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► **To cite this version:**

Manuela Royer. BSDEs Backward Stochastic Differential Equations with with jumps and related non linear expectations. Stochastic Processes and their Applications, 2006, 116, pp.1357-1376. <hal-01266359>

**HAL Id: hal-01266359**

**<https://hal.science/hal-01266359v1>**

Submitted on 16 Feb 2022

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# Backward Stochastic Differential Equations with jumps and related non linear expectations

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*Abstract*

*In this paper, we are interested in real-valued backward stochastic differential equations with jumps together with their applications to nonlinear expectations. The notion of nonlinear expectations has been studied only when the underlying filtration is given by a Brownian motion and in this work the filtration will be generated by both a Brownian motion and a Poisson random measure. We study at first backward stochastic differential equations driven by a Brownian motion and a Poisson random measure and then introduce the notions of  $f$ -expectations and of nonlinear expectations in this setup.*

**Keywords:** Backward stochastic differential equations, Jumps, Non linear expectation, Doob-Meyer decomposition.

## 1 Introduction

Linear backward stochastic differential equations (BSDEs in short for the remaining of the paper) first appeared ago as adjoint processes in the maximum principle for stochastic control problems and in the *Black-Scholes* formula for the pricing of options. In 1990, E. Pardoux and S. Peng introduced the notion of nonlinear BSDE in [19].

Since then, the interest in BSDEs has increased. Indeed, BSDEs provide connections with a large range of domains such as mathematical finance (we refer to [10]), stochastic control (see [21]) and partial differential equations. In our paper, we mainly focus on the latest.

Let us first consider a BSDE in the classical framework namely when the terminal time is deterministic, as introduced in [19].

We consider a Brownian motion  $(W_t)_{t \geq 0}$  and we denote by  $(\mathcal{G}_t)_{t \geq 0}$  its natural filtration. We look for a couple of processes  $(Y_t, Z_t)_{t \geq 0}$ ,  $(\mathcal{G}_t)_{t \geq 0}$ -adapted and satisfying the following equation

$$Y_t = \eta + \int_t^T g(s, Y_s, Z_s) ds - \int_t^T Z_s dW_s, \quad 0 \leq t \leq T, \quad (1)$$

where  $\eta$  is a  $\mathcal{G}_T$ -measurable random variable called the terminal condition and  $g$  is the generator.

Of course, we need suitable assumptions on  $\eta$  and  $g$  in order to obtain the existence of a solution.

In [19], E. Pardoux and S. Peng stated the following theorem: in any dimension there exists a unique solution to (1) as soon as  $g$  is Lipschitz w.r.t.  $y, z$ , and  $\eta$  is square-integrable.

Since then, some improvements have been obtained. For instance, S. Peng first introduced monotone generators in [20]. Besides, in the one-dimensional case, J. San Martin and J.-P. Lepeltier described the case of BSDEs with continuous generators and M. Kobylanski considered generators with quadratic growth in  $z$ .

S. Peng introduced in 1997 the notion of  $g$ -expectation in [22]. More precisely, the  $g$ -expectation for a random variable  $\eta$  is defined as the initial value of a classical BSDE driven by a generator

$g$  and with a terminal value  $\eta$ . In other words, if we consider the BSDE (1), then we call by  $g$ -expectation the operator  $\mathcal{E}_g$ , where  $\mathcal{E}_g(\eta) = Y_0$ . Let us note that simple conditions on  $g$  provide essential theorems for associated BSDEs : existence and uniqueness theorem, comparison theorem. Consequently, we derive existence and comparison theorems for  $g$ -expectations. A deeper study shows that  $g$ -expectations preserve all properties of classical expectation (except the linearity). And similarly to the classical case, we can define a related conditional  $g$ -expectation with respect to  $(\mathcal{G}_t)_{0 \leq t \leq T}$ , the natural filtration generated by the Brownian motion  $W$ . We also naturally motivate the notion of  $g$ -martingale by the economic theory (the wealth process of an investor in a stock market is a  $g$ -martingale if its consumption is negligible). In [23], S. Peng states that any càdlàg  $g$ -supermartingale bounded in  $\mathcal{L}^2$  is a  $g$ -supersolution.

A general non linear expectation is introduced in [6]. This notion is very close to  $g$ -expectations as particular cases. It is defined as an operator preserving monotonicity and constants. The notion of non linear expectation is very close to the classical expectation. Indeed, it satisfies almost all the classical properties, except the linearity. Thus, we can define concepts such as conditional non linear expectations and nonlinear martingales . Even though the linearity is missing, the theory can be carried on: for instance, under mild assumptions, the Doob–Meyer decomposition is proved. But the main result is an inverse theorem which allows any non linear expectation to be expressed as a  $g$ -expectation, *i.e.* as a solution of a BSDE.

The notion of nonlinear expectations has been studied in [6] under the assumption that the underlying filtration is generated by a Brownian motion. One of the main goal of this paper is to obtain results for nonlinear expectations when the filtration is generated by a Brownian motion together with a Poisson random measure. In particular, this will lead to nonlinear martingales with jumps.

The main results of this paper are the proof of a strict comparison theorem for BSDEs with jumps, and a representation theorem that identifies, under reasonable conditions, any abstract non linear expectation as a solution of a BSDE with jumps.

The paper is organised as follows. In section 2, we study BSDEs with jumps and we prove a comparison theorem. In section 3, we treat  $f$ -expectations and thanks to the theory of BSDEs, we give a Doob–Meyer decomposition for  $f$  supermartingales. In section 4, we extend the result to any non linear expectation, which can then be identified as an operator associating the initial value of a BSDE to its terminal value.

## 2 BSDEs with jumps

### 2.1 Context

We fix  $T > 0$  and consider a filtration generated by the two mutually independent processes

1. a  $d$ -dimensional standard Brownian motion  $(W_t)_{0 \leq t \leq T}$  on the probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ ,
2. a Poisson random measure  $\mu$  on  $(\mathbb{R}^+ \times B)$ , where  $B = \mathbb{R}^*$  is equipped with its Borel field  $\mathcal{B}^*$ , with compensator  $\nu(dt dx) = dt * \lambda(dx)$  such that  $\{\tilde{\mu}([0, T] \times A) = (\mu - \nu)([0, T] \times A)\}_{t \geq 0}$  is a martingale for all  $A \in \mathcal{B}^*$  satisfying  $\lambda(A) < \infty$ .  
 $\lambda$  is assumed to be a  $\sigma$ -finite measure on  $B$  such that  $\int_B (1 \wedge x^2) \lambda(dx) < \infty$ .

We introduce  $(\mathcal{F}_t)_{0 \leq t \leq T}$  the natural filtration generated both by  $W$  and  $\mu$ . It is a complete right continuous filtration.

We define the following sets of  $(\mathcal{F}_t)$ -progressively measurable  $\mathbb{R}$ -valued processes,

$$\mathcal{S}^2 = \left\{ \text{càdlàg processes } \psi ; \quad \mathbb{E} \left( \sup_{0 \leq t \leq T} |\psi_t|^2 \right) < \infty \right\},$$

$$\mathcal{L}^2(\tilde{\mu}) = \left\{ \mathcal{P} \otimes \mathcal{B}^* \text{-measurable processes } v ; \quad \mathbb{E} \left( \int_0^T \int_B |v_s(x)|^2 \lambda(dx) ds \right) < \infty \right\},$$

and the set of  $(\mathcal{F}_t)$ -progressively measurable  $\mathbb{R}$ -valued processes,

$$\mathcal{L}^2(W) = \left\{ \text{predictable processes } \theta ; \quad \mathbb{E} \left( \int_0^T \|\theta_s\|^2 ds \right) < \infty \right\},$$

where  $\mathcal{P}$  denotes the  $\sigma$ -field of predictable sets on  $[0, T] \times \Omega$ .

Let  $\eta \in \mathcal{L}^2(\mathcal{F}_T)$  be the terminal condition.

We consider  $f : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \times \mathcal{L}^2(B, \mathcal{B}^*, \lambda; \mathbb{R}) \longrightarrow \mathbb{R}$ .

### Definition 2.1

*Solution of the equation  $(f, \eta)$  is a triple of processes  $(Y_t, Z_t, U_t)_{0 \leq t \leq T} \in \mathcal{S}^2 \times \mathcal{L}^2(W) \times \mathcal{L}^2(\tilde{\mu})$  such that  $\forall t \in [0, T]$*

$$Y_t = \eta + \int_t^T f(s, Y_s, Z_s, U_s) ds - \int_t^T Z_s dW_s - \int_t^T \int_B U_s(x) \tilde{\mu}(ds dx).$$

Remark that processes  $Z$  and  $U$  have to be predictable. In the classical case, notions of optionality and predictability are equivalent whereas in our context a càdlàg adapted process is not necessarily predictable. So, we are led to use predictable stopping times instead of (optional) stopping times. For details about these notions, see [14] or [7].

Specific conditions are needed on generator  $f$  in order to ensure existence of a solution. S. Tang and X. Li stated such a theorem for Lipschitz generators in [25]. But this result was improved by E. Pardoux in [17], who proved that in the  $k$ -dimensional case, ( $k \in \mathbb{N}^*$ ), a unique solution to the equation  $(f, \eta)$  exists under the following assumptions, denoted by  $(\mathbf{H}_{ex})$ :

- (i)  $f$  is Lipschitz w.r.t.  $z, u$   
 $\exists K \geq 0$  such that  $\forall t \in [0, T], \forall y \in \mathbb{R}, \forall z, z' \in \mathbb{R}^d, \forall u, u' \in \mathcal{L}^2(B, \mathcal{B}^*, \lambda)$   
 $|f(t, y, z, u) - f(t, y, z', u')| \leq K \|z - z'\| + K \left( \int_B |u(x) - u'(x)|^2 \lambda(dx) \right)^{1/2}$
- (ii)  $f$  is continuous w.r.t.  $y$  and  
 $\exists$  an  $\mathbb{R}^+$ -valued adapted process  $(\varphi_t)_{0 \leq t \leq T}$  such that  $\mathbb{E} \left( \int_0^T \varphi_s^2 ds \right) < \infty$   
and  $|f(t, y, z, u)| \leq \varphi_t + K \left( |y| + \|z\| + \int_B |u(x)|^2 \lambda(dx) \right)^{1/2}$
- (iii)  $f$  is monotone w.r.t.  $y$ :  
 $\exists \alpha \in \mathbb{R}$  such that  $\forall t \geq 0, \forall y, y' \in \mathbb{R}, \forall z \in \mathbb{R}^d, \forall u, u' \in \mathcal{L}^2(B, \mathcal{B}^*, \lambda)$   
 $(y - y')(f(t, y, z, u) - f(t, y', z, u)) \leq \alpha |y - y'|^2 \quad \mathbb{P} p.s.$

Let us give an a priori estimate.

### Proposition 2.2

*Let  $f$  and  $f'$  be generators satisfying  $(\mathbf{H}_{ex})$ . We consider two terminal values  $\eta$  and  $\eta'$  in  $\mathcal{L}^2(\mathcal{F}_T)$ . We denote by  $(Y_t, Z_t, U_t)_{0 \leq t \leq T}$  and  $(Y'_t, Z'_t, U'_t)_{0 \leq t \leq T}$  the respective solution of each BSDE.*

*Then, for any constants  $\alpha \geq 0$  and  $\varepsilon > 0$ ,*

$$\begin{aligned} & \mathbb{E} \left( \int_t^T e^{\alpha s} \|Z_s - Z'_s\|^2 ds \right) + \mathbb{E} \left( \int_t^T \int_B e^{\alpha s} |U_s(x) - U'_s(x)|^2 \lambda(dx) ds \right) \\ & + \mathbb{E} \left( e^{\alpha t} |Y_t - Y'_t|^2 \right) \leq (1/\varepsilon - \alpha) \mathbb{E} \left( \int_t^T e^{\alpha s} |Y_s - Y'_s|^2 ds \right) \\ & + e^{\alpha T} \mathbb{E} \left( |\eta - \eta'|^2 \right) + \varepsilon \mathbb{E} \left( \int_t^T e^{\alpha s} |f(s, Y_s, Z_s, U_s) - f'(s, Y'_s, Z'_s, U'_s)|^2 ds \right). \end{aligned}$$

## 2.2 Comparison theorem

Let us denote by  $\mathcal{M}^2$  the set of square integrable martingales. Thanks to Martingale representation theorem, we can define the mapping  $\Phi : \mathcal{M}^2 \longrightarrow \mathcal{L}^2(W) \times \mathcal{L}^2(\tilde{\mu})$ ,  $M \mapsto (\theta, \nu)$  such that  $M_t = \int_0^t \theta_s dW_s + \int_0^t \int_B \nu_s(x) \tilde{\mu}(dsdx)$ . We then denote by  $\mathcal{M}$  the following subset:

$$\mathcal{M} = \left\{ (M_t)_{0 \leq t \leq T} \in \mathcal{M}^2 \mid \|\theta_s\| \leq C, \quad |\nu_s(x)| \leq C(1 \wedge |x|), \quad \nu_s(x) > -1, \quad \text{where } (\theta, \nu) = \Phi(M) \right\}.$$

$\forall M \in \mathcal{M}$ , we call Doléans-Dale exponential  $\mathcal{E}_T(M) = e^{M_T - \frac{1}{2} \langle M^c \rangle_T} \prod_{0 < s \leq T} (1 + \Delta M_s) e^{-\Delta M_s}$ .

### Theorem 2.3 Girsanov's theorem

Let  $(Z, U) \in \mathcal{L}^2(W) \times \mathcal{L}^2(\tilde{\mu})$ . We denote  $K_t = \int_0^t Z_s dW_s + \int_0^t \int_B U_s(x) \tilde{\mu}(dsdx)$ .

We consider  $M \in \mathcal{M}$ , with  $(\theta, \nu) = \Phi(M)$  and we compute  $\tilde{K} = K - \langle K, M \rangle$ .

Then process  $\tilde{K}$  is a martingale under the probability measure  $\tilde{\mathbb{P}} := \mathcal{E}_T(M) \cdot \mathbb{P}$ .

In order to introduce the notion of non linear expectation (operator preserving monotonicity and constants), we need a tool to compare solutions of BSDEs. Let us recall a comparison theorem stated by G. Barles, R. Buckdahn and E. Pardoux in [2].

### Theorem 2.4 Barles, Buckdahn and Pardoux's theorem

Let the generator be of the kind:  $f(\omega, t, y, z, u) = h(\omega, t, y, z, \int_B u(x) \bar{\gamma}_s(x) \lambda(dx))$ .

where  $\bar{\gamma} : \Omega \times [0, T] \times B \longrightarrow \mathbb{R}$  is  $\mathcal{P} \otimes \mathcal{B}^*$ -measurable and satisfies  $0 \leq \bar{\gamma}_t(x) \leq C(1 \wedge |x|)$ ,  $\forall x \in B$

and  $h : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \longrightarrow \mathbb{R}$  is  $\mathcal{P} \otimes \mathcal{B} \otimes \mathcal{B}^d \otimes \mathcal{B}$ -measurable and satisfies

$$(\omega, t, y, z, q) \mapsto h(\omega, t, y, z, q)$$

- (**H<sub>BBP</sub>**) (i)  $\mathbb{E} \left( \int_0^T |h(s, 0, 0, 0)|^2 ds \right) < \infty$ ,  
(ii)  $h$  is Lipschitz w.r.t.  $y, z, q$ ,  
(iii)  $h$  is non-decreasing w.r.t.  $q$ .

Let  $\eta_1$  and  $\eta_2$  be two terminal conditions in  $\mathcal{F}_T$  for BSDEs driven by the same generator  $f$ .

Denote by  $(Y^1, Z^1, U^1)$  and  $(Y^2, Z^2, U^2)$  the respective solutions.

If  $\eta_1 \leq \eta_2$ , then  $\forall t \in [0, T]$ ,  $Y_t^1 \leq Y_t^2$ .

In order to weaken the previous assumptions on  $f$ , we introduce the following hypothesis :

- (**A<sub>γ</sub>**) There exists  $-1 < C_1 \leq 0$  and  $C_2 \geq 0$  such that  
 $\forall y \in \mathbb{R}, \forall z \in \mathbb{R}^d, \forall u, u' \in \mathcal{L}^2(B, \mathcal{B}^*, \lambda; \mathbb{R})$ , we have  
 $f(t, y, z, u) - f(t, y, z, u') \leq \int_B (u(x) - u'(x)) \gamma_t^{y, z, u, u'}(x) \lambda(dx)$ ,  
where  $\gamma^{y, z, u, u'} : \Omega \times [0, T] \times B \longrightarrow \mathbb{R}$  is  $\mathcal{P} \otimes \mathcal{B}^*$ -measurable  
and satisfies  $C_1(1 \wedge |x|) \leq \gamma_t(x) \leq C_2(1 \wedge |x|)$ .

Furthermore, condition (**A<sub>γ</sub>**) implies that  $f$  is Lipschitz in  $u$ , and we denote by  $\Gamma$  its coefficient:

$$|f(s, y, z, u) - f(s, y, z, u')| \leq C \int_B |u(x) - u'(x)| (1 \wedge |x|) \lambda(dx) \leq \Gamma \left( \int_B |u(x) - u'(x)|^2 \lambda(dx) \right)^{1/2}.$$

This property will be useful to guarantee existence of solutions.

We can notice that the mapping  $\gamma$  will play the same role as  $\bar{\gamma}$  in Theorem 2.4, but we allow  $\gamma$  to depend on  $y, z, u, u'$  and it can be also negative as soon as it remains larger than  $C_1(1 \wedge |x|)$ .

We will consider the following list of conditions, weaker than (**H<sub>BBP</sub>**):

- (**H<sub>comp</sub>**) (i)  $\mathbb{E}\left(\int_0^T |f(s, 0, 0, 0)|^2 ds\right) < \infty$ ,  
(ii)  $f$  is Lipschitz w.r.t.  $y, z$  (with a constant denoted by  $K$ ),  
(iii)  $f$  satisfies (**A<sub>γ</sub>**).

**Theorem 2.5**

We consider a generator  $f_1$  satisfying (**H<sub>ex</sub>**) and we ask  $f_2$  to verify (**H<sub>comp</sub>**).  
Let  $\eta_1$  and  $\eta_2$  be two terminal conditions in  $\mathcal{F}_T$  for BSDEs driven respectively by  $f_1$  and  $f_2$ .  
Denote by  $(Y^1, Z^1, U^1)$  and  $(Y^2, Z^2, U^2)$  the respective solutions.  
If  $\eta_1 \leq \eta_2$  and  $f_1(t, Y_t^1, Z_t^1, U_t^1) \leq f_2(t, Y_t^1, Z_t^1, U_t^1)$ , then  $\forall t \in [0, T]$ ,  $Y_t^1 \leq Y_t^2$ .  
If moreover  $Y_0^1 = Y_0^2$ , then  $\forall t \in [0, T]$ ,  $Y_t^1 = Y_t^2$ , hence  $Z^1 = Z^2$  and  $U^1 = U^2$ .

**Proof.**

Note that if we assume (**H<sub>comp</sub>**), then a fortiori conditions (**H<sub>ex</sub>**) are verified. It allows us to consider the unique solution  $(Y^1, Z^1, U^1)$  and  $(Y^2, Z^2, U^2)$  of each BSDE.

We set  $\hat{\eta} = \eta_1 - \eta_2$ ,  $\hat{Y} = Y^1 - Y^2$ ,  $\hat{Z} = Z^1 - Z^2$ ,  $\hat{U} = U^1 - U^2$ .

Given that  $f$  is Lipschitz in  $y$  and in  $z$ , then we can define the real process  $\alpha$ :

$$\alpha_s = \begin{cases} \frac{f_2(s, Y_s^1, Z_s^1, U_s^1) - f_2(s, Y_s^2, Z_s^1, U_s^1)}{Y_s^1 - Y_s^2} & \text{if } Y_s^1 \neq Y_s^2, \\ 0 & \text{otherwise.} \end{cases}$$

$$\text{and the } \mathbb{R}^d\text{-valued process } \beta: \beta_s = \begin{cases} \frac{f_2(s, Y_s^2, Z_s^1, U_s^1) - f_2(s, Y_s^2, Z_s^2, U_s^1)}{\|Z_s^1 - Z_s^2\|^2} (Z_s^1 - Z_s^2) & \text{if } Z_s^1 \neq Z_s^2, \\ 0 & \text{otherwise.} \end{cases}$$

Next, we denote by  $R_t = e^{\int_0^t \alpha_u du}$  and we apply Itô's formula to  $R_s \hat{Y}_s$  between  $t$  and  $T$ .

$$\begin{aligned} R_t \hat{Y}_t &= R_T \hat{\eta} + \int_t^T R_s \left( f_1(s, Y_s^1, Z_s^1, U_s^1) - f_2(s, Y_s^1, Z_s^1, U_s^1) \right) ds \\ &\quad + \int_t^T R_s \left( f_2(s, Y_s^2, Z_s^2, U_s^1) - f_2(s, Y_s^2, Z_s^2, U_s^2) \right) ds \\ &\quad + \int_t^T R_s \beta_s \cdot \hat{Z}_s ds - \int_t^T R_s \hat{Z}_s dW_s - \int_t^T \int_B R_s \hat{U}_s(x) \tilde{\mu}(dsdx). \end{aligned} \tag{2}$$

Taking into consideration the assumptions on generators and terminal values, and using (**A<sub>γ</sub>**), we obtain

$$\begin{aligned} R_t \hat{Y}_t &\leq \int_t^T \int_B R_s \gamma_s^{Y_s^2, Z_s^2, U_s^1, U_s^2}(x) \hat{U}_s(x) \lambda(dx) ds + \int_t^T R_s \beta_s \cdot \hat{Z}_s ds \\ &\quad - \int_t^T R_s \hat{Z}_s dW_s - \int_t^T \int_B R_s \hat{U}_s(x) \tilde{\mu}(dsdx). \end{aligned}$$

Let us denote by  $\tilde{K}_t$  the right hand side and set  $M_t = \int_0^t \beta_s dW_s + \int_0^t \int_B \gamma_s^{Y_s^2, Z_s^2, U_s^1, U_s^2}(x) \tilde{\mu}(dsdx)$ .

Girsanov's theorem ensures that the process  $\tilde{K}_t$  is a martingale under probability measure  $\tilde{\mathbb{P}} := \mathcal{E}_T(M)$ .  $\mathbb{P}$ . Taking conditional expectation with respect to  $\mathcal{F}_t$  under probability measure  $\tilde{\mathbb{P}}$ , we get that  $R_t \hat{Y}_t \leq 0$   $\tilde{\mathbb{P}}$  a.s., and so  $\mathbb{P}$ -almost surely.  
Hence  $\forall t \in [0, T]$ ,  $Y_t^1 \leq Y_t^2$   $\mathbb{P}$  a.s.

The first part of the proof is done, we have now to establish a strong version of the theorem. We add from now on the further assumption  $Y_0^1 = Y_0^2$ . Remember that we still have  $Y_0^1 \leq Y_0^2$ .

We repeat the previous computation with  $t = 0$  in (2), and we consider again the probability measure  $\tilde{\mathbb{P}}$  given by Girsanov's theorem where  $\tilde{\mathbb{E}}$  denotes the associated expectation. Then, we obtain the following inequality

$$0 \leq \tilde{\mathbb{E}}\left(R_T \hat{\eta}\right) + \tilde{\mathbb{E}}\left(\int_0^T R_s \left(f_1(s, Y_s^1, Z_s^1, U_s^1) - f_2(s, Y_s^1, Z_s^1, U_s^1)\right) ds\right) \leq 0.$$

Taking into account equality (2), we deduce that

$$\begin{aligned} R_t \hat{Y}_t &= \int_t^T R_s \left(f_2(s, Y_s^2, Z_s^2, U_s^1) - f_2(s, Y_s^2, Z_s^2, U_s^2)\right) ds \\ &\quad + \int_t^T R_s \beta_s \cdot \hat{Z}_s ds - \int_t^T R_s \hat{Z}_s dW_s - \int_t^T \int_B R_s \hat{U}_s(x) \tilde{\mu}(dsdx). \end{aligned}$$

In order to prove that  $R_t \hat{Y}_t$  is nonnegative, we use assumption  $(\mathbf{A}_\gamma)$ . Indeed, we can consider a process  $\left(\bar{\gamma}_t^{Y_t^2, Z_t^2, U_t^1, U_t^2}(x)\right)_{0 \leq t \leq T}$  such that

$$\int_B \bar{\gamma}_s^{Y_s^2, Z_s^2, U_s^1, U_s^2}(x) \hat{U}_s(x) \lambda(dx) \leq f_2(s, Y_s^2, Z_s^2, U_s^1) - f_2(s, Y_s^2, Z_s^2, U_s^2).$$

Finally, applying Girsanov's theorem, we get that  $\forall t \in [0, T]$ ,  $R_t \hat{Y}_t \geq 0$   $\mathbb{P}a.s.$  Thus,  $Y^1 = Y^2$ .

Substituting in the initial BSDE solved by  $(\hat{Y}, \hat{Z}, \hat{U})$ , we get that the predictable process of finite variation  $\int_0^t \left(f_1(s, Y_s^1, Z_s^1, U_s^1) - f_2(s, Y_s^1, Z_s^2, U_s^2)\right) ds$  is equal to the martingale  $\int_0^t \hat{Z}_s dW_s + \int_0^t \int_B \hat{U}_s(x) \tilde{\mu}(dsdx)$ .

Consequently, we deduce that  $\mathbb{E}\left(\int_0^t \|\hat{Z}_s\|^2 ds\right) = 0$ , and  $\mathbb{E}\left(\int_0^t \int_B |\hat{U}_s(x)|^2 \lambda(dx) ds\right) = 0$ .

Hence,  $Z^1 = Z^2$  and  $U^1 = U^2$ . The proof is complete.

□

We can improve again the comparison Theorem 2.5 by considering a monotone generator. Nevertheless, we are not able to obtain a strict version of this result.

### Theorem 2.6

We consider a generator  $f_1$  satisfying  $(\mathbf{H}_{\mathbf{ex}})$ , and we suppose that  $f_2$  satisfies the following assumptions:

(i')  $f_2$  is continuous w.r.t.  $y$  and

$\exists$  an adapted  $\mathbb{R}^+$ -valued process  $(\varphi_t)_{0 \leq t \leq T}$  such that  $\mathbb{E}\left(\int_0^T \varphi_s^2 ds\right) < \infty$

and  $|f_2(t, y, z, u)| \leq \varphi_t + K\left(|y| + \|z\| + \int_B |u(x)|^2 \lambda(dx)\right)^{1/2}$ ,

(ii')  $f_2$  is monotone w.r.t.  $y$ , with a monotonicity constant  $\alpha \leq 0$ ,

(iii')  $f_2$  satisfies  $(\mathbf{A}_\gamma)$ ,

(iv')  $f_2$  is Lipschitz w.r.t.  $z$ , with a Lipschitz constant denoted by  $K$ .

Let  $\eta_1$  and  $\eta_2$  be two terminal conditions in  $\mathcal{F}_T$  for BSDEs driven respectively by  $f_1$  and  $f_2$ .

Denote by  $(Y^1, Z^1, U^1)$  and  $(Y^2, Z^2, U^2)$  the respective solutions.

If  $\eta_1 \leq \eta_2$  and  $f_1(t, Y_t^1, Z_t^1, U_t^1) \leq f_2(t, Y_t^1, Z_t^1, U_t^1)$ , then  $\forall t \in [0, T]$ ,  $Y_t^1 \leq Y_t^2$ .

**Proof.**

Similarly as in the proof of Theorem 2.5, we consider the equation solved by  $(\widehat{Y}, \widehat{Z}, \widehat{U})$ , and we define the process  $\beta$ .

Next, we apply Tanaka-Meyer's formula between  $t$  and  $T$ . Taking into consideration the assumptions on generators and terminal values, and using  $(\mathbf{A}_\gamma)$ , we obtain that

$$\begin{aligned} (\widehat{Y}_t)^+ &\leq \int_t^T \int_B \mathbb{1}_{\widehat{Y}_{s^-} > 0} \gamma_s^{Y_s^2, Z_s^2, U_s^1, U_s^2}(x) \widehat{U}_s(x) \lambda(dx) ds + \int_t^T \mathbb{1}_{\widehat{Y}_{s^-} > 0} \beta_s \cdot \widehat{Z}_s ds \\ &\quad - \int_t^T \mathbb{1}_{\widehat{Y}_{s^-} > 0} \widehat{Z}_s dW_s - \int_t^T \int_B \mathbb{1}_{\widehat{Y}_{s^-} > 0} \widehat{U}_s(x) \tilde{\mu}(dsdx). \end{aligned}$$

Girsanov's theorem provides that  $\forall t \in [0, T]$ ,  $(\widehat{Y}_t)^+ \leq 0$ , hence  $Y_t^1 \leq Y_t^2$   $\mathbb{P}$  a.s.

□

Whereas we required in Theorems 2.5 and 2.6 hypothesis  $(\mathbf{H}_{ex})$  for  $f_1$ , the proofs only needed existence for  $(Y_t^1, Z_t^1, U_t^1)_{0 \leq t \leq T}$ , and no particular additional condition on  $f_1$  is really necessary.

Actually, it is sufficient that there exists process  $(Y_t^1, Z_t^1, U_t^1)_{0 \leq t \leq T} \in \mathcal{S}^2 \times \mathcal{L}^2(W) \times \mathcal{L}^2(\tilde{\mu})$  which satisfy the equation  $(f_1, \eta_1)$ . Furthermore, G. Barles, R. Buckdahn and E. Pardoux give in [2] a counterexample which underlines the necessity of monotony for  $h$ . Studying the same example provides a contradiction if the assumption  $(\mathbf{A}_\gamma)$  is not satisfied.

We consider a Dirac measure  $\lambda(dx) = \delta_1(dx)$ , and the generator  $f(t, \omega, y, z, u) = cu(1)$ , with some real  $c < -1$ . Then  $f$  satisfies  $(\mathbf{H}_{ex})$  but not  $(\mathbf{H}_{comp})$  since  $\gamma_t(1) \leq c < -1$ .

We denote by  $N_t = \int_0^t \int_B \mathbb{1}_{\{x=1\}} \mu(dsdx)$  a standard Poisson process.

We consider two terminal values and we associate the respective solution of each BSDE.

On the one hand, we take  $\xi_1 = 0$  and  $f_1$  the null generator, then  $(Y_t^1, Z_t^1, U_t^1) = (0, 0, 0)$ .

On the other hand,  $\xi_2 = N_T$  and  $f_2 = f$ , then  $(Y_t^2, Z_t^2, U_t^2) = (N_t + (c+1)(T-t), 0, \mathbb{1}_{\{x=1\}})$ .

As a consequence, we have  $\xi_1 \leq \xi_2$ , and  $f_1(t, Y_t^1, Z_t^1, U_t^1) = f_2(t, Y_t^1, Z_t^1, U_t^1) = 0$ ,

whereas  $\mathbb{P}(Y_t^1 > Y_t^2) = \mathbb{P}(N_t < |c+1|(T-t)) > 0$ , which contradicts the conclusion of the comparison Theorem 2.5.

### 3 Non linear expectations

We look for a natural generalization of the classical expectation preserving as many properties as possible, except linearity. The concept of non linear expectation introduced in [6] answers to this issue. We recall that the notion of non linear expectation was introduced in [6], in the context of a filtration generated by a Brownian motion. In this particular case, the non linear conditional expectation is continuous. Here, we consider a filtration generated by both a  $d$ -dimensional Brownian motion  $(W_t)_{0 \leq t \leq T}$  and a Poisson random measure  $\mu$ .

#### 3.1 Definition

**Definition 3.1**

We say that an operator  $\mathcal{E} : \mathcal{L}^2(\mathcal{F}_T) \longrightarrow \mathbb{R}$  is a non linear expectation if

- $\forall c \in \mathbb{R}, \quad \mathcal{E}(c) = c$
- If  $\eta_1 \leq \eta_2$   $\mathbb{P}$  a.s. , then  $\mathcal{E}(\eta_1) \leq \mathcal{E}(\eta_2)$ . If moreover  $\mathcal{E}(\eta_1) = \mathcal{E}(\eta_2)$ , then  $\eta_1 = \eta_2$   $\mathbb{P}$  a.s.

Obviously, any classical expectation is a non linear expectation. For example, an operator associating the price of an option of strike  $K$  at maturity  $T$  to the payoff of this option is a non linear expectation. A lot of examples of non linear expectations appear in finance, with the notion of risk measure or utility function in particular.

*Example*

Consider a function  $h : \mathbb{R} \rightarrow \mathbb{R}$  such that  $h$  is a continuous, strictly increasing function  $|h(x)| \leq x^2 \quad \forall |x| \geq x_0, \quad h(0) = 0$

Let  $\eta$  be in  $\mathcal{L}^2(\mathcal{F}_T)$ , then, we can define  $\mathcal{E}'(\eta) = h^{-1}\left(\mathbb{E}(h(\eta))\right)$ .

The operator  $\mathcal{E}'$  provides a non linear expectation, where  $\mathcal{E}'(\eta)$  represents the amount of money an agent will accept to pay at initial time in order to get  $\eta$  at terminal time  $T$ .

Comparison Theorem 2.5 allows us to define the main case of non linear expectations represented by BSDEs.

### Proposition 3.2

Consider a BSDE driven by a generator  $f$  such that

- (i)  $f(s, y, 0, 0) = 0, \quad \forall y \in \mathbb{R},$
- (ii)  $f$  is Lipschitz in  $y, z,$
- (iii)  $f$  satisfies  $(A_\gamma)$ .

Conditions  $(\mathbf{H}_{ex})$ , and  $(\mathbf{H}_{comp})$  being satisfied, for any  $\eta$  fixed in  $\mathcal{L}^2(\mathcal{F}_T)$ , we denote the unique solution of the related BSDE by  $(Y^\eta, Z^\eta, U^\eta)$ . And we set  $\mathcal{E}_f(\eta) = Y_0^\eta$ , the initial value of the solution.

Then  $\mathcal{E}_f$  is a non linear expectation called  $f$ -expectation.

Generally,  $f$ -expectations are not linear, except when, for instance,  $f$  has a linear form :

$$f(t, y, z, u) = \alpha z + \beta \int_B u(x) \lambda(dx).$$

## 3.2 Filtration-consistent expectations

### Definition 3.3

A non linear expectation  $\mathcal{E}$  is said to be filtration-consistent if

$$\forall \eta \in \mathcal{L}^2(\mathcal{F}_T), \quad \forall t \in [0, T], \quad \exists \xi_t \in \mathcal{L}^2(\mathcal{F}_t) \text{ such that } \mathcal{E}(\eta 1_A) = \mathcal{E}(\xi_t 1_A), \quad \forall A \in \mathcal{F}_t.$$

In this case, we denote  $\mathcal{E}(\eta|\mathcal{F}_t) = \xi_t$  which is called the non linear conditional expectation of  $\eta$  with respect to  $\mathcal{F}_t$ .

### Proposition 3.4

Any  $f$ -expectation is filtration-consistent and, with the same notations as in the Proposition 3.2,  $\mathcal{E}_f(\eta|\mathcal{F}_t) = Y_t^\eta$ . Moreover,  $\mathbb{E}\left(\sup_{0 \leq t \leq T} |\mathcal{E}_f(\eta|\mathcal{F}_t)|^2\right) < \infty$ .

*Example:* The operator  $\mathcal{E}'$  is a non linear filtration-consistent expectation, with

$$\mathcal{E}'(\eta|\mathcal{F}_t) = h^{-1}\left(\mathbb{E}(h(\eta)|\mathcal{F}_t)\right).$$

### 3.3 Further properties

#### 3.3.1 Additivity

##### Definition 3.5

Let  $\mathcal{E}$  be a filtration-consistent expectation. We call additivity the following property:  
 $\forall \eta \in \mathcal{L}^2(\mathcal{F}_T), \forall t \in [0, T], \forall \xi \in \mathcal{L}^2(\mathcal{F}_t) \quad \mathcal{E}(\xi + \eta | \mathcal{F}_t) = \xi + \mathcal{E}(\eta | \mathcal{F}_t).$

#### 3.3.2 $\mathcal{E}^{\mu, C_1}$ -domination

Let  $\mu \in \mathbb{R}$  and  $-1 < C_1 \leq 0$ . We denote by  $\mathcal{E}^{\mu, C_1} = \mathcal{E}_{f_{\mu, C_1}}$  and  $\bar{\mathcal{E}}^{\mu, C_1} = \mathcal{E}_{\bar{f}_{\mu, C_1}}$  two particular non linear expectations, with  $f_{\mu, C_1}$  and  $\bar{f}_{\mu, C_1}$  respectively defined by

$$\begin{aligned} f_{\mu, C_1}(t, z, u) &= \mu \|z\| + |\mu| \int_B (1 \wedge |x|) u^+(x) \lambda(dx) - C_1 \int_B (1 \wedge |x|) u^-(x) \lambda(dx), \\ \bar{f}_{\mu, C_1}(t, z, u) &= -\mu \|z\| - |\mu| \int_B (1 \wedge |x|) u^-(x) \lambda(dx) + C_1 \int_B (1 \wedge |x|) u^+(x) \lambda(dx). \end{aligned}$$

##### Proposition 3.6

We fix  $\mu > 0$  and  $-1 < C_1 \leq 0$ .

We consider the set of probabilities  $\tilde{\mathcal{Q}} = \left\{ \text{probabilities } \tilde{Q} := \mathcal{E}_T(\tilde{M}) \cdot \mathbb{P} \mid \tilde{M} \in \tilde{\mathcal{M}} \right\}$ , with

$$\tilde{\mathcal{M}} = \left\{ \tilde{M} \in \mathcal{M}^2 \mid \|\tilde{\theta}_s\| \leq \mu, \tilde{v}_s^+(x) \leq \mu(1 \wedge |x|), \tilde{v}_s^-(x) \leq C_1(1 \wedge |x|) \text{ where } (\tilde{\theta}, \tilde{v}) = \Phi(\tilde{M}) \right\}.$$

Then  $\bar{\mathcal{E}}^{\mu, C_1}(\eta | \mathcal{F}_t) = \inf_{\tilde{Q} \in \tilde{\mathcal{Q}}} \mathbb{E}_{\tilde{Q}}(\eta | \mathcal{F}_t)$  and  $\mathcal{E}^{\mu, C_1}(\eta | \mathcal{F}_t) = \sup_{\tilde{Q} \in \tilde{\mathcal{Q}}} \mathbb{E}_{\tilde{Q}}(\eta | \mathcal{F}_t).$

##### Proof.

We fix  $\eta \in \mathcal{L}^2(\mathcal{F}_T).$

We sketch the proof for  $\mathcal{E}^{\mu, C_1}$ , and the result for  $\bar{\mathcal{E}}^{\mu, C_1}$  can be derived easily thanks to the property (issued from the Proposition 2.2) that links both operators :

$$\bar{\mathcal{E}}^{\mu, C_1}(\eta | \mathcal{F}_t) = -\mathcal{E}^{\mu, C_1}(-\eta | \mathcal{F}_t).$$

We denote by  $(Y, Z, U)$  the solution of the BSDE  $(f_{\mu, C_1}, \eta).$

On the one hand, Girsanov's Theorem allows us to identify  $\mathcal{E}^{\mu, C_1}$  as an expectation  $\mathbb{E}_Q$ , with respect to the probability  $Q = \mathcal{E}_T(M) \cdot \mathbb{P}$  where  $M = \Phi(\theta, v)$  with

$$\theta_{i,s} = \begin{cases} \mu \frac{\|Z_{i,s}\|}{Z_{i,s}} & \text{if } Z_{i,s} \neq 0 \\ \mu & \text{otherwise} \end{cases} \quad \text{and} \quad v_s(x) = \begin{cases} \frac{|\mu| U_s^+(x) - C_1 U_s^-(x)}{U_s(x)} (1 \wedge |x|) & \text{if } U_s(x) \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

We recall that for any  $\mathbb{R}^d$ -valued vector  $z$ ,  $z_i$  denotes its  $i_{th}$  coordinate.

On the other hand, we apply the comparison Theorem 2.5 to the generators  $f_{\mu, C_1}$  and  $\tilde{f}$ , with  $\tilde{f}(t, y, z, u) = \tilde{\theta}_s \cdot z + \tilde{v}_s(x) u(x).$   $\square$

##### Definition 3.7

Let  $\mathcal{E}$  be a filtration-consistent expectation.

We say that it is  $\mathcal{E}^{\mu, C_1}$ -dominated if  $\exists \mu > 0$  and  $-1 < C_1 \leq 0$  such that

$$\forall \eta, \eta' \in \mathcal{L}^2(\mathcal{F}_T), \quad \mathcal{E}(\eta + \eta') - \mathcal{E}(\eta) \leq \mathcal{E}^{\mu, C_1}(\eta').$$

##### Example

Any  $f$ -expectation with a generator  $f$  independent of  $y$  is additive and  $\mathcal{E}^{\mu, C_1}$ -dominated.

##### Proposition 3.8

If  $\mathcal{E}$  is an  $\mathcal{E}^{\mu, C_1}$ -dominated filtration-consistent expectation, then

$$\forall \eta, \eta' \in \mathcal{L}^2(\mathcal{F}_T) \quad \bar{\mathcal{E}}^{\mu, C_1}(\eta' | \mathcal{F}_t) \leq \mathcal{E}(\eta + \eta' | \mathcal{F}_t) - \mathcal{E}(\eta | \mathcal{F}_t) \leq \mathcal{E}^{\mu, C_1}(\eta' | \mathcal{F}_t).$$

**Corollary 3.9**

Let  $\mathcal{E}$  be an  $\mathcal{E}^{\mu, C_1}$ -dominated filtration-consistent expectation. Consider  $\eta$  and  $\eta'$  in  $\mathcal{L}^2(\mathcal{F}_T)$ . Then,  $\mathbb{E}\left(|\mathcal{E}(\eta|\mathcal{F}_t) - \mathcal{E}(\eta'|\mathcal{F}_t)|^2\right) \leq e^{(3\gamma^2+1)(T-t)} \mathbb{E}(|\eta - \eta'|^2)$  with  $\gamma = (|\mu| - C_1)(1 + \Gamma)$ .

**3.4 Down-crossing**

In the current subsection, we assume that  $\mathcal{E}$  is an  $\mathcal{E}^{\mu, C_1}$ -dominated filtration-consistent expectation.

**3.4.1 Regularity of an  $\mathcal{E}$ -martingale****Definition 3.10**

Let  $(X_t)_{0 \leq t \leq T}$  be a càdlàg process in  $\mathcal{L}_{\mathcal{F}}^2([0, T])$ .

We say that  $(X_t)_{0 \leq t \leq T}$  is an  $\mathcal{E}$ -martingale (resp.  $\mathcal{E}$ -supermartingale,  $\mathcal{E}$ -submartingale) if

$$\forall 0 \leq s \leq t \leq T \quad X_s = \mathcal{E}(X_t|\mathcal{F}_s) \quad (\text{resp. } \geq, \leq).$$

We shall denote it respectively  $\mathcal{E}$ -mg,  $\mathcal{E}$ -Smg and  $\mathcal{E}$ -smg.

**Lemma 3.11**

Let  $(X_t)_{0 \leq t \leq T}$  be an  $\mathcal{E}$ -supermartingale and  $D$  be a countable dense subset of  $\mathbb{R}^+$ .

Then for almost all  $\omega$ , for any  $t \in \mathbb{R}^+$ ,  $\lim_{s \in D, s \downarrow t} X_s(\omega)$  exists and is finite.

**Proof.** We can refer to [5] for a proof in the Brownian case.  $\square$

**Theorem 3.12**

Let  $(X_t)_{0 \leq t \leq T}$  be an  $\mathcal{E}$ -supermartingale such that  $\mathbb{E}\left(\sup_{0 \leq t \leq T} |X_t|^2\right) < \infty$ ,

then the process  $X$  possesses a càdlàg modification.

**Proof.**

Since  $X$  is an  $\mathcal{E}$ -supermartingale, then we can consider the right-limit process  $\widehat{X}$ , as defined in Lemma 3.11. By construction,  $\widehat{X}$  is a right-continuous process.

First, we prove that  $\forall t \in [0, T], X_t \geq \widehat{X}_t$ . Given that  $X$  is an  $\mathcal{E}$ -supermartingale, we get that for any countable dense subset  $D$  of  $\mathbb{R}^+$ ,  $\forall s \in [0, t] \cap D$ , we have  $X_t \geq \mathcal{E}(X_s|\mathcal{F}_t)$ . Moreover, the

dominated convergence Theorem applied in the Corollary 3.9 provides that  $\mathcal{E}(X_s|\mathcal{F}_t) \xrightarrow[s \in D, s \downarrow t]{\mathcal{L}^2} \widehat{X}_t$ ,

hence  $X_t \geq \widehat{X}_t$ .

Then, according to the definition of a non linear expectation, stating that  $\mathcal{E}(X_t) \leq \mathcal{E}(\widehat{X}_t)$  is sufficient to ensure that  $X_t = \widehat{X}_t$  *Pa.s.* The inequality stated in the Corollary 3.9 provides that

$$\mathbb{E}\left(|\mathcal{E}(\widehat{X}_t) - X_0|^2\right) \leq e^{(3\gamma^2+1)T} \mathbb{E}\left(|\widehat{X}_t - X_s|^2\right) \quad \forall s > t, s \in D.$$

Passing to the limit on  $s \in D$ , we obtain that  $\mathcal{E}(\widehat{X}_t) = X_0$ . And we derive that  $X$  possesses a càdlàg modification.  $\square$

**Corollary 3.13**

Let  $\eta \in \mathcal{L}^2(\mathcal{F}_T)$ . We set  $X_t = \mathcal{E}(\eta|\mathcal{F}_t)$ .

Then  $X$  possesses a modification in the space  $\mathcal{S}^2$ . In particular, it possesses a càdlàg modification.

From now on, we will always consider  $X$  as its càdlàg modification.

### 3.4.2 Doob's sampling Theorem

#### Theorem 3.14

Let  $\mathcal{E}$  be an  $\mathcal{E}^{\mu, C_1}$ -dominated and additive filtration-consistent expectation.

Let  $\tau$  and  $\sigma$  be two stopping times such that  $0 \leq \sigma \leq \tau \leq T$ .

If  $(X_t)_{0 \leq t \leq T}$  is an  $\mathcal{E}$ -supermartingale satisfying  $\mathbb{E} \left( \sup_{0 \leq t \leq T} |X_t|^2 \right) < \infty$ , then  $\mathcal{E}(X_\tau | \mathcal{F}_\sigma) \leq X_\sigma$ .

#### Proof.

We begin with the construction of sequences of finite valued stopping times decreasing to  $\tau$  and  $\sigma$  respectively.

We set

$$\begin{aligned} \tau_n &= \sum_{k=1}^{2^n} \frac{kT}{2^n} \mathbb{1}_{\left\{ \frac{(k-1)T}{2^n} \leq \tau < \frac{kT}{2^n} \right\}} + T \mathbb{1}_{\{\tau=T\}}, \\ \sigma_n &= \sum_{k=1}^{2^n} \frac{kT}{2^n} \mathbb{1}_{\left\{ \frac{(k-1)T}{2^n} \leq \sigma < \frac{kT}{2^n} \right\}} + T \mathbb{1}_{\{\sigma=T\}}. \end{aligned}$$

Let  $A$  be in  $\mathcal{F}_\sigma$ . Then  $\mathcal{E}(X_{\tau_n} \mathbb{1}_A) \leq \mathcal{E}(X_{\sigma_n} \mathbb{1}_A)$  (see [5]). Since the process  $(X_t)_{0 \leq t \leq T}$  is càdlàg, then we are ensured of the almost sure convergences  $X_{\tau_n} \xrightarrow[n \rightarrow +\infty]{\mathbb{P} \text{ a.s.}} X_\tau$  and  $X_{\sigma_n} \xrightarrow[n \rightarrow +\infty]{\mathbb{P} \text{ a.s.}} X_\sigma$ .

Referring to Corollary 3.9, dominated convergence Theorem provides that  $\left( \mathcal{E}(\mathbb{1}_A X_{\tau_n}) \right)_{n \geq 0}$  converges towards  $\mathcal{E}(\mathbb{1}_A X_\tau)$  in  $\mathcal{L}^2$ .

Consequently, we can extract a subsequence that converges almost surely. We do the same for stopping time  $\sigma$  and after extracting a common subsequence, we can pass to the limit in the inequality and finally get that  $\mathcal{E}(X_\tau | \mathcal{F}_\sigma) \leq X_\sigma$ .  $\square$

## 4 Inverse Theorem

### 4.1 Doob-Meyer decomposition

#### 4.1.1 Results for $f$ -expectations

#### Theorem 4.1 Doob-Meyer's decomposition for $\mathcal{E}_f$ -Smg

Suppose that (i)  $f(s, y, 0, 0) = 0, \quad \forall y \in \mathbb{R}$

(ii)  $f$  is Lipschitz w.r.t.  $y, z$

(iii)  $f$  satisfies  $(A_\gamma)$

Let  $(Y_t)_{0 \leq t \leq T}$  be in  $\mathcal{S}^2$ . If  $(Y_t)_{0 \leq t \leq T}$  is an  $\mathcal{E}_f$ -supermartingale, then there exists processes  $(Z_t, U_t)_{0 \leq t \leq T} \in \mathcal{L}^2(W) \times \mathcal{L}^2(\tilde{\mu})$ , and an increasing càdlàg predictable process  $(A_t)_{0 \leq t \leq T}$  with  $A_0 = 0, \mathbb{E}(A_T^2) < \infty$  such that

$$Y_t = Y_T + \int_t^T f(s, Y_s, Z_s, U_s) ds + (A_T - A_t) - \int_t^T Z_s dW_s - \int_t^T \int_B U_s(x) \tilde{\mu}(ds dx).$$

Moreover, processes  $Z, U$  and  $A$  are unique in their respective spaces.

#### Proof.

S. Peng had already studied this question in [23] in the context of a Brownian filtration. We follow the same approach. The main idea is to apply a penalization method introduced in [9] (so as to solve the problem of option pricing in incomplete markets).

The difficulty here lies in the fact that a stopping time is not necessarily predictable. See the proof in Annex.  $\square$

**Theorem 4.2 Doob-Meyer's decomposition for  $\mathcal{E}_f$ -smg**

Under the same hypotheses as in the Theorem 4.1, we consider  $(Y'_t)_{0 \leq t \leq T}$  in  $\mathcal{S}^2$ . If  $(Y'_t)_{0 \leq t \leq T}$  is an  $\mathcal{E}_f$ -submartingale, then there exists unique processes  $(Z'_t, U'_t)_{0 \leq t \leq T} \in \mathcal{L}^2(W) \times \mathcal{L}^2(\tilde{\mu})$  and an increasing càdlàg predictable process  $(A'_t)_{0 \leq t \leq T}$  with  $A'_0 = 0$ ,  $\mathbb{E}(A_T'^2) < \infty$  such that

$$Y'_t = Y'_T + \int_t^T f(s, Y'_s, Z'_s, U'_s) ds + (A'_t - A'_T) - \int_t^T Z'_s dW_s - \int_t^T \int_B U'_s(x) \tilde{\mu}(dsdx).$$

**Corollary 4.3**

Let  $\mathcal{E}$  be an  $\mathcal{E}^{\mu, C_1}$ -dominated filtration-consistent expectation.

Consider  $(Y_t)_{0 \leq t \leq T} \in \mathcal{S}^2$ . If  $(Y_t)_{0 \leq t \leq T}$  an  $\mathcal{E}$ -Smg (resp.  $\mathcal{E}$ -smg), then there exists an increasing predictable process  $A$  (resp.  $A'$ ) such that  $Y + A$  (resp.  $Y - A'$ ) is an  $\bar{\mathcal{E}}^{\mu, C_1}$ -mg (resp.  $\mathcal{E}^{\mu, C_1}$ -mg).

**4.1.2 Decomposition Theorem for an  $\mathcal{E}$ -Smg**

In this subsection, we assume that  $\mathcal{E}$  is a filtration-consistent expectation which satisfies both properties of  $\mathcal{E}^{\mu, C_1}$ -domination and additivity.

**Proposition 4.4**

Let  $(\tilde{Y}_t)_{0 \leq t \leq T}$  and  $(\hat{Y}_t)_{0 \leq t \leq T} \in \mathcal{S}^2$  be  $\mathcal{E}$ -martingales.

Then there exists functions  $\tilde{g}, \hat{g}$  and couples of processes  $(\tilde{Z}, \tilde{U}), (\hat{Z}, \hat{U}) \in \mathcal{L}^2(W) \times \mathcal{L}^2(\tilde{\mu})$  such that

$$\begin{aligned} \tilde{Y}_t &= \tilde{Y}_T + \int_t^T \tilde{g}_s ds - \int_t^T \tilde{Z}_s dW_s - \int_t^T \int_B \tilde{U}_s(x) \tilde{\mu}(dsdx), \\ \hat{Y}_t &= \hat{Y}_T + \int_t^T \hat{g}_s ds - \int_t^T \hat{Z}_s dW_s - \int_t^T \int_B \hat{U}_s(x) \tilde{\mu}(dsdx). \end{aligned}$$

Moreover,  $\bar{f}_{\mu, C_1}(s, \tilde{Z}_s - \hat{Z}_s, \tilde{U}_s - \hat{U}_s) \leq \tilde{g}_s - \hat{g}_s \leq f_{\mu, C_1}(s, \tilde{Z}_s - \hat{Z}_s, \tilde{U}_s - \hat{U}_s)$ .

**Proof.**

We apply Corollary 4.3 to  $\tilde{Y}$ , that gives processes  $A$  and  $A'$ . Next, with  $dA_t = a_t dt$  and  $dA'_t = a'_t dt$ , we construct  $\tilde{g}_s = \frac{|\mu| + C_1}{2} \int_B (1 \wedge |x|) \tilde{U}_s(x) \lambda(dx) + \frac{1}{2}(a_s - a'_s)$ . We proceed similarly for  $\hat{Y}$ .  $\square$

**Theorem 4.5 Doob-Meyer's decomposition for  $\mathcal{E}$ -Smg**

Let  $(Y_t)_{0 \leq t \leq T} \in \mathcal{S}^2$ . If  $(Y_t)_{0 \leq t \leq T}$  is an  $\mathcal{E}$ -supermartingale, i.e.  $Y_t \geq \mathcal{E}(Y_s | \mathcal{F}_t) \quad \forall s \geq t$ , then there exists an increasing càdlàg predictable process  $(A_t)_{0 \leq t \leq T}$  such that  $A_0 = 0$ ,  $\mathbb{E}(A_T^2) < \infty$ , and  $(Y_t + A_t)_{0 \leq t \leq T}$  is an  $\mathcal{E}$ -martingale, i.e.,  $Y_t + A_t = \mathcal{E}(Y_T + A_T | \mathcal{F}_t)$ .

**Proof.** It is important to point out that even if this theorem has a familiar formulation, we cannot use the classical proof which is fundamentally based on the linearity of expectation  $\mathbb{E}$ . The proof is given in Annex.  $\square$

**4.2 Inverse Theorem**

We defined  $f$ -expectations as particular cases of filtration-consistent expectations. It means that a large range of non linear expectations can be seen as solutions of BSDEs driven by some generator  $f$ . Actually, the aim of this study is to prove that any filtration-consistent expectation can be expressed as an  $f$ -expectation, under reasonable conditions.

**Theorem 4.6 Inverse theorem**

Let  $\mathcal{E}$  be a filtration-consistent expectation which satisfies both properties of  $\mathcal{E}^{\mu, C_1}$ -domination and additivity. Then there exists a function  $f : \Omega \times [0, T] \times \mathbb{R}^d \times \mathcal{L}^2(\tilde{\mu}) \longrightarrow \mathbb{R}$  such that  $\mathcal{E} = \mathcal{E}_f$ .

Moreover,  $f$  satisfies

a)  $\bar{f}_{\mu, C_1}(t, z, u) \leq f(t, z, u) \leq f_{\mu, C_1}(t, z, u),$

b) for any processes  $\zeta \in \mathcal{L}^2(W)$  and  $\vartheta \in \mathcal{L}^2(\tilde{\mu}), \quad \forall r \leq t \in [0, T],$

$$\mathcal{E} \left( - \int_r^t f(s, \zeta_s, \vartheta_s) ds + \int_r^t \zeta_s dW_s + \int_r^t \int_B \vartheta_s(x) \tilde{\mu}(dsdx) \mid \mathcal{F}_r \right) = 0.$$

**Proof.**

The idea of the proof is to construct a function  $f$  from an  $\mathcal{E}$ -martingale.

For any fixed  $z_0 \in \mathbb{R}^d$  and  $u_0 \in \mathcal{L}^2(B, \mathbb{R}; \lambda)$ , we set

$$Y_t^{z_0, u_0} = -t f_{\mu, C_1}(t, z_0, u_0) + z_0 W_t + \int_0^t \int_B u_0(x) \tilde{\mu}(dsdx).$$

We consider  $(Y_t, Z_t, U_t)_{0 \leq t \leq T}$  the unique solution of the BSDE  $(f_{\mu, C_1}, Y_T^{z_0, u_0})$ .

The uniqueness of solutions of such BSDE provides that

$$(Y_t, Z_t, U_t) = (Y_t^{z_0, u_0}, z_0, u_0) \text{ in the space } \mathcal{S}^2 \times \mathcal{L}^2(W) \times \mathcal{L}^2(\tilde{\mu}).$$

As a consequence, we apply Theorem 4.5 to the process  $Y$  which is an  $\mathcal{E}^{\mu, C_1}$ -martingale, hence an  $\mathcal{E}$ -supermartingale in  $\mathcal{S}^2$ . Consequently, there exists an increasing càdlàg predictable process  $(A_t)_{0 \leq t \leq T}$  such that  $A_0 = 0, \mathbb{E}(A_T^2) < \infty$  and  $(Y_t + A_t)_{0 \leq t \leq T}$  is an  $\mathcal{E}$ -martingale.

Recall that from Proposition 4.4, there exists a function, which can be denoted by  $f(s, z_0, u_0)$ , and processes  $\tilde{Z} \in \mathcal{L}^2(W), \tilde{U} \in \mathcal{L}^2(\tilde{\mu})$  satisfying

$$Y_t + A_t = Y_T^{z_0, u_0} + A_T + \int_t^T f(s, z_0, u_0) ds - \int_t^T \tilde{Z}_s dW_s - \int_t^T \int_B \tilde{U}_s(x) \tilde{\mu}(dsdx),$$

with  $\bar{f}_{\mu, C_1}(s, \tilde{Z}_s, \tilde{U}_s) \leq f(s, z_0, u_0) \leq f_{\mu, C_1}(s, \tilde{Z}_s, \tilde{U}_s).$

We derive an equality between the càdlàg finite variations process and the càdlàg martingale, which provides that  $(Z, U) = (z_0, u_0)$  in the space  $\mathcal{L}^2(W) \times \mathcal{L}^2(\tilde{\mu})$ .

Although we are ensured that  $f$  is well-defined, we need to check that the operator  $\mathcal{E}_f$  exists.

- i) The inequality  $\bar{f}_{\mu, C_1}(s, z_0, u_0) \leq f(s, z_0, u_0) \leq f_{\mu, C_1}(s, z_0, u_0)$  yields that  $f(s, 0, 0) = 0$ .
- ii) Proposition 4.4 also provides that  $f$  is Lipschitz w.r.t.  $z$ .
- iii) Using once more Proposition 4.4 we get that  $f$  satisfies assumption  $(A_\gamma)$  with  $\gamma_s^{z_0, u_0, u'_0}(x) = (|\mu| \mathbb{1}_{u_0 \geq u'_0}(x) + C_1 \mathbb{1}_{u_0 \leq u'_0}(x)) (1 \wedge |x|).$

Thus  $\mathcal{E}_f$  defines a non linear expectation. Furthermore, it is additive and  $\mathcal{E}_{\mu, C_1}$ -dominated.

Now we can prove point b). Indeed, for any  $r \leq t \in [0, T]$ , by construction of  $f$ ,

$$Y_t + A_t = Y_r + A_r + \int_r^t f(s, z_0, u_0) ds - \int_r^t z_0 dW_s - \int_r^t \int_B u_0(x) \tilde{\mu}(dsdx).$$

The additive property of  $\mathcal{E}$  and the martingale property of  $Y + A$  gives that the result holds for simple  $\mathcal{F}_r$ -measurable functions  $\zeta(\omega) = \sum_{i=1}^N z_i \mathbb{1}_{A_i}(\omega)$  and  $\vartheta(x)(\omega) = \sum_{i=1}^N u_i(x) \mathbb{1}_{A_i}(\omega), \forall \omega \in \Omega$ , where  $\{A_i\}_{i=1}^N$  is a  $\mathcal{F}_r$ -measurable partition of  $\Omega$ . Next we successively construct convenient approaching sequences. Corollary 3.9 provides the convergences and we are allowed to deduce that the result remains true for any processes  $\zeta \in \mathcal{L}^2(W)$  and  $\vartheta \in \mathcal{L}^2(\tilde{\mu})$ .

The last step deals with proving that it coincides with  $\mathcal{E}$ . Let us fix  $\eta \in \mathcal{L}^2(\mathcal{F}_T)$ .

We set  $\widehat{Y}_t = \mathcal{E}(\eta|\mathcal{F}_t)$  and we can also define  $(\widehat{Y}_t, \widehat{Z}_t, \widehat{U}_t)_{0 \leq t \leq T}$  as the unique solution of the BSDE  $(f, \eta)$ . By construction  $(\widehat{Y}, \widehat{Z}, \widehat{U}) \in \mathcal{S}^2 \times \mathcal{L}^2(W) \times \mathcal{L}^2(\tilde{\mu})$ .

Taking the non linear expectation of this BSDE and applying property b), we obtain that

$$\mathcal{E}(\eta - \widehat{Y}_0) = \mathcal{E}\left(-\int_0^T f(s, \widehat{Z}_s, \widehat{U}_s) ds + \int_0^T \widehat{Z}_s dW_s + \int_0^T \int_B \widehat{U}_s(x) \tilde{\mu}(dsdx)\right) = 0.$$

From the additive property of  $\mathcal{E}$ , we deduce that  $\widehat{Y}_0 = \mathcal{E}(\eta)$ . Hence,  $\mathcal{E} = \mathcal{E}_f$ .

□

Interest of inverse Theorem 4.6 lies in the fact that it allows to transpose a problem on a general non linear expectation into a study of BSDEs.

In this way, we obtain the following result: since process  $A$  in Doob-Meyer's decomposition of Theorem 4.1 is unique, then we also get uniqueness of decomposition for Theorem 4.5.

Article [6] also states an inverse theorem in the framework of a Brownian filtration. The authors express any non linear filtration-consistent expectation, as soon as it is additive and  $\mathcal{E}^\mu$ -dominated, as a solution of a BSDE with the simple form

$$Y_t = \eta + \int_t^T g(s, Y_s, Z_s) ds - \int_t^T Z_s dW_s.$$

In our context, we obtain a representation for any variable in  $\mathcal{L}^2(\mathcal{F}_T)$ , with  $(\mathcal{F}_t)_{0 \leq t \leq T}$  generated by the two mutually independant processes: a Brownian motion and a Poisson random measure. Consequently, we express a non linear expectation as the solution of

$$Y_t = \eta + \int_t^T f(s, Y_s, Z_s, U_s) ds - \int_t^T Z_s dW_s - \int_t^T \int_B U_s(x) \tilde{\mu}(dsdx).$$

Thus, the first case refers to continuous conditional expectations whereas current Theorem 4.6 allows to identify discontinuous conditional expectations.

## 5 Appendix

### 5.1 Proof of Doob-Meyer's decomposition for $\mathcal{E}_f$ -Smg

#### Proof of Theorem 4.1

The key of the proof lies in approaching process  $Y$  by a penalized sequence  $(Y^n)_{n \in \mathbb{N}}$ , where  $(Y_t^n, Z_t^n, U_t^n)_{0 \leq t \leq T}$  is defined as the unique solution of the BSDE with terminal value  $Y_T$  and driven by  $f_n(t, y, z, u) = f(t, y, z, u) + n(Y_t - y)^+$ , see [23].

Then  $(Y_t^n)_{n \in \mathbb{N}}$  is a nondecreasing sequence such that  $\forall n \in \mathbb{N}, \forall t \in [0, T]$ ,

$$\mathcal{E}_f(Y_T|\mathcal{F}_t) \leq Y_t^n \leq Y_t.$$

Hence sequences  $(Y_t^n)_{n \in \mathbb{N}}$ ,  $(Z_t^n)_{n \in \mathbb{N}}$  and  $(U_t^n)_{n \in \mathbb{N}}$  are bounded in respective spaces  $\mathcal{S}^2$ ,  $\mathcal{L}^2(W)$  and  $\mathcal{L}^2(\tilde{\mu})$ . We denote by  $C$  the common bound. We then deduce the following estimate :

$\mathbb{E}(\int_0^T |Y_s - Y_s^n|^2 ds) \leq \frac{C}{n^2}$ , which allows to identify the almost sure limit of  $(Y_t^n)_{n \in \mathbb{N}}$  as  $Y_t$ . And we even derive convergence in  $\mathcal{L}^2$  thanks to Lebesgue's dominated convergence Theorem. We also get that  $\mathbb{E}(\int_0^T |f(s, Y_s^n, Z_s^n, U_s^n)|^2 ds) \leq C$ . Note that boundedness of all these sequences, only

needs following conditions :

$$|f(s, Y_s^n, Z_s^n, U_s^n)| \leq \mu \|Z_s^n\| + \Gamma \left( \int_B |U_s^n(x)|^2 \lambda(dx) \right)^{1/2} \quad \text{and} \quad f(s, Y_s^n, 0, 0) = 0.$$

Then we can extract subsequences which weakly converge in the related spaces, and we call  $Z_t$ ,  $U_t$  and  $g_t$  respective weak limits of subsequences from  $(Z_t^n)_{n \in \mathbb{N}}$ ,  $(U_t^n)_{n \in \mathbb{N}}$  and  $(f(t, Y_t^n, Z_t^n, U_t^n))_{n \in \mathbb{N}}$ . Let us consider a stopping time  $\tau$  such that  $0 \leq \tau \leq T$ . By uniqueness of weak adherence value, entire sequences weakly converge to their respective limits in  $\mathcal{L}^2(\mathcal{F}_\tau)$ . By consequence, weak convergence also holds for  $(A_t^n)_{n \in \mathbb{N}}$ , where we set  $A_t^n = n \int_0^t |Y_s - Y_s^n| ds$ . We denote by  $A$  its limit; it is an increasing process with  $A_0 = 0$  and  $\mathbb{E}(|A_T|^2) < \infty$ . Given that  $A$  is equal to its dual predictable projection, then it is predictable.

Hence

$$A_\tau = Y_0 - Y_\tau - \int_0^\tau g_s ds + \int_0^\tau Z_s dW_s + \int_0^\tau \int_B U_s(x) \tilde{\mu}(dsdx).$$

Since the processes are *càdlàg* and equal for any stopping time  $\tau$ , then they are indistinguishable.

It remains to express  $g_s$  as  $f(s, Y_s, Z_s, U_s)$ . For this purpose, we need to prove strong convergence for sequences  $(Z_t^n)_{n \in \mathbb{N}}$  and  $(U_t^n)_{n \in \mathbb{N}}$ .

It's formula yields that for any stopping times  $\sigma$  and  $\tau$  such that  $0 \leq \sigma \leq \tau \leq T$ ,

$$\begin{aligned} & \mathbb{E} \left( \int_\sigma^\tau \|Z_s^n - Z_s\|^2 ds \right) + \mathbb{E} \left( \int_\sigma^\tau \int_B |U_s^n(x) - U_s(x)|^2 \lambda(dx) ds \right) \\ & \leq \mathbb{E}(|Y_\tau^n - Y_\tau|^2) + 2 \mathbb{E} \left( \int_\sigma^\tau (f(s, Y_s^n, Z_s^n, U_s^n) - g_s) (Y_s^n - Y_s) ds \right) \\ & + 2 \mathbb{E} \left( \int_\sigma^\tau \Delta(Y_{s-}^n - Y_{s-}) dA_s \right) + 2 \mathbb{E} \left( \int_\sigma^\tau (Y_{s-}^n - Y_{s-}) dA_s \right), \end{aligned} \quad (3)$$

where for any *càdlàg* process  $X$ ,  $\Delta X_s$  denotes the jump  $X_s - X_{s-}$ .

From now and on, we have to face new difficulties because neither processes  $Y^n$  nor  $Y$  are continuous. Their jumps come from Poisson stochastic integral and also process  $A$  (whereas each  $A^n$  is continuous).

Let us denote by  $N_t$  the following integral term  $N_t = \int_0^t \int_B (U_s^n(x) - U_s(x)) \tilde{\mu}(dsdx)$ .

Boundedness of the sequence  $(U_t^n)_{n \in \mathbb{N}}$  in  $\mathcal{L}^2(\tilde{\mu})$  implies that  $\mathbb{E}(\sup_{0 \leq t \leq T} |N_t|^2) \leq 4C$ . Consequently, we can apply the following Lemma.

### Lemma 5.1

Let  $A$  be an increasing predictable process. We consider its decomposition as a sum of a continuous and a purely discontinuous process (see [11]) :  $A_t = A_t^c + A_t^d$ .

We also consider a *càdlàg* martingale  $N$ , bounded in  $\mathcal{L}^2$ .

(a) For any stopping time  $\tau$  such that  $0 \leq \tau \leq T$ ,  $\mathbb{E} \left( \int_0^\tau \Delta N_s dA_s^c \right) = 0$ .

(b) For any predictable stopping time  $\tau$  such that  $0 \leq \tau \leq T$ ,

$$\mathbb{E} \left( \int_0^\tau \Delta N_s dA_s^d \right) = \mathbb{E} \left( \sum_{0 \leq s \leq \tau} \Delta N_s \Delta A_s^d \right).$$

Now, the key of the proof is a convenient choice of predictable stopping times. We may require that the union of intervals  $]\sigma_k, \tau_k]$  fulfill enough  $[0, T]$ , and also that jumps of process  $A^d$  are controlled on each interval  $]\sigma_k, \tau_k]$ . In other words, we fix  $\varepsilon$  and  $\delta$  in  $[0, 1[$  and we look for predictable times  $\sigma_k, \tau_k$ ,  $k = 0, 1, \dots, N$  such that  $]\sigma_j, \tau_j] \cap ]\sigma_i, \tau_i] = \emptyset$ ,  $\forall j \neq i$ ,

and  $\mathbb{E} \left( \sum_{k=0}^N (\tau_k - \sigma_k)(\omega) \right) \geq T - \varepsilon$ ,  $\sum_{k=0}^N \mathbb{E} \left( \sum_{\sigma_k < t \leq \tau_k} |\Delta A_t^d|^2 \right) \leq \delta$ .

We recall that S. Peng had already constructed a sequence of stopping times with such properties, in [23]. But in our framework, we need every  $\sigma_k, \tau_k$  to be predictable. Besides when filtration is only generated by a Brownian motion, notions of optionality and predictability are equivalent, whereas when the filtration is generated both by a Brownian motion and a Poisson random measure, an adapted càdlàg process is not necessarily predictable.

Nevertheless, we overcome the difficulty because  $A$  is a predictable process. Furthermore, we approach suitable predictable stopping times by announcing sequences (nondecreasing sequences of predictable stopping times).

Now, we set  $\delta' = \frac{\varepsilon^2 \delta^2}{64(C+1)}$ . We obtain

$$\mathbb{E} \left( \sum_{k=0}^N \int_{\sigma_k}^{\tau_k} \Delta(Y_s^n - Y_s) dA_s \right) \leq \mathbb{E} \left( \sum_{k=0}^N \int_{\sigma_k}^{\tau_k} \Delta A_s dA_s \right) + \delta'^{1/2} \mathbb{E} \left( \sum_{k=0}^N \sum_{\sigma_k < s \leq \tau_k} |\Delta N_s|^2 \right)^{1/2}.$$

and

$$\mathbb{E} \left( \sum_{k=0}^N \int_{\sigma_k}^{\tau_k} \|Z_s^n - Z_s\|^2 ds \right) + \mathbb{E} \left( \sum_{k=0}^N \int_{\sigma_k}^{\tau_k} \int_B |U_s^n(x) - U_s(x)|^2 \lambda(dx) ds \right) \leq \varepsilon \delta.$$

Therefore, it provides a convergence in measure for processes  $Z$  and  $U$ . Denoting by  $m$  Lebesgue's measure on  $[0, T]$ , from the construction of the predictable times  $\sigma_k$  and  $\tau_k$ , we can state that

$$\mathbb{P} \times m \left\{ (\omega, s) \in \Omega \times [0, T] \mid \|Z_s^n - Z_s\|^2 \geq \delta \right\} \leq 2\varepsilon,$$

$$\mathbb{P} \times m \left\{ (\omega, s, x) \in \Omega \times [0, T] \mid \int_B |U_s^n(x) - U_s(x)|^2 \lambda(dx) \geq \delta \right\} \leq 2\varepsilon.$$

We finally get strong convergence for sequences  $Z$  and  $U$  because  $Z^q$  and  $U^q$  are uniformly integrable for any real  $q$  in  $[1, 2[$ .

Once strong convergence is proved for sequences  $Z$  and  $U$ , we can explicit the generator of the equation. Indeed, Minkovski's inequality provides strong convergence of  $(f(\cdot, Y^n, Z^n, U^n))_{n \geq 0}$  towards  $f(\cdot, Y, Z, U)$  in  $\mathcal{L}^q_{\mathcal{F}}([0, T])$  for any  $q$  in  $[1, 2[$ . Taking into account uniqueness of weak limit, we identify  $\int_0^t g_s ds = \int_0^t f(s, Y_s, Z_s, U_s) ds$ . Finally, we conclude that

$$Y_t = Y_T + \int_t^T f(s, Y_s, Z_s, U_s) ds - \int_t^T Z_s dW_s - \int_t^T \int_B U_s(x) \tilde{\mu}(ds dx) - A_t + A_T.$$

We observe that when  $f$  does not depend on  $y$ , the triple of processes

$(Y_t + A_t, Z_t, U_t)_{0 \leq t \leq T}$  can be seen as the unique solution of BSDE driven by  $f$  and with terminal value  $Y_T + A_T$ . And so  $Y_t + A_t = \mathcal{E}_f(Y_T + A_T | \mathcal{F}_t)$ .  $\square$

## 5.2 Proof of Doob-Meyer's decomposition for $\mathcal{E}$ -Smg

### Proof of Theorem 4.5

We recall that in Theorem 4.1, we approached process  $Y$  by the way of penalization method. But if we look at this proof from another point of view, we guess that sequence  $(Y_t^n)_{n \geq 0}$  can be expressed as a non linear conditional expectation. Indeed, still denoting  $A_t^n = n \int_0^t (Y_s - Y_s^n)^+ ds$ , we defined  $Y_t^n = \mathcal{E}_f(Y_T + A_T^n - A_t^n | \mathcal{F}_t)$ . Existence Theorem in [17] ensured existence of such a sequence whereas comparison Theorem 2.5 provided its monotony.

Hence, we are led to construct the following sequence  $Y_t^n = \mathcal{E}(Y_T + A_T^n - A_t^n | \mathcal{F}_t)$  with once again  $A_t^n = n \int_0^t (Y_s - Y_s^n) ds$ , provided that such a sequence exists. Moreover, we may need a comparison theorem in order to prove that sequence  $(Y_t^n)_{n \geq 0}$  is nondecreasing.

By consequence, we study equations of a new kind:  $\bar{Y}_t = \mathcal{E}(\eta + \int_t^T h(s, \bar{Y}_s) ds | \mathcal{F}_t)$ , with  $h(s, y) = n(Y_s - y)$  and  $\eta = Y_T$ .

**Lemma 5.2**

We consider  $h : \Omega \times [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$  Lipschitz w.r.t.  $y$  and such that for any fixed  $y$ ,  $h(\cdot, y) \in \mathcal{L}^2(W)$ . For any  $\eta \in \mathcal{L}^2(\mathcal{F}_T)$ , there exists a process  $\bar{Y}$  such that  $\bar{Y}_t = \mathcal{E}(\eta + \int_t^T h(s, \bar{Y}_s) ds | \mathcal{F}_t)$ . Moreover,  $\bar{Y}$  is unique in  $\mathcal{S}^2$ . And for any  $0 \leq t \leq r \leq T$ , we have  $\bar{Y}_t = \mathcal{E}(\bar{Y}_r + \int_t^r h(s, \bar{Y}_s) ds | \mathcal{F}_t)$ .

Lemma 5.2 guarantees existence of sequence  $(Y_t^n)_{n \geq 0}$  and next one provides a comparison theorem for such a sequence.

**Lemma 5.3**

Let  $\eta_1, \eta_2 \in \mathcal{L}^2(\mathcal{F}_T)$  and  $\Phi : [0, T] \rightarrow \mathbb{R}$  a nonnegative function.

We denote  $h_1(s, y) = h(s, y)$  and  $h_2(s, y) = h(s, y) + \Phi(s)$

Consider  $\bar{Y}^1$  and  $\bar{Y}^2$  respective solutions of equations  $\bar{Y}_t^i = \mathcal{E}(\eta_i + \int_t^T h_i(s, \bar{Y}_s^i) ds | \mathcal{F}_t)$ .

If  $\eta_1 \leq \eta_2$ , then  $\bar{Y}^1 \leq \bar{Y}^2$ . Moreover,  $\bar{Y}^1 = \bar{Y}^2$  if and only if  $\eta_1 = \eta_2$  and  $\Phi = 0$   $d\mathbb{P} \times dt$  a.e.

**Proof of Lemma 5.3**

First part of the proof can be found in [6]. Nevertheless we cannot prove strong version of comparison theorem in the same way as in [6] because it lies on the continuity of process  $\bar{Y}$ .

Suppose that  $\eta_1 \leq \eta_2$ ,  $\Phi \geq 0$  and  $\bar{Y}_0^1 = \bar{Y}_0^2$ .

Then,

$$\mathcal{E}\left(\eta_1 + \int_0^T h(s, \bar{Y}_s) ds\right) = \mathcal{E}\left(\eta_2 + \int_0^T (h(s, \bar{Y}_s) + \Phi(s)) ds\right).$$

Given that  $\eta_1 + \int_0^T h(s, \bar{Y}_s) ds \leq \eta_2 + \int_0^T (h(s, \bar{Y}_s) + \Phi(s)) ds$ ,

then by strict monotonicity property of non linear expectation, we get that

$$\eta_1 + \int_0^T h(s, \bar{Y}_s) ds = \eta_2 + \int_0^T (h(s, \bar{Y}_s) + \Phi(s)) ds.$$

Consequently,  $\eta_1 = \eta_2$  and  $\Phi = 0$   $d\mathbb{P} \times dt$  a.e. This proves Lemma 5.3.

□

It yields that  $(Y_t^n)_{n \geq 0}$  is an increasing sequence. Next, we adopt the same strategy as in the proof of Theorem 4.1 and we show that  $\forall n \in \mathbb{N}, \forall t \in [0, T], Y_t^n \leq Y_t$ . Then increasing sequence  $(Y_t^n)_{n \geq 0}$  almost surely converges to some process and we need a further study to ensure that it equals  $Y$ .

Lemma 5.2 provides that for any  $t, r$  such that  $0 \leq t \leq r \leq T$ , we also have  $Y_t^n = \mathcal{E}(Y_r^n + \int_t^r |Y_s^n - Y_s| ds | \mathcal{F}_t)$ . In other words,  $Y_t^n + A_t^n = \mathcal{E}(Y_r^n + A_r^n | \mathcal{F}_t)$ . This corresponds to the definition of an  $\mathcal{E}$ -martingale.

Next, Proposition 4.4 allows us to express  $Y^n + A^n$  as a solution of a BSDE. Thus for any integer  $n \in \mathbb{N}$ , there exists a function  $g^n$  and processes  $(Z^n, U^n) \in \mathcal{L}^2(W) \times \mathcal{L}^2(\tilde{\mu})$  such that

$$Y_t^n + A_t^n = Y_T + A_T^n + \int_t^T g_s^n ds - \int_t^T Z_s^n dW_s - \int_t^T \int_B U_s^n(x) \tilde{\mu}(dsdx)$$

with  $\bar{f}_{\mu, C_1}(s, Z_s^n, U_s^n) \leq g_s^n \leq f_{\mu, C_1}(s, Z_s^n, U_s^n)$ .

We point out that we come to the same framework as for penalization method in the proof of Theorem 4.1, except that initial generator  $g^n$  also depends on  $n$ .

$$Y_t^n = Y_T + \int_t^T (n|Y_s - Y_s^n| + g_s^n) ds - \int_t^T Z_s^n dW_s - \int_t^T \int_B U_s^n(x) \tilde{\mu}(dsdx).$$

Nevertheless, we can keep the same approach. Indeed, in the proof of Theorem 4.1, Step 3, we only used the fact that  $|f(s, Y_s^n, Z_s^n, U_s^n)| \leq K_1 \|Z_s^n\| + K_2 \left(\int_B |U_s^n(x)|^2 \lambda(dx)\right)^{1/2}$ , where it is sufficient that constants  $K_1$  and  $K_2$  do not depend on  $n$ .

And in current context,  $|g_s^n| \leq |\mu| \|Z_s^n\| + \beta \left( \int_B |U_s^n(x)|^2 \lambda(dx) \right)^{1/2}$ ,

with  $\beta = \max(|\mu|, -C_1) \left( \int_B (1 \wedge |x|^2) \lambda(dx) \right)^{1/2}$ .

In particular, there exists an increasing predictable process  $A$  such that

$$|Y_t^n - \mathcal{E}(Y_T + A_T - A_t | \mathcal{F}_t)| = |\mathcal{E}(Y_T + A_T^n - A_t^n | \mathcal{F}_t) - \mathcal{E}(Y_T + A_T - A_t | \mathcal{F}_t)|$$

Corollary 3.9 provides convergence in  $\mathcal{L}^2$ , and by uniqueness of the limit, we obtain  $Y_t = \mathcal{E}(Y_T + A_T - A_t | \mathcal{F}_t)$ .  $\square$

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