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On estimates for the $\bar{\partial}$ equation in Stein manifolds.

Eric Amar

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Abstract

We generalize to strictly c -convex domains in Stein manifold, $L^r - L^s$ and Lipschitz estimates for the solutions of the $\bar{\partial}$ equation done by Ma and Vassiliadou for domains in \mathbb{C}^n . For this we use a Docquier-Grauert holomorphic retraction plus the raising steps method I introduce earlier. This gives results in the case of domains with low regularity, \mathcal{C}^3 , for their boundary.

1 Introduction.

The solutions with L^r and Lipschitz estimates of the equation $\bar{\partial}u = \omega$, $\bar{\partial}\omega = 0$ revealed to be very important in complex analysis and geometry.

The first results of this kind were done by the use of solving kernels: Grauert-Lieb [8], Henkin [9], Ovreid [15], Skoda [17], Krantz [12], in the case of strictly pseudo-convex domains with \mathcal{C}^∞ smooth

boundary in \mathbb{C}^n , with the exception of Kerzman [11] in the case of $(0, 1)$ forms in strictly pseudoconvex domains with \mathcal{C}^4 smooth boundary in Stein manifolds.

Here we shall be interested in strictly c -convex, s.c.c. for short, domain D in a complex manifold. Such a domain is defined by a function ρ of class \mathcal{C}^3 in a neighbourhood U of \bar{D} and such that $\partial\bar{\partial}\rho$ has at least $n - c + 1$ strictly positive eigenvalues in U .

These domains in \mathbb{C}^n have been studied in the case of smooth \mathcal{C}^∞ boundary by Fisher and Lieb [7]. They still use kernels method.

Definitive results were obtained in a very general framework by Beals, Greiner et Stanton [5]. They used the technology, heavy but extremely powerful, of pseudo-differentials operators. They study domains relatively compact with smooth \mathcal{C}^∞ boundary in complex manifold which are more general than s.c.c..

Ma and Vassiliadou [14] got very nice estimates even in the case of intersections of s.c.c. domains with \mathcal{C}^3 boundary. I shall use their results here.

Quite recently C. Laurent-Thiébaut [13] got this kind of result for s.c.c. domains with smooth \mathcal{C}^∞ boundary in complex manifold by use of the method of "bumps", as was done by Kerzman [11].

Let us state our main result which is completely analogous to the one Ma and Vassiliadou [14] got for domains in \mathbb{C}^n .

Theorem 1.1 *Let Ω a Stein manifold of dimension n and a strictly c -convex (s.c.c.) domain D such that D is relatively compact with smooth \mathcal{C}^3 boundary in Ω . Let ω a (p, q) form in $L^r_{p,q}(D)$, $\bar{\partial}\omega = 0$ with $1 < r < 2n + 2$, $c \leq q \leq n$. Then there is a $(p, q - 1)$ form u in $L^s(D)$, with $\frac{1}{s} = \frac{1}{r} - \frac{1}{2n + 2}$, such that $\bar{\partial}u = \omega$.*

If ω is in $L^r_{p,q}(D)$, $\bar{\partial}\omega = 0$ with $r \geq 2n + 2$, $c \leq q \leq n$, then there is a $(p, q - 1)$ form u in $\Lambda^\epsilon_{(p,q-1)}(\bar{D})$ such that $\bar{\partial}u = \omega$ with $\epsilon = \frac{1}{2} - \frac{n + 1}{r}$.

The spaces $\Lambda^\epsilon_{(p,q-1)}(\bar{D})$ are the (isotropic) Lipschitz spaces of order ϵ and we set $\Lambda^0_{(p,q-1)}(\bar{D}) := L^\infty_{(p,q-1)}(D)$.

It has to be noticed that the boundary regularity is just \mathcal{C}^3 , which is allowed by kernels. So it seems that this is a new result in a Stein manifold for such a low regularity. In the case of a \mathcal{C}^∞ boundary regularity then this result is contained in Beals, Greiner et Stanton [5] one, but the proof is completely different and, in some sense, "lighter" because it uses for the analytic part kernels methods plus essentially geometric ones.

The first result we get is a non optimal theorem.

Theorem 1.2 *Let M be a closed submanifold of dimension d of a Stein domain U_0 in \mathbb{C}^n . Let D be a s.c.c. domain relatively compact in M ($\bar{D} \subset M$) with \mathcal{C}^3 boundary. Then, with $r \geq 2n + 2$, we can solve in D , $\bar{\partial}u = \omega$ when $\bar{\partial}\omega = 0$ and with $u \in \Lambda^\epsilon_{(p,q-1)}(\bar{D})$ if $\omega \in L^r_{(p,q)}(D)$, $\bar{\partial}\omega = 0$, $c \leq q \leq n$, with $\epsilon = \frac{1}{2} - \frac{n + 1}{r}$.*

The results of Ma and Vassiliadou [14] gives good estimates in case of domains in \mathbb{C}^n . The point here was to pass from \mathbb{C}^n to a submanifold of \mathbb{C}^n . To do this I was inspired by a nice paper of H. Rossi [16] on Docquier Grauert holomorphic retraction.

Then we use the raising steps method [2] (see also [4], [3]) and to use it we need a global result and we use these $L^r - \Lambda^\epsilon$ estimates as a threshold. We get the same optimal results as for domains in \mathbb{C}^n .

So we get

Theorem 1.3 *Let M be a complex submanifold of dimension d in \mathbb{C}^n and a s.c.c. domain D such that D is relatively compact with smooth boundary of class \mathcal{C}^3 in M . Let ω a (p, q) form in $L^r_{p,q}(D)$, $\bar{\partial}\omega = 0$, $c \leq q \leq n$, with $1 < r < 2d + 2$. Then there is a $(p, q - 1)$ form u in $L^s(D)$, with $\frac{1}{s} = \frac{1}{r} - \frac{1}{2d + 2}$, such that $\bar{\partial}u = \omega$.*

If $r \geq 2n + 2$ then there is a $(p, q - 1)$ form u in $\Lambda^\epsilon_{(p,q-1)}(\bar{D})$ such that $\bar{\partial}u = \omega$ with $\epsilon = \frac{1}{2} - \frac{d + 1}{r}$.

To pass to Stein manifold, we use an embedding theorem of Bishop and Narashiman (see theorem 5.3.9. of Hörmander [10]) to see an abstract Stein manifold of dimension d as a submanifold of \mathbb{C}^{2d+1} . So we get our main result.

This work will be presented in the following way.

- First we recall the estimates in the case of strictly c -convex domains in \mathbb{C}^n done by Ma and Vassiliadou [14].
- We recall the Docquier Grauert holomorphic retraction on a complex submanifold M of \mathbb{C}^n .
- We extend a form ω from a domain D s.c.c. in M to a domain E s.c.c. in \mathbb{C}^n by use of a generalization of a theorem of H. Rossi [16]. We then solve the form in E by the known estimates in \mathbb{C}^n .
- We show that the solution in E can be restricted to D to get a solution in D with good enough estimates, for $r \geq 2n + 2$. This gives theorem 1.2.
- We use the raising steps theorem with the threshold given by theorem 1.2. So we have theorem 1.3 for the case of a submanifold of \mathbb{C}^n .
- By use of a theorem of Bishop and Narashiman, i.e. the proper embedding of a Stein manifold of dimension d in \mathbb{C}^{2d+1} , we get our main theorem 1.1 for any Stein manifold.

2 Strictly c -convex domain in \mathbb{C}^n .

We shall use the nice estimates for a smoothly \mathcal{C}^3 bounded c convex domains in \mathbb{C}^n obtained by Ma and Vassiliadou [14], lemma 5.3.

Theorem 2.1 *Let D be a bounded s.c.c. domain in \mathbb{C}^n with a \mathcal{C}^3 defining function. Then*

$$\forall \omega \in L^r_{(p,q)}, \bar{\partial}\omega = 0, c \leq q \leq n, 1 \leq r < 2n + 2,$$

there exists $u \in L^s_{(p,q-1)}(D)$, $\frac{1}{s} = \frac{1}{r} - \frac{1}{2n + 2}$, with the following properties:

i) $\bar{\partial}u = \omega$ in the sense of currents in D ,

ii) if $r = 1$, $u \in L^{\frac{2n+2}{2n+1}-\eta}_{(p,q-1)}$ for any $\eta > 0$.

iii) if $2n + 2 \leq r \leq \infty$, $u \in \Lambda^\epsilon_{(p,q-1)}(\bar{D})$ with $\epsilon = \frac{1}{2} - \frac{n + 1}{r}$.

3 The Docquier - Grauert holomorphic retraction.

We have the Docquier-Grauert lemma [6] :

Lemma 3.1 *Let K be a compact subset of a closed complex submanifold M of \mathbb{C}^n . There is a neighbourhood U of K and a holomorphic map $\pi : U \rightarrow U \cap M$ such that $\pi(\zeta) = \zeta$ for $\zeta \in U \cap M$.*

In fact we have more (Rossi [16], p 172) from the argument of Docquier-Grauert we have that the fibers $\pi^{-1}\pi\zeta$ of π intersect M transversally at all points of M and are of dimension $n - d$.

Let M be a complex manifold of dimension d in \mathbb{C}^n and D a relatively compact domain strictly c -convex in M . We have the following lemma.

Lemma 3.2 *Let $\zeta \in \bar{D}$, there is a neighborhood U of ζ in \mathbb{C}^n and a bi-holomorphic application (U, φ) , $\varphi : U \rightarrow T$ such that, with $z = (z_1, \dots, z_n)$ the coordinates in T , we have: $\varphi(D) \cap T = \{z_{d+1} = \dots = z_n = 0\}$ and the retraction $\tilde{\pi} := \varphi \circ \pi \circ \varphi^{-1}$ read in the application φ is given by $\tilde{\pi}(z) = (z_1, \dots, z_d, 0, \dots, 0)$, i.e. this is the orthogonal projection onto the subspace of $z' := (z_1, \dots, z_d)$. Moreover one can choose for T a tube around $\varphi(M)$ of width $\delta > 0$.*

Proof.

The manifold M is given, by use of the retraction π , by the functions $f_k(\zeta) := \zeta_k - \pi_k(\zeta)$, $k = 1, \dots, n$. We have if $\zeta \in M$, $\zeta - \pi(\zeta) = 0$; if $\zeta \notin M$, $\zeta - \pi(\zeta) \neq 0$, because $\pi(\zeta) \in M$. The transversality of the fibers with respect to M at all points of \bar{D} insures that the Jacobian of the application $f = (f_1, \dots, f_n)$ has rank $n - d$, which is the complex co-dimension of M . Take a point $\zeta^0 \in \bar{D}$, there are $n - d$ functions f_j which are independent in a neighborhood U of ζ^0 . Re-numerating the functions f_j and the variables ζ_k , we may suppose that the determinant $(\frac{\partial f_j}{\partial \zeta_k})_{j,k=d+1,\dots,n}$ is different from zero.

Now we shall make the change of variables $z = \varphi(\zeta)$ with $z_j = \zeta_j$, $j = 1, \dots, d$; $z_j = f_j(\zeta)$, $j = d + 1, \dots, n$. This is actually a change of variables because the Jacobian of φ is different from zero in the open set U . We have that the application φ is a bi-holomorphism from the open set U onto the open set $T := \varphi(U)$.

Let $z' = (z_1, \dots, z_d)$ and $z'' = (z_{d+1}, \dots, z_n)$; we have in T that:

$$N := \varphi(M) = \{z = (z', z'') \in T :: z'' = 0\}.$$

Now take a tube around N centered in ζ^0 , $T = \{z = (z', z'') :: z'' \in B((z', 0), \delta)\}$, we call it again T , and we still denote $U = \varphi^{-1}(T)$.

We cover \bar{D} by a finite number of these bi-holomorphisms (U_j, φ_j) . We note N_j the manifold $N_j := \varphi_j(M \cap U_j) \subset T_j$ and, diminishing a little bit the U_j if necessary, we can suppose that the width of the tubes T_j around the N_j is constant and equals $\delta > 0$. We know that there is a constant $\mu > 0$ such that $\mu^{-1} < J_j < \mu$, where J_j is the Jacobian of φ_j , because there is a finite number of charts (U_j, φ_j) .

We denote dV the Lebesgue measure on the manifold M .

We have the following basic lemma.

Lemma 3.3 *Let f be a function in $L^1(U_j)$ and \tilde{f} this function read in the application φ_j , i.e. $\tilde{f} := f \circ \varphi_j^{-1}$, we get*

$$\int_{U_j} f(\zeta) dm(\zeta) = \int_{N_j} \int_{B((z',0),\delta)} \tilde{f}(z', z'') J_j(z', z'') dm(z'') dV(z').$$

Proof.

This is simply the change of variables formula because $\varphi_j(U_j) = T_j = N_j \times B(\cdot, \delta)$ and the Jacobian of φ_j is J_j .

Lemma 3.4 *Let f be a measurable function, positive on M , then*

$$\int_{U_j} f \circ \pi(\zeta) dm(\zeta) \leq \mu c(\delta) \int_{N_j} f(z', 0) dV(z'),$$

with dm the Lebesgue measure on \mathbb{C}^n , dV the Lebesgue measure on M and $c(\delta) := |B(x, \delta)|$.

Proof.

We can apply lemma 3.4 with the notation $z = (z', z'')$, z' the coordinates in N_j , z'' the coordinates in the fibers:

$$\int_{U_j} f \circ \pi(\zeta) dm(\zeta) = \int_{N_j} \left\{ \int_{B(z', \delta)} \tilde{f}(z', z'') J_j(z', z'') dm(z'') \right\} dV(z'),$$

but here we have $\tilde{f}(z', z'') = f(z', 0)$ because $\tilde{\pi}(z) = (z', 0)$ hence the formula is now:

$$\int_{U_j} f \circ \pi(z) dm(z) \leq \mu \int_{N_j} f(z', 0) |B(z', \delta)| dV(z') = \mu c(\delta) \int_{N_j} f(z', 0) dV(z').$$

So we get the lemma. ■

We notice that the open set $U := \bigcup_{j=1}^N U_j$ contains \bar{D} .

The theorem 7.4 in the appendix, which generalizes to s.c.c. domains a theorem by Rossi [16] done for strictly pseudo convex domains, gives us the existence of the strictly c -convex domain E in \mathbb{C}^n such that $\pi : \bar{E} \rightarrow \bar{D}$. Now on we fix this s.c.c. domain E .

4 Extension of the form ω .

Let ω be a (p, q) form in $L^r(D)$, $\bar{\partial}$ closed ; we extend it in E by use of the retraction π in the following manner : $\tilde{\omega} := \pi^* \omega$.

Lemma 4.1 *We have $\bar{\partial} \tilde{\omega} = 0$. Moreover if $\omega \in L^r(D)$ we have $\tilde{\omega} \in L^r(E)$ with $\|\tilde{\omega}\|_{L^r(E)} \leq \mu c(\delta) \|\omega\|_{L^r(D)}$.*

Proof.

Because the retraction π is holomorphic we get $\bar{\partial} \tilde{\omega} = \pi^* \bar{\partial} \omega = 0$. Moreover the lemma 4.2 gives that $\tilde{\omega}$ is still in $L^r(E)$; we start by extending ω to $U \cap M$ by zero outside \bar{D} ; we had that the coefficients of $\tilde{\omega}$ can be written $f \circ \pi$ hence, applying lemma 4.2 to the functions $|f \circ \pi|^r$ we get $\|\tilde{\omega}\|_{L^r(U_j)} \leq \mu c(\delta) \|\omega\|_{L^r(N_j)}$. We have only a finite number of open sets U_j , so we get $\|\tilde{\omega}\|_{L^r(U)} \lesssim \mu c(\delta) \|\omega\|_{L^r(D)}$. Because $E \subset U$ we get $\|\tilde{\omega}\|_{L^r(E)} \leq \mu c(\delta) \|\omega\|_{L^r(D)}$. ■

Now E is s.c.c. in \mathbb{C}^n and, with $r \geq 2n + 2$, we can solve the $\bar{\partial}$ in the space $L^\infty(E) : \bar{\partial}\tilde{u} = \tilde{\omega}$, $\tilde{u} \in L^\infty(E)$ by the theorem 2.1. Fix $\omega \in L^r_{(p,q)}(D)$, $\bar{\partial}\omega = 0$, $\tilde{\omega}$ as above then we have $\tilde{u} \in L^\infty(E)$ also fixed.

We shall need the lemma

Lemma 4.2 *In an open set U_j of our covering, there is (p, q) form \tilde{v}_j such that $\bar{\partial}\tilde{v}_j = \bar{\partial}_M\tilde{v}_j = \tilde{\omega}$. This means that the $\bar{\partial}$ of the form \tilde{v}_j read in (U_j, φ_j) does not contain any $d\bar{z}''_k$. Moreover the coefficients of \tilde{v}_j are bounded in U_j and holomorphic in the fibers of π .*

Proof.

We work directly in $T = T_j$ by use of the bi-holomorphism φ_j .

The first part is coming from the fact that $\tilde{\omega}$ read in (U, φ) does not contain any form $d\bar{z}''_l$. Suppose that \tilde{u} contains such a form, we have : $\tilde{u} = v_l d\bar{z}''_l + \Gamma_l$ with Γ_l not containing $d\bar{z}''_l$; notice that the Γ_l are unique and so linear in \tilde{u} . Hence, keeping the notation \tilde{u} for its reading in (U, φ) ,

with the notation $\bar{\partial}_{z'_k}\tilde{u} = \frac{\partial\tilde{u}}{\partial\bar{z}''_k}$,

$$\forall k = 1, \dots, d, \bar{\partial}_{z'_k}\tilde{u} = \bar{\partial}_{z'_k}v_l \wedge d\bar{z}''_l + \bar{\partial}_{z'_k}\Gamma_l$$

cannot have terms in $d\bar{z}''_l$ because $\tilde{\omega}$ has not, hence we get $\bar{\partial}_{z'_k}v_l \wedge d\bar{z}''_l = 0$. So we get :

$$\bar{\partial}_{z'}(\tilde{u} - \sum_{l=d+1}^n v_l \wedge d\bar{z}''_l) = \tilde{\omega}.$$

on the other hand we get : $\bar{\partial}_{z''_k}\tilde{u} = \bar{\partial}_{z''_k}v_l \wedge d\bar{z}''_l + \bar{\partial}_{z''_k}\Gamma_l$ and, because there are no terms of the form $d\bar{z}''_k \wedge d\bar{z}''_l$ in $\tilde{\omega}$, we get necessarily $\forall l, k = d+1, \dots, n$, $\bar{\partial}_{z''_k}v_l \wedge d\bar{z}''_l = 0$. So the Γ_l are holomorphic in

the variables z'' . Hence we showed $\tilde{v}_j := \tilde{u} - \sum_{l=d+1}^n v_l \wedge d\bar{z}''_l = \sum_{l=d+1}^n \Gamma_l$ is still a solution of $\bar{\partial}\tilde{v}_j = \tilde{\omega}$

and, because the coefficients of \tilde{u} are bounded, we get all the properties stated in the lemma. We still notice that \tilde{v}_j is linear with respect to \tilde{u} . ■

Lemma 4.3 *There is a (p, q) form \tilde{v} such of that, in E , $\bar{\partial}\tilde{v} = \bar{\partial}_M\tilde{v} = \tilde{\omega}$. Moreover the coefficients of \tilde{v} are bounded in E and holomorphic on the fibers $\forall x \in D$, $F_x := \pi^{-1}(x)$.*

Proof.

Let us take two open sets of our covering: U_j, U_k ; the lemma 4.2 gives us:

$$\tilde{u} = v_j \wedge d\bar{z}'' + \tilde{v}_j \text{ in } U_j \text{ with } \tilde{v}_j \text{ without any } d\bar{z}''_l.$$

The same way in the open set U_k we have :

$$\tilde{u} = v_k \wedge d\bar{z}'' + \tilde{v}_k \text{ in } U_k \text{ with } \tilde{v}_k \text{ without any } d\bar{z}''_l.$$

So in the intersection of the two sets, because \tilde{u} is global, choosing one of these two systems of coordinates and with clear notations, we get:

$$(v_j - v_k) \wedge d\bar{z}'' + (\tilde{v}_j - \tilde{v}_k) = 0 \text{ in } U_j \cap U_k .$$

Because the \tilde{v}_j do not contain the $d\bar{z}''_l$, we necessarily have $\tilde{v}_j = \tilde{v}_k$ and $v_j = v_k$. So the forms \tilde{v}_j make a global form \tilde{v} just posing $\tilde{v} := \tilde{v}_j$ in U_j . ■

5 Estimates in the case of a submanifold of \mathbb{C}^n .

We shall show the following theorem :

Theorem 5.1 *Let M be a complex submanifold of dimension d in \mathbb{C}^n and a s.c.c. domain D such that D is relatively compact with smooth \mathcal{C}^3 boundary in M . Let again ω be a (p, q) form in $L^r_{p,q}(D)$, $\bar{\partial}\omega = 0$ with $r \geq 2n + 2$, $c \leq q \leq n$. Then there is a $(p, q - 1)$ form u in $\Lambda^{\epsilon}_{(p,q-1)}(\bar{D})$ such that $\bar{\partial}u = \omega$.*

Proof.

We restrict the function \tilde{v} got in the lemma 4.3 to D , $u := \tilde{v}|_D$, and we need to see that, on D , $\bar{\partial}u = \omega$. It is enough to see this in an open set U_j of our covering.

We shall use that $\bar{\partial}\tilde{v} = \tilde{\omega}$ in the distributions sense, with $\tilde{v} \in L^\infty(E)$. So let $\chi \in \mathcal{D}_{(n-p, n-q)}(U_j \cap D)$; we get

$$\langle \bar{\partial}u, \chi \rangle_D = (-1)^{q+1} \langle u, \bar{\partial}\chi \rangle_D;$$

with the scalar product of the manifold M . We read these data in (U_j, φ_j) and we keep the same names.

We extend the form χ from $N_j = U_j \cap M$ to the whole U_j in making it constant with respect to z'' . Let us denote $\tilde{\chi}$ this extended form.

Because the form $\tilde{v} = \tilde{v}_j$ in T_j is holomorphic in the variables z'' , its values in (z', z'') is the mean value of its values in the ball centered at $(z', 0) \in N_j$ and with radius δ in z'' . The lemma 3.4 gives:

$$\int_{T_j} (\tilde{v}, \bar{\partial}\tilde{\chi}) dm = \int_{N_j} \left\{ \int_{B(z', \delta)} (\tilde{v}, \bar{\partial}\tilde{\chi})(z', z'') dm(z'') \right\} dV(z),$$

but

$$\int_{B(z', \delta)} (\tilde{v}, \bar{\partial}\tilde{\chi})(z', z'') dm(z'') = \int_{B(z', \delta)} \sum_{I,J} \tilde{v}_{I,J}(z', z'') \tilde{\chi}'_{I^c J^c}(z', z'') dm(z'')$$

where $\tilde{\chi}' = \bar{\partial}\tilde{\chi}$. Because the form $\tilde{\chi}$ is compactly supported in N_j and is constant on the fibers, we get

$$\int_{B(z', \delta)} \sum_{I,J} \tilde{v}_{I,J}(z', z'') \tilde{\chi}'_{I^c J^c}(z', z'') dm(z'') = \sum_{I,J} \chi'_{I^c J^c}(z') \int_{B(z', \delta)} \tilde{v}_{I,J}(z', z'') dm(z'');$$

So we get, because $\int_{B(z', \delta)} \tilde{v}_{I,J}(z', z'') dm(z'') = c(\delta) u_{I,J}(z')$, with $c(\delta) := |B(z', \delta)|$,

$$\langle \tilde{v}, \bar{\partial}\tilde{\chi} \rangle_{T_j} := \int_{T_j} (\tilde{v}, \bar{\partial}\tilde{\chi}) dm = c(\delta) \int_{N_j} (u, \bar{\partial}\chi)(x) dV(x) = c(\delta) \langle u, \bar{\partial}\chi \rangle_{N_j}.$$

keeping the same notations, we come back to the open set U_j and, because χ is compactly supported in $N_j = M \cap U_j$, we get, because $T_j = \varphi_j(U_j)$ is a tube around N_j :

$$\langle \tilde{v}, \bar{\partial}\tilde{\chi} \rangle := c(\delta) \langle u, \bar{\partial}\chi \rangle_D.$$

exactly by the same way we get:

$$\langle \bar{\partial}\tilde{v}_j, \tilde{\chi} \rangle = c(\delta) \langle \bar{\partial}u, \chi \rangle_D,$$

And, because $\tilde{\omega}$ is constant on the fibers, we get

$$\langle \tilde{\omega}, \tilde{\chi} \rangle = c(\delta) \langle \omega, \chi \rangle_D.$$

Recall that $\bar{\partial}\tilde{v} = \tilde{\omega}$ in the distributions sense, and, because $\tilde{\omega}$ is a current in $L^r_{p,q}(E)$, this is also true for $\bar{\partial}\tilde{v}$. Hence we have, because $\tilde{\chi} \in L^{r'}_{n-p, n-q}(E)$, with r' the conjugate exponent of r , that $\langle \bar{\partial}\tilde{v}, \tilde{\chi} \rangle = \langle \tilde{\omega}, \tilde{\chi} \rangle$ is well defined, even if $\tilde{\chi}$ is not compactly supported. The previous equalities give $\langle \bar{\partial}u, \chi \rangle_D = \langle \omega, \chi \rangle_D$. Because this is true for all \mathcal{C}^∞ functions compactly supported χ in an open set T_j of a covering of \bar{D} , we get $\bar{\partial}u = \omega$ in the distributions sense on D , with $u \in \Lambda^{\epsilon}_{(p,q-1)}(\bar{D})$. ■

Remark 5.2 *We have no such estimates in the case $r < 2n + 2$ because the mean value in a ball of the fiber of a function in L^s in E is no longer in $L^s(D)$ for $s < \infty$.*

Now we are in position to apply the raising steps theorem.

Theorem 5.3 *Let M be a complex submanifold of dimension d in \mathbb{C}^n and D be a s.c.c. domain which is a relatively compact with smooth \mathcal{C}^3 boundary in M . Let ω be a (p, q) form in $L_{p,q}^r(D)$, $\bar{\partial}\omega = 0$ with $1 < r < 2d + 2$, $c \leq q \leq n$. Then there is a $(p, q - 1)$ form u in $L^s(D)$, with $\frac{1}{s} = \frac{1}{r} - \frac{1}{2d + 2}$, such that $\bar{\partial}u = \omega$.*

Proof.

In order to have the local result for all points in \bar{D} we use the same method as in [2], but with the proposition 7.2 and the results of Ma and Vassiliadou [14].

We have the global result : if $\omega \in L_{p,q}^{2n+2}(D)$, $\bar{\partial}\omega = 0$ then we have a solution u in $L_{(p,q-1)}^\infty(D)$ such that $\bar{\partial}u = \omega$ by use of theorem 5.1. Now we take $\omega \in L_{p,q}^r(D)$, $\bar{\partial}\omega = 0$, then we have that the optimal exponent for the solution u is s such that $\frac{1}{s} = \frac{1}{r} - \frac{1}{2d + 2}$, then we choose any $r_0 > \max(2n + 2, s)$ as a threshold and, because the correction to the solution u is in $L_{p,q-1}^\infty(D)$, and D is bounded, we have that $u \in L_{(p,q-1)}^s(D)$, by the raising steps theorem [2], and this ends the proof of the theorem. ■

6 Estimates in the case of a Stein manifold.

We can apply a theorem of Bishop and Narashiman (see theorem 5.3.9. of Hörmander [10]) which tells us that, if Ω is a Stein manifold of dimension d , there is an element $f \in \mathcal{H}(\Omega)^{2d+1}$ which defines a regular injective and proper application from Ω in \mathbb{C}^{2d+1} . Denote $M := f(\Omega)$; if D' is the strictly c -convex domain in Ω , relatively compact in Ω , then its image $D = f(D')$ is a strictly c -convex domain in M . We can apply theorem 5.3 and we get:

Theorem 6.1 *Let Ω be a Stein manifold of dimension n and a strictly c -convex domain D such that D is relatively compact with smooth \mathcal{C}^3 boundary in Ω . Let ω a (p, q) form in $L_{p,q}^r(D)$, $\bar{\partial}\omega = 0$ with $1 < r < 2n + 2$, $c \leq q \leq n$. Then there is a $(p, q - 1)$ form u in $L^s(D)$, with $\frac{1}{s} = \frac{1}{r} - \frac{1}{2n + 2}$, such that $\bar{\partial}u = \omega$.*

Let again ω be a (p, q) form in $L_{p,q}^r(D)$, $\bar{\partial}\omega = 0$ with $r \geq 2n + 2$, $c \leq q \leq n$. Then there is a $(p, q - 1)$ form u in $\Lambda_{(p,q-1)}^\epsilon(\bar{D})$ such that $\bar{\partial}u = \omega$.

We notice that there is no hypothesis here in the case $r > 2$ on the compactness of the support of the form ω , in contrast to the previous results we had in [2].

7 Appendix.

Lemma 7.1 *Let A, B be two self adjoint matrices such that A has at least $n - c + 1$ eigenvalues strictly positive and B is positive. Then $A + B$ has at least $n - c + 1$ eigenvalues strictly positive.*

Proof.

Let E the space generated by the eigenvectors associated to the strictly positive eigenvalues of A . Then E has dimension at least $n - c + 1$. Let $S := A + B$, because B is positive, we get

$$\forall x \in E, \langle Sx, x \rangle = \langle Ax, x \rangle + \langle Bx, x \rangle > 0.$$

Now let e_1, \dots, e_k be the eigenvectors associated to the negative eigenvalues of S . We set $F = \text{span}\{e_1, \dots, e_k\}$ and we have that F is invariant by S and we have $\forall x \in F, \langle Sx, x \rangle \leq 0$. If the space $G := E \cap F$ is of non zero dimension, we get $\forall x \in G, x \neq 0, \langle Sx, x \rangle > 0$ and $\langle Sx, x \rangle \leq 0$ so a contradiction. Hence $\dim G = 0$ and $\dim F \leq \text{codim} E = c - 1$, which means that S has a least $n - c + 1$ strictly positive eigenvalues. ■

The next proposition generalizes the one in [1], proposition 1.1, done in the pseudo convex case.

Proposition 7.2 *Let D be a strictly c -convex domain with \mathcal{C}^3 boundary in \mathbb{C}^n . Let $\zeta \in \partial D$, U a neighbourhood of ζ in \mathbb{C}^n and $B(\zeta, r)$ a ball centered at ζ and of radius r ; then there is a domain \tilde{D} , c -convex and with \mathcal{C}^3 boundary such that we have $\tilde{D} \subset U$ and $\partial D \cap B(\zeta, r) = \partial \tilde{D} \cap B(\zeta, r)$.*

Proof.

Let ρ be a defining function for D . Let $\zeta \in \partial D$ and U a neighbourhood of ζ in \mathbb{C}^n . Consider now a positive convex increasing function χ defined on \mathbb{R}^+ , \mathcal{C}^∞ such that $\chi = 0$ in $(0, r)$. Set $\tilde{\rho}(z) := \rho(z) + a\chi(|z - \zeta|^2)$; we have $\partial\bar{\partial}\tilde{\rho} = \partial\bar{\partial}\rho + a\partial\bar{\partial}\chi$. But, as is easily seen, $\partial\bar{\partial}\chi$ is positive at each point z , hence, setting $A = i\partial\bar{\partial}\rho$, $B = ai\partial\bar{\partial}\chi$, we can apply lemma 7.1 and we have that the domain $\tilde{D} := \{\tilde{\rho} < 0\}$ is also c -convex with smooth \mathcal{C}^3 boundary.

Now we choose r small enough to have $B(\zeta, 3r) \subset U$. We have $\tilde{\rho}(z) < 0 \Rightarrow \rho(z) < -a\chi(|z - \zeta|^2)$; so we set :

$\alpha := \sup_{z \in D} -\rho(z) < \infty$, by the compactness of \bar{D} and $\beta := \inf_{z \in U \setminus B(\zeta, 2r)} \chi(|z - \zeta|^2) = 4r^2$ then with $a := \frac{\alpha + 1}{\beta}$ we get that $\{\tilde{\rho}(z) < 0\} \subset U$ because if not $\exists z \notin B(\zeta, 3r) :: \rho(z) < -a\chi(|z - \zeta|^2) < -(\alpha + 1)$ which is not possible.

Of course in the ball $B(\zeta, r)$ we have $\partial D \cap B(\zeta, r) = \partial \tilde{D} \cap B(\zeta, r)$. ■

On a theorem of H. Rossi.

We shall use the following lemma.

Lemma 7.3 *Let A, B two self adjoint $n \times n$ matrices such that A has at least $d - c + 1$ eigenvalues strictly positive and $\ker A$ is of dimension $n - d$ and B is positive and has $n - d$ eigenvectors in $\ker A$ associated to strictly positive eigenvalues. Then $A + B$ has at least $n - c + 1$ strictly positive eigenvalues.*

Proof.

Because A is self adjoint, the spaces $\ker A$ and $H := \ker A^\perp$ are invariant for A . Because $\ker A$ has dimension $n - d$ and there is $n - d$ eigenvectors of B in it, then $\ker A$ is generated by these eigenvectors hence because B is self adjoint this means that $\ker A$ and H are also invariant for B . Set $S := A + B$.

Let $v \in \ker A$ such that $Bv = \lambda v$, $\lambda > 0$, then $Sv = Av + Bv = Bv = \lambda v$ hence on $\ker A$, S has $n - d$ strictly positive eigenvalues.

On H we have $B \geq 0$ and A has at least $d - c + 1$ strictly positive eigenvalues, hence on H we can apply lemma 7.1 and we have that S has at least $d - c + 1$ strictly positive eigenvalues on H . Because H and $\ker A$ have an intersection reduced to $\{0\}$, S has $d - c + 1 + n - d = n - c + 1$ strictly positive eigenvalues. ■

The aim is to extend a theorem by H. Rossi [16] where we replace strictly pseudo convex by strictly c -convex.

Theorem 7.4 *Let M a closed sub manifold of a Stein domain U_0 in \mathbb{C}^n . Suppose there is a neighbourhood U of M and an holomorphic retraction $\pi : U \rightarrow M$. Let D be a strictly c -convex domain in M , $\bar{D} \subset M$.*

Then there is a strictly c -convex domain E in \mathbb{C}^n such that :

(A) $\bar{E} \subset U \cap U_0$

(B) $E \cap M = D$

(C) ∂E cuts M transversaly along ∂D

(D) $\pi : \bar{E} \rightarrow \bar{D}$.

Proof.

I shall copy the main points in the proof by H. Rossi making the necessary changes.

Docquier and Grauert (see [16]) give us a neighbourhood U of \bar{D} in \mathbb{C}^n and a retraction $\pi : U \rightarrow M \cap U$ such that the fibers of π cut transversely $M \cap U$ and are of dimension $n - d$.

We set for $z \in U$ and $j = 1, \dots, n$, $f_j(z) = z_j - \pi_j(z)$. The equations $z - \pi(z) = 0$ define the sub manifold M :

if $z \in M$, $\pi(z) = z$ because π is a retraction on M ; if $z \notin M$, because $\pi(z) \in M$, $z - \pi(z) \neq 0$. Moreover, because the fibers of π cut transversely M at any point ζ of \bar{D} , we have that the jacobian matrix contains a $(n - d) \times (n - d)$ sub determinant which is not 0 at ζ , hence not 0 in a neighbourhood of this point. This means that, by a change of variables, the set $(f_j)_{j=1, \dots, n}$ contains a coordinates system for the fibers of π at any point of \bar{D} , hence at all points of a neighbourhood U_1 of \bar{D} in \mathbb{C}^n . These "explicite" functions replace the one generating the idealsheaf of M used by H. Rossi.

Let ρ be a defining function for D in M , we still follow H. Rossi and we set:

$$\sigma(z) := \rho \circ \pi + A \sum_{j=1}^n |f_j|^2,$$

where the constant A will be chosen later. Because $F(z) := \sum_{j=1}^n |f_j(z)|^2 = 0$ on $M \cap U$, it exists a

$\epsilon_0 > 0$ such that $\{F(z) < \epsilon_0\} \cap U \subset U_1$.

It remains to see that σ is strictly c -convex, i.e. $i\partial\bar{\partial}\sigma$ has at least $n - c + 1$ strictly positive eigenvalues.

Fix $\zeta \in \bar{D}$; because D is strictly c -convex $i\partial\bar{\partial}\rho \circ \pi(\zeta)$ has at least $d - c + 1$ strictly positive eigenvalues on the tangent space to M at ζ . Because the set $(f_j)_{j=1, \dots, n}$ contains a coordinates system for the fibers of π we have $i\partial\bar{\partial}(\sum_{j=1}^n |f_j|^2)$ has all, i.e. $n - d$, strictly positive eigenvalues on the tangent space to the fiber $\pi^{-1}\pi(\zeta)$ at ζ .

Because the kernel of $i\partial\bar{\partial}\rho \circ \pi$ is the tangent space to the fiber $\pi^{-1}\pi(\zeta)$, we get, by lemma 7.3, that $i\partial\bar{\partial}\sigma = i\partial\bar{\partial}\rho \circ \pi + i\partial\bar{\partial}(\sum_{j=1}^n |f_j|^2)$ has at least $n - c + 1$ strictly positive eigenvalues. So we have at least $n - c + 1$ strictly positive eigenvalues at any point of \bar{D} hence also in a neighbourhood V of \bar{D} in \mathbb{C}^n . Now we take $A\epsilon_0 > \sup_{z \in D} |\rho(z)|$ and we set $E := \{z \in U \cap V :: \sigma(z) < 0\}$; we get exactly as H. Rossi, that E is strictly c -convex and we have all properties of the theorem. ■

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