j)$ and, diminishing a little bit the U_j if necessary, we can suppose that the width of the tubes T_j around the N_j is constant and equals $\delta > 0$. We know that there is a constant $\mu > 0$

such that $\mu^{-1} < J_j < \mu$, where J_j is the Jacobian of φ_j , because there is a finite number of charts (U_j, φ_j) . ■

In fact this lemma is a slightly more explicit local version of the proof of the Docquier-Grauert theorem in the book by Gunning and Rossi ([9], Theorem 8, p.257) as was noticed by the referee.

We shall need this local version in order to get estimates in Lebesgue norms.

We denote dV the Lebesgue measure on the manifold M and by dm the Lebesgue measure in \mathbb{C}^k .

We have the following basic lemma.

Lemma 3.3. *Let f be a measurable function, positive on M , then*

$$\int_{U_j} f \circ \pi(\zeta) dm(\zeta) \leq \mu c(\delta) \int_{N_j} f(z', 0) dV(z'),$$

where $c(\delta) := |B(x, \delta)|$ is the volume of the ball $B(x, \delta)$.

Proof.

Let f be a function in $L^1(U_j)$ and \tilde{f} this function read in the map φ_j , i.e. $\tilde{f} := f \circ \varphi_j^{-1}$, we get

$$(3.1) \quad \int_{U_j} f(\zeta) dm(\zeta) = \int_{N_j} \left\{ \int_{B((z', 0), \delta)} \tilde{f}(z', z'') J_j(z', z'') dm(z'') \right\} dV(z').$$

This is simply the change of variables formula because $\varphi_j(U_j) = T_j = N_j \times B(\cdot, \delta)$ and the Jacobian of φ_j is J_j .

With the notation $z = (z', z'')$, z' the coordinates in N_j , z'' the coordinates in the fibers, equation (3.1) gives:

$$\int_{U_j} f \circ \pi(\zeta) dm(\zeta) = \int_{N_j} \left\{ \int_{B(z', \delta)} \tilde{f}(z', z'') J_j(z', z'') dm(z'') \right\} dV(z').$$

Here we have $\tilde{f}(z', z'') = f(z', 0)$ because $\tilde{\pi}(z) = (z', 0)$ hence the formula is now:

$$\int_{U_j} f \circ \pi(z) dm(z) \leq \mu \int_{N_j} f(z', 0) |B(z', \delta)| dV(z') = \mu c(\delta) \int_{N_j} f(z', 0) dV(z'). \quad \blacksquare$$

We notice that the open set $U := \bigcup_{j=1}^N U_j$ contains \bar{D} .

4. ESTIMATES IN THE CASE OF A SUBMANIFOLD OF \mathbb{C}^n .

We shall show the following theorem:

Theorem 4.1. *Let M be a complex submanifold of dimension d in \mathbb{C}^n and a s.c.c. domain D such that D is relatively compact with smooth \mathcal{C}^3 boundary in M . Let ω be a (p, q) form in $L^r_{p,q}(D)$, $\bar{\partial}\omega = 0$ with $r > 2n + 2$, $c \leq q \leq n$. Then there is a $(p, q - 1)$ form u in $\Lambda^c_{(p, q - 1)}(\bar{D})$, $\epsilon = \frac{1}{2} - \frac{n + 1}{r}$, such that $\bar{\partial}u = \omega$.*

Proof.

The idea is the following one:

first we extend the form ω from D to E by $\tilde{\omega} = \pi^*\omega$. Then we solve the equation $\bar{\partial}\tilde{u} = \tilde{\omega}$ in E by theorem 2.1 with the estimates on \tilde{u} . Then we restrict \tilde{u} to D to get the solution u .

Let us see that.

The theorem 7.5 in the appendix, which generalizes to s.c.c. domains a theorem by Rossi [21] obtained for strictly pseudo convex domains, gives us the existence of a strictly c -convex domain

$E \subset U$ in \mathbb{C}^n such that $\pi : \bar{E} \rightarrow \bar{D}$. The open set U here is $U := \bigcup_{j=1}^N U_j$ where (U_j, φ_j) is the covering from lemma 3.2. Now on we fix this s.c.c. domain E .

Let ω be a (p, q) form in $L_{p,q}^r(D)$, $\bar{\partial}$ closed ; we extend it in E by use of the retraction π in the following manner : $\tilde{\omega} := \pi^*\omega$. With the notations of lemma 3.2 we start by extending ω to $U \cap M$ by zero outside \bar{D} ; the coefficients of $\tilde{\omega}$ can be written $f \circ \pi$ hence, applying lemma 3.3 to the functions $|f \circ \pi|^r$ we get $\|\tilde{\omega}\|_{L_{p,q}^r(U_j)} \leq \mu c(\delta) \|\omega\|_{L_{p,q}^r(N_j)}$. We have only a finite number of open sets U_j to cover \bar{D} , so we get $\|\tilde{\omega}\|_{L_{p,q}^r(U)} \lesssim \mu c(\delta) \|\omega\|_{L_{p,q}^r(D)}$. Because $E \subset U$ we get $\|\tilde{\omega}\|_{L_{p,q}^r(E)} \leq \mu c(\delta) \|\omega\|_{L_{p,q}^r(D)}$. Now with $r > 2n + 2$, we can solve the equation $\bar{\partial}$ in the Lipschitz space by the theorem 2.1, iii):

$$\exists \tilde{u} \in \Lambda_{(p,q-1)}^\epsilon(\bar{E}) \text{ with } \epsilon = \frac{1}{2} - \frac{n+1}{r} \text{ such that}$$

$$(4.2) \quad \bar{\partial} \tilde{u} = \tilde{\omega}.$$

Let $j : \bar{D} \rightarrow \bar{E}$ denote the inclusion map which is holomorphic. Notice that $\pi \circ j$ is the identity map on \bar{D} .

Set $u := j^*\tilde{u}$. Then $\bar{\partial}u = j^*\bar{\partial}\tilde{u} = j^*\tilde{\omega}$ by equation (4.2). So

$$\bar{\partial}u = j^*\tilde{\omega} = j^*\pi^*\omega = \omega$$

because $j^*\pi^*$ is the identity map on forms on D .

Because \tilde{u} has its coefficients in $\Lambda^\epsilon(\bar{E})$ then u has its coefficients in $\Lambda_{(p,q-1)}^\epsilon(\bar{D})$ because the restriction of a Lipschitz function is a Lipschitz function. ■

Remark 4.2. *The theorem just proved with $r > 2n + 2$, is enough to apply the raising steps method, and its proof is natural and simple. The case $\omega \in L_{p,q}^{2n+2}(D)$, $\bar{\partial}\omega = 0$, with a solution u of the equation $\bar{\partial}u = \omega$ in $L_{p,q-1}^\infty(D)$ is not stated here, the problem being that the restriction to D of a bounded function in E is in general not even defined, opposite to the case of a Lipschitz function. Nevertheless this result is true and is a special case of theorem 5.3 done for s.c.c. intersection, with a more involved proof.*

Now we are in position to apply the raising steps theorem.

Theorem 4.3. *Let M be a complex submanifold of dimension d in \mathbb{C}^n and D be a s.c.c. domain which is relatively compact with smooth \mathcal{C}^3 boundary in M . Let ω be a (p, q) form in $L_{p,q}^r(D)$, $\bar{\partial}\omega = 0$ with $1 < r < 2d + 2$, $c \leq q \leq n$. Then there is a $(p, q-1)$ form u in $L_{p,q-1}^s(D)$, with $\frac{1}{s} = \frac{1}{r} - \frac{1}{2d+2}$, such that $\bar{\partial}u = \omega$.*

Proof.

In order to have the local result for all points in \bar{D} we use the same method as in [2], but with the proposition 7.2 and the results of Ma and Vassiliadou [18]. Let us see it.

Let $\zeta \in \partial D$ and (V, φ) be a chart in a neighbourhood of ζ in M and ω a (p, q) form in $L_{p,q}^r(D)$, $\bar{\partial}\omega = 0$ with $1 < r < 2d + 2$, $c \leq q \leq n$. We read this situation in \mathbb{C}^d via the chart (V, φ) so we have an open set $W := \varphi(V) \subset \mathbb{C}^d$ and a piece of s.c.c. domain $\varphi(V \cap D)$ near the point $\eta := \varphi(\zeta) \in \mathbb{C}^d$ because the bi-holomorphic map φ keeps the s.c.c. property. By use of the localizing proposition 7.2 there exist a s.c.c. domain $E \subset \varphi(V \cap D)$, with \mathcal{C}^3 boundary, which shares a part of its boundary near η with the boundary of $\varphi(V \cap D)$. We read the form ω by φ , which gives us a $\tilde{\omega} := \varphi^*\omega$ still in $L_{p,q}^r(\varphi(V \cap D))$, $\bar{\partial}\tilde{\omega} = 0$ hence $\tilde{\omega} \in L_{p,q}^r(E)$. Now we apply the results

of Ma and Vassiliadou [18], theorem 2.1 here, to get a $(p, q - 1)$ form \tilde{u} solution of the equation $\bar{\partial}\tilde{u} = \tilde{\omega}$, $\tilde{u} \in L^s(E)$, with $\frac{1}{s} = \frac{1}{r} - \frac{1}{2d+2}$. Back to D via φ^{-1} we have our local estimates : set $u := (\varphi^{-1})^*\tilde{u}$, then $\bar{\partial}u = \omega$, $u \in L_{p,q-1}^s(\varphi^{-1}(E))$.

Hence we have the (i) of the raising steps theorem 1.5.

We have the global result, i.e. the (ii) of the raising steps theorem 1.5 : set $t > 2n + 2$; if $\mu \in L_{p,q}^t(D)$, $\bar{\partial}\mu = 0$ then we have a solution v in $\Lambda_{p,q}^\epsilon(\bar{D}) \subset L_{p,q}^\infty(D)$, with $\epsilon = t - (2n + 2)$, such that $\bar{\partial}v = \mu$ by use of theorem 4.1.

Now we take $\omega \in L_{p,q}^r(D)$, $\bar{\partial}\omega = 0$, then we have that the optimal exponent for the solution u of the equation $\bar{\partial}u = \omega$ is s such that $\frac{1}{s} = \frac{1}{r} - \frac{1}{2d+2}$; we choose any real t such that $t > \max(2n + 2, s)$ as a threshold and, because $L_{p,q-1}^\infty(D) \subset L_{(p,q-1)}^t(D)$, for D is a bounded domain, we have a global solution to $\bar{\partial}v = \mu$ in $L_{(p,q-1)}^t(D)$ if $\mu \in L_{(p,q)}^t(D)$. Now $s < t$ gives that $u \in L_{(p,q-1)}^s(D)$, by the raising steps theorem 1.5, and this ends the proof. ■

5. THE CASE OF \mathcal{C}^3 c -CONVEX INTERSECTION.

We proceed exactly the same way than for just one s.c.c. domain.

For the local estimates we use the localizing proposition 7.3 and we repeat the proof above. This is the point where we need to have at most $N = n - 3$ domains in a Stein manifold of dimension n , compare to $n - 2$ domains in \mathbb{C}^n . By use of Ma and Vassiliadou main theorem 2.2 and with $\nu \in \mathbb{N}^+$ defined there, we get :

Theorem 5.1. *Let M be a complex submanifold of \mathbb{C}^n of dimension d and a \mathcal{C}^3 c -convex intersection domain D such that D is relatively compact in M . Let ω be a (p, q) form in $L_{p,q}^r(D)$, $\bar{\partial}\omega = 0$ with $q \geq c$, $1 < r < 2d + 2\nu$. Then there is finite covering $\{U_j\}_{j=1,\dots,N}$ of \bar{D} and $(p, q - 1)$ form u_j in $L^s(U_j \cap D)$ such that $\bar{\partial}u_j = \omega$ in $U_j \cap D$ with $\frac{1}{s} = \frac{1}{r} + \frac{1}{\lambda} - 1$ where $1 \leq \lambda < \frac{2d + 2\nu}{2d - 1 + 2\nu}$.*

More precisely,

i) For any $1 < r < 2d + 2\nu$, there exists $c_r(D)$, a positive constant such that

$$\|u_j\|_{L_{(p,q-1)}^s(U_j \cap D)} \leq c_r(D) \|\omega\|_{L_{(p,q)}^r(U_j \cap D)}$$

with $\frac{1}{s} = \frac{1}{r} - \frac{1}{2d + 2\nu}$.

ii) For $r \geq 2d + 2\nu$, we have $\|u_j\|_{L_{(p,q-1)}^\infty(U_j \cap D)} \leq a_r(D) \|\omega\|_{L_{(p,q)}^r(U_j \cap D)}$ for some positive constant $a_r(D)$.

Now for the global threshold, we copy the proof of theorem 4.1, replacing $r_0 > \max(2n + 2, s)$ by $r_0 > \max(2n + 2\nu, s)$. We remark that we have not the Lipschitz estimates here but only L^∞ estimates and this is why we have to use the next lemma 5.2.

Let E be \mathcal{C}^3 c -convex intersection domain in \mathbb{C}^n , $E \subset U$ and, with $r \geq 2n + 2$, we can solve the $\bar{\partial}$ in the space $L^\infty(E)$: $\bar{\partial}\tilde{u} = \tilde{\omega}$, $\tilde{u} \in L_{p,q-1}^\infty(E)$ by the theorem 5.1. Fix $\omega \in L_{(p,q)}^r(D)$, $\bar{\partial}\omega = 0$, with $\tilde{\omega}$ as above, we have $\tilde{u} \in L_{p,q-1}^\infty(E)$ also fixed.

Lemma 5.2. *We have, with $j : D \rightarrow E$ the canonical injection,*

$$\|j^*\tilde{\omega} - j^*\tilde{\omega}_\epsilon\|_{L^r(D_\epsilon)} \xrightarrow{\epsilon \rightarrow 0} 0.$$

Proof.

Recall that the coefficients $\tilde{a}_{I,J}$ of $\tilde{\omega}$ verify $\tilde{a}_{I,J}(z) = a_{I,J} \circ \pi(z)$, if $a_{I,J}$ is the corresponding coefficient of ω . We have, by definition of the convolution, noting $\tilde{a}_{I,J}^\epsilon$ a coefficient of $\tilde{\omega}_\epsilon$,

$$\tilde{a}_{I,J}^\epsilon(z) := \int_{\mathbb{C}^n} a_{I,J} \circ \pi(z - \zeta) \chi_\epsilon(\zeta) dm(\zeta),$$

so

$$\tilde{a}_{I,J}^\epsilon(z) - \tilde{a}_{I,J}(z) = \int_{\mathbb{C}^n} (a_{I,J} \circ \pi(z - \zeta) - a_{I,J} \circ \pi(z)) \chi_\epsilon(\zeta) dm(\zeta).$$

Now j^* is the operator of restriction to D so take $z \in D$, then in a chart (U_j, φ_j) , keeping the same notations for the functions read in this chart, with $z = (z', z'')$, $\zeta = (\zeta', \zeta'')$,

$$\tilde{a}_{I,J}^\epsilon(z', 0) - \tilde{a}_{I,J}(z', 0) = \int_{\mathbb{C}^n} (a_{I,J}(z' - \zeta') - a_{I,J}(z')) \chi_\epsilon(\zeta) dm(\zeta),$$

because here π is the orthogonal projection on $(z', 0)$, hence $\tilde{a}_{I,J}(z) = a_{I,J}(z')$.

So, decomposing the measure, we set

$$\rho_\epsilon(\zeta') := \int_{\mathbb{C}^{n-d}} \chi_\epsilon(\zeta', \zeta'') dm(\zeta''),$$

and $\rho_\epsilon(\zeta')$ is an approximate identity in \mathbb{C}^d because

$$\|\rho_\epsilon\|_{L^1(\mathbb{C}^d)} = \int_{\mathbb{C}^n} \chi_\epsilon(\zeta', \zeta'') dm(\zeta) = 1$$

and

$$\forall \varphi \in \mathcal{C}(\mathbb{C}^d), \forall \zeta' \in \mathbb{C}^d, \varphi * \rho_\epsilon(\zeta') \xrightarrow{\epsilon \rightarrow 0} \varphi(\zeta').$$

So we have a convolution on $a_{I,J}(z')$ hence

$$\tilde{a}_{I,J}^\epsilon(z', 0) - \tilde{a}_{I,J}(z', 0) = \int_{\mathbb{C}^d} (a_{I,J}(z' - \zeta') - a_{I,J}(z')) \rho_\epsilon(\zeta) dm(\zeta'),$$

and, because the convolution is continuous on L^r , $\forall r \geq 1$,

$$\|\tilde{a}_{I,J}^\epsilon - \tilde{a}_{I,J}\|_{L^r(U_j \cap D)} \lesssim \|a_{I,J}^\epsilon - a_{I,J}\|_{L^r(U_j \cap D)} \xrightarrow{\epsilon \rightarrow 0} 0.$$

Because we have only a finite number of charts (U_j, φ_j) to cover E , the proof is complete. \blacksquare

Now we get, still with $\nu \in \mathbb{N}^+$ given by Ma and Vassiliadou main theorem 2.2:

Theorem 5.3. *Let M be a complex submanifold of dimension d in \mathbb{C}^n and a \mathcal{C}^3 c -convex intersection domain D such that D is relatively compact in M . Let ω be a (p, q) form in $L^r_{p,q}(D)$, $\bar{\partial}\omega = 0$ with $r \geq 2n + 2\nu$, $c \leq q \leq n$. Then there is a $(p, q - 1)$ form v in $L^\infty_{(p,q-1)}(D)$ such that $\bar{\partial}v = \omega$.*

Proof.

The only difference with the proof of theorem 4.1 is that the restriction of a $L^\infty(E)$ function to D is not even defined a priori.

So we regularise the solution $\tilde{u} \in L^\infty(E)$ given by Ma and Vassiliadou main theorem 2.2 by convolution with a smooth function χ such that $\chi(t) \in \mathcal{C}_c^\infty(]0, 1[)$.

As usual we choose χ such that $\int_{\mathbb{C}^n} \chi(|z|^2) dm(z) = 1$ and we set $\chi_\epsilon(z) := \frac{1}{\epsilon^{2n}} \chi\left(\frac{|z|^2}{\epsilon^2}\right)$ and $\tilde{u}_\epsilon(z) := (\tilde{u} * \chi_\epsilon)(z)$, which means that the convolution is done on the coefficients.

Now we have that $\tilde{u}_\epsilon \in \mathcal{C}^\infty_{p,q-1}(E_\epsilon)$ where $E_\epsilon := \{z \in E :: d(z, E^c) > \epsilon\}$ and, because $\bar{\partial}\tilde{u} = \omega$,

$$(5.3) \quad \bar{\partial}\tilde{u}_\epsilon = \tilde{\omega}_\epsilon$$

with $\tilde{\omega}_\epsilon := (\tilde{\omega} * \chi_\epsilon)(z)$.

Moreover we have

$$\forall \epsilon > 0, \|\tilde{u}_\epsilon\|_{L^\infty_{p,q-1}(E_\epsilon)} \leq \|\tilde{u}\|_{L^\infty_{p,q-1}(E)}$$

and

$$(5.4) \quad \|\tilde{\omega} - \tilde{\omega}_\epsilon\|_{L^r_{p,q}(E_\epsilon)} \xrightarrow{\epsilon \rightarrow 0} 0.$$

Let $j : \bar{D} \rightarrow \bar{E}$ denote the inclusion map which is holomorphic. Notice that $\pi \circ j$ is the identity map on \bar{D} .

Set $u_\epsilon := j^* \tilde{u}_\epsilon$. Then $\bar{\partial} u_\epsilon = j^* \bar{\partial} \tilde{u}_\epsilon = j^* \tilde{\omega}_\epsilon$ by equation (5.3).

Now by (5.4) we have, by use of lemma 5.2, which is necessary because in general the restriction is not a continuous operator on L^r ,

$$\|j^* \tilde{\omega} - j^* \tilde{\omega}_\epsilon\|_{L^r(D_\epsilon)} \xrightarrow{\epsilon \rightarrow 0} 0.$$

So

$$(5.5) \quad \bar{\partial} u_\epsilon = j^* \tilde{\omega}_\epsilon = j^*(\tilde{\omega}_\epsilon - \tilde{\omega}) + j^* \pi^* \omega = j^*(\tilde{\omega}_\epsilon - \tilde{\omega}) + \omega,$$

because $j^* \pi^*$ is the identity map on forms on D .

Take the sequence $\{u_{1/k}\}_{k \in \mathbb{N}}$, then, because $L^\infty(D)$ is the dual of $L^1(D)$, there is a sub-sequence $\{v_k\}_{k \in \mathbb{N}}$ of $\{u_{1/k}\}_{k \in \mathbb{N}}$ *-weakly converging, i.e. converging against $L^1_{n-p, n-q+1}(D)$ forms to $v \in L^\infty_{p, q-1}(D)$.

Let $\varphi \in \mathcal{C}^\infty_{n-p, n-q+1}(D) \subset L^1_{n-p, n-q+1}(D)$ with compact support in D , i.e. a test form. Then for $k \geq k_0$ big enough we have that $\text{Supp } \varphi \subset E_{1/k_0}$ and by (5.5) we have:

$$\forall k \geq k_0, (-1)^{p+q-1} \langle v_k, \bar{\partial} \varphi \rangle = \langle \bar{\partial} v_k, \varphi \rangle = \langle j^*(\tilde{\omega}_{1/k} - \tilde{\omega}), \varphi \rangle + \langle \omega, \varphi \rangle,$$

and this is well defined because $\text{Supp } \varphi \subset E_{1/k_0}$. Letting $k \rightarrow \infty$ we get

$$(-1)^{p+q-1} \langle v_k, \bar{\partial} \varphi \rangle \rightarrow \langle \omega, \varphi \rangle.$$

But $\langle v_k, \bar{\partial} \varphi \rangle \xrightarrow{k \rightarrow \infty} \langle v, \bar{\partial} \varphi \rangle$, because $\bar{\partial} \varphi \in L^1_{n-p, n-q+1}(D)$, so we get $(-1)^{p+q-1} \langle v, \bar{\partial} \varphi \rangle = \langle \omega, \varphi \rangle$

which means that $\bar{\partial} v = \omega$ in the distributions sense. ■

Remark 5.4. *We have no such estimates in the case $r < 2n + 2$ because the limit of mean values in balls of a function in L^s in E , which is the case of v , is no longer in $L^s(D)$ for $s < \infty$, in general, as can be easily seen.*

So we can apply the raising steps theorem to get the analogous results in the case of a \mathcal{C}^3 c -convex intersection domain D such that D is relatively compact in M , a closed complex submanifold of \mathbb{C}^n .

6. ESTIMATES IN THE CASE OF A STEIN MANIFOLD.

We can apply a theorem of Bishop and Narashiman (see theorem 5.3.9. of Hörmander [13]) which tells us that, if Ω is a Stein manifold of dimension d there is an element $f \in \mathcal{H}(\Omega)^{2d+1}$ which defines a regular injective and proper map from Ω in \mathbb{C}^{2d+1} . Denote $M := f(\Omega)$; if D' is the strictly c -convex domain in Ω relatively compact in Ω , then its image $D = f(D')$ is a strictly c -convex domain in M . We can apply theorems 4.1 and 4.3.

Of course the same is true for \mathcal{C}^3 c -convex intersection in Stein manifold M and we get our main theorems.

We get an easy corollary of our main theorems, (see Ma and Vassiliadou [18], corollary 1.). We have, because D is relatively compact, the estimate $L^2 - L^2$, and this gives :

Corollary 6.1. *Let Ω be a Stein manifold of dimension n and a strictly c -convex domain D , or a \mathcal{C}^3 c -convex intersection, such that D is relatively compact with smooth \mathcal{C}^3 boundary in Ω . Then, for $q \geq c$, the operator $\bar{\partial} : L^2_{(p,q-1)}(D) \rightarrow L^2_{(p,q)}(D)$ has closed range.*

Proof.

This is fairly well known: for instance theorem 1.1.1 in [12]. ■

Because the L^2 norm of the canonical solution of $\bar{\partial}u = \omega$ is smaller than the solution we obtain, this implies the L^2 existence of the $\bar{\partial}$ -Neumann operator on strictly c -convex domains. We also have that the strictly c -convex condition implies the $Z(q)$ condition of Beals, Greiner and Stanton [5] for $c \leq q \leq n$, hence we get an automatic improvement of regularity in the case of a \mathcal{C}^∞ smoothly bounded s.c.c. domain, by theorems 2 and 4 in Beals, Greiner and Stanton [5].

Theorem 6.2. *Let Ω be a Stein manifold of dimension n and a strictly c -convex (s.c.c.) domain D such that D is relatively compact with smooth \mathcal{C}^∞ boundary in Ω . Let $k \in \mathbb{N}$ and ω a (p, q) form in $W^{k,r}(D)$, $\bar{\partial}\omega = 0$ with $1 < r < 2n + 2$, $c \leq q \leq n$. Then there is a $(p, q - 1)$ form u in $W^{k+1/2,r}(D)$, such that $\bar{\partial}u = \omega$.*

If $\epsilon > 0$ and ω is in $\Lambda_{p,q}^\epsilon(D)$, $\bar{\partial}\omega = 0$ with $c \leq q \leq n$, then there is a $(p, q - 1)$ form u in $\Lambda_{(p,q-1)}^{\epsilon+1/2}(\bar{D})$ such that $\bar{\partial}u = \omega$.

Here we use the notation $W^{k,r}(D)$ for the Sobolev space of functions whose derivatives of order less than k are in L^r .

We notice that there is no hypothesis here in the case $r > 2$ on the compactness of the support of the form ω , in contrast to the previous results we had in [2].

7. APPENDIX.

Lemma 7.1. *Let A, B be two self adjoint matrices such that A has at least $n - c + 1$ strictly positive eigenvalues and B is positive. Then $A + B$ has at least $n - c + 1$ strictly positive eigenvalues.*

Proof.

Let E be the space generated by the eigenvectors associated to the strictly positive eigenvalues of A . Then E has dimension at least $n - c + 1$. Let $S := A + B$, because B is positive, we get

$$\forall x \in E, \langle Sx, x \rangle = \langle Ax, x \rangle + \langle Bx, x \rangle > 0.$$

Now let e_1, \dots, e_k be the eigenvectors associated to the negative eigenvalues of S . We set $F = \text{span}\{e_1, \dots, e_k\}$, we have that F is invariant by S and we have $\forall x \in F, \langle Sx, x \rangle \leq 0$. If the space $G := E \cap F$ is of non zero dimension, we get $\forall x \in G, x \neq 0, \langle Sx, x \rangle > 0$ and $\langle Sx, x \rangle \leq 0$ so a contradiction. Hence $\dim G = 0$ and $\dim F \leq \text{codim} E = c - 1$, which means that S has at least $n - c + 1$ strictly positive eigenvalues. ■

The next proposition generalizes the one in [1], proposition 1.1, done for the pseudo convex case.

Proposition 7.2. *(Localizing s.c.c. domain) Let D be a strictly c -convex domain with \mathcal{C}^3 boundary in \mathbb{C}^n . Let $\zeta \in \partial D$, U a neighbourhood of ζ in \mathbb{C}^n and $B(\zeta, r)$ a ball centered at ζ and of radius r such that $B(\zeta, 3r) \subset U$; then there is a domain \tilde{D} , s.c.c. and with \mathcal{C}^3 boundary such that we have $\tilde{D} \subset U$ and $\partial D \cap B(\zeta, r) = \partial \tilde{D} \cap B(\zeta, r)$.*

Proof.

Let ρ be a defining function for D . Let $\zeta \in \partial D$ and U a neighbourhood of ζ in \mathbb{C}^n . Consider a positive convex increasing function χ defined on \mathbb{R}^+ , \mathcal{C}^∞ and such that $\chi = 0$ in $(0, r)$. Set $\tilde{\rho}(z) := \rho(z) + a\chi(|z - \zeta|^2)$; we have $\partial \bar{\partial} \tilde{\rho} = \partial \bar{\partial} \rho + a\partial \bar{\partial} \chi$. But, as is easily seen, $i\partial \bar{\partial} \chi$ is positive at

each point z , hence, setting $A = i\partial\bar{\partial}\rho$, $B = ai\partial\bar{\partial}\chi$, we can apply lemma 7.1 and we have that the domain $\tilde{D} := \{\tilde{\rho} < 0\}$ is also s.c.c. with smooth \mathcal{C}^3 boundary.

Now we choose r small enough to have $B(\zeta, 3r) \subset U$. We have $\tilde{\rho}(z) < 0 \Rightarrow \rho(z) < -a\chi(|z - \zeta|^2)$; so we set :

$$\alpha := \sup_{z \in \tilde{D}} -\rho(z) < \infty, \text{ by the compactness of } \tilde{D} \text{ and } \beta := \inf_{z \in U \setminus B(\zeta, 2r)} \chi(|z - \zeta|^2) = 4r^2.$$

Then with $a := \frac{\alpha + 1}{\beta}$ we get that $\{\tilde{\rho}(z) < 0\} \subset U$ because if not $\exists z \notin B(\zeta, 3r) :: \rho(z) < -a\chi(|z - \zeta|^2) < -(\alpha + 1)$ which is not possible.

Of course in the ball $B(\zeta, r)$ we have $\partial D \cap B(\zeta, r) = \partial\tilde{D} \cap B(\zeta, r)$. ■

We shall need to extend this proposition to the case of \mathcal{C}^3 c -convex intersection.

Proposition 7.3. (*Localizing s.c.c. intersection*) Let D be a \mathcal{C}^3 c -convex intersection in \mathbb{C}^n . Let $\zeta_0 \in \partial D$, U a neighbourhood of ζ_0 in \mathbb{C}^n and $B(\zeta_0, r)$ a ball centered at ζ_0 and of radius r ; then there is a domain \tilde{D} , \mathcal{C}^3 c -convex intersection, such that we have $\tilde{D} \subset U$ and $\partial D \cap B(\zeta_0, r) = \partial\tilde{D} \cap B(\zeta_0, r)$.

Proof.

By assumption the 1-forms $\{d\rho_j(\zeta)\}_{j \in I}$ are linearly independent in $\bigcap_{j \in I} \{\rho_j(z) \leq 0\}$ and $|I| \leq n - 3$.

Take a point $\zeta_0 \in \bigcap_{j \in I} \{\rho_j(z) = 0\}$, by translation in \mathbb{C}^n , we may suppose that $\zeta_0 = 0$, and we have to define the domain \tilde{D} with the properties stated in the proposition.

Take a vector $h \in \mathbb{C}^n$ of norm 1 and set, with $a \cdot b := \sum_{k=1}^n a_k b_k$,

$$\rho(z) := (|z|^2 - r^2)(1 + h \cdot z + \bar{h} \cdot \bar{z}).$$

Because $1 + h \cdot z + \bar{h} \cdot \bar{z} > 0$ near the origin, $\rho(z)$ is a defining function for the ball centered at 0 and of radius r .

Now the claim is: we can choose the vector h in such a way that $d\rho_j(0)$ and $d\rho(0)$ are linearly independent for $j \in I$.

We have $d\rho(0) = (-r^2)(h \cdot dz + \bar{h} \cdot d\bar{z})$.

We already know that the $d\rho_j(0)$ are linearly independent and span a space of dimension less than $n - 3$, so we take h in such a way that the form $(h \cdot dz + \bar{h} \cdot d\bar{z})$ is not in the span of the $d\rho_j(0)$ for $j \in I$. This is independent of the choice of $r > 0$. By continuity this is still true for z in a neighbourhood V of 0 with V independent of $r > 0$, so we now choose $r > 0$ in order that $B(0, r) \subset V$. We extend ρ outside of the ball $B(0, r)$ to be a \mathcal{C}^∞ function $\tilde{\rho}$ in \mathbb{C}^n , $\tilde{\rho} = \rho$ in $B(0, r)$ and strictly positive in $\bar{B}(0, r)^c$ in order for this $\tilde{\rho}$ to be a genuine defining function for $B(0, r)$. So we get the condition (i) in the definition 1.2.

The condition (ii) is easier because we have that

$$\partial\bar{\partial}\tilde{\rho} = (1 + h \cdot z + \bar{h} \cdot \bar{z}) \sum_{k=1}^n dz_k \wedge d\bar{z}_k + h \cdot dz \wedge z \cdot d\bar{z} + \bar{z} \cdot dz \wedge \bar{h} \cdot d\bar{z} = \sum_{k=1}^n dz_k \wedge d\bar{z}_k + \mathcal{O}(|z|).$$

Hence there is a subspace T_z^I such that for $i \in I$ the Levi forms $L\rho_i$ restricted on T_z^I are positive definite by hypothesis and, because $L\tilde{\rho}$ is positive definite everywhere, we have that it is also positive definite on T_z^I which has the right dimension $n - c + 1$.

It is at this point that we need $N \leq n - 3$, because we add the new domain $B(0, r)$. ■

On a theorem of H. Rossi.

We shall use the following lemma.

Lemma 7.4. *Let A, B two self adjoint $n \times n$ matrices such that A has at least $d - c + 1$ strictly positive eigenvalues and $\ker A$ is of dimension $n - d$ and B is positive and has $n - d$ eigenvectors in $\ker A$ associated to strictly positive eigenvalues. Then $A + B$ has at least $n - c + 1$ strictly positive eigenvalues.*

Proof.

Because A is self adjoint, the spaces $\ker A$ and $H := \ker A^\perp$ are invariant for A . Because $\ker A$ has dimension $n - d$ and there is $n - d$ eigenvectors of B in it, then $\ker A$ is generated by these eigenvectors. Hence, because B is self adjoint, this means that $\ker A$ and H are also invariant for B . Set $S := A + B$.

Let $v \in \ker A$ be such that $Bv = \lambda v$, $\lambda > 0$, then $Sv = Av + Bv = Bv = \lambda v$; hence on $\ker A$, S has $n - d$ strictly positive eigenvalues.

On H we have $B \geq 0$ and A has at least $d - c + 1$ strictly positive eigenvalues, hence on H we can apply lemma 7.1 and we have that S has at least $d - c + 1$ strictly positive eigenvalues on H . Because H and $\ker A$ have an intersection reduced to $\{0\}$, S has $d - c + 1 + n - d = n - c + 1$ strictly positive eigenvalues. ■

The aim is to extend a theorem by H. Rossi [21] where we replace strictly pseudo convex by strictly c -convex.

Theorem 7.5. *Let M be a closed submanifold of a Stein domain U_0 in \mathbb{C}^n . Suppose there is a neighbourhood U of M and an holomorphic retraction $\pi : U \rightarrow M$. Let D be a strictly c -convex domain in M , $\bar{D} \subset M$.*

Then there is a strictly c -convex domain E in \mathbb{C}^n such that :

- (A) $\bar{E} \subset U \cap U_0$
- (B) $E \cap M = D$
- (C) ∂E cuts M transversely along ∂D
- (D) $\pi : \bar{E} \rightarrow \bar{D}$.

Proof.

I shall copy the main points in the proof by H. Rossi making the necessary changes.

Docquier and Grauert (see [21]) give us a neighbourhood U of \bar{D} in \mathbb{C}^n and a retraction $\pi : U \rightarrow M \cap U$ such that the fibers of π cut transversely $M \cap U$ and are of dimension $n - d$.

We set for $z \in U$ and $j = 1, \dots, n$, $f_j(z) = z_j - \pi_j(z)$. The equations $z - \pi(z) = 0$ define the sub manifold M :

if $z \in M$, $\pi(z) = z$ because π is a retraction on M ; if $z \notin M$, because $\pi(z) \in M$, $z - \pi(z) \neq 0$. Moreover, because the fibers of π cut transversely M at any point ζ of \bar{D} , we have that the jacobian matrix contains a $(n-d) \times (n-d)$ sub determinant which is not 0 at ζ , hence not 0 in a neighbourhood of this point. This means that, by a change of variables, the set $(f_j)_{j=1, \dots, n}$ contains a coordinates system for the fibers of π at any point of \bar{D} , hence at all points of a neighbourhood U_1 of \bar{D} in \mathbb{C}^n . These "explicit" functions replace the one generating the idealsheaf of M used by H. Rossi.

Let ρ be a defining function for D in M , we still follow H. Rossi and we set:

$$\sigma(z) := \rho \circ \pi + A \sum_{j=1}^n |f_j|^2,$$

where the constant A will be chosen later. Because $F(z) := \sum_{j=1}^n |f_j(z)|^2 = 0$ on $M \cap U$, it exists a

$\epsilon_0 > 0$ such that $\{F(z) < \epsilon_0\} \cap U \subset U_1$.

It remains to see that σ is strictly c -convex, i.e. $i\partial\bar{\partial}\sigma$ has at least $n - c + 1$ strictly positive eigenvalues.

Fix $\zeta \in \bar{D}$; because D is strictly c -convex, $i\partial\bar{\partial}\rho \circ \pi(\zeta)$ has at least $d - c + 1$ strictly positive eigenvalues on the tangent space to M at ζ . Because the set $(f_j)_{j=1, \dots, n}$ contains a coordinates system for the fibers of π we have $i\partial\bar{\partial}(\sum_{j=1}^n |f_j|^2)$ has all, i.e. $n - d$, strictly positive eigenvalues on the tangent space to the fiber $\pi^{-1}\pi(\zeta)$ at ζ .

Because the kernel of $i\partial\bar{\partial}\rho \circ \pi$ is the tangent space to the fiber $\pi^{-1}\pi(\zeta)$, we get, by lemma 7.4, that $i\partial\bar{\partial}\sigma = i\partial\bar{\partial}\rho \circ \pi + i\partial\bar{\partial}(\sum_{j=1}^n |f_j|^2)$ has at least $n - c + 1$ strictly positive eigenvalues. So we have at

least $n - c + 1$ strictly positive eigenvalues at any point of \bar{D} hence also in a neighbourhood V of \bar{D} in \mathbb{C}^n . Now we take $A\epsilon_0 > \sup_{z \in D} |\rho(z)|$ and we set $E := \{z \in U \cap V : \sigma(z) < 0\}$; we get exactly as H. Rossi, that E is strictly c -convex and we have all properties of the theorem. ■

We have to get an analogous result in the case where D is a \mathcal{C}^3 c -convex intersection in M .

Theorem 7.6. *Let M be a closed submanifold of a Stein domain U_0 in \mathbb{C}^n . Suppose there is a neighbourhood U of M and an holomorphic retraction $\pi : U \rightarrow M$. Let $D := \bigcap_{k=1}^N D_k$ be a \mathcal{C}^3 c -convex intersection in M , $\bar{D} \subset M$.*

Then there is a \mathcal{C}^3 c -convex intersection $E := \bigcap_{k=1}^N \tilde{D}_k$ in \mathbb{C}^n such that :

- (A) $\bar{E} \subset U \cap U_0$
- (B) $E \cap M = D$
- (C) ∂E cuts M transversely along ∂D
- (D) $\pi : \bar{E} \rightarrow \bar{D}$.

Proof.

By theorem 7.5, and with the same notations, we can extend each D_k by $\tilde{D}_k := \{\tilde{\rho}_k < 0\}$ in \mathbb{C}^n , with $\tilde{\rho}_k := \rho_k \circ \pi + AF$, where ρ_k are the defining function for D_k and $F(z) := \sum_{j=1}^n |f_j(z)|^2$, such that

they fulfil the \mathcal{C}^3 c -convex intersection requirements in M . The point is to see that we can choose A in such a way that $\tilde{D} := \bigcap_{k=1}^N \tilde{D}_k$ fulfils the \mathcal{C}^3 c -convex intersection requirements in \mathbb{C}^n .

First we choose $A\epsilon_0 > \sup_{k=1, \dots, N, z \in D_k} |\rho_k(z)|$ in order to have that all \tilde{D}_k are in the domain of the retraction π as for theorem 7.5.

Fix a point $z_0 \in \tilde{D}$ and take a vector X in \mathbb{C}^n ; then we can decompose it as $X = X_M \oplus X_F$ where X_M is tangent at z_0 to the manifold $\{z : z - \pi(z) = z_0 - \pi(z_0)\}$, "parallel to M ", and X_F is tangent to the fiber passing through z_0 , $\{z : \pi(z) = \pi(z_0)\}$, because we know that the fibers are transverse to M , which is still true in a neighbourhood V of \tilde{D} in \mathbb{C}^n . Choose A big enough to have all the \tilde{D}_k in V , the same way we did it in the proof of theorem 7.5.

Now if $z \in \bar{D} \subset M$ we already have that the $\{d\tilde{\rho}_j(z)\}_{j \in I}$ are linearly independent because there we have $F(z) = dF(z) = 0$ hence $d\tilde{\rho}_j(z) = d\rho_j(z)$. So we make the assumption that $z \notin M$. Let $I = (i_1, \dots, i_l)$ and suppose that the $\{d\tilde{\rho}_j\}_{j \in I}$ are not linearly independent, then there is $\lambda \in \mathbb{R}^{|I|}$ such that

$$\exists z \in \tilde{D} :: 0 = \sum_{j \in I} \lambda_j d\tilde{\rho}_j(z) = \sum_{j \in I} \lambda_j d(\rho_j \circ \pi(z)) + \left(\sum_{j \in I} \lambda_j \right) AdF(z).$$

This means that

$$(7.6) \quad -\left(\sum_{j \in I} \lambda_j\right) AdF(z) = \sum_{j \in I} \lambda_j d(\rho_j \circ \pi(z)).$$

Take any vector X tangent to \mathbb{C}^n at z ; then we have $X = X_M \oplus X_F$ and $\langle d(\rho_j \circ \pi)(z), X_F \rangle = 0$ because $\rho_j \circ \pi(\zeta)$ is constant along the fiber $\{\zeta :: \pi(\zeta) = \pi(z)\}$. The same way $\langle dF(z), X_M \rangle = 0$ because $F(z) = |z - \pi(z)|^2$ is constant along $\{\zeta :: \zeta - \pi(\zeta) = z - \pi(z)\}$. So

$$\begin{aligned} -\left(\sum_{j \in I} \lambda_j\right) A \langle dF(z), X \rangle &= -\left(\sum_{j \in I} \lambda_j\right) A \langle dF(z), X_F \rangle = \\ &= \sum_{j \in I} \lambda_j \langle d(\rho_j \circ \pi)(z), X_F \rangle = 0. \end{aligned}$$

Hence for any $X \in \mathbb{C}^n$, $\left(\sum_{j \in I} \lambda_j\right) A \langle dF(z), X \rangle = 0$ which means that the 1-form $\left(\sum_{j \in I} \lambda_j\right) AdF(z) = 0$

hence, because $dF(z) \neq 0$ for $z \notin M$, we have that $\left(\sum_{j \in I} \lambda_j\right) = 0$.

But by (7.6) this implies $\sum_{j \in I} \lambda_j d(\rho_j \circ \pi(z)) = 0$ which means that $\lambda_j = 0$ because the $d\rho_j(\zeta)$ are independent at all points and in particular at the point $\zeta = \pi(z)$. So a contradiction which proves that the $d\tilde{\rho}_k$ are linearly independent.

To have the ii) fix $z \in \bigcap_{j \in I} \{\tilde{\rho}_j \leq 0\}$, and set $\zeta := \pi(z) \in \bar{D}$. The points z, ζ belongs to an open set $U := U_j$ of the covering (U_j, φ_j) done via lemma 3.2, so reading by $\varphi := \varphi_j$ we are in the following situation (I keep the same notations) : we have $z = (z', z'')$, $\zeta = (z', 0)$ and the retraction π is the orthogonal projection $w \rightarrow (w', 0)$ where $w' := (w_1, \dots, w_d)$; $w'' := (w_{d+1}, \dots, w_n)$. The tangent space $T_\zeta(M)$ is just $\{w :: w'' = 0\}$ and by the hypotheses on the $\rho_j(w) = \rho_j(w')$ we know that there is a subspace T_ζ^I , of dimension at least $d - c + 1$, of the tangent space $T_\zeta(M)$ on which the Levi forms $L\rho_j(\zeta)$ are positive definite. Lifting this space $T_\zeta(M)$ at the point z keeping it parallel to itself, call it T_z^I , it still have dimension $d - c + 1$, and because the ρ_j do not depend on w'' , we still have that the Levi form $L(\rho_j \circ \pi)(z)$ on T_z^I is the same as the Levi form $L\rho_j(\zeta)$ on T_ζ^I , so it is positive definite.

Now we have $\tilde{\rho}_k := \rho_k \circ \pi + AF$ and $i\partial\bar{\partial}F$ has all its eigenvalues positive so on T_z^I the Levi form $L\rho_j(z)$ is positive definite by the proof of lemma 7.1. ■

REFERENCES

- [1] E. Amar. Cohomologie complexe et applications. *J. London Math. Soc.*, 2(29):127–140, 1984.
- [2] E. Amar. The raising steps method. Application to the $\bar{\partial}$ equation in Stein manifolds. *J. Geometric Analysis*, 26(2):898–913, 2016. DOI 10.1007/s12220-015-9576-8.
- [3] Eric Amar. On the L^r Hodge theory in complete non compact riemannian manifolds. *HAL-01168927*, 2015.
- [4] Eric Amar. The raising steps method. Applications to the L^r Hodge theory in a compact riemannian manifold. *HAL-01158323*, 2015.
- [5] R. Beals, P. Greiner, and N. Stanton. L^p and Lipschitz estimates for the $\bar{\partial}$ -equation and the $\bar{\partial}$ -Neumann problem. *Math. Ann.* 277, 185-196 (1987), 277:185–196, 1987.
- [6] F. Docquier and H. Grauert. Levisches problem und Rungescher satz fur teilgebiete Steinscher mannigfaltigkeiten. *Math. Ann.*, 140:94–123, 1960.
- [7] W. Fischer and Lieb. Lokale kerne und beschr. lssungen fir den $\bar{\partial}$ -operator auf q-konvexen gebieten. *Math. Ann.*, 208:249–265, 1974.

- [8] H. Grauert and I. Lieb. Das Ramirezsche integral und die lösung der gleichung $\bar{\partial}f = \alpha$ im bereich der beschränkten formen. *Rice Univ. Stud.*, 56(2):29–50, 1970.
- [9] R.C. Gunning and H. Rossi. *Analytic functions of several complex variables*. AMS Chelsea Publishing, Providence, RI, 1965.
- [10] G. M. Henkin and J. Leiterer. Theory of functions on strictly pseudoconvex sets with non-smooth boundary. *Report R-MATH, I. Math. der Akad. der DDR, Berlin*, 1981.
- [11] G.M. Henkin. Integral representations of functions in strictly pseudoconvex domains and applications to the $\bar{\partial}$ -problem. *Math. USSR Sb.*, 11:181–273, 1970.
- [12] L. Hörmander. L^2 estimates and existence theorems for the $\bar{\partial}$ operator. *Acta Math.*, 113:89–152, 1965.
- [13] L. Hörmander. *An introduction to complex analysis in several variables*. North-Holland/American Elsevier, 1994.
- [14] N. Kerzman. Hölder and L^p estimates for solutions of $\bar{\partial}u = f$ in strongly pseudoconvex domains. *Comm. Pure. Appl. Math.*, 24:301–379, 1971.
- [15] S. Krantz. Optimal Lipschitz and L^p regularity for the equation $\bar{\partial}u = f$ on strongly pseudo-convex domains. *Math. Ann.*, 219(3):233–260, 1976.
- [16] Christine Laurent-Thiébaud. Théorie L^p pour l'équation de Cauchy-Riemann. *arXiv:1301.1611*, 2013.
- [17] J. Leiterer and C. Laurent-Thiébaud. Uniform estimates for the Cauchy-Riemann equation on q-convex wedges. *Ann. Inst. Fourier*, 43:383–436, 1993.
- [18] L. Ma and S. Vassiliadou. L^p estimates for Cauchy-Riemann operator on q-convex intersections in \mathbb{C}^n . *Manuscripta math*, 103:413–433, 2000.
- [19] C. Menini. Estimations pour la résolution du $\bar{\partial}$ sur une intersection d'ouverts strictement pseudoconvexes. *Math. Z.*, 1:87–93, 1997.
- [20] N. Ovreid. Integral representation formulas and L^p estimates for the $\bar{\partial}$ equation. *Math. Scand.*, 29:137–160, 1971.
- [21] H. Rossi. A Docquier-Grauert lemma for strongly pseudo convex domains in complex manifolds. *Rocky mountain journal of mathematics*, 6(1):171–176, 1976.
- [22] G. Schmalz. Solution of the $\bar{\partial}$ -equation with uniform estimates on strictly q-convex domains with non-smooth boundary. *Math. Z.*, 202:409–430, 1989.
- [23] H. Skoda. Valeurs au bord pour les solutions de l'opérateur d'' et caractérisation des zéros de la classe de Nevanlinna. *Bull. Soc. Math. France*, 104:225–299, 1976.