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Comments on the Chernoff \sqrt{n} -Lemma

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Dedicated to Pavel Exner in occasion of his 70th Birthday

Abstract

The Chernoff \sqrt{n} -Lemma is revised. This concerns two aspects: an improvement of the Chernoff estimate in the strong operator topology and an operator-norm estimate for quasi-sectorial contractions. Applications to the Lie-Trotter product formula approximation for semigroups is presented.

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1 Introduction: \sqrt{n} -Lemma

The Chernoff \sqrt{n} -Lemma is a key point in the theory of semigroup approximations proposed in [3]. For the reader convenience we recall this lemma below.

Lemma 1.1. *Let C be a contraction on a Banach space \mathfrak{X} . Then $\{e^{t(C-\mathbf{1})}\}_{t \geq 0}$ is a norm-continuous contraction semigroup on \mathfrak{X} and one has the estimate*

$$\|(C^m - e^{n(C-\mathbf{1})})x\| \leq \sqrt{n} \|(C - \mathbf{1})x\| . \quad (1.1)$$

for all $x \in \mathfrak{X}$ and $n \in \mathbb{N}$.

Proof. To prove the inequality (1.1) we use the representation

$$C^m - e^{n(C-\mathbf{1})} = e^{-n} \sum_{m=0}^{\infty} \frac{n^m}{m!} (C^m - C^m) . \quad (1.2)$$

To proceed we insert

$$\|(C^m - C^m)x\| \leq \|(C^{|n-m|} - \mathbf{1})x\| \leq |m - n| \|(C - \mathbf{1})x\| , \quad (1.3)$$

into (1.2) to obtain by the Cauchy-Schwarz inequality the estimate:

$$\begin{aligned} \|(C^m - e^{n(C-\mathbf{1})})x\| &\leq \|(C - \mathbf{1})x\| e^{-n} \sum_{m=0}^{\infty} \frac{n^m}{m!} |m - n| \leq \\ &\left\{ \sum_{m=0}^{\infty} e^{-n} \frac{n^m}{m!} |m - n|^2 \right\}^{1/2} \|(C - \mathbf{1})x\| , \quad x \in \mathfrak{X} , \end{aligned} \quad (1.4)$$

Note that the sum in the right-hand side of (1.4) can be calculated explicitly. This gives the value n , which yields (1.1). \square

The aim of the present note is to revise the Chernoff \sqrt{n} -Lemma in two directions. First, we improve the \sqrt{n} -estimate (1.1) for contractions. Then we apply this new estimate to the proof of the Trotter product formula in the *strong* operator topology (Section 2).

Second, we use the idea of Section 2 to lift these results in Section 3 to the *operator-norm* estimates for a special class of contractions: the *quasi-sectorial* contractions.

2 Revised \sqrt{n} -Lemma and Lie-Trotter product formula

We start by a technical lemma. It is a *revised* version of the Chernoff \sqrt{n} -Lemma 1.1. Our estimate (2.1) in $\sqrt[3]{n}$ -Lemma 2.1 is better than (1.1). The scheme of the proof will be useful later (Section 3), when we use it for proving the convergence of Lie-Trotter product formula in the *operator-norm* topology.

Lemma 2.1. *Let C be a contraction on a Banach space \mathfrak{X} . Then $\{e^{t(C-1)}\}_{t \geq 0}$ is a norm-continuous contraction semigroup on \mathfrak{X} and one has the estimate*

$$\|(C^n - e^{n(C-1)})x\| \leq \left[\frac{1}{n^{2\delta}} + n^{\delta+1/2} \right] \|(\mathbf{1} - C)x\|, \quad n \in \mathbb{N}. \quad (2.1)$$

for all $x \in \mathfrak{X}$ and $\delta \in \mathbb{R}$.

Proof. Since the operator C is bounded and $\|C\| \leq 1$, $(\mathbf{1} - C)$ is a generator of the norm-continuous semigroup, which is also a contraction:

$$\|e^{t(C-1)}\| \leq e^{-t} \left\| \sum_{m=0}^{\infty} \frac{t^m}{m!} C^m \right\| \leq 1, \quad t \geq 0.$$

To estimate (2.1) we use the representation

$$C^n - e^{n(C-1)} = e^{-n} \sum_{m=0}^{\infty} \frac{n^m}{m!} (C^n - C^m). \quad (2.2)$$

Let $\epsilon_n := n^{\delta+1/2}$, $n \in \mathbb{N}$. We split the sum (2.2) into two parts: the *central* part for $|m - n| \leq \epsilon_n$ and *tails* for $|m - n| > \epsilon_n$.

To estimate the *tails* we use the *Tchebychev inequality*. Let X_n be a *Poisson random variable* of the parameter n , i.e., the probability $\mathbb{P}\{X_n = m\} = n^m e^{-n} / m!$. One obtains for the expectation $\mathbb{E}(X_n) = n$ and for the variance $\text{Var}(X_n) = n$. Then by the Tchebychev inequality:

$$\mathbb{P}\{|X_n - \mathbb{E}(X_n)| > \epsilon\} \leq \frac{\text{Var}(X_n)}{\epsilon^2}, \quad \text{for any } \epsilon > 0.$$

Now to estimate (2.2) we note that

$$\begin{aligned} \|(C^n - C^m)x\| &= \|C^{n-k}(C^k - C^{m-n+k})x\| \\ &\leq |m - n| \|C^{n-k}(\mathbf{1} - C)x\|, \quad k = 0, 1, \dots, n. \end{aligned}$$

Put in this inequality $k = [\epsilon_n]$, here $[\cdot]$ denotes the integer part. Then by $\|C\| \leq 1$ and by the Tchebychev inequality we obtain the estimate for *tails*:

$$e^{-n} \sum_{|m-n|>\epsilon_n} \frac{n^m}{m!} \|(C^n - C^m)x\| \leq \tag{2.3}$$

$$\|(\mathbf{1} - C)x\| e^{-n} \sum_{|m-n|>\epsilon_n} \frac{n^m}{m!} |m - n| \leq \frac{n}{\epsilon_n^2} \|(\mathbf{1} - C)x\| = \frac{1}{n^{2\delta}} \|(\mathbf{1} - C)x\| .$$

To estimate the *central* part of the sum (2.2), when $|m - n| \leq \epsilon_n$, note that:

$$\begin{aligned} \|(C^m - C^n)x\| &\leq |m - n| \|C^{n-[\epsilon_n]}(\mathbf{1} - C)x\| \\ &\leq \epsilon_n \|(\mathbf{1} - C)x\| . \end{aligned} \tag{2.4}$$

Then we obtain:

$$e^{-n} \sum_{|m-n|\leq\epsilon_n} \frac{n^m}{m!} \|(C^n - C^m)x\| \leq n^{\delta+1/2} \|(\mathbf{1} - C)x\| ,$$

for $n \in \mathbb{N}$. This last estimate together with (2.3) yield (2.1). \square

Note that for $\delta = 0$ the estimate (2.1) gives for large n the same asymptotic as the Chernoff \sqrt{n} -Lemma, whereas for optimal value $\delta = (-1/6)$ the asymptotic $2\sqrt[3]{n}$ is better than (1.1). We call this result the $\sqrt[3]{n}$ -Lemma.

Theorem 2.2. *Let $\Phi : t \mapsto \Phi(t)$ be a function from \mathbb{R}^+ to contractions on \mathfrak{X} such that $\Phi(0) = \mathbf{1}$. Let $\{U_A(t)\}_{t \geq 0}$ be a contraction semigroup, and let $D \subset \text{dom}(A)$ be a core of the generator A . If the function $\Phi(t)$ has a strong right-derivative $\Phi'(+0)$ at $t = 0$ and*

$$\Phi'(+0)u := \lim_{t \rightarrow +0} \frac{1}{t} (\Phi(t) - \mathbf{1})u = -Au ,$$

for all $u \in D$, then

$$\lim_{n \rightarrow \infty} [\Phi(t/n)]^n x = U_A(t)x , \tag{2.5}$$

for all $t \in \mathbb{R}^+$ and $x \in \mathfrak{X}$

Proof. Consider the bounded approximation A_n of generator A :

$$A_n(s) := \frac{\mathbf{1} - \Phi(s/n)}{s/n} . \quad (2.6)$$

This operator is *accretive*: $(A_n(s) + \zeta \mathbf{1})^{-1} \in \mathcal{L}(\mathfrak{X})$ and $\|(A_n(s) + \zeta \mathbf{1})^{-1}\| \leq (\operatorname{Re}(\zeta))^{-1}$ for $\operatorname{Re}(\zeta) > 0$, and

$$\lim_{n \rightarrow \infty} A_n(s) u = A u , \quad (2.7)$$

for all $u \in D$ and for bounded s . Therefore, by virtue of the Trotter-Neveu-Kato generalised strong convergence theorem one gets:

$$\lim_{n \rightarrow \infty} e^{-t A_n(s)} x = U_A(t) x , \quad (2.8)$$

i.e., the strong and the uniform in t and s convergence (2.8) of the approximants $\{e^{-t A_n(s)}\}_{n \geq 1}$ for $s \in (0, s_0]$. By Lemma 2.1 for contraction $C := \Phi(t/n)$ and for $A_n(s)|_{s=t}$ we obtain

$$\begin{aligned} \|[\Phi(t/n)]^n x - e^{-t A_n(t)} x\| &= \|([\Phi(t/n)]^n - e^{n(\Phi(t/n) - \mathbf{1})}) x\| \leq \\ &\leq \frac{2}{n^{2\delta}} \|x\| + n^{\delta+1/2} \|(\mathbf{1} - \Phi(t/n)) x\| . \end{aligned} \quad (2.9)$$

Since for any $u \in D$ and uniformly on $[0, t_0]$ one gets

$$\lim_{n \rightarrow \infty} n^{\delta+1/2} \|(\mathbf{1} - \Phi(t/n)) u\| = \lim_{n \rightarrow \infty} t n^{\delta-1/2} \|A_n(t) u\| = 0 , \quad (2.10)$$

for $\delta < 1/2$, equations (2.9) and (2.10) imply

$$\lim_{n \rightarrow \infty} \|[\Phi(t/n)]^n u - e^{-t A_n(t)} u\| = 0, \quad u \in D . \quad (2.11)$$

Then (2.8) and (2.11) together with estimate $\|[\Phi(t/n)]^n - e^{-t A_n(t)}\| \leq 2$ yield uniformly in $t \in [0, t_0]$:

$$\lim_{n \rightarrow \infty} [\Phi(t/n)]^n x = U_A(t) x ,$$

which by density of D is extended to all $x \in \mathfrak{X}$, cf (2.5). \square

We call (2.5) the (strong) *Chernoff approximation formula* for the semi-group $\{U_A(t)\}_{t \geq 0}$.

Proposition 2.3. [3](Lie-Trotter product formula) *Let A, B and C be generators of contraction semigroups on \mathfrak{X} . Suppose that algebraic sum*

$$Cu = Au + Bu , \quad (2.12)$$

is valid for all vectors u in a core $D \subset \text{dom } C$. Then the semigroup $\{U_C(t)\}_{t \geq 0}$ can be approximated on \mathfrak{X} in the strong operator topology (2.9) by the Lie-Trotter product formula:

$$e^{-tC} x = \lim_{n \rightarrow \infty} (e^{-tA/n} e^{-tB/n})^n x , \quad x \in \mathfrak{X} , \quad (2.13)$$

for all $t \in \mathbb{R}^+$ and for $C := \overline{(A + B)}$, which is the closure of the algebraic sum (2.12).

Proof. Let us define the contraction $\mathbb{R}^+ \ni t \mapsto \Phi(t)$, $\Phi(0) = \mathbf{1}$, by

$$\Phi(t) := e^{-tA} e^{-tB} . \quad (2.14)$$

Note that if $u \in D$, then derivative

$$\Phi'(+0)u = \lim_{t \rightarrow +0} \frac{1}{t} (\Phi(t) - \mathbf{1}) u = -(A + B) u . \quad (2.15)$$

Now we are in position to apply Theorem 2.2. This yields (2.13) for $C := \overline{(A + B)}$. \square

Corollary 2.4. *Extension of the strong convergent Lie-Trotter product formula of Proposition 2.3 to quasi-bounded and holomorphic semigroups goes through verbatim.*

3 Quasi-sectorial contractions: $(\sqrt[3]{n})^{-1}$ -Theorem

Definition 3.1. [2] *A contraction C on the Hilbert space \mathfrak{H} is called quasi-sectorial with semi-angle $\alpha \in [0, \pi/2)$ with respect to the vertex at $z = 1$, if its numerical range $W(C) \subseteq D_\alpha$. Here*

$$D_\alpha := \{z \in \mathbb{C} : |z| \leq \sin \alpha\} \cup \{z \in \mathbb{C} : |\arg(1 - z)| \leq \alpha \text{ and } |z - 1| \leq \cos \alpha\}. \quad (3.1)$$

We comment that $D_{\alpha=\pi/2} = \mathbb{D}$ (unit disc) and recall that a general contraction C verifies the weaker condition: $W(C) \subseteq \mathbb{D}$.

Note that if operator C is a quasi-sectorial contraction, then $\mathbf{1} - C$ is an m -sectorial operator with vertex $z = 0$ and semi-angle α . Then for C the limits: $\alpha = 0$ and $\alpha = \pi/2$, correspond respectively to self-adjoint and to standard contractions whereas for $\mathbf{1} - C$ they give a non-negative self-adjoint and an m -accretive (bounded) operators.

The resolvent of an m -sectorial operator A , with semi-angle $\alpha \in [0, \alpha_0]$, $\alpha_0 < \pi/2$, and vertex at $z = 0$, gives an example of the quasi-sectorial contraction.

Proposition 3.2. [2, 6] *If C is a quasi-sectorial contraction on a Hilbert space \mathfrak{H} with semi-angle $0 \leq \alpha < \pi/2$, then*

$$\|C^n(\mathbf{1} - C)\| \leq \frac{K}{n+1}, \quad n \in \mathbb{N}. \quad (3.2)$$

The property (3.2) implies that the quasi-sectorial contractions belong to the class of the so-called *Ritt operators* [5]. This allows to go beyond the $\sqrt[3]{n}$ -Lemma 2.1 to the $(\sqrt[3]{n})^{-1}$ -Theorem.

Theorem 3.3. ($(\sqrt[3]{n})^{-1}$ -Theorem) *Let C be a quasi-sectorial contraction on \mathfrak{H} with numerical range $W(C) \subseteq D_\alpha$, $0 \leq \alpha < \pi/2$. Then*

$$\|C^n - e^{n(C-\mathbf{1})}\| \leq \frac{M}{n^{1/3}}, \quad n = 1, 2, 3, \dots \quad (3.3)$$

where $M = 2K + 2$ and K is defined by (3.2).

Proof. Note that with help of inequality (3.2) we can improve the estimate (2.4) in Lemma 2.1:

$$\|C^n - C^m\| \leq |m - n| \|C^{n-[\epsilon_n]}(\mathbf{1} - C)\| \leq \epsilon_n \frac{K}{n - [\epsilon_n] + 1},$$

for $\epsilon_n = n^{\delta+1/2}$. Then for $\delta < 1/2$ there the above inequality together with (2.3) give instead of (2.1) (or (1.1)) the operator-norm estimate

$$\|C^n - e^{n(C-\mathbf{1})}\| \leq \frac{2}{n^{2\delta}} + \frac{2K}{n^{1/2-\delta}}, \quad n \in \mathbb{N}. \quad (3.4)$$

Then the estimate $M/n^{1/3}$ of the Theorem 3.3 results from the optimal choice of the value: $\delta = 1/6$, in (3.4). \square

Similar to $(\sqrt[3]{n})$ -Lemma, the $(\sqrt[3]{n})^{-1}$ -Theorem is the first step in developing the *operator-norm* approximation formula à la Chernoff. To this end one needs an operator-norm analogue of Theorem 2.2. Since the last includes the Trotter-Neveu-Kato strong convergence theorem, we need the operator-norm extension of this assertion to quasi-sectorial contractions.

Proposition 3.4. [2] *Let $\{X(s)\}_{s>0}$ be a family of m -sectorial operators in a Hilbert space \mathfrak{H} with $W(X(s)) \subseteq S_\alpha$ for some $0 < \alpha < \pi/2$ and for all $s > 0$. Let X_0 be an m -sectorial operator defined in a closed subspace $\mathfrak{H}_0 \subseteq \mathfrak{H}$, with $W(X_0) \subseteq S_\alpha$. Then the two following assertions are equivalent:*

- (a) $\lim_{s \rightarrow +0} \|(\zeta \mathbf{1} + X(s))^{-1} - (\zeta \mathbf{1} + X_0)^{-1} P_0\| = 0$, for $\zeta \in S_{\pi-\alpha}$,
- (b) $\lim_{s \rightarrow +0} \|e^{-tX(s)} - e^{-tX_0} P_0\| = 0$, for $t > 0$.

Here P_0 denotes the orthogonal projection from \mathfrak{H} onto \mathfrak{H}_0 .

Now $(\sqrt[3]{n})^{-1}$ -Theorem 3.3 and Proposition 3.4 yield a desired generalisation of the operator-norm approximation formula:

Proposition 3.5. [2] *Let $\{\Phi(s)\}_{s \geq 0}$ be a family of uniformly quasi-sectorial contractions on a Hilbert space \mathfrak{H} , i.e. such that there exists $0 \leq \alpha < \pi/2$ and $W(\Phi(s)) \subseteq D_\alpha$, for all $s \geq 0$. Let*

$$X(s) := (\mathbf{1} - \Phi(s))/s, \quad (3.5)$$

and let X_0 be a closed operator with non-empty resolvent set, defined in a closed subspace $\mathfrak{H}_0 \subseteq \mathfrak{H}$. Then the family $\{X(s)\}_{s>0}$ converges, when $s \rightarrow +0$, in the uniform resolvent sense to the operator X_0 if and only if

$$\lim_{n \rightarrow \infty} \|\Phi(t/n)^n - e^{-tX_0} P_0\| = 0, \quad \text{for } t > 0. \quad (3.6)$$

Here P_0 denotes the orthogonal projection onto the subspace \mathfrak{H}_0 .

Let A be an m -sectorial operator with semi-angle $0 < \alpha < \pi/2$ and with vertex at 0, which means that numerical range $W(A) \subseteq S_\alpha = \{z \in \mathbb{C} : |\arg(z)| \leq \alpha\}$. Then $\{\Phi(t) := (\mathbf{1} + tA)^{-1}\}_{t \geq 0}$ is the family of quasi-sectorial contractions, i.e. $W(\Phi(t)) \subseteq D_\alpha$. Let $X(s) := (\mathbf{1} - \Phi(s))/s$, $s > 0$, and $X_0 := A$. Then $X(s)$ converges, when $s \rightarrow +0$, to X_0 in the uniform resolvent sense with the asymptotic

$$\|(\zeta \mathbf{1} + X(s))^{-1} - (\zeta \mathbf{1} + X_0)^{-1}\| = s \left\| \frac{A}{\zeta \mathbf{1} + A + \zeta s A} \cdot \frac{A}{\zeta \mathbf{1} + A} \right\| = O(s),$$

for any $\zeta \in S_{\pi-\alpha}$, since we have the estimate:

$$\left\| \frac{A}{\zeta \mathbf{1} + A + \zeta s A} \cdot \frac{A}{\zeta \mathbf{1} + A} \right\| \leq \left(1 + \frac{|\zeta|}{\text{dist}(\zeta(1 + s\zeta)^{-1}, -S_\alpha)} \right) \left(1 + \frac{|\zeta|}{\text{dist}(\zeta, -S_\alpha)} \right).$$

Therefore, the family $\{\Phi(t)\}_{t \geq 0}$ satisfies the conditions of Proposition 3.5. This implies the operator-norm approximation of the exponential function, i.e. the semigroup for m -sectorial generator, by the powers of resolvent (*the Euler approximation formula*):

Corollary 3.6. *If A is an m -sectorial operator in a Hilbert space \mathfrak{H} , with semi-angle $\alpha \in (0, \pi/2)$ and with vertex at 0, then*

$$\lim_{n \rightarrow \infty} \left\| (\mathbf{1} + tA/n)^{-n} - e^{-tA} \right\| = 0, \quad t \in S_{\pi/2-\alpha}. \quad (3.7)$$

4 Conclusion

Summarising we note that for the quasi-sectorial contractions instead of *divergent* Chernoff's estimate (1.1) we find the estimate (3.4), which converges for $n \rightarrow \infty$ to zero in the *operator-norm* topology. Note that the rate $O(1/n^{1/3})$ of this convergence is obtained with help of the Poisson representation and the Tchebychev inequality in the spirit of the proof of Lemma 2.1, and that it is not optimal.

The estimate $M/n^{1/3}$ in the $(\sqrt[3]{n})^{-1}$ -Theorem 3.3 can be improved by a more refined lines of reasoning.

For example, by scrutinising our probabilistic arguments one can find a more precise Tchebychev-type bound for the tail probabilities. This improves the estimate (3.4) to the rate $O(\sqrt{\ln(n)/n})$, see [4].

On the other hand, a careful analysis of localisation the numerical range of quasi-sectorial contractions [6, 1], allows to lift the estimate in Theorem 3.3 and in Corollary 3.6 to the ultimate optimal rate $O(1/n)$.

Note that the optimal estimate $O(1/n)$ in (3.4) one can easily obtain with help of the spectral representation for a particular case of the *self-adjoint* quasi-sectorial contractions, i.e. for $\alpha = 0$. This also concerns the optimal $O(1/n)$ rate of convergence of the self-adjoint Euler approximation formula (3.7).

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