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# On estimates for the $\bar{\partial}$ equation in Stein manifolds.

Eric Amar

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## Abstract

We generalize to strictly  $c$ -convex domains in Stein manifold,  $L^r - L^s$  and Lipschitz estimates for the solutions of the  $\bar{\partial}$  equation done by Ma and Vassiliadou for domains in  $\mathbb{C}^n$ . For this we use a Docquier-Grauert holomorphic retraction plus the raising steps method I introduce earlier. This gives results in the case of domains with low regularity,  $\mathcal{C}^3$ , for their boundary.

## 1 Introduction.

The solutions with  $L^r$  and Lipschitz estimates of the equation  $\bar{\partial}u = \omega$ ,  $\bar{\partial}\omega = 0$  revealed to be very important in complex analysis and geometry.

The first results of this kind were done by the use of solving kernels: Grauert-Lieb [8], Henkin [9], Ovreid [15], Skoda [17], Krantz [12], in the case of strictly pseudo-convex domains with  $\mathcal{C}^\infty$  smooth

boundary in  $\mathbb{C}^n$ , with the exception of Kerzman [11] in the case of  $(0, 1)$  forms in strictly pseudoconvex domains with  $\mathcal{C}^4$  smooth boundary in Stein manifolds.

Here we shall be interested in strictly  $c$ -convex, s.c.c. for short, domain  $D$  in a complex manifold. Such a domain is defined by a function  $\rho$  of class  $\mathcal{C}^3$  in a neighbourhood  $U$  of  $\bar{D}$  and such that  $\partial\bar{\partial}\rho$  has at least  $n - c + 1$  strictly positive eigenvalues in  $U$ .

These domains in  $\mathbb{C}^n$  have been studied in the case of smooth  $\mathcal{C}^\infty$  boundary by Fisher and Lieb [7]. They still use kernels method.

Definitive results were obtained in a very general framework by Beals, Greiner et Stanton [5]. They used the technology, heavy but extremely powerful, of pseudo-differentials operators. They study domains relatively compact with smooth  $\mathcal{C}^\infty$  boundary in complex manifold which are more general than s.c.c..

Ma and Vassiliadou [14] got very nice estimates even in the case of intersections of s.c.c. domains with  $\mathcal{C}^3$  boundary. I shall use their results here.

Quite recently C. Laurent-Thiébaut [13] got this kind of result for s.c.c. domains with smooth  $\mathcal{C}^\infty$  boundary in complex manifold by use of the method of "bumps", as was done by Kerzman [11].

Let us state our main result which is completely analogous to the one Ma and Vassiliadou [14] got for domains in  $\mathbb{C}^n$ .

**Theorem 1.1** *Let  $\Omega$  a Stein manifold of dimension  $n$  and a strictly  $c$ -convex (s.c.c.) domain  $D$  such that  $D$  is relatively compact with smooth  $\mathcal{C}^3$  boundary in  $\Omega$ . Let  $\omega$  a  $(p, q)$  form in  $L^r_{p,q}(D)$ ,  $\bar{\partial}\omega = 0$  with  $1 < r < 2n + 2$ ,  $c \leq q \leq n$ . Then there is a  $(p, q - 1)$  form  $u$  in  $L^s(D)$ , with  $\frac{1}{s} = \frac{1}{r} - \frac{1}{2n + 2}$ , such that  $\bar{\partial}u = \omega$ .*

*If  $\omega$  is in  $L^r_{p,q}(D)$ ,  $\bar{\partial}\omega = 0$  with  $r \geq 2n + 2$ ,  $c \leq q \leq n$ , then there is a  $(p, q - 1)$  form  $u$  in  $\Lambda^\epsilon_{(p,q-1)}(\bar{D})$  such that  $\bar{\partial}u = \omega$  with  $\epsilon = \frac{1}{2} - \frac{n + 1}{r}$ .*

The spaces  $\Lambda^\epsilon_{(p,q-1)}(\bar{D})$  are the (isotropic) Lipschitz spaces of order  $\epsilon$  and we set  $\Lambda^0_{(p,q-1)}(\bar{D}) := L^\infty_{(p,q-1)}(D)$ .

It has to be noticed that the boundary regularity is just  $\mathcal{C}^3$ , which is allowed by kernels. So it seems that this is a new result in a Stein manifold for such a low regularity. In the case of a  $\mathcal{C}^\infty$  boundary regularity then this result is contained in Beals, Greiner et Stanton [5] one, but the proof is completely different and, in some sense, "lighter" because it uses for the analytic part kernels methods plus essentially geometric ones.

The first result we get is a non optimal theorem.

**Theorem 1.2** *Let  $M$  be a closed submanifold of dimension  $d$  of a Stein domain  $U_0$  in  $\mathbb{C}^n$ . Let  $D$  be a s.c.c. domain relatively compact in  $M$  ( $\bar{D} \subset M$ ) with  $\mathcal{C}^3$  boundary. Then, with  $r \geq 2n + 2$ , we can solve in  $D$ ,  $\bar{\partial}u = \omega$  when  $\bar{\partial}\omega = 0$  and with  $u \in \Lambda^\epsilon_{(p,q-1)}(\bar{D})$  if  $\omega \in L^r_{(p,q)}(D)$ ,  $\bar{\partial}\omega = 0$ ,  $c \leq q \leq n$ , with  $\epsilon = \frac{1}{2} - \frac{n + 1}{r}$ .*

The results of Ma and Vassiliadou [14] gives good estimates in case of domains in  $\mathbb{C}^n$ . The point here was to pass from  $\mathbb{C}^n$  to a submanifold of  $\mathbb{C}^n$ . To do this I was inspired by a nice paper of H. Rossi [16] on Docquier Grauert holomorphic retraction.

Then we use the raising steps method [2] (see also [4], [3]) and to use it we need a global result and we use these  $L^r - \Lambda^\epsilon$  estimates as a threshold. We get the same optimal results as for domains in  $\mathbb{C}^n$ .

So we get

**Theorem 1.3** *Let  $M$  be a complex submanifold of dimension  $d$  in  $\mathbb{C}^n$  and a s.c.c. domain  $D$  such that  $D$  is relatively compact with smooth boundary of class  $\mathcal{C}^3$  in  $M$ . Let  $\omega$  a  $(p, q)$  form in  $L^r_{p,q}(D)$ ,  $\bar{\partial}\omega = 0$ ,  $c \leq q \leq n$ , with  $1 < r < 2d + 2$ . Then there is a  $(p, q - 1)$  form  $u$  in  $L^s(D)$ , with  $\frac{1}{s} = \frac{1}{r} - \frac{1}{2d + 2}$ , such that  $\bar{\partial}u = \omega$ .*

*If  $r \geq 2n + 2$  then there is a  $(p, q - 1)$  form  $u$  in  $\Lambda^\epsilon_{(p,q-1)}(\bar{D})$  such that  $\bar{\partial}u = \omega$  with  $\epsilon = \frac{1}{2} - \frac{d + 1}{r}$ .*

To pass to Stein manifold, we use an embedding theorem of Bishop and Narashiman (see theorem 5.3.9. of Hörmander [10]) to see an abstract Stein manifold of dimension  $d$  as a submanifold of  $\mathbb{C}^{2d+1}$ . So we get our main result.

This work will be presented in the following way.

- First we recall the estimates in the case of strictly  $c$ -convex domains in  $\mathbb{C}^n$  done by Ma and Vassiliadou [14].
- We recall the Docquier Grauert holomorphic retraction on a complex submanifold  $M$  of  $\mathbb{C}^n$ .
- We extend a form  $\omega$  from a domain  $D$  s.c.c. in  $M$  to a domain  $E$  s.c.c. in  $\mathbb{C}^n$  by use of a generalization of a theorem of H. Rossi [16]. We then solve the form in  $E$  by the known estimates in  $\mathbb{C}^n$ .
- We show that the solution in  $E$  can be restricted to  $D$  to get a solution in  $D$  with good enough estimates, for  $r \geq 2n + 2$ . This gives theorem 1.2.
- We use the raising steps theorem with the threshold given by theorem 1.2. So we have theorem 1.3 for the case of a submanifold of  $\mathbb{C}^n$ .
- By use of a theorem of Bishop and Narashiman, i.e. the proper embedding of a Stein manifold of dimension  $d$  in  $\mathbb{C}^{2d+1}$ , we get our main theorem 1.1 for any Stein manifold.

## 2 Strictly $c$ -convex domain in $\mathbb{C}^n$ .

We shall use the nice estimates for a smoothly  $\mathcal{C}^3$  bounded  $c$  convex domains in  $\mathbb{C}^n$  obtained by Ma and Vassiliadou [14], lemma 5.3.

**Theorem 2.1** *Let  $D$  be a bounded s.c.c. domain in  $\mathbb{C}^n$  with a  $\mathcal{C}^3$  defining function. Then*

$$\forall \omega \in L^r_{(p,q)}, \bar{\partial}\omega = 0, c \leq q \leq n, 1 \leq r < 2n + 2,$$

*there exists  $u \in L^s_{(p,q-1)}(D)$ ,  $\frac{1}{s} = \frac{1}{r} - \frac{1}{2n + 2}$ , with the following properties:*

*i)  $\bar{\partial}u = \omega$  in the sense of currents in  $D$ ,*

*ii) if  $r = 1$ ,  $u \in L^{\frac{2n+2}{2n+1}-\eta}_{(p,q-1)}$  for any  $\eta > 0$ .*

*iii) if  $2n + 2 \leq r \leq \infty$ ,  $u \in \Lambda^\epsilon_{(p,q-1)}(\bar{D})$  with  $\epsilon = \frac{1}{2} - \frac{n + 1}{r}$ .*

### 3 The Docquier - Grauert holomorphic retraction.

We have the Docquier-Grauert lemma [6] :

**Lemma 3.1** *Let  $K$  be a compact subset of a closed complex submanifold  $M$  of  $\mathbb{C}^n$ . There is a neighbourhood  $U$  of  $K$  and a holomorphic map  $\pi : U \rightarrow U \cap M$  such that  $\pi(\zeta) = \zeta$  for  $\zeta \in U \cap M$ .*

In fact we have more (Rossi [16], p 172) from the argument of Docquier-Grauert we have that the fibers  $\pi^{-1}\pi\zeta$  of  $\pi$  intersect  $M$  transversally at all points of  $M$  and are of dimension  $n - d$ .

Let  $M$  be a complex manifold of dimension  $d$  in  $\mathbb{C}^n$  and  $D$  a relatively compact domain strictly  $c$ -convex in  $M$ . We have the following lemma.

**Lemma 3.2** *Let  $\zeta \in \bar{D}$ , there is a neighborhood  $U$  of  $\zeta$  in  $\mathbb{C}^n$  and a bi-holomorphic application  $(U, \varphi)$ ,  $\varphi : U \rightarrow T$  such that, with  $z = (z_1, \dots, z_n)$  the coordinates in  $T$ , we have:  $\varphi(D) \cap T = \{z_{d+1} = \dots = z_n = 0\}$  and the retraction  $\tilde{\pi} := \varphi \circ \pi \circ \varphi^{-1}$  read in the application  $\varphi$  is given by  $\tilde{\pi}(z) = (z_1, \dots, z_d, 0, \dots, 0)$ , i.e. this is the orthogonal projection onto the subspace of  $z' := (z_1, \dots, z_d)$ . Moreover one can choose for  $T$  a tube around  $\varphi(M)$  of width  $\delta > 0$ .*

Proof.

The manifold  $M$  is given, by use of the retraction  $\pi$ , by the functions  $f_k(\zeta) := \zeta_k - \pi_k(\zeta)$ ,  $k = 1, \dots, n$ . We have if  $\zeta \in M$ ,  $\zeta - \pi(\zeta) = 0$  ; if  $\zeta \notin M$ ,  $\zeta - \pi(\zeta) \neq 0$ , because  $\pi(\zeta) \in M$ . The transversality of the fibers with respect to  $M$  at all points of  $\bar{D}$  insures that the Jacobian of the application  $f = (f_1, \dots, f_n)$  has rank  $n - d$ , which is the complex co-dimension of  $M$ . Take a point  $\zeta^0 \in \bar{D}$ , there are  $n - d$  functions  $f_j$  which are independent in a neighborhood  $U$  of  $\zeta^0$ . Re-numerating the functions  $f_j$  and the variables  $\zeta_k$ , we may suppose that the determinant  $(\frac{\partial f_j}{\partial \zeta_k})_{j,k=d+1,\dots,n}$  is different from zero.

Now we shall make the change of variables  $z = \varphi(\zeta)$  with  $z_j = \zeta_j$ ,  $j = 1, \dots, d$  ;  $z_j = f_j(\zeta)$ ,  $j = d + 1, \dots, n$ . This is actually a change of variables because the Jacobian of  $\varphi$  is different from zero in the open set  $U$ . We have that the application  $\varphi$  is a bi-holomorphism from the open set  $U$  onto the open set  $T := \varphi(U)$ .

Let  $z' = (z_1, \dots, z_d)$  and  $z'' = (z_{d+1}, \dots, z_n)$  ; we have in  $T$  that:

$$N := \varphi(M) = \{z = (z', z'') \in T :: z'' = 0\}.$$

Now take a tube around  $N$  centered in  $\zeta^0$ ,  $T = \{z = (z', z'') :: z'' \in B((z', 0), \delta)\}$ , we call it again  $T$ , and we still denote  $U = \varphi^{-1}(T)$ .

We cover  $\bar{D}$  by a finite number of these bi-holomorphisms  $(U_j, \varphi_j)$ . We note  $N_j$  the manifold  $N_j := \varphi_j(M \cap U_j) \subset T_j$  and, diminishing a little bit the  $U_j$  if necessary, we can suppose that the width of the tubes  $T_j$  around the  $N_j$  is constant and equals  $\delta > 0$ . We know that there is a constant  $\mu > 0$  such that  $\mu^{-1} < J_j < \mu$ , where  $J_j$  is the Jacobian of  $\varphi_j$ , because there is a finite number of charts  $(U_j, \varphi_j)$ .

We denote  $dV$  the Lebesgue measure on the manifold  $M$ .

We have the following basic lemma.

**Lemma 3.3** *Let  $f$  be a function in  $L^1(U_j)$  and  $\tilde{f}$  this function read in the application  $\varphi_j$ , i.e.  $\tilde{f} := f \circ \varphi_j^{-1}$ , we get*

$$\int_{U_j} f(\zeta) dm(\zeta) = \int_{N_j} \int_{B((z',0),\delta)} \tilde{f}(z', z'') J_j(z', z'') dm(z'') dV(z').$$

Proof.

This is simply the change of variables formula because  $\varphi_j(U_j) = T_j = N_j \times B(\cdot, \delta)$  and the Jacobian of  $\varphi_j$  is  $J_j$ .

**Lemma 3.4** *Let  $f$  be a measurable function, positive on  $M$ , then*

$$\int_{U_j} f \circ \pi(\zeta) dm(\zeta) \leq \mu c(\delta) \int_{N_j} f(z', 0) dV(z'),$$

*with  $dm$  the Lebesgue measure on  $\mathbb{C}^n$ ,  $dV$  the Lebesgue measure on  $M$  and  $c(\delta) := |B(x, \delta)|$ .*

Proof.

We can apply lemma 3.4 with the notation  $z = (z', z'')$ ,  $z'$  the coordinates in  $N_j$ ,  $z''$  the coordinates in the fibers:

$$\int_{U_j} f \circ \pi(\zeta) dm(\zeta) = \int_{N_j} \left\{ \int_{B(z', \delta)} \tilde{f}(z', z'') J_j(z', z'') dm(z'') \right\} dV(z'),$$

but here we have  $\tilde{f}(z', z'') = f(z', 0)$  because  $\tilde{\pi}(z) = (z', 0)$  hence the formula is now:

$$\int_{U_j} f \circ \pi(z) dm(z) \leq \mu \int_{N_j} f(z', 0) |B(z', \delta)| dV(z') = \mu c(\delta) \int_{N_j} f(z', 0) dV(z').$$

So we get the lemma. ■

We notice that the open set  $U := \bigcup_{j=1}^N U_j$  contains  $\bar{D}$ .

The theorem 7.4 in the appendix, which generalizes to s.c.c. domains a theorem by Rossi [16] done for strictly pseudo convex domains, gives us the existence of the strictly  $c$ -convex domain  $E$  in  $\mathbb{C}^n$  such that  $\pi : \bar{E} \rightarrow \bar{D}$ . Now on we fix this s.c.c. domain  $E$ .

## 4 Extension of the form $\omega$ .

Let  $\omega$  be a  $(p, q)$  form in  $L^r(D)$ ,  $\bar{\partial}$  closed ; we extend it in  $E$  by use of the retraction  $\pi$  in the following manner :  $\tilde{\omega} := \pi^* \omega$ .

**Lemma 4.1** *We have  $\bar{\partial} \tilde{\omega} = 0$ . Moreover if  $\omega \in L^r(D)$  we have  $\tilde{\omega} \in L^r(E)$  with  $\|\tilde{\omega}\|_{L^r(E)} \leq \mu c(\delta) \|\omega\|_{L^r(D)}$ .*

Proof.

Because the retraction  $\pi$  is holomorphic we get  $\bar{\partial} \tilde{\omega} = \pi^* \bar{\partial} \omega = 0$ . Moreover the lemma 4.2 gives that  $\tilde{\omega}$  is still in  $L^r(E)$  ; we start by extending  $\omega$  to  $U \cap M$  by zero outside  $\bar{D}$  ; we had that the coefficients of  $\tilde{\omega}$  can be written  $f \circ \pi$  hence, applying lemma 4.2 to the functions  $|f \circ \pi|^r$  we get  $\|\tilde{\omega}\|_{L^r(U_j)} \leq \mu c(\delta) \|\omega\|_{L^r(N_j)}$ . We have only a finite number of open sets  $U_j$ , so we get  $\|\tilde{\omega}\|_{L^r(U)} \lesssim \mu c(\delta) \|\omega\|_{L^r(D)}$ . Because  $E \subset U$  we get  $\|\tilde{\omega}\|_{L^r(E)} \leq \mu c(\delta) \|\omega\|_{L^r(D)}$ . ■

Now  $E$  is s.c.c. in  $\mathbb{C}^n$  and, with  $r \geq 2n + 2$ , we can solve the  $\bar{\partial}$  in the space  $L^\infty(E)$  :  $\bar{\partial}\tilde{u} = \tilde{\omega}$ ,  $\tilde{u} \in L^\infty(E)$  by the theorem 2.1. Fix  $\omega \in L^r_{(p,q)}(D)$ ,  $\bar{\partial}\omega = 0$ ,  $\tilde{\omega}$  as above then we have  $\tilde{u} \in L^\infty(E)$  also fixed.

We shall need the lemma

**Lemma 4.2** *In an open set  $U_j$  of our covering, there is  $(p, q)$  form  $\tilde{v}_j$  such that  $\bar{\partial}\tilde{v}_j = \bar{\partial}_M\tilde{v}_j = \tilde{\omega}$ . This means that the  $\bar{\partial}$  of the form  $\tilde{v}_j$  read in  $(U_j, \varphi_j)$  does not contain any  $d\bar{z}''_k$ . Moreover the coefficients of  $\tilde{v}_j$  are bounded in  $U_j$  and holomorphic in the fibers of  $\pi$ .*

Proof.

We work directly in  $T = T_j$  by use of the bi-holomorphism  $\varphi_j$ .

The first part is coming from the fact that  $\tilde{\omega}$  read in  $(U, \varphi)$  does not contain any form  $d\bar{z}''_l$ . Suppose that  $\tilde{u}$  contains such a form, we have :  $\tilde{u} = v_l d\bar{z}''_l + \Gamma_l$  with  $\Gamma_l$  not containing  $d\bar{z}''_l$  ; notice that the  $\Gamma_l$  are unique and so linear in  $\tilde{u}$ . Hence, keeping the notation  $\tilde{u}$  for its reading in  $(U, \varphi)$ ,

with the notation  $\bar{\partial}_{z'_k}\tilde{u} = \frac{\partial\tilde{u}}{\partial\bar{z}''_k}$ ,

$$\forall k = 1, \dots, d, \bar{\partial}_{z'_k}\tilde{u} = \bar{\partial}_{z'_k}v_l \wedge d\bar{z}''_l + \bar{\partial}_{z'_k}\Gamma_l$$

cannot have terms in  $d\bar{z}''_l$  because  $\tilde{\omega}$  has not, hence we get  $\bar{\partial}_{z'_k}v_l \wedge d\bar{z}''_l = 0$ . So we get :

$$\bar{\partial}_{z'}(\tilde{u} - \sum_{l=d+1}^n v_l \wedge d\bar{z}''_l) = \tilde{\omega}.$$

on the other hand we get :  $\bar{\partial}_{z''_k}\tilde{u} = \bar{\partial}_{z''_k}v_l \wedge d\bar{z}''_l + \bar{\partial}_{z''_k}\Gamma_l$  and, because there are no terms of the form  $d\bar{z}''_k \wedge d\bar{z}''_l$  in  $\tilde{\omega}$ , we get necessarily  $\forall l, k = d+1, \dots, n$ ,  $\bar{\partial}_{z''_k}v_l \wedge d\bar{z}''_l = 0$ . So the  $\Gamma_l$  are holomorphic in

the variables  $z''$ . Hence we showed  $\tilde{v}_j := \tilde{u} - \sum_{l=d+1}^n v_l \wedge d\bar{z}''_l = \sum_{l=d+1}^n \Gamma_l$  is still a solution of  $\bar{\partial}\tilde{v}_j = \tilde{\omega}$

and, because the coefficients of  $\tilde{u}$  are bounded, we get all the properties stated in the lemma. We still notice that  $\tilde{v}_j$  is linear with respect to  $\tilde{u}$ . ■

**Lemma 4.3** *There is a  $(p, q)$  form  $\tilde{v}$  such of that, in  $E$ ,  $\bar{\partial}\tilde{v} = \bar{\partial}_M\tilde{v} = \tilde{\omega}$ . Moreover the coefficients of  $\tilde{v}$  are bounded in  $E$  and holomorphic on the fibers  $\forall x \in D$ ,  $F_x := \pi^{-1}(x)$ .*

Proof.

Let us take two open sets of our covering:  $U_j, U_k$  ; the lemma 4.2 gives us:

$$\tilde{u} = v_j \wedge d\bar{z}'' + \tilde{v}_j \text{ in } U_j \text{ with } \tilde{v}_j \text{ without any } d\bar{z}''_l.$$

The same way in the open set  $U_k$  we have :

$$\tilde{u} = v_k \wedge d\bar{z}'' + \tilde{v}_k \text{ in } U_k \text{ with } \tilde{v}_k \text{ without any } d\bar{z}''_l.$$

So in the intersection of the two sets, because  $\tilde{u}$  is global, choosing one of these two systems of coordinates and with clear notations, we get:

$$(v_j - v_k) \wedge d\bar{z}'' + (\tilde{v}_j - \tilde{v}_k) = 0 \text{ in } U_j \cap U_k .$$

Because the  $\tilde{v}_j$  do not contain the  $d\bar{z}''_l$ , we necessarily have  $\tilde{v}_j = \tilde{v}_k$  and  $v_j = v_k$ . So the forms  $\tilde{v}_j$  make a global form  $\tilde{v}$  just posing  $\tilde{v} := \tilde{v}_j$  in  $U_j$ . ■

## 5 Estimates in the case of a submanifold of $\mathbb{C}^n$ .

We shall show the following theorem :

**Theorem 5.1** *Let  $M$  be a complex submanifold of dimension  $d$  in  $\mathbb{C}^n$  and a s.c.c. domain  $D$  such that  $D$  is relatively compact with smooth  $\mathcal{C}^3$  boundary in  $M$ . Let again  $\omega$  be a  $(p, q)$  form in  $L^r_{p,q}(D)$ ,  $\bar{\partial}\omega = 0$  with  $r \geq 2n + 2$ ,  $c \leq q \leq n$ . Then there is a  $(p, q - 1)$  form  $u$  in  $\Lambda^{\epsilon}_{(p,q-1)}(\bar{D})$  such that  $\bar{\partial}u = \omega$ .*

Proof.

We restrict the function  $\tilde{v}$  got in the lemma 4.3 to  $D$ ,  $u := \tilde{v}|_D$ , and we need to see that, on  $D$ ,  $\bar{\partial}u = \omega$ . It is enough to see this in an open set  $U_j$  of our covering.

We shall use that  $\bar{\partial}\tilde{v} = \tilde{\omega}$  in the distributions sense, with  $\tilde{v} \in L^\infty(E)$ . So let  $\chi \in \mathcal{D}_{(n-p, n-q)}(U_j \cap D)$ ; we get

$$\langle \bar{\partial}u, \chi \rangle_D = (-1)^{q+1} \langle u, \bar{\partial}\chi \rangle_D;$$

with the scalar product of the manifold  $M$ . We read these data in  $(U_j, \varphi_j)$  and we keep the same names.

We extend the form  $\chi$  from  $N_j = U_j \cap M$  to the whole  $U_j$  in making it constant with respect to  $z''$ . Let us denote  $\tilde{\chi}$  this extended form.

Because the form  $\tilde{v} = \tilde{v}_j$  in  $T_j$  is holomorphic in the variables  $z''$ , its values in  $(z', z'')$  is the mean value of its values in the ball centered at  $(z', 0) \in N_j$  and with radius  $\delta$  in  $z''$ . The lemma 3.4 gives:

$$\int_{T_j} (\tilde{v}, \bar{\partial}\tilde{\chi}) dm = \int_{N_j} \left\{ \int_{B(z', \delta)} (\tilde{v}, \bar{\partial}\tilde{\chi})(z', z'') dm(z'') \right\} dV(z),$$

but

$$\int_{B(z', \delta)} (\tilde{v}, \bar{\partial}\tilde{\chi})(z', z'') dm(z'') = \int_{B(z', \delta)} \sum_{I,J} \tilde{v}_{I,J}(z', z'') \tilde{\chi}'_{I^c J^c}(z', z'') dm(z'')$$

where  $\tilde{\chi}' = \bar{\partial}\tilde{\chi}$ . Because the form  $\tilde{\chi}$  is compactly supported in  $N_j$  and is constant on the fibers, we get

$$\int_{B(z', \delta)} \sum_{I,J} \tilde{v}_{I,J}(z', z'') \tilde{\chi}'_{I^c J^c}(z', z'') dm(z'') = \sum_{I,J} \chi'_{I^c J^c}(z') \int_{B(z', \delta)} \tilde{v}_{I,J}(z', z'') dm(z'');$$

So we get, because  $\int_{B(z', \delta)} \tilde{v}_{I,J}(z', z'') dm(z'') = c(\delta) u_{I,J}(z')$ , with  $c(\delta) := |B(z', \delta)|$ ,

$$\langle \tilde{v}, \bar{\partial}\tilde{\chi} \rangle_{T_j} := \int_{T_j} (\tilde{v}, \bar{\partial}\tilde{\chi}) dm = c(\delta) \int_{N_j} (u, \bar{\partial}\chi)(x) dV(x) = c(\delta) \langle u, \bar{\partial}\chi \rangle_{N_j}.$$

keeping the same notations, we come back to the open set  $U_j$  and, because  $\chi$  is compactly supported in  $N_j = M \cap U_j$ , we get, because  $T_j = \varphi_j(U_j)$  is a tube around  $N_j$ :

$$\langle \tilde{v}, \bar{\partial}\tilde{\chi} \rangle := c(\delta) \langle u, \bar{\partial}\chi \rangle_D.$$

exactly by the same way we get:

$$\langle \bar{\partial}\tilde{v}_j, \tilde{\chi} \rangle = c(\delta) \langle \bar{\partial}u, \chi \rangle_D,$$

And, because  $\tilde{\omega}$  is constant on the fibers, we get

$$\langle \tilde{\omega}, \tilde{\chi} \rangle = c(\delta) \langle \omega, \chi \rangle_D.$$

Recall that  $\bar{\partial}\tilde{v} = \tilde{\omega}$  in the distributions sense, and, because  $\tilde{\omega}$  is a current in  $L^r_{p,q}(E)$ , this is also true for  $\bar{\partial}\tilde{v}$ . Hence we have, because  $\tilde{\chi} \in L^{r'}_{n-p, n-q}(E)$ , with  $r'$  the conjugate exponent of  $r$ , that  $\langle \bar{\partial}\tilde{v}, \tilde{\chi} \rangle = \langle \tilde{\omega}, \tilde{\chi} \rangle$  is well defined, even if  $\tilde{\chi}$  is not compactly supported. The previous equalities give  $\langle \bar{\partial}u, \chi \rangle_D = \langle \omega, \chi \rangle_D$ . Because this is true for all  $\mathcal{C}^\infty$  functions compactly supported  $\chi$  in an open set  $T_j$  of a covering of  $\bar{D}$ , we get  $\bar{\partial}u = \omega$  in the distributions sense on  $D$ , with  $u \in \Lambda^{\epsilon}_{(p,q-1)}(\bar{D})$ . ■

**Remark 5.2** *We have no such estimates in the case  $r < 2n + 2$  because the mean value in a ball of the fiber of a function in  $L^s$  in  $E$  is no longer in  $L^s(D)$  for  $s < \infty$ .*



Now we are in position to apply the raising steps theorem.

**Theorem 5.3** *Let  $M$  be a complex submanifold of dimension  $d$  in  $\mathbb{C}^n$  and  $D$  be a s.c.c. domain which is a relatively compact with smooth  $\mathcal{C}^3$  boundary in  $M$ . Let  $\omega$  be a  $(p, q)$  form in  $L_{p,q}^r(D)$ ,  $\bar{\partial}\omega = 0$  with  $1 < r < 2d + 2$ ,  $c \leq q \leq n$ . Then there is a  $(p, q - 1)$  form  $u$  in  $L^s(D)$ , with  $\frac{1}{s} = \frac{1}{r} - \frac{1}{2d + 2}$ , such that  $\bar{\partial}u = \omega$ .*

Proof.

In order to have the local result for all points in  $\bar{D}$  we use the same method as in [2], but with the proposition 7.2 and the results of Ma and Vassiliadou [14].

We have the global result : if  $\omega \in L_{p,q}^{2n+2}(D)$ ,  $\bar{\partial}\omega = 0$  then we have a solution  $u$  in  $L_{(p,q-1)}^\infty(D)$  such that  $\bar{\partial}u = \omega$  by use of theorem 5.1. Now we take  $\omega \in L_{p,q}^r(D)$ ,  $\bar{\partial}\omega = 0$ , then we have that the optimal exponent for the solution  $u$  is  $s$  such that  $\frac{1}{s} = \frac{1}{r} - \frac{1}{2d + 2}$ , then we choose any  $r_0 > \max(2n + 2, s)$  as a threshold and, because the correction to the solution  $u$  is in  $L_{p,q-1}^\infty(D)$ , and  $D$  is bounded, we have that  $u \in L_{(p,q-1)}^s(D)$ , by the raising steps theorem [2], and this ends the proof of the theorem. ■

## 6 Estimates in the case of a Stein manifold.

We can apply a theorem of Bishop and Narashiman (see theorem 5.3.9. of Hörmander [10]) which tells us that, if  $\Omega$  is a Stein manifold of dimension  $d$ , there is an element  $f \in \mathcal{H}(\Omega)^{2d+1}$  which defines a regular injective and proper application from  $\Omega$  in  $\mathbb{C}^{2d+1}$ . Denote  $M := f(\Omega)$ ; if  $D'$  is the strictly  $c$ -convex domain in  $\Omega$ , relatively compact in  $\Omega$ , then its image  $D = f(D')$  is a strictly  $c$ -convex domain in  $M$ . We can apply theorem 5.3 and we get:

**Theorem 6.1** *Let  $\Omega$  be a Stein manifold of dimension  $n$  and a strictly  $c$ -convex domain  $D$  such that  $D$  is relatively compact with smooth  $\mathcal{C}^3$  boundary in  $\Omega$ . Let  $\omega$  a  $(p, q)$  form in  $L_{p,q}^r(D)$ ,  $\bar{\partial}\omega = 0$  with  $1 < r < 2n + 2$ ,  $c \leq q \leq n$ . Then there is a  $(p, q - 1)$  form  $u$  in  $L^s(D)$ , with  $\frac{1}{s} = \frac{1}{r} - \frac{1}{2n + 2}$ , such that  $\bar{\partial}u = \omega$ .*

*Let again  $\omega$  be a  $(p, q)$  form in  $L_{p,q}^r(D)$ ,  $\bar{\partial}\omega = 0$  with  $r \geq 2n + 2$ ,  $c \leq q \leq n$ . Then there is a  $(p, q - 1)$  form  $u$  in  $\Lambda_{(p,q-1)}^\epsilon(\bar{D})$  such that  $\bar{\partial}u = \omega$ .*

We notice that there is no hypothesis here in the case  $r > 2$  on the compactness of the support of the form  $\omega$ , in contrast to the previous results we had in [2].

## 7 Appendix.

**Lemma 7.1** *Let  $A, B$  be two self adjoint matrices such that  $A$  has at least  $n - c + 1$  eigenvalues strictly positive and  $B$  is positive. Then  $A + B$  has at least  $n - c + 1$  eigenvalues strictly positive.*

Proof.

Let  $E$  the space generated by the eigenvectors associated to the strictly positive eigenvalues of  $A$ . Then  $E$  has dimension at least  $n - c + 1$ . Let  $S := A + B$ , because  $B$  is positive, we get

$$\forall x \in E, \langle Sx, x \rangle = \langle Ax, x \rangle + \langle Bx, x \rangle > 0.$$

Now let  $e_1, \dots, e_k$  be the eigenvectors associated to the negative eigenvalues of  $S$ . We set  $F = \text{span}\{e_1, \dots, e_k\}$  and we have that  $F$  is invariant by  $S$  and we have  $\forall x \in F, \langle Sx, x \rangle \leq 0$ . If the space  $G := E \cap F$  is of non zero dimension, we get  $\forall x \in G, x \neq 0, \langle Sx, x \rangle > 0$  and  $\langle Sx, x \rangle \leq 0$  so a contradiction. Hence  $\dim G = 0$  and  $\dim F \leq \text{codim} E = c - 1$ , which means that  $S$  has a least  $n - c + 1$  strictly positive eigenvalues. ■

The next proposition generalizes the one in [1], proposition 1.1, done in the pseudo convex case.

**Proposition 7.2** *Let  $D$  be a strictly  $c$ -convex domain with  $\mathcal{C}^3$  boundary in  $\mathbb{C}^n$ . Let  $\zeta \in \partial D$ ,  $U$  a neighbourhood of  $\zeta$  in  $\mathbb{C}^n$  and  $B(\zeta, r)$  a ball centered at  $\zeta$  and of radius  $r$ ; then there is a domain  $\tilde{D}$ ,  $c$ -convex and with  $\mathcal{C}^3$  boundary such that we have  $\tilde{D} \subset U$  and  $\partial D \cap B(\zeta, r) = \partial \tilde{D} \cap B(\zeta, r)$ .*

Proof.

Let  $\rho$  be a defining function for  $D$ . Let  $\zeta \in \partial D$  and  $U$  a neighbourhood of  $\zeta$  in  $\mathbb{C}^n$ . Consider now a positive convex increasing function  $\chi$  defined on  $\mathbb{R}^+$ ,  $\mathcal{C}^\infty$  such that  $\chi = 0$  in  $(0, r)$ . Set  $\tilde{\rho}(z) := \rho(z) + a\chi(|z - \zeta|^2)$ ; we have  $\partial\bar{\partial}\tilde{\rho} = \partial\bar{\partial}\rho + a\partial\bar{\partial}\chi$ . But, as is easily seen,  $\partial\bar{\partial}\chi$  is positive at each point  $z$ , hence, setting  $A = i\partial\bar{\partial}\rho$ ,  $B = ai\partial\bar{\partial}\chi$ , we can apply lemma 7.1 and we have that the domain  $\tilde{D} := \{\tilde{\rho} < 0\}$  is also  $c$ -convex with smooth  $\mathcal{C}^3$  boundary.

Now we choose  $r$  small enough to have  $B(\zeta, 3r) \subset U$ . We have  $\tilde{\rho}(z) < 0 \Rightarrow \rho(z) < -a\chi(|z - \zeta|^2)$ ; so we set :

$\alpha := \sup_{z \in D} -\rho(z) < \infty$ , by the compactness of  $\bar{D}$  and  $\beta := \inf_{z \in U \setminus B(\zeta, 2r)} \chi(|z - \zeta|^2) = 4r^2$  then with  $a := \frac{\alpha + 1}{\beta}$  we get that  $\{\tilde{\rho}(z) < 0\} \subset U$  because if not  $\exists z \notin B(\zeta, 3r) :: \rho(z) < -a\chi(|z - \zeta|^2) < -(\alpha + 1)$  which is not possible.

Of course in the ball  $B(\zeta, r)$  we have  $\partial D \cap B(\zeta, r) = \partial \tilde{D} \cap B(\zeta, r)$ . ■

### On a theorem of H. Rossi.

We shall use the following lemma.

**Lemma 7.3** *Let  $A, B$  two self adjoint  $n \times n$  matrices such that  $A$  has at least  $d - c + 1$  eigenvalues strictly positive and  $\ker A$  is of dimension  $n - d$  and  $B$  is positive and has  $n - d$  eigenvectors in  $\ker A$  associated to strictly positive eigenvalues. Then  $A + B$  has at least  $n - c + 1$  strictly positive eigenvalues.*

Proof.

Because  $A$  is self adjoint, the spaces  $\ker A$  and  $H := \ker A^\perp$  are invariant for  $A$ . Because  $\ker A$  has dimension  $n - d$  and there is  $n - d$  eigenvectors of  $B$  in it, then  $\ker A$  is generated by these eigenvectors hence because  $B$  is self adjoint this means that  $\ker A$  and  $H$  are also invariant for  $B$ . Set  $S := A + B$ .

Let  $v \in \ker A$  such that  $Bv = \lambda v$ ,  $\lambda > 0$ , then  $Sv = Av + Bv = Bv = \lambda v$  hence on  $\ker A$ ,  $S$  has  $n - d$  strictly positive eigenvalues.

On  $H$  we have  $B \geq 0$  and  $A$  has at least  $d - c + 1$  strictly positive eigenvalues, hence on  $H$  we can apply lemma 7.1 and we have that  $S$  has at least  $d - c + 1$  strictly positive eigenvalues on  $H$ . Because  $H$  and  $\ker A$  have an intersection reduced to  $\{0\}$ ,  $S$  has  $d - c + 1 + n - d = n - c + 1$  strictly positive eigenvalues. ■

The aim is to extend a theorem by H. Rossi [16] where we replace strictly pseudo convex by strictly  $c$ -convex.

**Theorem 7.4** *Let  $M$  a closed sub manifold of a Stein domain  $U_0$  in  $\mathbb{C}^n$ . Suppose there is a neighbourhood  $U$  of  $M$  and an holomorphic retraction  $\pi : U \rightarrow M$ . Let  $D$  be a strictly  $c$ -convex domain in  $M$ ,  $\bar{D} \subset M$ .*

*Then there is a strictly  $c$ -convex domain  $E$  in  $\mathbb{C}^n$  such that :*

- (A)  $\bar{E} \subset U \cap U_0$
- (B)  $E \cap M = D$
- (C)  $\partial E$  cuts  $M$  transversaly along  $\partial D$
- (D)  $\pi : \bar{E} \rightarrow \bar{D}$ .

Proof.

I shall copy the main points in the proof by H. Rossi making the necessary changes.

Docquier and Grauert (see [16]) give us a neighbourhood  $U$  of  $\bar{D}$  in  $\mathbb{C}^n$  and a retraction  $\pi : U \rightarrow M \cap U$  such that the fibers of  $\pi$  cut transversely  $M \cap U$  and are of dimension  $n - d$ .

We set for  $z \in U$  and  $j = 1, \dots, n$ ,  $f_j(z) = z_j - \pi_j(z)$ . The equations  $z - \pi(z) = 0$  define the sub manifold  $M$  :

if  $z \in M$ ,  $\pi(z) = z$  because  $\pi$  is a retraction on  $M$  ; if  $z \notin M$ , because  $\pi(z) \in M$ ,  $z - \pi(z) \neq 0$ . Moreover, because the fibers of  $\pi$  cut transversely  $M$  at any point  $\zeta$  of  $\bar{D}$ , we have that the jacobian matrix contains a  $(n - d) \times (n - d)$  sub determinant which is not 0 at  $\zeta$ , hence not 0 in a neighbourhood of this point. This means that, by a change of variables, the set  $(f_j)_{j=1, \dots, n}$  contains a coordinates system for the fibers of  $\pi$  at any point of  $\bar{D}$ , hence at all points of a neighbourhood  $U_1$  of  $\bar{D}$  in  $\mathbb{C}^n$ . These "explicite" functions replace the one generating the idealsheaf of  $M$  used by H. Rossi.

Let  $\rho$  be a defining function for  $D$  in  $M$ , we still follow H. Rossi and we set:

$$\sigma(z) := \rho \circ \pi + A \sum_{j=1}^n |f_j|^2,$$

where the constant  $A$  will be chosen later. Because  $F(z) := \sum_{j=1}^n |f_j(z)|^2 = 0$  on  $M \cap U$ , it exists a

$\epsilon_0 > 0$  such that  $\{F(z) < \epsilon_0\} \cap U \subset U_1$ .

It remains to see that  $\sigma$  is strictly  $c$ -convex, i.e.  $i\partial\bar{\partial}\sigma$  has at least  $n - c + 1$  strictly positive eigenvalues.

Fix  $\zeta \in \bar{D}$  ; because  $D$  is strictly  $c$ -convex  $i\partial\bar{\partial}\rho \circ \pi(\zeta)$  has at least  $d - c + 1$  strictly positive eigenvalues on the tangent space to  $M$  at  $\zeta$ . Because the set  $(f_j)_{j=1, \dots, n}$  contains a coordinates system for the fibers of  $\pi$  we have  $i\partial\bar{\partial}(\sum_{j=1}^n |f_j|^2)$  has all, i.e.  $n - d$ , strictly positive eigenvalues on the tangent space to the fiber  $\pi^{-1}\pi(\zeta)$  at  $\zeta$ .

Because the kernel of  $i\partial\bar{\partial}\rho \circ \pi$  is the tangent space to the fiber  $\pi^{-1}\pi(\zeta)$ , we get, by lemma 7.3, that  $i\partial\bar{\partial}\sigma = i\partial\bar{\partial}\rho \circ \pi + i\partial\bar{\partial}(\sum_{j=1}^n |f_j|^2)$  has at least  $n - c + 1$  strictly positive eigenvalues. So we have at least  $n - c + 1$  strictly positive eigenvalues at any point of  $\bar{D}$  hence also in a neighbourhood  $V$  of  $\bar{D}$  in  $\mathbb{C}^n$ . Now we take  $A\epsilon_0 > \sup_{z \in D} |\rho(z)|$  and we set  $E := \{z \in U \cap V :: \sigma(z) < 0\}$ ; we get exactly as H. Rossi, that  $E$  is strictly  $c$ -convex and we have all properties of the theorem. ■

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