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On estimates for the $\bar{\partial}$ equation in Stein manifolds.

Eric Amar

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Abstract

We generalize to intersection of strictly c -convex domains in Stein manifold, $L^r - L^s$ and Lipschitz estimates for the solutions of the $\bar{\partial}$ equation done by Ma and Vassiliadou for domains in \mathbb{C}^n . For this we use a Docquier-Grauert holomorphic retraction plus the raising steps method I introduce earlier. This gives results in the case of intersection of domains with low regularity, \mathcal{C}^3 , for their boundary.

1 Introduction.

The solutions with L^r and Lipschitz estimates of the equation $\bar{\partial}u = \omega$, $\bar{\partial}\omega = 0$ revealed to be very important in complex analysis and geometry.

The first results of this kind were done by the use of solving kernels: Grauert-Lieb [8], Henkin [10], Ovrelid [19], Skoda [22], Krantz [14], in the case of strictly pseudo-convex domains with \mathcal{C}^∞ smooth boundary in \mathbb{C}^n , with the exception of Kerzman [13] in the case of $(0, 1)$ forms in strictly pseudo-convex domains with \mathcal{C}^4 smooth boundary in Stein manifolds.

Here we shall be interested in strictly c -convex, s.c.c. for short, domain D in a complex manifold. Such a domain is defined by a function ρ of class \mathcal{C}^3 in a neighbourhood U of \bar{D} and such that $i\partial\bar{\partial}\rho$ as at least $n - c + 1$ strictly positive eigenvalues in U .

These domains in \mathbb{C}^n have been studied in the case of smooth \mathcal{C}^∞ boundary by Fisher and Lieb [7].

Ma and Vassiliadou [17] got very nice estimates even in the case of intersections of s.c.c. domains with \mathcal{C}^3 boundary. I shall use their results here.

Quite recently C. Laurent-Thiébaud [15] get this kind of result for s.c.c. domains with smooth \mathcal{C}^∞ boundary in complex manifold by use of the Grauert's method of "bumps".

Concerning the study of transverse intersection of domains, one can cite the works of Henkin and Leiterer [9], Menini [18] for strictly pseudo convex domains and G. Schmalz [21] and Ma and Vassiliadou [17] for c -convex domains. C. Laurent-Thiébaud and J. Leiterer [16] solve the $\bar{\partial}$ equation in a case of intersection of s.c.c. domains more general than the one considered by Ma and Vassiliadou [17] but for bounded forms and they got solutions in Lipschitz spaces. It seems that the L^p case is still open for their case.

Let us state our first result which is completely analogous to the one Ma and Vassiliadou [17] got for domains in \mathbb{C}^n .

Theorem 1.1 *Let Ω be a Stein manifold of dimension n and a strictly c -convex (s.c.c.) domain D such that D is relatively compact with smooth \mathcal{C}^3 boundary in Ω . Let ω be a (p, q) form in $L^r_{p,q}(D)$, $\bar{\partial}\omega = 0$ with $1 < r < 2n + 2$, $c \leq q \leq n$. Then there is a $(p, q - 1)$ form u in $L^s(D)$, with $\frac{1}{s} = \frac{1}{r} - \frac{1}{2n + 2}$, such that $\bar{\partial}u = \omega$.*

If ω is in $L^r_{p,q}(D)$, $\bar{\partial}\omega = 0$ with $r \geq 2n + 2$, $c \leq q \leq n$, then there is a $(p, q - 1)$ form u in $\Lambda^\epsilon_{(p,q-1)}(\bar{D})$ such that $\bar{\partial}u = \omega$ with $\epsilon = \frac{1}{2} - \frac{n + 1}{r}$.

The spaces $\Lambda^\epsilon_{(p,q-1)}(\bar{D})$ are the (isotropic) Lipschitz spaces of order ϵ and we set $\Lambda^0_{(p,q-1)}(\bar{D}) := L^\infty_{(p,q-1)}(D)$.

It has to be noticed that the boundary regularity is just \mathcal{C}^3 , so it seems that this is a new result in a Stein manifold for such a low regularity.

In the case of a \mathcal{C}^∞ boundary regularity then this result is contained in C. Laurent-Thiébaud [15] corollary 2.11, but the proof here is completely different and, in some sense, "lighter" because it never uses Beals, Greiner et Stanton [5] heavy technology. We use for the analytic part kernels methods plus essentially geometric ones. Nevertheless we can recover the Sobolev estimates by a direct use of Beals, Greiner et Stanton [5] in the case of a \mathcal{C}^∞ boundary regularity by theorem 7.2. This avoid the use of the "bumps method" but this is valid only in Stein manifolds although C. Laurent-Thiébaud [15] results are valid in any complex manifold.

To state our next result, we need the definition of a \mathcal{C}^3 c convex domain, still taken from [17].

Definition 1.2 A relatively compact domain D in a Stein manifold Ω shall be called a \mathcal{C}^3 c -convex intersection if there exists a relatively compact neighbourhood W in Ω of \bar{D} and a finite number of real \mathcal{C}^3 functions ρ_1, \dots, ρ_N where $n \geq N + 3$ defined on W such that $D = \{z \in W :: \rho_1(z) < 0, \dots, \rho_N(z) < 0\}$ and the following are true:

- i) For $1 \leq i_1 < \dots < i_l \leq N$ the 1-forms $d\rho_{i_1}, \dots, d\rho_{i_l}$ are \mathbb{R} -linearly independent on $\bigcap_{j=1}^l \{\rho_{i_j} \leq 0\}$.
- ii) For $1 \leq i_1 < \dots < i_l \leq N$, for every $z \in \bigcap_{j=1}^l \{\rho_{i_j} \leq 0\}$, if we set $I := (i_1, \dots, i_l)$, there exists a linear subspace T_z^I of Ω of complex dimension at least $n - c + 1$ such that for $i \in I$ the Levi forms $L\rho_i$ restricted on T_z^I are positive definite.

We notice that in \mathbb{C}^n Ma and Vassiliadou need $N \leq n - 2$ and here we need $N \leq n - 3$. Now we can state:

Theorem 1.3 Let Ω be a Stein manifold of dimension n and a \mathcal{C}^3 c -convex intersection D such that D is relatively compact in Ω . There exists a $\nu \in \mathbb{N}^+$ (which depends on the maximal number of non empty intersections of $\{\rho_j = 0\}$) such that :

if ω is a (p, q) form in $L_{p,q}^r(D)$, $\bar{\partial}\omega = 0$ with $q \geq c$, $1 < r < 2n + 2$, then there is a $(p, q - 1)$ form u in $L^s(D)$, such that $\bar{\partial}u = \omega$ with $\frac{1}{s} = \frac{1}{r} + \frac{1}{\lambda} - 1$, where $1 \leq \lambda < \frac{2n + 2\nu}{2n - 1 + 2\nu}$.

More precisely,

- i) For any $1 < r < 2n + 2\nu$, there exists $c_r(D)$ positive constant such that

$$\|u\|_{L_{(p,q-1)}^s(D)} \leq c_r(D) \|\omega\|_{L_{(p,q)}^r(D)}$$

with $\frac{1}{s} = \frac{1}{r} - \frac{1}{2n + 2\nu}$.

- ii) For $r \geq 2n + 2\nu$, we have $\|u\|_{L_{(p,q-1)}^\infty(D)} \leq a_r(D) \|\omega\|_{L_{(p,q)}^r(D)}$ for some positive constant $a_r(D)$.

This also seems to be new in case \mathbb{C}^n is replaced by a Stein manifold.

The results of Ma and Vassiliadou [17] give good estimates in case of domains in \mathbb{C}^n . The point here was to pass from \mathbb{C}^n to a submanifold of \mathbb{C}^n . To do this I was inspired by a nice paper of H. Rossi [20] on Docquier Grauert holomorphic retraction. The first result is based on it and is the following non optimal theorem.

Theorem 1.4 Let M be a closed submanifold of dimension d of a Stein domain U_0 in \mathbb{C}^n . Let D be a s.c.c. domain relatively compact in M ($\bar{D} \subset M$) with \mathcal{C}^3 boundary. Then, with $r \geq 2n + 2$, we can solve in D $\bar{\partial}u = \omega$ when $\bar{\partial}\omega = 0$ and with $u \in \Lambda_{(p,q-1)}^\epsilon(\bar{D})$ if $\omega \in L_{(p,q)}^r(D)$, $c \leq q \leq n$, with

$$\epsilon = \frac{1}{2} - \frac{n+1}{r}.$$

Then we use the raising steps method [2] (see also [4] for more general operators than $\bar{\partial}$ and [3] in the non compact case). Let me recall it in this specific case.

Theorem 1.5 *Let M be a closed complex manifold and D a relatively compact domain in M . Suppose there is $\delta > 0$ and a finite covering $\{U_j\}_{j=1,\dots,N}$ of \bar{D} such that :*

(i) $\forall r > 1, \forall \omega \in L^r_{(p,q)}(D), \bar{\partial}\omega = 0, \exists u_j \in L^t_{(p,q-1)}(D \cap U_j) :: \bar{\partial}u_j = \omega$ in $D \cap U_j$

and $\|u_j\|_{L^t(D \cap U_j)} \lesssim \|\omega\|_{L^r(D \cap U_j)}$, with $\frac{1}{t} = \frac{1}{r} - \delta$.

(ii) $\exists s > 1, \forall \omega \in L^s_{(p,q)}(\Omega), \bar{\partial}\omega = 0, \exists w \in L^s_{(p,q-1)}(\Omega) :: \bar{\partial}w = \omega$ and $\|w\|_{L^s(\Omega)} \lesssim \|\omega\|_{L^s(\Omega)}$.

Then there is a constant $c > 0$ such that, for $r \leq s$, if $\omega \in L^r_{(p,q)}(D), \bar{\partial}\omega = 0$, it exists $u \in L^t_{(p,q-1)}(D)$ with $\lambda := \min(\delta, \frac{1}{r} - \frac{1}{s})$ and $\frac{1}{t} = \frac{1}{r} - \lambda$, such that $\bar{\partial}u = \omega, u \in L^t_{(p,q-1)}(D), \|u\|_{L^t(D)} \leq c\|\omega\|_{L^r(D)}$.

The local estimates (i) are given by "localizing s.c.c. domain", proposition 8.2 plus the results of Ma and Vassiliadou, theorem 2.1 here. The global estimate (ii), the threshold, is given by the $L^r - \Lambda^\epsilon$ estimates done in theorem 1.4. We get the same optimal results as for domains in \mathbb{C}^n .

Theorem 1.6 *Let M be a complex submanifold of dimension d in \mathbb{C}^n and a s.c.c. domain D such that D is relatively compact with smooth boundary of class \mathcal{C}^3 in M . Let ω a (p, q) form in $L^r_{(p,q)}(D), \bar{\partial}\omega = 0, c \leq q \leq n$, with $1 < r < 2d + 2$. Then there is a $(p, q - 1)$ form u in $L^s(D)$, with $\frac{1}{s} = \frac{1}{r} - \frac{1}{2d + 2}$, such that $\bar{\partial}u = \omega$.*

If $r \geq 2n + 2$ then there is a $(p, q - 1)$ form u in $\Lambda^\epsilon_{(p,q-1)}(\bar{D})$ such that $\bar{\partial}u = \omega$ with $\epsilon = \frac{1}{2} - \frac{d + 1}{r}$.

We follow exactly the same path to work with \mathcal{C}^3 c -convex intersection with again local estimates given by "localizing s.c.c. intersection", proposition 8.3 plus the results of Ma and Vassiliadou, theorem 2.2. The global estimate (ii), the threshold, is given by the $L^r - L^\infty$ estimates done in theorem 2.2 plus the generalization of a theorem of Rossi done in theorem 8.6.

To pass to Stein manifold, we use an embedding theorem of Bishop and Narashiman (see theorem 5.3.9. of Hörmander [12]) to see an abstract Stein manifold of dimension d as a submanifold of \mathbb{C}^{2d+1} . So we get our main results.

This work will be presented in the following way.

- First we recall the estimates in the case of strictly c -convex domains in \mathbb{C}^n done by Ma and Vassiliadou [17].
- We recall the Docquier Grauert holomorphic retraction on a complex submanifold M of \mathbb{C}^n .
- We extend a form ω from a domain D s.c.c. in M to a domain E s.c.c. in \mathbb{C}^n by use of a generalization of a theorem of H. Rossi [20]. We then solve the form in E by the known estimates in \mathbb{C}^n .
- We show that the solution in E can be restricted to D to get a solution in D with good enough estimates, for $r \geq 2n + 2$. This gives theorem 1.4.
- We use the raising steps theorem with the threshold given by theorem 1.4. So we have theorem 1.6 for the case of a submanifold of \mathbb{C}^n .
- Then by the same way, using ad-hoc modifications of propositions in the appendix, we get theorem 1.3 in the case of a submanifold of \mathbb{C}^n .
- By use of a theorem of Bishop and Narashiman, i.e. the proper embedding of a Stein manifold of dimension d in \mathbb{C}^{2d+1} , we get our main theorems 1.1, 1.3 for any Stein manifold.
- Finally we prove technical results we need in the appendix.

I am indebted to C. Laurent-Thiébaud who pointed to me the precise link between the work of Beals, Greiner et Stanton [5] and the existence of actual solutions for the $\bar{\partial}$ Neuman problem.

2 Strictly c -convex domain in \mathbb{C}^n .

We shall use the nice estimates for a smoothly \mathcal{C}^3 bounded c convex domains in \mathbb{C}^n obtained by Ma and Vassiliadou [17] , lemma 5.3. in their paper.

Theorem 2.1 *Let D be a bounded s.c.c. domain in \mathbb{C}^n with a \mathcal{C}^3 defining function. Then*

$$\forall \omega \in L^r_{(p,q)}, \bar{\partial}\omega = 0, \quad c \leq q \leq n, \quad 1 \leq r < 2n + 2,$$

there exists $u \in L^s_{(p,q-1)}(D)$, $\frac{1}{s} = \frac{1}{r} - \frac{1}{2n+2}$, with the following properties:

i) $\bar{\partial}u = \omega$ in the sense of currents in D ,

ii) if $r = 1$, $u \in L^{\frac{2n+2}{2n+1}-\eta}_{(p,q-1)}$ for any $\eta > 0$.

iii) if $2n + 2 \leq r \leq \infty$, $u \in \Lambda^{\epsilon}_{(p,q-1)}(\bar{D})$ with $\epsilon = \frac{1}{2} - \frac{n+1}{r}$.

They also prove results in the case of intersections.

Theorem 2.2 *Let a \mathcal{C}^3 c -convex intersection domain D such that D is relatively compact in \mathbb{C}^n . Then there exists a $\nu \in \mathbb{N}^+$ (which depends on the maximal number of non empty intersections of $\{\rho_j = 0\}$) such that :*

if ω a (p, q) form in $L^r_{p,q}(D)$, $\bar{\partial}\omega = 0$ with $q \geq c$, $1 < r < 2n + 2\nu$, there is a $(p, q - 1)$ form u in $L^s(D)$, such that $\bar{\partial}u = \omega$ with $\frac{1}{s} = \frac{1}{r} + \frac{1}{\lambda} - 1$, where $1 \leq \lambda < \frac{2n + 2\nu}{2n - 1 + 2\nu}$.

More precisely

i) For any $1 < r < 2n + 2\nu$, there exists $c_r(D)$ positive constant such that

$$\|u\|_{L^s_{(p,q-1)}(D)} \leq c_r(D) \|\omega\|_{L^r_{(p,q)}(D)}$$

with $\frac{1}{s} = \frac{1}{r} - \frac{1}{2n + 2\nu}$.

ii) For $r \geq 2n + 2\nu$, we have $\|u\|_{L^\infty_{(p,q-1)}(D)} \leq a_r(D) \|\omega\|_{L^r_{(p,q)}(D)}$ for some positive constant $a_r(D)$.

3 The Docquier - Grauert holomorphic retraction.

We have the Docquier-Grauert lemma [6] :

Lemma 3.1 *Let K be a compact subset of a closed complex submanifold M of \mathbb{C}^n . There is a neighbourhood U of K and a holomorphic map $\pi : U \rightarrow U \cap M$ such that $\pi(\zeta) = \zeta$ for $\zeta \in U \cap M$.*

In fact we have more (Rossi [20], p 172) from the argument of Docquier-Grauert we have that the fibers $\pi^{-1}\pi\zeta$ of π intersect M transversally at all points of M and are of dimension $n - d$.

Let M be a complex submanifold of dimension d in \mathbb{C}^n and D a relatively compact domain strictly c -convex in M . We have the following lemma.

Lemma 3.2 *Let $\zeta \in \bar{D}$, there is a neighborhood U of ζ in \mathbb{C}^n and a bi-holomorphic application (U, φ) , $\varphi : U \rightarrow T$ such that, with $z = (z_1, \dots, z_n)$ the coordinates in T , we have: $\varphi(D) \cap T = \{z_{d+1} = \dots = z_n = 0\}$ and the retraction $\tilde{\pi} := \varphi \circ \pi \circ \varphi^{-1}$ read in the application φ is given by $\tilde{\pi}(z) = (z_1, \dots, z_d, 0, \dots, 0)$, i.e. this is the orthogonal projection onto the subspace of $z' := (z_1, \dots, z_d)$. Moreover one can choose for T a tube around $\varphi(M)$ of width $\delta > 0$.*

Proof.

The manifold M is given, by use of the retraction π , by the functions $f_k(\zeta) := \zeta_k - \pi_k(\zeta)$, $k = 1, \dots, n$. We have if $\zeta \in M$, $\zeta - \pi(\zeta) = 0$; if $\zeta \notin M$, $\zeta - \pi(\zeta) \neq 0$, because $\pi(\zeta) \in M$. The transversality of the fibers with respect to M at all points of \bar{D} insures that the Jacobian of the application $f = (f_1, \dots, f_n)$ has rank $n - d$, which is the complex co-dimension of M . Take a point $\zeta^0 \in \bar{D}$, there are $n - d$ functions f_j which are independent in a neighborhood U of ζ^0 . Re-numerating the functions f_j and the variables ζ_k , we may suppose that the determinant $(\frac{\partial f_j}{\partial \zeta_k})_{j,k=d+1,\dots,n}$ is different from zero.

Now we shall make the change of variables $z = \varphi(\zeta)$ with $z_j = \zeta_j$, $j = 1, \dots, d$; $z_j = f_j(\zeta)$, $j = d + 1, \dots, n$. This is actually a change of variables because the Jacobian of φ is different from zero in the open set U . We have that the application φ is a bi-holomorphism from the open set U onto the open set $T := \varphi(U)$.

Let $z' = (z_1, \dots, z_d)$ and $z'' = (z_{d+1}, \dots, z_n)$; we have in T that:

$$N := \varphi(M) = \{z = (z', z'') \in T :: z'' = 0\}.$$

Now take a tube around N centered in ζ^0 , $T = \{z = (z', z'') :: z'' \in B((z', 0), \delta)\}$, we call it again T , and we still denote $U = \varphi^{-1}(T)$.

We cover \bar{D} by a finite number of these bi-holomorphisms (U_j, φ_j) . We note N_j the manifold $N_j := \varphi_j(M \cap U_j) \subset T_j$ and, diminishing a little bit the U_j if necessary, we can suppose that the width of the tubes T_j around the N_j is constant and equals $\delta > 0$. We know that there is a constant $\mu > 0$ such that $\mu^{-1} < J_j < \mu$, where J_j is the Jacobian of φ_j , because there is a finite number of charts (U_j, φ_j) .

We denote dV the Lebesgue measure on the manifold M .

We have the following basic lemma.

Lemma 3.3 *Let f be a function in $L^1(U_j)$ and \tilde{f} this function read in the application φ_j , i.e. $\tilde{f} := f \circ \varphi_j^{-1}$, we get*

$$\int_{U_j} f(\zeta) dm(\zeta) = \int_{N_j} \int_{B((z', 0), \delta)} \tilde{f}(z', z'') J_j(z', z'') dm(z'') dV(z').$$

Proof.

This is simply the change of variables formula because $\varphi_j(U_j) = T_j = N_j \times B(\cdot, \delta)$ and the Jacobian of φ_j is J_j .

Lemma 3.4 *Let f be a measurable function, positive on M , then*

$$\int_{U_j} f \circ \pi(\zeta) dm(\zeta) \leq \mu c(\delta) \int_{N_j} f(z', 0) dV(z'),$$

with dm the Lebesgue measure on \mathbb{C}^n , dV the Lebesgue measure on M and $c(\delta) := |B(x, \delta)|$.

Proof.

We can apply lemma 3.3 with the notation $z = (z', z'')$, z' the coordinates in N_j , z'' the coordinates in the fibers:

$$\int_{U_j} f \circ \pi(\zeta) dm(\zeta) = \int_{N_j} \left\{ \int_{B(z', \delta)} \tilde{f}(z', z'') J_j(z', z'') dm(z'') \right\} dV(z'),$$

but here we have $\tilde{f}(z', z'') = f(z', 0)$ because $\tilde{\pi}(z) = (z', 0)$ hence the formula is now:

$$\int_{U_j} f \circ \pi(z) dm(z) \leq \mu \int_{N_j} f(z', 0) |B(z', \delta)| dV(z') = \mu c(\delta) \int_{N_j} f(z', 0) dV(z'). \blacksquare$$

We notice that the open set $U := \bigcup_{j=1}^N U_j$ contains \bar{D} .

The theorem 8.5 in the appendix, which generalizes to s.c.c. domains a theorem by Rossi [20] done for strictly pseudo convex domains, gives us the existence of the strictly c -convex domain E in \mathbb{C}^n such that $\pi : \bar{E} \rightarrow \bar{D}$. Now on we fix this s.c.c. domain E .

4 Extension of the form ω .

Let ω be a (p, q) form in $L^r(D)$, $\bar{\partial}$ closed ; we extend it in E by use of the retraction π in the following manner : $\tilde{\omega} := \pi^* \omega$.

Lemma 4.1 *We have $\bar{\partial} \tilde{\omega} = 0$. Moreover if $\omega \in L^r(D)$ we have $\tilde{\omega} \in L^r(E)$ with $\|\tilde{\omega}\|_{L^r(E)} \leq \mu c(\delta) \|\omega\|_{L^r(D)}$.*

Proof.

Because the retraction π is holomorphic we get $\bar{\partial} \tilde{\omega} = \pi^* \bar{\partial} \omega = 0$. Moreover the lemma 3.4 gives that $\tilde{\omega}$ is still in $L^r(E)$; we start by extending ω to $U \cap M$ by zero outside \bar{D} ; we had that the coefficients of $\tilde{\omega}$ can be written $f \circ \pi$ hence, applying lemma 3.4 to the functions $|f \circ \pi|^r$ we get $\|\tilde{\omega}\|_{L^r(U_j)} \leq \mu c(\delta) \|\omega\|_{L^r(N_j)}$. We have only a finite number of open sets U_j , so we get $\|\tilde{\omega}\|_{L^r(U)} \lesssim \mu c(\delta) \|\omega\|_{L^r(D)}$. Because $E \subset U$ we get $\|\tilde{\omega}\|_{L^r(E)} \leq \mu c(\delta) \|\omega\|_{L^r(D)}$. \blacksquare

Now E is s.c.c. in \mathbb{C}^n and, with $r \geq 2n + 2$, we can solve the $\bar{\partial}$ in the space $L^\infty(E)$: $\bar{\partial} \tilde{u} = \tilde{\omega}$, $\tilde{u} \in L^\infty(E)$ by the theorem 2.1. Fix $\omega \in L^r_{(p,q)}(D)$, $\bar{\partial} \omega = 0$, with $\tilde{\omega}$ as above, we have $\tilde{u} \in L^\infty(E)$ also fixed.

We shall need the lemma

Lemma 4.2 *In an open set U_j of our covering, there is (p, q) form \tilde{v}_j such that $\bar{\partial} \tilde{v}_j = \bar{\partial}_M \tilde{v}_j = \tilde{\omega}$. This means that the $\bar{\partial}$ of the form \tilde{v}_j read in (U_j, φ_j) does not contain any $d\bar{z}''_k$. Moreover the coefficients of \tilde{v}_j are bounded in U_j and holomorphic in the fibers of π .*

Proof.

We work directly in $T = T_j$ by use of the bi-holomorphism φ_j and we set $U := U_j$, $\varphi := \varphi_j$.

The first part is coming from the fact that $\tilde{\omega}$ read in (U, φ) does not contain any form $d\bar{z}''_l$. Suppose that \tilde{u} contains such a form, we have : $\tilde{u} = v_l d\bar{z}''_l + \Gamma_l$ with Γ_l not containing $d\bar{z}''_l$; notice that the Γ_l are unique and so linear in \tilde{u} . Hence, keeping the notation \tilde{u} for its reading in (U, φ) , with the notation $\bar{\partial}_{z'_k} \tilde{u} = \frac{\partial \tilde{u}}{\partial \bar{z}'_k}$,

$$\forall k = 1, \dots, d, \quad \bar{\partial}_{z'_k} \tilde{u} = \bar{\partial}_{z'_k} v_l \wedge d\bar{z}''_l + \bar{\partial}_{z'_k} \Gamma_l$$

cannot have terms in $d\bar{z}''_l$ because $\tilde{\omega}$ has not, hence we get $\bar{\partial}_{z'_k} v_l \wedge d\bar{z}''_l = 0$. So we get :

$$\bar{\partial}_{z'}(\tilde{u} - \sum_{l=d+1}^n v_l \wedge d\bar{z}''_l) = \tilde{\omega}.$$

on the other hand we get : $\bar{\partial}_{z''_k} \tilde{u} = \bar{\partial}_{z''_k} v_l \wedge d\bar{z}''_l + \bar{\partial}_{z''_k} \Gamma_l$ and, because there are no terms of the form $d\bar{z}''_k \wedge d\bar{z}''_l$ in $\tilde{\omega}$, we get necessarily $\forall l, k = d+1, \dots, n$, $\bar{\partial}_{z''_k} v_l \wedge d\bar{z}''_l = 0$. So the Γ_l are holomorphic in the variables z'' . Hence we showed $\tilde{v}_j := \tilde{u} - \sum_{l=d+1}^n v_l \wedge d\bar{z}''_l = \sum_{l=d+1}^n \Gamma_l$ is still a solution of $\bar{\partial}\tilde{v}_j = \tilde{\omega}$ and, because the coefficients of \tilde{u} are bounded, we get all the properties stated in the lemma. We still notice that \tilde{v}_j is linear with respect to \tilde{u} . ■

Lemma 4.3 *There is a (p, q) form \tilde{v} such of that, in E , $\bar{\partial}\tilde{v} = \bar{\partial}_M \tilde{v} = \tilde{\omega}$. Moreover the coefficients of \tilde{v} are bounded in E and holomorphic on the fibers $\forall \zeta \in D$, $F_\zeta := \pi^{-1}(\zeta)$.*

Proof.

Let us take two open sets of our covering: U_j, U_k ; the lemma 4.2 gives us:

$$\tilde{u} = v_j \wedge d\bar{z}'' + \tilde{v}_j \text{ in } U_j \cap E \text{ with } \tilde{v}_j \text{ without any } d\bar{z}''_l.$$

The same way in the open set U_k we have :

$$\tilde{u} = v_k \wedge d\bar{z}'' + \tilde{v}_k \text{ in } U_k \cap E \text{ with } \tilde{v}_k \text{ without any } d\bar{z}''_l.$$

So in the intersection of the two sets, because \tilde{u} is global, choosing one of these two systems of coordinates and with clear notations, we get:

$$(v_j - v_k) \wedge d\bar{z}'' + (\tilde{v}_j - \tilde{v}_k) = 0 \text{ in } U_j \cap U_k \cap E.$$

Because the \tilde{v}_j do not contain the $d\bar{z}''_l$, we necessarily have $\tilde{v}_j = \tilde{v}_k$ and $v_j = v_k$. So the forms \tilde{v}_j make a global form \tilde{v} just posing $\tilde{v} := \tilde{v}_j$ in U_j . ■

5 Estimates in the case of a submanifold of \mathbb{C}^n .

We shall show the following theorem :

Theorem 5.1 *Let M be a complex submanifold of dimension d in \mathbb{C}^n and a s.c.c. domain D such that D is relatively compact with smooth \mathcal{C}^3 boundary in M . Let again ω be a (p, q) form in $L^r_{p,q}(D)$, $\bar{\partial}\omega = 0$ with $r \geq 2n + 2$, $c \leq q \leq n$. Then there is a $(p, q - 1)$ form u in $\Lambda^{\epsilon}_{(p,q-1)}(\bar{D})$, $\epsilon = \frac{1}{2} - \frac{n+1}{r}$, such that $\bar{\partial}u = \omega$.*

Proof.

We restrict the form \tilde{v} got in the lemma 4.3 to D , $u := \tilde{v}|_D$, and we need to see that, on D , $\bar{\partial}u = \omega$. It is enough to see this in an open set U_j of our covering.

We shall use that $\bar{\partial}\tilde{v} = \tilde{\omega}$ in the distributions sense, with $\tilde{v} \in L^\infty(E)$. So let $\chi \in \mathcal{D}_{(n-p, n-q)}(U_j \cap D)$; we get

$$\langle \bar{\partial}u, \chi \rangle_M = (-1)^{q+1} \langle u, \bar{\partial}\chi \rangle_M;$$

with the scalar product of the manifold M . We read these data in (U_j, φ_j) and we keep the same names.

We extend the form χ from $N_j = U_j \cap M$ to the whole U_j in making it constant with respect to z'' . Let us denote $\tilde{\chi}$ this extended form.

Because the form $\tilde{v} = \tilde{v}_j$ in T_j is holomorphic in the variables z'' , its values in (z', z'') is the mean value of its values in the ball centered at $(z', 0) \in N_j$ and with radius δ in z'' . The lemma 3.3 gives:

$$\int_{T_j} (\tilde{v}, \bar{\partial}\tilde{\chi}) dm = \int_{N_j} \left\{ \int_{B(z', \delta)} (\tilde{v}, \bar{\partial}\tilde{\chi})(z', z'') dm(z'') \right\} dV(z),$$

but

$$\int_{B(z', \delta)} (\tilde{v}, \bar{\partial}\tilde{\chi})(z', z'') dm(z'') = \int_{B(z', \delta)} \sum_{I, J} \tilde{v}_{I, J}(z', z'') \tilde{\chi}'_{I^c J^c}(z', z'') dm(z'')$$

where $\tilde{\chi}' := \bar{\partial}\tilde{\chi}$. Because the form $\tilde{\chi}$ is compactly supported in N_j and is constant on the fibers, we get

$$\int_{B(z', \delta)} \sum_{I, J} \tilde{v}_{I, J}(z', z'') \tilde{\chi}'_{I^c J^c}(z', z'') dm(z'') = \sum_{I, J} \chi'_{I^c J^c}(z') \int_{B(z', \delta)} \tilde{v}_{I, J}(z', z'') dm(z'');$$

So we get, because $\int_{B(z', \delta)} \tilde{v}_{I, J}(z', z'') dm(z'') = c(\delta) u_{I, J}(z')$, with $c(\delta) := |B(z', \delta)|$,

$$\langle \tilde{v}, \bar{\partial}\tilde{\chi} \rangle_{T_j} := \int_{T_j} (\tilde{v}, \bar{\partial}\tilde{\chi}) dm = c(\delta) \int_{N_j} (u, \bar{\partial}\chi)(x) dV(x) = c(\delta) \langle u, \bar{\partial}\chi \rangle_{N_j}.$$

keeping the same notations, we come back to the open set U_j and, because χ is compactly supported in $N_j = M \cap U_j$, we get, because $T_j = \varphi_j(U_j)$ is a tube around N_j :

$$\langle \tilde{v}, \bar{\partial}\tilde{\chi} \rangle := c(\delta) \langle u, \bar{\partial}\chi \rangle_M.$$

exactly by the same way we get:

$$\langle \bar{\partial}\tilde{v}_j, \tilde{\chi} \rangle = c(\delta) \langle \bar{\partial}u, \chi \rangle_M,$$

And, because $\tilde{\omega}$ is constant on the fibers, we get

$$\langle \tilde{\omega}, \tilde{\chi} \rangle = c(\delta) \langle \omega, \chi \rangle_M.$$

Recall that $\bar{\partial}\tilde{v} = \tilde{\omega}$ in the distributions sense, and, because $\tilde{\omega}$ is a current in $L^r_{p, q}(E)$, this is also true for $\bar{\partial}\tilde{v}$. Hence we have, because $\tilde{\chi} \in L^{r'}_{n-p, n-q}(E)$, with r' the conjugate exponent of r , that $\langle \bar{\partial}\tilde{v}, \tilde{\chi} \rangle = \langle \tilde{\omega}, \tilde{\chi} \rangle$ is well defined, even if $\tilde{\chi}$ is not compactly supported. The previous equalities give $\langle \bar{\partial}u, \chi \rangle_D = \langle \omega, \chi \rangle_D$. Because this is true for all C^∞ functions compactly supported χ in an open set T_j of a covering of \bar{D} , we get $\bar{\partial}u = \omega$ in the distributions sense on D , with $u \in \Lambda^c_{(p, q-1)}(\bar{D})$. ■

Remark 5.2 *We have no such estimates in the case $r < 2n + 2$ because the mean value in a ball of the fiber of a function in L^s in E and holomorphic on the fibers is no longer in $L^s(D)$ for $s < \infty$, which can be easily seen.*

Now we are in position to apply the raising steps theorem.

Theorem 5.3 *Let M be a complex submanifold of dimension d in \mathbb{C}^n and D be a s.c.c. domain which is relatively compact with smooth C^3 boundary in M . Let ω be a (p, q) form in $L^r_{p, q}(D)$, $\bar{\partial}\omega = 0$ with $1 < r < 2d + 2$, $c \leq q \leq n$. Then there is a $(p, q - 1)$ form u in $L^s(D)$, with $\frac{1}{s} = \frac{1}{r} - \frac{1}{2d + 2}$, such that $\bar{\partial}u = \omega$.*

Proof.

In order to have the local result for all points in \bar{D} we use the same method as in [2], but with the proposition 8.2 and the results of Ma and Vassiliadou [17]. Let us see it.

Let $\zeta \in \partial D$ and (V, φ) be a chart in a neighbourhood of ζ in M and ω a (p, q) form in $L_{p,q}^r(D)$, $\bar{\partial}\omega = 0$ with $1 < r < 2d + 2$, $c \leq q \leq n$. We read this situation in \mathbb{C}^d via the chart (V, φ) so we have an open set $W := \varphi(V) \subset \mathbb{C}^d$ and a piece of s.c.c. domain $\varphi(V \cap D)$ near the point $\eta := \varphi(\zeta) \in \mathbb{C}^d$ because the bi-holomorphic map φ keeps the s.c.c. property. By use of the localizing proposition 8.2 there exist a s.c.c. domain $E \subset \varphi(V \cap D)$, with \mathcal{C}^3 boundary, which shares a part of its boundary near η with the boundary of $\varphi(V \cap D)$. We read the form ω by φ , which gives us a $\tilde{\omega} := \varphi^*\omega$ still in $L_{p,q}^r(\varphi(V \cap D))$, $\bar{\partial}\tilde{\omega} = 0$ hence $\tilde{\omega} \in L_{p,q}^r(E)$. Now we apply the results of Ma and Vassiliadou [17], theorem 2.1 here, to get a $(p, q - 1)$ form \tilde{u} solution of $\bar{\partial}\tilde{u} = \tilde{\omega}$, $\tilde{u} \in L^s(E)$, with $\frac{1}{s} = \frac{1}{r} - \frac{1}{2d+2}$. Back to D via φ^{-1} we have our local estimates : set $u := (\varphi^{-1})^*\tilde{u}$, then $\bar{\partial}u = \omega$, $u \in L^s(\varphi^{-1}(E))$.

Hence we have the (i) of the raising steps theorem 1.5.

We have the global result, i.e. the (ii) of the raising steps theorem 1.5 : if $\mu \in L_{p,q}^{2n+2}(D)$, $\bar{\partial}\mu = 0$ then we have a solution v in $L_{(p,q-1)}^\infty(D)$ such that $\bar{\partial}v = \mu$ by use of theorem 4.3. Now we take $\omega \in L_{p,q}^r(D)$, $\bar{\partial}\omega = 0$, then we have that the optimal exponent for the solution u is s such that $\frac{1}{s} = \frac{1}{r} - \frac{1}{2d+2}$, then we choose any $r_0 > \max(2n+2, s)$ as a threshold and, because $L_{p,q-1}^\infty(D) \subset L_{(p,q-1)}^{r_0}(D)$, for D is bounded, we have a global solution to $\bar{\partial}v = \mu$ in $L_{(p,q-1)}^{r_0}(D)$ if $\mu \in L_{(p,q)}^{r_0}(D)$. Now $s < r_0$ gives that $u \in L_{(p,q-1)}^s(D)$, by the raising steps theorem 1.5, and this ends the proof. \blacksquare

6 The case of \mathcal{C}^3 c -convex intersection.

We proceed exactly the same way than for just one s.c.c. domain.

For the local estimates we use the localizing proposition 8.3 and we repeat the proof above. This is the point where we need to have at most $N = n - 3$ domains in a Stein manifold of dimension n , compare to $n - 2$ domains in \mathbb{C}^n . By use of Ma and Vassiliadou main theorem 2.2 and with $\nu \in \mathbb{N}^+$ defined there, we get :

Theorem 6.1 *Let M be a complex submanifold of \mathbb{C}^n of dimension d and a \mathcal{C}^3 c -convex intersection domain D such that D is relatively compact in M . Let ω be a (p, q) form in $L_{p,q}^r(D)$, $\bar{\partial}\omega = 0$ with $q \geq c$, $1 < r < 2n + 2\nu$. Then there is finite covering $\{U_j\}_{j=1,\dots,N}$ of \bar{D} and $(p, q - 1)$ form u_j in $L^s(U_j \cap D)$, such that $\bar{\partial}u_j = \omega$ in $U_j \cap D$ with $\frac{1}{s} = \frac{1}{r} + \frac{1}{\lambda} - 1$, where $1 \leq \lambda < \frac{2n + 2\nu}{2n - 1 + 2\nu}$.*

More precisely,

i) For any $1 < r < 2n + 2\nu$, there exists $c_r(D)$ positive constant such that

$$\|u_j\|_{L_{(p,q-1)}^s(U_j \cap D)} \leq c_r(D) \|\omega\|_{L_{(p,q)}^r(U_j \cap D)}$$

with $\frac{1}{s} = \frac{1}{r} - \frac{1}{2n + 2\nu}$.

ii) For $r \geq 2n + 2\nu$, we have $\|u_j\|_{L_{(p,q-1)}^\infty(U_j \cap D)} \leq a_r(D) \|\omega\|_{L_{(p,q)}^r(U_j \cap D)}$ for some positive constant $a_r(D)$.

Now for the global threshold, we copy the proof of theorem 4.3, replacing $r_0 > \max(2n + 2, s)$ by $r_0 > \max(2n + 2\nu, s)$. We remark that we have not the Lipschitz estimates here. So we get, still with $\nu \in \mathbb{N}^+$ given by Ma and Vassiliadou main theorem 2.2:

Theorem 6.2 *Let M be a complex submanifold of dimension d in \mathbb{C}^n and a \mathcal{C}^3 c -convex intersection domain D such that D is relatively compact in M . Let ω be a (p, q) form in $L^r_{(p,q)}(D)$, $\bar{\partial}\omega = 0$ with $r \geq 2n + 2\nu$, $c \leq q \leq n$. Then there is a $(p, q - 1)$ form u in $L^\infty_{(p,q-1)}(D)$ such that $\bar{\partial}u = \omega$.*

So we can apply the raising steps theorem to get the analogous results in the case of a \mathcal{C}^3 c -convex intersection domain D such that D is relatively compact in M , a closed complex submanifold of \mathbb{C}^n .

7 Estimates in the case of a Stein manifold.

We can apply a theorem of Bishop and Narashiman (see theorem 5.3.9. of Hörmander [12]) which tells us that, if Ω is a Stein manifold of dimension d , there is an element $f \in \mathcal{H}(\Omega)^{2d+1}$ which defines a regular injective and proper application from Ω in \mathbb{C}^{2d+1} . Denote $M := f(\Omega)$; if D' is the strictly c -convex domain in Ω , relatively compact in Ω , then its image $D = f(D')$ is a strictly c -convex domain in M . We can apply theorem 5.3.

Of course the same is true for \mathcal{C}^3 c -convex intersection in Stein manifold M and we get our main theorems.

We get a easy corollary of our main theorems 1.1, (see Ma and Vassiliadou [17], corollary 1.):

Corollary 7.1 *Let Ω be a Stein manifold of dimension n and a strictly c -convex domain D , or a \mathcal{C}^3 c -convex intersection, such that D is relatively compact with smooth \mathcal{C}^3 boundary in Ω . Then, for $q \geq c$, the operator $\bar{\partial} : L^2_{(p,q-1)}(D) \rightarrow L^2_{(p,q)}(D)$ has closed range.*

Proof.

This is fairly well known: for instance theorem 1.1.1 in [11]. ■

The above will imply the L^2 existence of the $\bar{\partial}$ -Neumann operator on c -convex domains so we get an automatic improvement of regularity in the case of a \mathcal{C}^∞ smoothly bounded s.c.c. domain, by theorems 2 and 4 in Beals, Greiner et Stanton [5]:

Theorem 7.2 *Let Ω be a Stein manifold of dimension n and a strictly c -convex (s.c.c.) domain D such that D is relatively compact with smooth \mathcal{C}^∞ boundary in Ω . Let $k \in \mathbb{N}$ and ω a (p, q) form in $W^{k,r}(D)$, $\bar{\partial}\omega = 0$ with $1 < r < 2n + 2$, $c \leq q \leq n$. Then there is a $(p, q - 1)$ form u in $W^{k+1/2,r}(D)$, such that $\bar{\partial}u = \omega$.*

If $\epsilon > 0$ and ω is in $\Lambda^\epsilon_{(p,q)}(D)$, $\bar{\partial}\omega = 0$ with $c \leq q \leq n$, then there is a $(p, q - 1)$ form u in $\Lambda^{\epsilon+1/2}_{(p,q-1)}(\bar{D})$ such that $\bar{\partial}u = \omega$.

Here we use the notation $W^{k,r}(D)$ for the Sobolev space of functions whose derivatives of order less than k are in L^r .

We notice that there is no hypothesis here in the case $r > 2$ on the compactness of the support of the form ω , in contrast to the previous results we had in [2].

8 Appendix.

Lemma 8.1 *Let A, B be two self adjoint matrices such that A has at least $n - c + 1$ strictly positive eigenvalues and B is positive. Then $A + B$ has at least $n - c + 1$ strictly positive eigenvalues.*

Proof.

Let E be the space generated by the eigenvectors associated to the strictly positive eigenvalues of A . Then E has dimension at least $n - c + 1$. Let $S := A + B$, because B is positive, we get

$$\forall x \in E, \langle Sx, x \rangle = \langle Ax, x \rangle + \langle Bx, x \rangle > 0.$$

Now let e_1, \dots, e_k be the eigenvectors associated to the negative eigenvalues of S . We set $F = \text{span}\{e_1, \dots, e_k\}$, we have that F is invariant by S and we have $\forall x \in F, \langle Sx, x \rangle \leq 0$. If the space $G := E \cap F$ is of non zero dimension, we get $\forall x \in G, x \neq 0, \langle Sx, x \rangle > 0$ and $\langle Sx, x \rangle \leq 0$ so a contradiction. Hence $\dim G = 0$ and $\dim F \leq \text{codim} E = c - 1$, which means that S has a least $n - c + 1$ strictly positive eigenvalues. ■

The next proposition generalizes the one in [1], proposition 1.1, done for the pseudo convex case.

Proposition 8.2 *(Localizing s.c.c. domain) Let D be a strictly c -convex domain with \mathcal{C}^3 boundary in \mathbb{C}^n . Let $\zeta \in \partial D$, U a neighbourhood of ζ in \mathbb{C}^n and $B(\zeta, r)$ a ball centered at ζ and of radius r such that $B(\zeta, 3r) \subset U$; then there is a domain \tilde{D} , s.c.c. and with \mathcal{C}^3 boundary such that we have $\tilde{D} \subset U$ and $\partial D \cap B(\zeta, r) = \partial \tilde{D} \cap B(\zeta, r)$.*

Proof.

Let ρ be a defining function for D . Let $\zeta \in \partial D$ and U a neighbourhood of ζ in \mathbb{C}^n . Consider a positive convex increasing function χ defined on \mathbb{R}^+ , \mathcal{C}^∞ such that $\chi = 0$ in $(0, r)$. Set $\tilde{\rho}(z) := \rho(z) + a\chi(|z - \zeta|^2)$; we have $\partial \bar{\partial} \tilde{\rho} = \partial \bar{\partial} \rho + a \partial \bar{\partial} \chi$. But, as is easily seen, $i \partial \bar{\partial} \chi$ is positive at each point z , hence, setting $A = i \partial \bar{\partial} \rho$, $B = a i \partial \bar{\partial} \chi$, we can apply lemma 8.1 and we have that the domain $\tilde{D} := \{\tilde{\rho} < 0\}$ is also s.c.c. with smooth \mathcal{C}^3 boundary.

Now we choose r small enough to have $B(\zeta, 3r) \subset U$. We have $\tilde{\rho}(z) < 0 \Rightarrow \rho(z) < -a\chi(|z - \zeta|^2)$; so we set :

$$\alpha := \sup_{z \in \bar{D}} -\rho(z) < \infty, \text{ by the compactness of } \bar{D} \text{ and } \beta := \inf_{z \in U \setminus B(\zeta, 2r)} \chi(|z - \zeta|^2) = 4r^2.$$

Then with $a := \frac{\alpha + 1}{\beta}$ we get that $\{\tilde{\rho}(z) < 0\} \subset U$ because if not $\exists z \notin B(\zeta, 3r) :: \rho(z) < -a\chi(|z - \zeta|^2) < -(\alpha + 1)$ which is not possible.

Of course in the ball $B(\zeta, r)$ we have $\partial D \cap B(\zeta, r) = \partial \tilde{D} \cap B(\zeta, r)$. ■

We shall need to extend this proposition to the case of \mathcal{C}^3 c -convex intersection.

Proposition 8.3 *(Localizing s.c.c. intersection) Let D be a \mathcal{C}^3 c -convex intersection in \mathbb{C}^n . Let $\zeta_0 \in \partial D$, U a neighbourhood of ζ_0 in \mathbb{C}^n and $B(\zeta_0, r)$ a ball centered at ζ_0 and of radius r ; then there is a domain \tilde{D} , \mathcal{C}^3 c -convex intersection such that we have $\tilde{D} \subset U$ and $\partial D \cap B(\zeta_0, r) = \partial \tilde{D} \cap B(\zeta_0, r)$.*

Proof.

By assumption the 1-forms $\{d\rho_j(\zeta)\}_{j \in I}$ are linearly independent in $\bigcap_{j \in I} \{\rho_j(z) \leq 0\}$ and $|I| \leq n - 3$.

Take a point $\zeta_0 \in \bigcap_{j \in I} \{\rho_j(z) = 0\}$, by translation in \mathbb{C}^n , we may suppose that $\zeta_0 = 0$, and we have to define the domain \tilde{D} with the properties stated in the proposition.

Take a vector $h \in \mathbb{C}^n$ of norm 1 and set, with $a \cdot b := \sum_{k=1}^n a_k b_k$,

$$\rho(z) := (|z|^2 - r^2)(1 + h \cdot z + \bar{h} \cdot \bar{z}).$$

Because $1 + h \cdot z + \bar{h} \cdot \bar{z} > 0$ near the origin, this is a defining function for the ball centered at 0 and of radius r .

Now the claim is: we can choose the vector h in such a way that $d\rho_j(0)$ and $d\rho(0)$ are linearly independent for $j \in I$.

We have $d\rho(0) = (-r^2)(h \cdot dz + \bar{h} \cdot d\bar{z})$.

We already know that the $d\rho_j(0)$ are linearly independent and span a space of dimension less than $n-3$, so we take h in such a way that the form $(h \cdot dz + \bar{h} \cdot d\bar{z})$ is not in the span of the $d\rho_j(0)$ for $j \in I$. This is independent of the choice of $r > 0$. By continuity this is still true for z in a neighbourhood V of 0 with V independent of $r > 0$ so we now choose $r > 0$ in order that $B(0, r) \subset V$. We extend ρ outside of the ball $B(0, r)$ to be a \mathcal{C}^∞ function $\tilde{\rho}$ in \mathbb{C}^n , $\tilde{\rho} = \rho$ in $B(0, r)$ and strictly positive in $\bar{B}(0, r)^c$ in order for this $\tilde{\rho}$ to be a genuine defining function for $B(0, r)$. So we get the condition (i) in the definition 1.2.

The condition (ii) is easier because we have that

$$\partial\bar{\partial}\tilde{\rho} = (1 + h \cdot z + \bar{h} \cdot \bar{z}) \sum_{k=1}^n dz_k \wedge d\bar{z}_k + h \cdot dz \wedge z \cdot d\bar{z} + \bar{z} \cdot dz \wedge \bar{h} \cdot d\bar{z} = \sum_{k=1}^n dz_k \wedge d\bar{z}_k + \mathcal{O}(|z|).$$

Hence there is a subspace T_z^I such that for $i \in I$ the Levi forms $L\rho_i$ restricted on T_z^I are positive definite by hypothesis and, because $L\tilde{\rho}$ is positive definite everywhere, we have that it is also positive definite on T_z^I which has the right dimension $n - c + 1$.

It is at this point that we need $N \leq n - 3$, because we add the new domain $B(0, r)$. ■

On a theorem of H. Rossi.

We shall use the following lemma.

Lemma 8.4 *Let A, B two self adjoint $n \times n$ matrices such that A has at least $d - c + 1$ strictly positive eigenvalues and $\ker A$ is of dimension $n - d$ and B is positive and has $n - d$ eigenvectors in $\ker A$ associated to strictly positive eigenvalues. Then $A + B$ has at least $n - c + 1$ strictly positive eigenvalues.*

Proof.

Because A is self adjoint, the spaces $\ker A$ and $H := \ker A^\perp$ are invariant for A . Because $\ker A$ has dimension $n - d$ and there is $n - d$ eigenvectors of B in it, then $\ker A$ is generated by these eigenvectors. Hence, because B is self adjoint, this means that $\ker A$ and H are also invariant for B . Set $S := A + B$.

Let $v \in \ker A$ such that $Bv = \lambda v$, $\lambda > 0$, then $Sv = Av + Bv = Bv = \lambda v$; hence on $\ker A$, S has $n - d$ strictly positive eigenvalues.

On H we have $B \geq 0$ and A has at least $d - c + 1$ strictly positive eigenvalues, hence on H we can apply lemma 8.1 and we have that S has at least $d - c + 1$ strictly positive eigenvalues on H . Because H and $\ker A$ have an intersection reduced to $\{0\}$, S has $d - c + 1 + n - d = n - c + 1$ strictly positive eigenvalues. ■

The aim is to extend a theorem by H. Rossi [20] where we replace strictly pseudo convex by strictly c -convex.

Theorem 8.5 *Let M be a closed submanifold of a Stein domain U_0 in \mathbb{C}^n . Suppose there is a neighbourhood U of M and an holomorphic retraction $\pi : U \rightarrow M$. Let D be a strictly c -convex domain in M , $\bar{D} \subset M$.*

Then there is a strictly c -convex domain E in \mathbb{C}^n such that :

- (A) $\bar{E} \subset U \cap U_0$
- (B) $E \cap M = D$
- (C) ∂E cuts M transversely along ∂D
- (D) $\pi : \bar{E} \rightarrow \bar{D}$.

Proof.

I shall copy the main points in the proof by H. Rossi making the necessary changes.

Docquier and Grauert (see [20]) give us a neighbourhood U of \bar{D} in \mathbb{C}^n and a retraction $\pi : U \rightarrow M \cap U$ such that the fibers of π cut transversely $M \cap U$ and are of dimension $n - d$.

We set for $z \in U$ and $j = 1, \dots, n$, $f_j(z) = z_j - \pi_j(z)$. The equations $z - \pi(z) = 0$ define the sub manifold M :

if $z \in M$, $\pi(z) = z$ because π is a retraction on M ; if $z \notin M$, because $\pi(z) \in M$, $z - \pi(z) \neq 0$. Moreover, because the fibers of π cut transversely M at any point ζ of \bar{D} , we have that the jacobian matrix contains a $(n - d) \times (n - d)$ sub determinant which is not 0 at ζ , hence not 0 in a neighbourhood of this point. This means that, by a change of variables, the set $(f_j)_{j=1, \dots, n}$ contains a coordinates system for the fibers of π at any point of \bar{D} , hence at all points of a neighbourhood U_1 of \bar{D} in \mathbb{C}^n . These "explicit" functions replace the one generating the idealsheaf of M used by H. Rossi.

Let ρ be a defining function for D in M , we still follow H. Rossi and we set:

$$\sigma(z) := \rho \circ \pi + A \sum_{j=1}^n |f_j|^2,$$

where the constant A will be chosen later. Because $F(z) := \sum_{j=1}^n |f_j(z)|^2 = 0$ on $M \cap U$, it exists a $\epsilon_0 > 0$ such that $\{F(z) < \epsilon_0\} \cap U \subset U_1$.

It remains to see that σ is strictly c -convex, i.e. $i\partial\bar{\partial}\sigma$ has at least $n - c + 1$ strictly positive eigenvalues.

Fix $\zeta \in \bar{D}$; because D is strictly c -convex $i\partial\bar{\partial}\rho \circ \pi(\zeta)$ has at least $d - c + 1$ strictly positive eigenvalues on the tangent space to M at ζ . Because the set $(f_j)_{j=1, \dots, n}$ contains a coordinates system for the fibers of π we have $i\partial\bar{\partial}(\sum_{j=1}^n |f_j|^2)$ has all, i.e. $n - d$, strictly positive eigenvalues on the tangent space to the fiber $\pi^{-1}\pi(\zeta)$ at ζ .

Because the kernel of $i\partial\bar{\partial}\rho \circ \pi$ is the tangent space to the fiber $\pi^{-1}\pi(\zeta)$, we get, by lemma 8.4, that $i\partial\bar{\partial}\sigma = i\partial\bar{\partial}\rho \circ \pi + i\partial\bar{\partial}(\sum_{j=1}^n |f_j|^2)$ has at least $n - c + 1$ strictly positive eigenvalues. So we have at

least $n - c + 1$ strictly positive eigenvalues at any point of \bar{D} hence also in a neighbourhood V of \bar{D}

in \mathbb{C}^n . Now we take $A\epsilon_0 > \sup_{z \in D} |\rho(z)|$ and we set $E := \{z \in U \cap V :: \sigma(z) < 0\}$; we get exactly as H. Rossi, that E is strictly c -convex and we have all properties of the theorem. ■

We have to get an analogous result in the case where D is a \mathcal{C}^3 c -convex intersection in M .

Theorem 8.6 *Let M be a closed submanifold of a Stein domain U_0 in \mathbb{C}^n . Suppose there is a neighbourhood U of M and an holomorphic retraction $\pi : U \rightarrow M$. Let $D := \bigcap_{k=1}^N D_k$ be a \mathcal{C}^3 c -convex intersection in M , $\bar{D} \subset M$.*

Then there is a \mathcal{C}^3 c -convex intersection $E := \bigcap_{k=1}^N \tilde{D}_k$ in \mathbb{C}^n such that :

- (A) $\bar{E} \subset U \cap U_0$
- (B) $E \cap M = D$
- (C) ∂E cuts M transversely along ∂D
- (D) $\pi : \bar{E} \rightarrow \bar{D}$.

Proof.

By theorem 8.5, and with the same notations, we can extend each D_k by $\tilde{D}_k := \{\tilde{\rho}_k < 0\}$ in \mathbb{C}^n , with $\tilde{\rho}_k := \rho_k \circ \pi + AF$ where ρ_k are the defining function for D_k , such that they fulfil the \mathcal{C}^3 c -convex intersection requirements in M . The point is to see that we can choose A in such a way that $\tilde{D} := \bigcap_{k=1}^N \tilde{D}_k$ fulfils the \mathcal{C}^3 c -convex intersection requirements in \mathbb{C}^n .

First we choose $A\epsilon_0 > \sup_{k=1, \dots, N, z \in D_k} |\rho_k(z)|$ in order to have that all \tilde{D}_k are in the domain of the retraction π as for theorem 8.5.

Fix a point $z_0 \in \tilde{D}$ and take a vector X in \mathbb{C}^n ; then we can decompose it as $X = X_M \oplus X_F$ where X_M is tangent at z_0 to the manifold $\{z :: z - \pi(z) = z_0 - \pi(z_0)\}$, "parallel to M ", and X_F is tangent to the fiber passing through z_0 , $\{z :: \pi(z) = \pi(z_0)\}$, because we know that the fibers are transverse to M , which is still true in a neighbourhood V of \tilde{D} in \mathbb{C}^n . Choose A big enough to have all the \tilde{D}_k in V , the same way we did it in the proof of theorem 8.5.

Now if $z \in \bar{D} \subset M$ we already have that the $\{d\tilde{\rho}_j(z)\}_{j \in I}$ are linearly independent because there we have $F(z) = dF(z) = 0$ hence $d\tilde{\rho}_j(z) = d\rho_j(z)$. So we make the assumption that $z \notin M$. Let $I = (i_1, \dots, i_l)$ and suppose that the $\{d\tilde{\rho}_j\}_{j \in I}$ are not linearly independent, then there is $\lambda \in \mathbb{R}^{|I|}$ such that

$$\exists z \in \tilde{D} :: 0 = \sum_{j \in I} \lambda_j d\tilde{\rho}_j(z) = \sum_{j \in I} \lambda_j d(\rho_j \circ \pi(z)) + \left(\sum_{j \in I} \lambda_j\right) AdF(z).$$

This means that

$$-\left(\sum_{j \in I} \lambda_j\right) AdF(z) = \sum_{j \in I} \lambda_j d(\rho_j \circ \pi(z)). \quad (8.1)$$

Take any vector X tangent to \mathbb{C}^n at z ; then we have $X = X_M \oplus X_F$ and $\langle d(\rho_j \circ \pi)(z), X_F \rangle = 0$ because $\rho_j \circ \pi(\zeta)$ is constant along the fiber $\{\zeta :: \pi(\zeta) = \pi(z)\}$. The same way $\langle dF(z), X_M \rangle = 0$ because $F(z) = |z - \pi(z)|^2$ is constant along $\{\zeta :: \zeta - \pi(\zeta) = z - \pi(z)\}$. So

$$\begin{aligned} -\left(\sum_{j \in I} \lambda_j\right) A \langle dF(z), X \rangle &= -\left(\sum_{j \in I} \lambda_j\right) A \langle dF(z), X_F \rangle = \\ &= \sum_{j \in I} \lambda_j \langle d(\rho_j \circ \pi)(z), X_F \rangle = 0. \end{aligned}$$

Hence for any $X \in \mathbb{C}^n$, $(\sum_{j \in I} \lambda_j) A \langle dF(z), X \rangle = 0$ which means that the 1-form $(\sum_{j \in I} \lambda_j) AdF(z) = 0$

hence, because $dF(z) \neq 0$ for $z \notin M$, we have that $(\sum_{j \in I} \lambda_j) = 0$.

But by (8.1) this implies $\sum_{j \in I} \lambda_j d(\rho_j \circ \pi(z)) = 0$ which means that $\lambda_j = 0$ because the $d\rho_j(\zeta)$ are independent at all points and in particular at the point $\zeta = \pi(z)$. So a contradiction which proves that the $d\tilde{\rho}_k$ are linearly independent.

To have the ii) fix $z \in \bigcap_{j \in I} \{\tilde{\rho}_j \leq 0\}$, and set $\zeta := \pi(z) \in \bar{D}$. The points z, ζ belongs to an open set $U := U_j$ of the covering (U_j, φ_j) done via lemma 3.2, so reading by $\varphi := \varphi_j$ we are in the following situation (I keep the same notations) : we have $z = (z', z'')$, $\zeta = (z', 0)$ and the retraction π is the orthogonal projection $w \rightarrow (w', 0)$ where $w' := (w_1, \dots, w_d)$; $w'' := (w_{d+1}, \dots, w_n)$. The tangent space $T_\zeta(M)$ is just $\{w :: w'' = 0\}$ and by the hypotheses on the $\rho_j(w) = \rho_j(w')$ we know that there is a subspace T_ζ^I , of dimension at least $d - c + 1$, of the tangent space $T_\zeta(M)$ on which the Levi forms $L\rho_j(\zeta)$ are positive definite. Lifting this space $T_\zeta(M)$ at the point z keeping it parallel to itself, call it T_z^I , it still have dimension $d - c + 1$, and because the ρ_j do not depend on w'' , we still have that the Levi form $L(\rho_j \circ \pi)(z)$ on T_z^I is the same as the Levi form $L\rho_j(\zeta)$ on T_ζ^I , so it is positive definite.

Now we have $\tilde{\rho}_k := \rho_k \circ \pi + AF$ and $i\partial\bar{\partial}F$ has all its eigenvalues positive so on T_z^I the Levi form $L\rho_j(z)$ is positive definite by the proof of lemma 8.1. ■

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