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REFLECTED BROWNIAN MOTION: SELECTION, APPROXIMATION AND LINEARIZATION

MARC ARNAUDON AND XUE-MEI LI

ABSTRACT. We constructed a family of SDEs with parameter a whose solutions converge to a reflected Brownian flow as a approaches 0 in UCP. This selects a reflected Brownian 'flow'. We then prove the stochastic damped transports W_t^a along (Y_t^a) converge in a suitable sense. This selects a damped stochastic parallel transport and a probabilistic representation for the solution to the heat equation on differential 1-forms with absolute boundary conditions. The limiting process can be described by its jumps and the standard stochastic damped parallel transport equation with an additional shape operator driven by the local time. It can be constructed by a family of tangent space valued stochastic processes (W_t^ε), who evolves with the stochastic damped parallel transport equation in the interior, and whose normal part is erased at the end of an excursion of size greater or equal to ϵ. As $\epsilon \to 0$, this family of stochastic processes converge to a tangent space valued process (W_t) with jumps, in the topology of uniform convergence. For the half space, this is exactly the process constructed in N. Ikeda and S. Watanabe [21] by Poisson point processes. This allow us to approximate the local time on the boundary in UCP, but not in the semi-martingale topology. We also conclude W_t being the weak derivative of Y_t with respect to the starting point.

1. INTRODUCTION

A. Let M be a smooth connected d-dimensional Riemannian manifold with boundary ∂M and interior M^o. Both the interior and the boundary, not necessarily connected, are smooth manifolds without boundary. Denote Δ the Laplacian on functions and $\Delta^1 = -(d^*d + dd^*)$ the Hodge-Laplace-Beltrami operator restricted to differential 1-forms. The essentially self-adjoint operators satisfy $d\Delta = \Delta^1 d$ on smooth functions with compact supports.

Let us consider heat conduction in a perfect insulator. Let $u : [0,T] \times M \to \mathbf{R}$ be a solution to the following heat equation with initial value and Neumann boundary condition:

$$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u \ \text{ in } M^o, \quad du|_{\partial M}(\nu) = 0, \quad u(0,\cdot) = f \tag{1.1}$$

Then $u(t, Y_0) = \mathbf{E} f(Y_t)$ where Y_t is a reflected Brownian motion (RBM). How would we simulate a RBM? We might want to select one with desired properties, which also solves the heat equation for differential 1-forms. The spatial differential $du(t,\cdot)$ solves the heat equation on differential 1-forms:

$$\frac{\partial \phi}{\partial t} = \frac{1}{2}\Delta^1 \phi, \quad \phi(t,\nu) = 0, \quad \phi(0,\cdot) = df \tag{1.2}$$

where ν denotes the unit inward pointing normal vector on the boundary. This equation is under determined. We impose the absolute boundary condition $\phi(\nu) = 0$ and $d\phi|_{\partial M}(t,\nu) = 0$ where d denotes exterior differentiation. The absolute boundary condition is automatically satisfied by $\phi = du$, and is equivalent to $\nabla_\nu \phi - \phi(\nabla \nu) = 0$ given the Dirichlet boundary condition $\phi(\nu) = 0$. Solving heat equation for differential

1-forms helps us to understand differentiating solution to the heat equation for functions. But heat equations for differential forms are interesting of their own rights. For example de Rham cohomologies, defined by quotients of closed forms and exact forms are topological invariants. An L^2 Hodge decomposition theorem would relate this to the space of harmonic differential forms. The existence and vanishing of harmonic forms for manifolds with boundary are naturally interesting questions. A study of heat equation on differential forms for manifolds with boundary goes back to P. E. Conner [11]. P. Malliavin [24] and H. Airault [2] studied harmonic forms using perturbations to a boundary operator, see also A. Méritet [25] for a result on the vanishing of the first de Rham cohomology on manifolds with boundary. See also I. Shigekawa, N. Ueki, and S. Watanabe [27] who gave a proof of the Gauss-Bonnet-Chern Theorem on manifolds with boundaries.

To solve the heat equation on 1-forms with the absolute boundary condition, we seek a tangent space valued process W_t such that $\phi_t(W_0) = \mathbf{E}\phi(0, W_t)$. An ansatz is the stochastic covariant differential equation along a given RBM: $\frac{D}{dt}W_t = -\frac{1}{2}\mathrm{Ric}^{\#}(W_t)$ where for each ω, $\frac{D}{dt}$ denotes covariant differentiation of $W_t(\omega)$ along the path $Y_t(\omega)$ using the Levi-Civita connection ∇ on M, the parallel equation damped by the Ricci curvature. This is the case for a manifold without a boundary and is essentially true for the half plane. As we will explain, it is necessary to consider the shape of the boundary and add the shape operator $\mathcal{S}$:

$$\frac{D}{dt}W_t = -\frac{1}{2}\mathrm{Ric}^{\#}_{Y_t}(W_t) + \mathcal{S}_{Y_t}(W_t)dL_t$$

where L_t is the boundary time of the RBM? We use W_t to stand both for the tangent vector $W_t \in T_{Y_t}M$ and the linear map $W_t : T_{Y_0}M \to T_{Y_t}M$, in the latter sense (W_t) are 'damped parallel translations'. The solution of the equation is one of the solution, but our approximation will select a different one: the minimal one.

The RBM spends Lebesgue measure zero time on the boundary and there is a great deal of freedom in choosing a pathwise solution. Given a RBM (Y_t) there is more than one such process: if (Y_t, W_t) solves the absolute boundary problem, so would (Y_t, W'_t) if they agree on the interior and their tangential parts agree on the boundary. Since we allow for jumping on the boundary, the values of (W_t) at the boundary times do not determine the value of the process in the interior and vice versa. Our approximation selects the 'minimal' damped parallel translation along our RBM, which for the half space is exactly the process constructed by N. Ikeda and S. Watanabe using Poisson point processes where they did not need the shape operator. Denote R_t the distance of Y_t to the boundary. The Poisson point process of R_t determines the frequency of the projection. The 'local time' or the boundary time of the RBM is the local time of R_t at 0 which is essentially the δ-measure on the boundary, $\int_0^t \delta_{\partial M}(Y_s)ds$.

B. In case of M having no boundary, stochastic parallel translation goes back to K. Itô [22], J. Eells, D. Elworthy [12], P. Malliavin [24], H. Airault [2] and N. Ikeda and S. Watanabe [20]. The damped parallel translations along a Brownian motion x_t are constructed using stochastic parallel translation. Together with x_t it is an $\mathbb{L}(TM, TM)$-valued diffusion process along x_t with Markov generator $\frac{1}{2}\Delta^1$. It is well understood that $\mathbf{E}df(W_t)$, where $f \in BC^2(M)$, solves the heat equation for 1-forms. See e.g. D. Elworthy [13], J.-M. Bismut [9], N. Ikeda and S. Watanabe [20], and P.-A. Meyer [26]. Furthermore if P_t is the heat semi-group, the process $dP_{T-t}f(W_t)$ is a local martingale. See [23] for a systematic study of probabilistic representations of heat semigroups on differential forms using both the damped parallel translation and the derivative flow to a Brownian system. It is also well known that (W_t) can be obtained from the derivative process $d\phi_t$ of a Brownian flow by

conditioning, from which information on the derivative of the heat semigroup on functions can be obtained. This method was used by M. Arnaudon, B. Driver, K. D. Elworthy, Y. Le-Jan, X.-M. Li and A. Thalmaier [23, 15, 14, 16, 28, 5], and many other related works. This plays also a role in the study of the regularity of finely harmonic maps, c.f. M. Arnaudon, X.-M. Li and A. Thalmaier [8]. The stochastic Jacobi field point of view is developed in M. Arnaudon and A. Thalmaier [6], leading to estimates for the derivatives of harmonic maps between manifolds and to Liouville-type theorems.

The study of stochastic damped parallel transports on manifolds with boundary began with the upper half plane preluding which the construction of the reflecting Brownian motion (Y_t) by a canonical horizontal stochastic differential equation (SDE) with drift given by the horizontal lift of the 'reflecting' vector field. See N. Ikeda [19], Watanabe [30], and N. Ikeda and S. Watanabe [20]. The same SDE on the orthonormal frame bundle defines also the stochastic parallel transport process $/\!\!/_t(Y)$, a sample continuous stochastic process with values in the tangent bundle. Their construction of the damped parallel translation is harder than that of the parallel translation. The latter reduces to solving a 'reflected' stochastic differential equation while the former is a stochastic process with seemingly arbitrarily induced jumps. Our results shows that the introduction of the jumps are not at all arbitrary.

Since a manifold with boundary can be transformed to the upper half plane by choices of charts, as pointed out in N. Ikeda and the S. Watanabe [20], this implies the local existence of damped parallel transport on a general manifold. We note that H. Airault studied Dirichlet Neumann problems [1, 2] using a different approach, multiplicative functionals, which were followed up by E. Hsu [18] to give a neat treatment for the reflected SDE on the orthonormal frame bundle. In Appendix C, we extend and explain Ikeda-Watanabe's construction [21] for damped parallel transport, the formulation involves an additional curvature term, the shape operator of the normal vector ν. For the upper half plane, the shape operator vanishes identically.

C. Our aim is to construct a nice family of SDEs whose solutions are smooth and stay in the interior for all time, in particular there is no boundary time. The solutions Y_t^a and their derivative flows W_t^a select a RBM and a damped stochastic parallel transport along the RBM. Although Y_t^a and W_t^a are sample continuous, the process selected by (Y_t^a, W_t^a) has jumps on the boundary.

Our construction is motivated by the following well known facts for manifold without boundaries: (1) damped parallel translations can be obtained from perturbation of the Brownian motion with respect to its initial data and (2) a formula for the derivative of the heat semigroup follows naturally from differentiating a Brownian flow with respect to its initial value. More precisely if $d\phi_t(v)$ is the derivative of a Brownian flow $\phi_t(x)$ with respect to its initial data x, then W_t is the conditional expectation of $d\phi_t(v)$ with respect to the filtration of the Brownian flow. From our construction it would be trivial to see the formula $dQ_t f(v) = \frac{1}{t}\mathbf{E} f(\phi_t(x))M_t$ for manifolds with boundary, where (M_t) is a suitable local martingale.

Let us recall that a Brownian system is a stochastic differential equations (SDE)

$$dx_t = \sum_{k=1}^{m} \sigma_k(x_t) \circ dB_t^k + \sigma_0(x_t)dt,$$

with infinitesimal generator $\frac{1}{2}\Delta$ where $\circ$ denotes Stratonovich integration. Such equations always exist and is not unique. One particularly nice Brownian system is the gradient Brownian system by which we mean σ_i are the gradient vector fields given by an isometric

embedding. Formally a RBM is a solution to an SDE:

$$dY_t = \text{“}dx_t\text{”} + \nu \cdot \delta_{\partial M}(Y_t)dt.$$

Its approximations are given by

$$dY_t^a = \text{“}dx_t\text{”} + A^a(Y_t^a)dt,$$

We hope that $L_t^a = \int_0^t A^a(Y_s^a)ds$ approximate $\nu \cdot \delta_{\partial M}(Y_t)dt$ and the solutions to

$$\frac{D}{dt}W_t^a = -\frac{1}{2}\mathrm{Ric}^{\#}_{Y_t^a}(W_t^a) + \mathcal{S}_{Y_t^a}(W_t^a)dL_t^a$$

will have a limit. Note that dL_t^a and dL_t has mutually singular supports and it is clear not every choice of “dx_t” yield an approximation. Our choice is rather natural which we will elaborate later. Then W_t^a converges, in a sense which we specify later, to a minimal probability representation for the heat equation for one forms. This limits has jumps on the boundary.

2. Outline and Main Theorems

The paper is organised as following. In Section 3 we construct a family of SDEs whose smooth solution flows $(Y_t^a, a > 0)$ exist. Close to the boundary, they are Brownian motions with drift $A^a(x)$, the gradient of $\ln \tanh\left(\frac{R}{a}\right)$ where $R(x)$ is the distance of x to the boundary. Furthermore they satisfy the following properties.

Theorem 3.4.

(a) *For each $a > 0$, (Y_t^a) remains in M^0;*
(b) *As $a \to 0$, (Y_t^a) converges to the reflected Brownian motion process (Y_t), in the topology of uniform convergence in probability (UCP).*

The convergence is in $\mathscr{S}_p([0,T])$ if M is compact, and indeed if the boundary has a tubular neighbourhood with a uniform lower bound and if suitable bounds on the curvatures are imposed. The notation concerning the convergence of stochastic processes on manifolds is introduced in Appendix B where we also summarise the relevant estimates for stochastic integrals in the $\mathscr{S}_p$ and $\mathscr{H}_p$ norm, as well as relating different notions of convergences.

For $M = \mathbf{R}_+$, the solution to the Skorohod problem is selected while the solution given by Tanaka's formula is not, see Appendix A for detail where we illustrate the construction.

The main idea behind the construction is to ensure that $R(Y_t^a)$ involves one single real valued Brownian motion for all parameters a, and hence pathwise analysis is possible. The construction involves a tubular neighbourhood of the boundary and the product metric in this tubular neighbourhood is used for the estimation. By the construction, converges in the other directions, when restricted to the tubular neighbourhood are trivial. Outside of the tubular neighbourhood the drift $A^a(x)$ vanishes while inside it converges to zero exponentially fast as $a \to 0$ for every positive x.

Furthermore, there exists a family of stochastic processes L_t^a converging to the local time L_t in $\mathscr{S}_p([0,T])$ but not in the $\mathscr{H}_p([0,T])$ topology. The former convergence is a key for the convergence of the damped parallel translations while the lack of convergence in the latter makes it difficult to follow the standard methods for the convergence of solutions of SDEs.

Corollary 3.6 and Lemma 7.8 *Let M be compact. Then for any $p > 1$ and $T > 0$,*

$$\lim_{a \to 0} \|L_t^a - L_t\|_{\mathscr{S}_p([0,T])} = 0.$$

Despite of the above mentioned convergence, dL_t and dL_t^a are mutually singular measures. In particular the total variation norm $|d(L_t - L_t^a)| \to 2|dL_t|$ as a approaches zero. We observe also that $|L_t^a|_{\mathscr{H}_p([0,T])} = |L_t^a|_{\mathscr{S}_p[0,T]}$, $|L_t|_{\mathscr{H}_p[0,T]} = |L_t|_{\mathscr{S}_p[0,T]}$ and $|L_t - L_t^a|_{\mathscr{H}_p([0,T])} = |dL_t|_{\mathscr{H}_p([0,T])} + |dL_t^a|_{\mathscr{H}_p([0,T])}$.

Below we state our main theorem whose proof is the content of Section 7. A preliminary and easier result on the convergence of the stochastic parallel transport is given in §4. We conclude that there exists a stochastic parallel transport process (W_t) along the reflected Brownian motion (Y_t) constructed in Theorem 3.4 such that the tangential parts of W_t^a converge to the tangential part of (W_t) in $\mathscr{S}_p([0,T])$ and the normal parts of W_t^a converge to the normal part of W_t in $\mathscr{S}_p([0,T])$.

If $v \in T_yM$, where y is in the boundary, we denote v^T and v^ν respectively its tangential and normal component. Thus $(W_t^{a,T})$ is the tangential part of W_t^a and

$$W_t^a = W_t^{a,T} + f_a(t)\nu(Y_t^a).$$

Lemma 7.11, Lemma 7.12 and Theorem 5.6. Let M be a compact Riemannian manifold. Let $p \in [1,\infty)$. Then

(1) $\lim_{a\to 0} W_\cdot^{a,T} = W_\cdot^T$ in UCP.
(2)
$$\mathbf{E}\left[\int_0^T |f_a(t) - f(t)|^p\, dt\right] \to 0.$$
(3) For any C^2 differential 1-form ϕ such that $\langle \phi, \nu\rangle = 0$ on boundary,
$$\lim_{a\to 0} \sup_{s\le t} \mathbf{E}\, |\phi(W_s^a) - \phi(W_s)|^p = 0.$$

The family of sample continuous processes (W_t^a) cannot converges to the stochastic process (W_t) with jumps, in the topology of uniform convergence in probability, nor in the $\mathscr{S}_p$ norm nor in the Skorohod topology. The uniform distance between a continuous path and a path with jumps is at least half of the size of the largest jump. Two elementary results are the ideas behind the proof, Lemma 7.9 and Corollary 7.10. For example, it will be proven that there exists a real continuous process $c_a(u)$ such that $df_a(t) = -c_a(t)f_a(t)\,dt +$ *other terms*, and the limit as $a \to 0$ of $-\int_0^t c_a(u)f_a(u)\,du$ is the only one which creates jumps. More precisely, if $\alpha_t = \sup_{s\le t}\{s \le t : Y_s \in \partial M\}$ and $t \notin \mathscr{R}(\omega)$, then for all $s,t \in [0,S]$ satisfying $s < t$,

$$\lim_{a\to 0} e^{-\int_s^t c_a(u)\,du} = 1 \quad \text{if} \quad s > \alpha_t$$
$$\lim_{a\to 0} e^{-\int_s^t c_a(u)\,du} = 0 \quad \text{if} \quad s < \alpha_t.$$

Let us now explain the jumps in the limiting process W_t, see the Appendix for more detail. Let ρ denote the distance function to the boundary. Viewed by the distance process $R_t = \rho(Y_t, \partial M)$, the reflected Brownian motion reaches the boundary at a stopping time ζ where it makes excursions into the interior. These excursions are similar to those made by a real valued reflected Brownian motion. The compliment to the set of boundary points of R_t are disjoint open intervals. Let $\mathscr{R}(\omega)$ denote the set of the 'right most points' of the excursions of the distance function R_t and $\mathscr{R}_\epsilon(\omega)$ its subset coming from excursions of length at least ϵ. Denote by $\mathrm{Ric}_x^\#$ the Ricci curvature as a linear map from T_xM to T_xM and $\mathcal{S}$ the second fundamental form of the level sets of R, see (C.5).

The stochastic process $W_t \in L(T_{Y_0}M; T_{Y_t}M)$ is the unique càdlàg process along the reflected Brownian motion (Y_t) satisfying the following equation during an excursion to the interior:

$$\begin{aligned} W_t(v) =& v - \frac{1}{2}/\!\!/_t \int_0^t /\!\!/_s^{-1} \mathrm{Ric}^\sharp_{Y_{s-}}(W_{s-}(v))ds - /\!\!/_t \int_0^t /\!\!/_s^{-1} \mathcal{S}_{Y_{s-}}(W_{s-}(v))dL_s \\ & - \sum_{s \in \mathscr{R}(\omega)\cap[0,t]} \langle W_{s-}(v), \nu(Y_s)\rangle /\!\!/_{s,t} \nu(Y_s), \end{aligned} \tag{2.1}$$

where $/\!\!/_{s,t} = /\!\!/_{s,t}(Y_\cdot)$ denotes the parallel transport process along $(Y_\cdot)$ from $T_{Y_s}M$ to $T_{Y_t}M$. We abbreviate $/\!\!/_{0,t}$ to $/\!\!/_t$. If we remove only the normal part of W_t on excursion intervals equal or exceeding size ε, the above stated procedure makes sense and the resulting processes (W_t^ε) has a limit. The above equation is understood in this limiting sense. More precisely, let W_t^ε the solution to

$$\begin{aligned} DW_t^\varepsilon &= -\frac{1}{2}\mathrm{Ric}^\sharp(W_t^\varepsilon)\, dt - \mathcal{S}(W_t^\varepsilon)\, dL_t - \chi_{\{t \in \mathscr{R}_\varepsilon(\omega)\}} \langle W_{t-}^\varepsilon, \nu_{Y_t}\rangle \nu_{Y_t}, \\ W_0^\varepsilon &= \mathrm{Id}_{T_{Y_0}M}. \end{aligned} \tag{2.2}$$

There exists an adapted right continuous stochastic process W_t such that $\lim_{\epsilon\to 0} W_t^\epsilon = W_t$ in UCP and in $\mathscr{S}_2$ for M compact. Furthermore for any $\alpha \in \mathcal{T}$, the set of stochastic processes with values in bounded 1-forms above Y_t vanishing outside some compact set, c.f. (C.6),

$$\lim_{\varepsilon\to 0} \left(\int_0^{\cdot\wedge\tau_D} \alpha_s(DW_s^\varepsilon) \right) \overset{\mathscr{S}_2}{=} \left(\int_0^{\cdot\wedge\tau_D} \alpha_s(DW_s) \right).$$

This construction agrees with the derivative flow of the Skorohod reflected Brownian motion in the half line, extending stochastic damped parallel translation of Ikeda and Watanabe to general manifolds with boundary. This was further elaborated in Section 6 where we prove damped parallel translation is a weak derivative of the reflected Brownian flow. See also E. Hsu [18] and F. Wang [29] where two other constructions for damped parallel translations are given. However the shape operators are not explicitly mentioned. In Proposition 5.1 we explain that removing the normal part of the damped parallel translation at the beginning of excursions leads to the same object in the limit. In *Theorem 5.3* we observe that (W_t) has the local martingale property when composed with the differential of a solution to the heat equation with Neumann boundary condition, c.f. [21]. A stochastic representation for the semigroup on differential one forms with absolute boundary conditions follows.

Theorem 5.2 *Suppose that the tubular neighbourhood of the boundary has positive radius, the curvatures* $\mathrm{Ric}^\#$ *and* $\mathcal{S}$ *are bounded. If* $\phi(t,\cdot)$ *is a solution to the heat equation with absolute boundary conditions, then* $\phi(t,\cdot) = \mathbf{E}\phi(0, W_t(\cdot))$.

Finally we consider the problem of obtaining (W_t^a) by varying the initial value in a family of stochastic flows. We have previously obtained a family of processes $(Y_t^a, 0 < a \le 1)$ which converges as $a \to 0$ in to a reflected Brownian motion in L^p. For a fixed we consider a variation $(Y_t^a(u), u \in [0,1])$ satisfying $Y_0^a(u) = \varphi(u)$ where $\varphi(u)$ is a C^1 curve and $\partial_u Y_t^a(u) = W_t^a(u)\dot\varphi(u)$, see [4] for the construction.

Proposition 6.1, Theorem 6.2 and Theorem 6.3 *If M is compact, then the two parameter family of stochastic processes $\{Y_\cdot^a(u), a \in (0,1], u \in [0,1]\}$ is tight. Furthermore if $Y_\cdot^{a_k}(u)$ converges weakly to $\{Y_t(u), u \in [0,1]\}$ then*

(1) *for every $u \in [0,1]$, $Y_t(u)$ is a reflected Brownian motion on M started at $\varphi(u)$;*

(2) *for every* $p \in [1, \infty)$ *there exists a number* $C'(p, T)$ *s.t. for all* $0 \le u_1, u_2 \le 1$,

$$\sup_{0 \le t \le T} \mathbf{E}\left[\rho^p(Y_t(u_1), Y_t(u_2))\right] \le C'(p, T)\|\dot{\varphi}\|_\infty (u_2 - u_1)^p. \tag{2.3}$$

(3) *For all* $f \in C^2(M)$ *with* $df|_{\partial M}(\nu) = 0$,

$$\mathbf{E}\left[f(Y_t(u_2)) - f(Y_t(u_1)) - \int_{u_1}^{u_2} df\big(W_t(u)\dot{\varphi}(u)\big)\, du\right] = 0.$$

The last identity implies that if $\dot{\phi}(u) = v$, then $\frac{\partial}{\partial u} Y_t(\phi(u)) = W_t(u)(v)$. Thus the damped parallel translation can be interpreted as a derivative of the RBM with respect to the initial point, making connection with the study of K. Burdzy [10], and S. Anders [3]. for Euclidean domains.

3. A REFLECTED BROWNIAN FLOW

In [21], N. Ikeda and S. Watanabe constructed reflected Brownian motions by the method of orthonormal frames. The local theorem is as following, which implies in particular that a non-sticky SDE on a manifold has a local solution, which is unique. Denote by $C_b^{0,1}$ the space of bounded and Lipschitz continuous function in a domain D.

Theorem 3.1 ([21, Thm 7.2])**.** *Let* $D = \mathbf{R}_d^+ = \{(x_1, \dots, x^d), x^d \ge 0\}$. *For* $i = 1, \dots, m$, *let* $\sigma_k^i, b^i \in C_b^{0,1}(D)$ *and* $\delta \in C_b^{0,1}(\partial D)$. *Suppose that for some number* $C > 0$, $\sum_{k=1}^m (\sigma_k^d(x))^2 \ge C$ *and* $\delta(x) \ge C$. *Then there is a pair of adapted stochastic processes,* (x_t, L_t), *s.t.*

$$\begin{aligned} dx_t^i &= \sum_{k=1}^m \sigma_k^i(x_t)\chi_{D^0}(x_t) dB_t^k + b^i(x_t)\chi_{x_t \in D^0} dt, \quad 1 \le i \le d-1 \\ dx_t^d &= \sum_{k=1}^m \sigma_k^d(x_t)\chi_{D^0}(x_t) dB_t^k + b^d(x_t)\chi_{x_t \in D^0} dt + \delta(x_t) dL_t \\ &\int_0^t \chi_{x_s \in \partial D} dL_s = L_t, \quad \forall t \ge 0 \ \ a.s. \end{aligned} \tag{3.1}$$

Here (L_t) *is continuous, increasing process and such that* $l_0 = 0$. *Furthermore uniqueness in law holds.*

For the reflected problem in local coordinates, take $\delta \equiv 1$. Other reflected stochastic processes, with not necessarily normal reflection, can also be constructed. All reasonable constructions should lead to the same essential quantity: the local time of the Brownian motion on the boundary. This philosophy follows from the uniqueness to the associated sub-martingale problem. Denote $C^j(M)$ the space of real valued C^j smooth functions on M and $C_K^j(M)$ its subspace of C^j smooth functions with compact support. Let P be a probability measure on $C(M)$ and (X_t) the canonical process. A probability measure P on $C(M)$ is a sub-martingale problem associated with $\mathcal{L} = \frac{1}{2}\Delta + Z$ with initial value x, if

$$X_t^f := f(X_t) - f(x) - \int_0^t \mathcal{L} f(X_s) ds$$

is a martingale for all $f \in C_K^\infty(M^0)$, and a sub-martingale for all $f \in C_K^\infty(M)$ with $df(\nu) \ge 0$.

We define a reflected Brownian motion as a solution to the Skorohod problem. A solution to Skorohod problem is a sample continuous strong Markov process satisfying the

following properties: (1) its Markov generator $\mathcal{L}$ restricted to $C_K^2(M^0)$ is $\frac{1}{2}\Delta$; (2) it spends almost all the time in the interior of the manifold (with respect to Lebesgue measure on time set); and (3) its drift on the boundary is colinear with ν, the inward-pointing unit normal vector field on ∂M. By inward-pointing we mean the sign of ν is chosen so that the exponential map, for sufficiently small $t > 0$, $\exp(t\nu)$ belongs to M^0.

If $\{\sigma_1, \dots \sigma(x)\}$ is a family of vector fields spanning the tangent space at each point, we associate to it a bundle map $\sigma : M \times \mathbf{R}^m \to TM$ given by $\sigma(x)(e) = \sum_{k=1}^m \sigma_k(x)\langle e_k, e\rangle$ where $\{e_i\}$ is an orthonormal basis of $\mathbf{R}^m$.

Definition 3.2. A stochastic flow is a Riemannian reflected Brownian flow if it solves the Skorohod problem

$$dY_t = \sigma(Y_t) \circ dB_t + \sigma_0(Y_t)dt + A(Y_t)dL_t, \tag{3.2}$$

where σ is a smooth bundle map, σ_0 is a smooth vector field, A is a smooth vector field extending the inward normal vector field, and L_t is the local time of Y_t at ∂M, a non-decreasing process satisfying that

$$\int_0^t \chi_{(\partial M)^c}(Y_s)dL_s = 0.$$

Since dL_t is supported on the boundary of M, Y_t behaves exactly like a Brownian motion in the interior.

The aim of the section is to construct a family of Brownian systems on M with large drift A^a pushing away from the boundary, approximating a reflected Brownian motion in the topology of uniform convergence in probability. Furthermore we wish the convergence is 'uniform' and the limiting process is continuous with respect to the initial data.

3.1. We denote by ρ the Riemannian distance function on M and by R the distance function to the boundary, $R(x) = \inf\{\rho(x, y) : y \in \partial M\}$. By the tubular neighbourhood theorem, there exists a continuous function $\delta : \partial M \to (0, \infty)$ such that for $E_0 = \{(x, t) : x \in \partial M,\ t < \delta(x)\}$, the map

$$\begin{aligned} \Phi : E_0 &\to F_0 := \Phi(E_0) \subset M \\ (x, t) &\mapsto \exp_x(t\nu_x) \end{aligned} \tag{3.3}$$

is a diffeomorphism such that $\Psi(y) = (\pi(y), R(y))$ on F_0 where $\pi(y)$ is the boundary point given by $\rho(y, \pi(y)) = R(y)$ and $\Psi = \Phi^{-1}$. In other words, on F_0, $R\Big(\Phi(x, t)\Big) = t$ and the distance function R is smooth. For $0 \le c \le 1$, define

$$E_c = \{(x, t) : t < (1 - c)\delta(x)\}, \quad F_c = \Phi(E_c).$$

Since ∂M is a Riemannian manifold of its own right, we may represent $\Delta_{\partial M} = \sum_{j=2}^m \bar{\sigma}_j$ as the sum of squares of vector fields, for example by taking $\{\bar{\sigma}_j, j = 2, \dots, m\}$ to be a gradient system on ∂M.

In the tubular neighbourhood around a relatively compact set U of the boundary, the width $\delta(x)$ can be taken to be a positive constant $3\delta_0$. Let R be a real valued 1-Lipschitz smooth function on M which on $F_{1/3}$ agrees with the distance function to the boundary and such that $R \ge \delta_0$ on $F_{1/3}^c$. This can be obtained by modifying the distance function to the boundary.

Proposition 3.3. *Let $r \geq 1$ and $c \in (0,1)$. Let $\{\bar{\sigma}_0,\ \bar{\sigma}_j, j = 2, \ldots, m\}$ be a family of C^r vector fields on ∂M with the property that*

$$\frac{1}{2}\sum_{j=2}^{m} L_{\bar{\sigma}_j} L_{\bar{\sigma}_j} + L_{\bar{\sigma}_0} = \frac{1}{2}\Delta_{\partial M}.$$

Suppose that E_0 has strictly positive radius. Let δ_0 be a positive number such that $\delta \geq 3\delta_0$. Then there exist a finite number of C^r vector fields $\{\sigma_j, j = 0, \ldots, N\}$ on M such that

(1) $\frac{1}{2}\sum_{j=1}^{N} L_{\sigma_j} L_{\sigma_j} + L_{\sigma_0} = \frac{1}{2}\Delta$,
(2) $\sigma_1 = \nabla R$ *on* F_0,
(3) *For* $2 \leq j \leq m$, σ_j *extends* $\bar{\sigma}_j$,
(4) *For all* $p \in \Phi(\partial M \times [0, \delta_0])$, *for all* $j \geq 2$: $\langle \sigma_1, \sigma_j \rangle_p = 0$.

In the sequel we denote $A = \sigma_1$.

Proof. We first assume that there exist a family of locally defined vector fields $\sigma_j, 1 \leq j \leq m$ in the tubular neighbourhood F_0. Denote $\sigma : M \times \mathbf{R}^m \to TM$ the corresponding bundle map, as indicated earlier. Let us extend the construction to M and then return to the local construction.

Let $\tilde{\sigma}_i, m+1 \leq i \leq m+m'$ be a family of vector fields in M such that

$$\sum_{i=m+1}^{m+m'} L_{\tilde{\sigma}_i} L_{\tilde{\sigma}_i} = \Delta.$$

Let $\tilde{\sigma}$ be the corresponding bundle map from $M \times \mathbf{R}^{m'} \to TM$ and let $N = m + m'$. We take a real valued function $\beta \in C^\infty(M; [0,1])$ with the property that $\sqrt{\beta}$ and $\sqrt{1-\beta}$ are smooth, $\beta|_{F_{2/3}} = 1$and $\beta|_{M \setminus F_{1/3}} = 0$. Let us define a new bundle map $\hat{\sigma} : \mathbf{R}^m \times \mathbf{R}^{m'} \to TM$ as following:

$$\hat{\sigma}(e_1, e_2) = \sqrt{\beta}\,\sigma(e_1) + \sqrt{1-\beta}\,\tilde{\sigma}(e_2),$$

and prove that it is a surjection. Let

$$(\hat{\sigma})^*(v) = (\sqrt{\beta}\sigma^*(v), \sqrt{1-\beta}(\tilde{\sigma})^*(v)).$$

Then $(\hat{\sigma})^* : T_x M \to \mathbf{R}^N$ is the right inverse to $\hat{\sigma}$. Indeed

$$\hat{\sigma}\hat{\sigma}^* = \beta\sigma\sigma^* + (1-\beta)\tilde{\sigma}(\tilde{\sigma})^* = id,$$

and σ induces the Laplace-Beltrami operator.

Let us now construct σ on F_0. If $y \in m$, let $\pi(y)$ denote the projection of y to M, γ_y the geodesic from $\pi(y)$ to y, and $/\!/(\gamma_y)$ the parallel transport along γ_y which is a linear map from $T_{\pi(y)}M$ to $T_y M$. For $j \neq 1$, along a geodesic normal to the boundary, we may extend $\bar{\sigma}_j$ by parallel transport along the geodesic in the normal vector direction:

$$\sigma_j(y) = /\!/(\gamma_y)\bar{\sigma}_j(\pi(y)).$$

In other words it is constant along the geodesic. It is clear that $\langle \sigma_j(x), \sigma_1(x) \rangle = 0$ for all $j > 1$ and $x \in F_0$.

From the assumption, $\frac{1}{2}\sum_k \nabla^{\partial}_{\bar{\sigma}_k}\bar{\sigma}_k + \bar{\sigma}_0 = 0$ on ∂M and $\sum_{j=2}^{m} \bar{\sigma}_j(x)\bar{\sigma}_j^\flat(x) = \mathrm{Id}_{T_x \partial M}$. Here $\bar{\sigma}_j^\flat(x)$ denotes the 1-form $\langle \sigma_j(x), \cdot \rangle$. Let us prove that for all $x \in F_0$,

$$\sum_{j=1}^{m} \sigma_j(x)\sigma_j^\flat(x) = \mathrm{Id}_{T_x M}$$

where $\sigma_j^\flat(x)$ is defined in a similar way. Since the vectors $\{\sigma_k(x)\}$ generate T_xM, it is sufficient to prove that for all $k = 1, \dots, m$,

$$\left(\sum_{j=1}^m \sigma_j(x)\sigma_j^\flat(x)\right)(\sigma_k(x)) = \sigma_k(x).$$

For $k = 1$:

$$\left(\sum_{j=1}^m \sigma_j(x)\sigma_j^\flat(x)\right)(\sigma_1(x)) = \sigma_1(x)\langle\sigma_1(x), \sigma_1(x)\rangle = \sigma_1(x).$$

For $k \geq 2$:

$$\begin{aligned}\left(\sum_{j=1}^m \sigma_j(x)\sigma_j^\flat(x)\right)(\sigma_k(x)) &= \sum_{j=2}^m \sigma_j(x)\langle\sigma_j(x), \sigma_k(x)\rangle \\ &= \sum_{j=2}^m /\!\!/(\gamma_x)\bar{\sigma}_j(\pi(x))\langle\bar{\sigma}_j(\pi(x)), \bar{\sigma}_k(\pi(x))\rangle \\ &= /\!\!/(\gamma_x)\bar{\sigma}_k(\pi(x)) = \sigma_k(x).\end{aligned}$$

To complete the construction, we take the drift vector with the following property:

$$\sigma_0(x) = -\frac{1}{2}\sum_{j=1}^m \nabla_{\sigma_j(x)}\sigma_j = -\frac{1}{2}\sum_{j=2}^m \nabla_{\sigma_j(x)}\sigma_j, \qquad x \in F_0.$$

This completes the proof. □

The vector field $A = \sigma_1$, constructed in Proposition 3.3 extends the unit inward normal vector field, defined on ∂M, and coincides with ∇R on $F_{2/3}$. Off the cut locus of the R, ∇R exists almost everywhere. For the Skorohod problem, we will only need the information of A on $F := F_{2/3}$ and in particular we do not need to worry the effect of the cut locus.

Next we take a family of additional drift vector fields converging to 0 in the interior of M and to the local time on the boundary. We divide the manifold M into three regions: inner tubular neighbourhood, the middle region and the outer region. The inner region, a subset of E_0 with the product metric is quasi isometric to its image, i.e. there is a constant $C > 0$ such that for all $x, y \in \partial M$, for all $s, t \in [0, \delta_0]$, denoting $\bar{\rho}$ the distance in ∂M,

$$\frac{1}{C}\left(\bar{\rho}(x, y) + |s - r|\right) \leq \rho\left(\Phi(x, s)\Phi(y, r)\right) \leq C\left(\bar{\rho}(x, y) + |s - r|\right).$$

Outside of the tubular neighbourhood the drifts will be chosen to be uniformly bounded and to converge to zero uniformly. If x_n is a sequence of points in the outer region with limit x_0, we need to assume that the solution with initial value x_n converges in some sense. In the first region we have convergence in probability and in the second we will need a control on the rate of convergence that induces the property of a flow.

By the convergence of the manifold valued stochastic process (Y_t^a) to (Y_t) we mean that $H(Y_t^a)$ converges to $H(Y_t)$ where $H : M \to \mathbf{R}^k$ is an embedding, with the same notion of convergence. We recall that (Y_t^a) converges to Y_t in UCP implies that for any $\epsilon > 0$, the explosion times ξ^a of Y_t^a and its exit times from relatively compact sets, for sufficiently small a, are bounded below by the corresponding ones for $Y_t = Y_t^0$ minus ϵ (in other words $\liminf_{a\to 0} \xi^a \geq \xi^0$). Also, if we assume sufficient growth control on the

curvatures and the shape of the tubular neighbourhood, the convergence will be in $\mathscr{S}_p$. We only discuss this aspect for a compact manifold.

Let $R : M \to \mathbf{R}$ be a smooth function such that $R|F$ is the distance to the boundary, and $R|_{F^c} \geq \tilde{\delta}$ for some number $\tilde{\delta}$, and $R(x)$ is a constant on the complement of F^c.

Theorem 3.4. *Let a be a positive number. For σ_k, A satisfying properties stated in Proposition 3.3, let $F_t^a(x)$ and $F_t(x)$ denote respectively the maximal solution to the equations, with initial value x,*

$$dY_t^a = \sum_{k=1}^m \sigma_k(Y_t^a) \circ dB_t^k + \sigma_0(Y_t^a)dt + \nabla \ln\left(\tanh\left(\frac{R(Y_t^a)}{a}\right)\right) dt, \quad (3.4)$$

$$dY_t = \sum_{k=1}^m \sigma_k(Y_t) \circ dB_t^k + \sigma_0(Y_t)dt + A(Y_t)dL_t. \quad (3.5)$$

Let $Y_t^a = F_t(Y_0^a)$ and $Y_t = F_t^a(Y_0)$. Suppose that $\rho(Y_0^a, Y_0)$ converges to 0 in probability.

(1) *Then $\lim_{a\to 0} Y^a = Y$, in the topology of uniform convergence in probability.*
(2) *If M is compact, then the SDEs do not explode and for all $p \in [1, \infty)$ and for all $T > 0$, Y^a converges to Y in $\mathscr{S}_p([0, T])$, i.e.*

$$\lim_{a\to 0} \mathbf{E} \sup_{0\leq s\leq T} \rho(Y_s^a, Y)^p \to 0.$$

Proof. Since UCP convergence is local and is implied by local convergence in $\mathscr{S}_p$, (1) is a consequence of (2). See Corollary B.4. So we assume that M is compact and choose a constant $\delta_0 > 0$ such that the function δ is bounded below by $3\delta_0$.

We define $h^a(x) = \ln\left(\tanh\left(\frac{R(x)}{a}\right)\right)$ and

$$A^a(x) = \nabla h^a(x) = \frac{2\nabla R(x)}{a \sinh\left(\frac{2R(x)}{a}\right)}.$$

This is an approximation for a vector field that vanishes on M^0 and exerts an 'infinity' force in the direction of $\nabla R = A$ on the boundary.

Let $R_t^a = R(Y_t^a)$. Then

$$R_t^a = R_0^a + \sum_k \int_0^t \langle dR, \sigma_k(Y_s^a)\rangle dB_s^k + \frac{1}{2}\int_0^t \Delta R(Y_s^a)ds + \int_0^t \frac{2}{a\sinh(\frac{2R_s^a}{a})}ds.$$

Let us denote by β_t^a the stochastic term:

$$R_t^a = R_0^a + \beta_t^a + \frac{1}{2}\int_0^t \Delta R(Y_s^a)ds + \int_0^t \frac{2}{a\sinh(\frac{2R_s^a}{a})}ds.$$

For $Y_t^a \in F_{1/3}$ the tubular neighbourhood of ∂M, we have by Proposition 3.3 (2) that $d\beta_t^a = dB_t^1$ is independent of a and of Y_t^a, which will be crucial for the sequel:

$$dR_t^a = dB_t^1 + \frac{1}{2}\Delta R(Y_t^a)dt + \frac{2}{a\sinh(\frac{2R_t^a}{a})}dt. \quad (3.6)$$

Note that on the boundary $\nabla_\cdot(\nabla R)$ is the shape operator, $\mathrm{Hess}R(v_1, v_2) = \langle S(v_1), v_2\rangle$.

Since we assumed that M is compact, $|\Delta R|$ is bounded, so the drift is essentially $\frac{2}{a\sinh(\frac{2R_s^a}{a})}$ and R_t^a never touches the boundary and the equation is well defined.

Recall that π is the map that sends a point $x \in M$ to the nearest point on ∂M, it is defined on F_0. The tubular neighbourhood map $\Psi : F_0 \to E_0$ splits into two parts, $\Psi(x) =$

$(\pi(x), R(x))$. Since Ψ is a diffeomorphism onto its image, on $\{Y_t \in F_0\}$, the processes Y_t^a converges to Y_t in the Riemannian metric on M if and only if $\Psi(Y_t^a)$ converges to $\Psi(Y_t)$ in the product metric of $\partial M \times [0, \delta]$.

On any subset of M not intersecting the tubular neighbourhood that is distance $c\delta$ from ∂M for some $c < 1$, the functions $|\nabla A^a|$ is uniformly bounded in a and converge to zero as $a \to 0$. The local time does not charge any real time if Y_t is not on the boundary. For a C^3 embedding $\Phi : M \to \mathbf{R}$,

$$\begin{aligned}\Phi(Y_t^a) - \Phi(Y_t) =& \Phi(Y_0^a) - \Phi(Y_0) + \int_0^t \langle (\sigma^* \nabla \Phi)(Y_s^a) - (\sigma^* \nabla \Phi)(Y_s), dB_s \rangle \\ &+ \frac{1}{2} \int_0^t \left(\Delta \Phi(Y_s^a) - \Delta \Phi(Y_s) \right) ds.\end{aligned}$$

By standard estimates, if $Y_0^a \to Y_0$ in probability, the processes $\Phi(Y_t^a)$ started outside the closed tubular set $F_{\frac{1}{3}}$ and stopped at the first entrance time of $F_{\frac{2}{3}}$ converge to $\Phi(Y_t)$ in UCP. In particular this holds for isometric embeddings and since the intrinsic Riemannian distance is controlled by the extrinsic distance function, we see that the stochastic process $\rho(Y_t^a, Y_t)$ converges in UCP.

Splitting in a proper way the times, for the UCP topology it is enough to prove that the processes (Y_t^a) started inside the open set $F_{\frac{2}{3}}$ and stopped at exiting $F_{\frac{1}{3}}$ converge to (Y_t) whenever $Y_0^a \to Y_0$.

So we assume that Y_0^a and Y_0 belong to $F_{2/3}$ and $\rho(Y_0^a, Y_0)$ converges to 0 in probability. We let

$$\tau = \inf\{t \geq 0,\ R(Y_t) = 2\delta_0\}, \quad \tau^a = \inf\{t \geq 0,\ R(Y_t^a) = 2\delta_0\}.$$

We first prove that for all $T > 0$,

$$\forall T > 0,\ \lim_{a \to 0} \mathbf{E}\left[\sup_{t \leq \tau^a \wedge \tau \wedge T} \rho^2(Y_t^a, Y_t) \right] = 0. \tag{3.7}$$

Notice if (3.7) holds, $\sup_{s \leq t} R(Y_s) < 2\delta_0$ implies that $\sup_{s \leq t} R(Y_s^a) < 2\delta_0$ for sufficiently small a, consequently,

$$\lim_{a \to 0} \mathbf{E}\left[\sup_{t \leq \tau \wedge T} \rho^2(Y_t^a, Y_t) \right] = 0. \tag{3.8}$$

This in turn shows that

$$\liminf_{a \to \infty} \tau^a \wedge T \geq \tau \wedge T \tag{3.9}$$

and the convergence of Y_t^a to Y_t in the UCP topology follows.

Let $R_t^a = R(Y_t^a)$ and $R_t = R(Y_t)$. Denote $\bar{\rho}$ the Riemannian distance on ∂M. Using the tubular neighbourhood map, proving (3.7) will be equivalent to prove the following two limits:

$$\lim_{a \to 0} \mathbf{E}\left[\sup_{t \leq \tau^a \wedge \tau \wedge T} (R_t^a - R_t)^2 \right] = 0, \quad \lim_{a \to 0} \mathbf{E}\left[\sup_{t \leq \tau^a \wedge \tau \wedge T} (\bar{\rho})^2(\pi(Y_t^a), \pi(Y_t)) \right] = 0. \tag{3.10}$$

For $t \leq \tau^a \wedge \tau$ we have by (3.6) and (3.2),

$$R_t^a - R_t = R_0^a - R_0 + \int_0^t \frac{ds}{a \sinh\left(\frac{2R_s^a}{a}\right)} - L_t + \frac{1}{2} \int_0^t \left(\Delta R(Y_s^a) - \Delta R(Y_s) \right) ds. \tag{3.11}$$

We remark that in the above equation there is no martingale part. Let $\varepsilon > 0$, and

$$\tilde{L}^a_t = \int_0^t \frac{ds}{a \sinh\left(\frac{2R^a_s}{a}\right)} - L_t.$$

We apply Itô-Tanaka formula to the convex function $\max(y, \varepsilon)$ to obtain:

$$\begin{aligned}\varepsilon \vee |R^a_t - R_t| =& |R^a_0 - R_0| \vee \varepsilon + \int_0^t \chi_{\{R^a_s - R_s > \varepsilon\}}\, d\tilde{L}^a_s - \int_0^t \chi_{\{R^a_s - R_s < -\varepsilon\}}\, d\tilde{L}^a_s \\ &+ \frac{1}{2}\int_0^t \chi_{\{|R^a_s - R_s| > \varepsilon\}} \left(\Delta R(Y^a_s) - \Delta R(Y_s)\right) ds.\end{aligned}$$

It is vital to remark that $L_s > 0$ if and only if $R^a_s - R_s = R^a_s$. Also $R^a_s - R_s < -\varepsilon$, if only if $R_s \neq 0$, and so $-d\tilde{L}^a_s$ is a negative measure. We may ignore the third term on the right hand side of the identity. For each $\alpha > 0$ and $\varepsilon > 0$, there exists a number $a(\varepsilon, \alpha) > 0$ such that for all $a \leq a(\varepsilon, \alpha)$ and $r \geq \varepsilon$, $\frac{1}{a \sinh\left(\frac{2r}{a}\right)} < \alpha$. Hence,

$$\begin{aligned}&\int_0^t \chi_{\{R^a_s - R_s > \varepsilon, R_s = 0\}} d\tilde{L}^a_s \\ &\leq \int_0^t \frac{1}{a \sinh\left(\frac{2(R^a_s - R_s)}{a}\right)} \chi_{\{R^a_s - R_s > \varepsilon, R_s = 0\}} ds - L_t \leq \alpha t, \\ &\int_0^t \chi_{\{R^a_s - R_s > \varepsilon, R_s \neq 0\}} d\tilde{L}^a_s = \int_0^t \frac{1}{a \sinh\left(\frac{2R^a_s}{a}\right)} \chi_{\{R^a_s - R_s > \varepsilon, R_s \neq 0\}} ds \\ &\leq \int_0^t \frac{1}{a \sinh\left(\frac{2\varepsilon}{a}\right)} ds \leq \alpha t,\end{aligned}$$

It follows that

$$\varepsilon \vee |R^a_t - R_t| \leq |R^a_0 - R_0| + \varepsilon + 2\alpha t + \frac{1}{2}\|\nabla \Delta R\|_{L^\infty(F_0)} \int_0^t \sup_{r \leq s} \rho(Y^a_r, Y_r)\, ds. \quad (3.12)$$

So

$$\begin{aligned}\mathbf{E}\left[\sup_{t \leq \tau \wedge \tau^a \wedge T} (R^a_t - R_t)^2\right] \leq& 4\mathbf{E}\left[(R^a_0 - R_0)^2\right] + 4\varepsilon^2 + 8\alpha^2 t^2 + \\ &2\|\nabla \Delta R\|_{L^\infty(F_0)} \int_0^T \mathbf{E}\left[\sup_{s \leq \tau \wedge \tau^a \wedge t} \rho^2(Y^a_s, Y_s)\right] dt.\end{aligned} \quad (3.13)$$

Before continuing with the estimate above, we estimate $\bar{\rho}(\pi(Y^a_t, \pi(Y_t))$. The distance function $\bar{\rho}$ is not smooth on $\partial M \times \partial M$. So we will consider an isometric embedding $\imath : \partial M \to \mathbf{R}^{m'}$ (in fact since ∂M is compact any embedding would do) and instead of proving the second limit in (3.10) we will prove that

$$\lim_{a \to 0} \mathbf{E}\left[\sup_{t \leq \tau^a \wedge \tau \wedge T} (\imath(\pi(Y^a_t)) - \imath(\pi(Y_t)))^2\right] = 0 \quad (3.14)$$

We extend $\imath$ to F_0 to obtain $\tilde{\imath}(y) = (\imath \circ \pi)(y)$, then

$$\tilde{\imath}(Y_t^a) - \tilde{\imath}(Y_t) = \tilde{\imath}(Y_0^a) - \tilde{\imath}(Y_0) + \int_0^t \langle \sigma^* \nabla \tilde{\imath}(Y_s^a) - \sigma^* \nabla \tilde{\imath}(Y_s), dB_s \rangle + \frac{1}{2} \int_0^t \left(\Delta \tilde{\imath}(Y_s^a) - \Delta \tilde{\imath}(Y_s) \right) ds + \int_0^t d\tilde{\imath}(A^a(y_s^a)) ds - \int_0^t d\tilde{\imath}(A(y_s)) dL_s. \tag{3.15}$$

Since $d\pi(A) = 0$ and $d\pi(A^a) = 0$, the last two terms vanish. By standard calculation,

$$\begin{aligned} &\mathbf{E}\left[\sup_{t \le T \wedge \tau \wedge \tau^a} \|\tilde{\imath}(Y_t^a) - \tilde{\imath}(Y_t)\|^2 \right] \\ &\le 4\mathbf{E}\left[\|\tilde{\imath}(Y_0^a) - \tilde{\imath}(Y_0)\|^2 \right] + 16 \|\nabla \sigma^* \nabla \tilde{\imath}\|^2_{L^\infty(F_0)} \int_0^T \mathbf{E}\left[\sup_{s \le t \wedge \tau \wedge \tau^a} \rho^2(Y_s^a, Y_s) \right] dt \\ &+ 2\|\nabla \Delta \tilde{\imath}\|_{L^\infty(F_0)} \int_0^T \mathbf{E}\left[\sup_{s \le \tau \wedge \tau^a \wedge t} \rho^2(Y_s^a, Y_s) \right] dt. \end{aligned} \tag{3.16}$$

Since ∂M is compact, F_0 is compact. The quantities $\nabla \sigma^*$ and $\nabla \tilde{\imath} = \nabla \pi(\nabla i)$ are bounded. Similarly $\|\nabla \Delta \tilde{\imath}\|_{L^\infty(F_0)}$ is finite. For $x \in F_0$, set

$$H(x) = (\tilde{\imath}(x), R(x)) \in \mathbf{R}^{m'+1}. \tag{3.17}$$

Let $C_H > 0$ be a constant such that for all $x, x' \in F_0$,

$$\frac{1}{C_H} \|H(x) - H(x')\| \le \rho(x, x') \le C_H \|H(x) - H(x')\|. \tag{3.18}$$

Define

$$C = \left(16 \|\nabla \sigma^* \nabla \tilde{\imath}\|^2_{L^\infty(F_0)} + 2\|\nabla \Delta H\|_{L^\infty(F_0)} \right) (C_H)^2. \tag{3.19}$$

From (3.16), using Gronwall lemma that if $a < a(\varepsilon)$,

$$\mathbf{E}\left[\sup_{s \le T \wedge \tau \wedge \tau^a} \|H(Y_s^a) - H(Y_s)\|^2 \right] \le 4 \left(\|H(Y_0^a) - H(Y_0)\|^2 + \varepsilon^2 + \alpha^2 T^2 \right) e^{CT}. \tag{3.20}$$

Since ε and α can be chosen as small as we like and C is independent of $\varepsilon,\ \alpha,\ a, Y_0^a \to Y_0$ and H is bounded, we obtain that

$$\lim_{a \to 0} \mathbf{E}\left[\sup_{s \le T \wedge \tau \wedge \tau^a} \|H(Y_s^a) - H(Y_s)\|^2 \right] = 0. \tag{3.21}$$

Together with (3.13), we see that

$$\lim_{a \to 0} \mathbf{E}\left[\sup_{s \le T \wedge \tau \wedge \tau^a} \rho^2(Y_s^a, Y_s) \right] = 0. \tag{3.22}$$

This implies that $\tau^a \wedge \tau \to \tau$ almost surely, and since the distance is bounded,

$$\lim_{a \to 0} \mathbf{E}\left[\sup_{s \le T \wedge \tau} \rho^2(Y_s^a, Y_s) \right] = 0. \tag{3.23}$$

This completes the proof for the convergence of Y^a to Y in UCP, and also in $\mathscr{S}_p$ for compact manifold M. □

We emphasise that for any open relatively compact set U with $\tau^U(Y)$ (resp. $\tau^U(Y^a)$), the exit time of $Y_\cdot$ (resp. $Y_\cdot^a$) from U,

$$\liminf_{a\to 0} \tau^U(Y^a) \geq \tau^U(Y).$$

Corollary 3.5. *Let S_1, S_2 be stopping times such that $S_1 < S_2$ and a_0 a positive constant. Suppose that $Y_t^a \in F_0$ for $a \in (0, a_0]$ and $t \in [S_1, S_2]$. Then on the interval $[S_1, S_2]$, $\lim_{a\to 0} \pi(Y_\cdot^a) = \pi(Y_\cdot)$. The convergence is in the semi-martingale topology. If moreover M is compact and S_2 is bounded, then the convergence hold in $\mathscr{H}_p$ for all $p \in [1, \infty)$.*

Proof. Since the drifts A and A^a belong to the kernel of the differential $T\pi$, we obtain with Itô formula the following equations:

$$d(\pi(Y_t^a)) = \sum_{k=1}^m T\pi\sigma_k(Y_t^a) \circ dB_t^k + T\pi \circ \sigma_0(Y_t^a)\, dt$$

$$d(\pi(Y_t)) = \sum_{k=1}^m T\pi \circ \sigma_k(Y_t) \circ dB_t^k + T\pi \circ \sigma_0(Y_t^a)\, dt.$$

Since $\frac{1}{2}\sum_{k=1}^m \nabla_{\sigma_k}\sigma_k + \sigma_0 = 0$, we only need to be concerned with the following term from the Itô correction: $\frac{1}{2}\sum_{k=1}^m \nabla T\pi(\cdot)(\sigma_k, \sigma_k)$. By Theorem 3.4, both $T\pi \circ \sigma_k(Y_t^a)$ and $\nabla T\pi(Y_t^a)(\sigma_k, \sigma_k)$ converge in the UCP topology, and they are locally uniformly bounded. The limits are respectively $T\pi\sigma_k(Y_t)$ and $\nabla T\pi(Y_t)(\sigma_k, \sigma_k)$. By Theorem 2 in [17], see [7] for the manifold case, $\pi(Y_\cdot^a)$ converges to $\pi(Y_\cdot)$ in the semimartingale topology. □

Define $L_t^a = \int_0^t \frac{ds}{a\sinh\left(\frac{2R_s^a}{a}\right)}$. Then in the tubular neighbourhood,

$$A^a(Y_t^a) = \nabla R(Y_t^a)\frac{d}{dt}L_t^a.$$

Corollary 3.6. *Suppose that M is compact. Then for all $p \geq 1$ and $T > 0$,*

$$\lim_{a\to 0} \sup_{s\leq T} |L_s^a - L_s|^p = 0.$$

Moreover, letting $L^0 = L$, for all $\lambda > 0$, there exists $C(T, \lambda)$ such that for all $a \in [0, 1]$,

$$\mathbf{E}\left[e^{\lambda L_T^a}\right] \leq C(T, \lambda). \tag{3.24}$$

Proof. Firstly we take Y_0^a, Y_0 in $F_{2/3}$, the $\frac{2}{3}$ tubular neighbourhood of the boundary. Let $\tau = \inf\{R(Y_t) = 2\delta_0\}$ and $\tau^a = \inf\{R(Y_t^a) = 2\delta_0\}$ be respectively the first exit times of Y and Y^a from $\{x : R(x) < 2\delta_0\} \subset F_{\frac{2}{3}}$. On $\{t < \tau^a \wedge \tau\}$ we have (3.11):

$$L_t^a - L_t = -R_0^a + R_0 + R_t^a - R_t - \frac{1}{2}\int_0^t \left(\Delta R(Y_s^a) - \Delta R(Y_s)\right) ds. \tag{3.25}$$

By the convergence of Y^a to Y in $\mathscr{S}_p([0, T])$,

$$\mathbf{E} \sup_{t<\tau^a\wedge\tau} |L_t^a - L_t|^p < \infty.$$

Outside of the $2/3$ tubular neighbourhood $F_{\frac{2}{3}}$, $\frac{1}{a\sinh(\frac{2R(x)}{a})}$ converges to 0 uniformly in x and L_t vanishes. Note that $\lim_{a\to 0} A^a(r) \to 0$ for any $r > 0$. The required convergence result follows.

To prove (3.24) we write for $a \in [0, 1]$

$$L_t^a = R_t^a - R_0^a + \int_0^t \alpha_s^a\, dZ_s^a + \int_0^t \beta_s^a\, ds$$

where for all a, Z_t^a is a real valued Brownian motion, and $|R_t^a - R_0^a|$, α_t^a and β_t^a are uniformly bounded independently of a. The result immediately follows. □

4. Convergence of the parallel transports

Let (Y_t) and (Y_t^a) are respectively the solutions of (3.5) and (3.4). The parallel transport along (Y_t) and (Y_t^a) are respectively the solution to the canonical horizontal stochastic differential equations on the orthonormal frame bundle with drift the horizontal lift of the drift vector fields A and A^a respectively.

Denote by $/\!\!/_t^a$ the parallel transport along Y_t^a, $/\!\!/_s$ the parallel transport along Y_t. Recall that $\sigma_y^* : T_yM \to \mathbf{R}^m$ is the right inverse to σ_y. Take $v^a \in T_{Y_0^a}M$ and $v \in T_{Y_0}M$ with the property that $\lim_{a\to 0} \sigma^*(Y_t^a)(v^a) = \sigma^*(Y_t^a)(v)$. Let U be a continuous vector field. Then

$$\langle /\!\!/_t^a v^a, U(Y_t^a)\rangle = \langle \sigma^*(Y_t^a)(/\!\!/_t^a v^a), \sigma^*(Y_t^a)(U(Y_t))\rangle$$

Since $Y_t^a \to Y_t$ as $a \to 0$, so does $\sigma^*(Y_t^a)U(Y_t^a)$ to $\sigma^*(Y_t)U(Y_t)$. We prove below that $\sigma^*(Y_t^a)(/\!\!/_t^a v^a) \to \sigma^*(Y_t)(/\!\!/_t v)$.

Proposition 4.1. *Let $v^a \in T_{Y_0^a}M$ and $v \in T_{Y_0}M$. Suppose that $\sigma^*_{Y_0^a} v^a$ converges to $\sigma^*_{Y_0} v$ in probability as $a \to 0$. Then $\lim_{a\to 0} \sigma^*(Y_t^a)(/\!\!/_t^a v^a) = \sigma^*(Y_t)(/\!\!/_t v)$, with convergence in the semi-martingale topology. Also $\lim_{a\to 0} /\!\!/_t^a v^a \overset{UCP}{\to} /\!\!/_t v$.*

If M is compact, the convergences hold respectively in $\mathscr{H}_p([0,T])$ and $\mathscr{S}_p([0,T])$ for all $T > 0$ and $p \geq 1$.

Proof. Since $\nabla_{\circ dY_t^a}(/\!\!/_t^a v^a) = 0$ by the definition, the stochastic differential of $/\!\!/_t^a v^a$ satisfies the following equation:

$$d(\sigma^*(Y_t^a)/\!\!/_t^a v^a) = \nabla_{\circ dY_t^a}\sigma^*(Y_t^a)/\!\!/_t^a v^a = \sum_{j=2}^m (\nabla_{\sigma_j(Y_t^a)}\sigma^*)/\!\!/_t^a v^a \circ dB_t^j$$

where we used the fact that $\nabla_\nu \sigma^* = 0$. Upon converting the Stratonovich integral on the right hand side we see that

$$\begin{aligned} d(\sigma^*(Y_t^a)/\!\!/_t^a v^a) &= \sum_{j=2}^m (\nabla_{\sigma_j(Y_t^a)}\sigma^*)/\!\!/_t^a v^a dB_t^j + \frac{1}{2}\sum_{j=2}^m d(\nabla_{\sigma_j(Y_t^a)}\sigma^*)/\!\!/_t^a v^a dB_t^j \\ &= \sum_{j=2}^m (\nabla_{\sigma_j(Y_t^a)}\sigma^*)/\!\!/_t^a v^a dB_t^j + \frac{1}{2}\sum_{j=2}^m (\nabla_{\sigma_j(Y_t^a)}\nabla_{\sigma_j(Y_t^a)}\sigma^*)/\!\!/_t^a v^a dt. \end{aligned}$$

This can be rewritten as

$$\begin{aligned} d(\sigma^*(Y_t^a)/\!\!/_t^a v^a) = &\sum_{j=2}^m \left(\nabla_{\sigma_j(Y_t^a)}\sigma^*\right)\sigma(Y_t^a)\left(\sigma^*(Y_t^a)/\!\!/_t^a v^a\right) dB_t^j \\ &+ \frac{1}{2}\sum_{j=2}^m \left(\nabla_{\sigma_j(Y_t^a)}\nabla_{\sigma_j(Y_t^a)}\sigma^*\right)\sigma(Y_t^a)\left(\sigma^*(Y_t^a)/\!\!/_t^a v^a\right) dt. \end{aligned}$$

Since the coefficients of the SDE converges as $a \to 0$ uniformly in probability, we get that $\sigma^*(Y_t^a)/\!\!/_t^a v^a$ converges to $\sigma^*(Y_t)/\!\!/_t v$ in semi-martingale topology, see Theorem 2 in [17].

Finally, since the linear maps $\sigma(Y_t^a) : \mathbf{R}^m \to T_{Y_t^a}M$ converge to $\sigma(Y_t) : \mathbf{R}^m \to T_{Y_t}M$ in UCP topology and $/\!\!/_t^a = \sigma(Y_t^a)\sigma^*(Y_t^a)/\!\!/_t^a$, we see that $/\!\!/_t^a$ converges to $/\!\!/_t$ in the same topology.

□

5. Convergence of the Damped Parallel Translations

Let (Y_t) be the reflected Brownian motion and (Y_t^a) the approximate reflected Brownian motions, constructed by (3.5) and (3.4) respectively. Let $A^a = \nabla \ln \tanh(\frac{R}{a})$. Denote (W_t^a) the damped parallel translations (Y_t^a), solving the equation

$$\frac{DW_t^a}{dt} = -\frac{1}{2}\mathrm{Ric}^{\#}(W_t^a)dt + \nabla_{W_t^a} A^a, \qquad W_0^a = \mathrm{Id}. \tag{5.1}$$

Let (W_t) the the damped parallel translation along (Y_t). We take the version constructed by Theorem C.3 so (W_t) is an adapted right continuous stochastic process such that $\lim_{\varepsilon\to 0} W_t^\varepsilon = W_t$ in UCP where (W_t^ε) are solutions to the equations (2.2).

Our aim is to prove that W_t^a converges to W_t. It is fairly easy to see the convergence when Y_t^a and Y_t are in M^0. When they are in a a neighbourhood of ∂M, we use the pathwise construction for W. Let $(\varepsilon_n)_{n\geq 0}$ be a sequence of positive numbers converging to 0. As soon as a continuous version of (Y_t) and parallel translations along (Y_t) are chosen, each W^{ε_n} is constructed pathwise. Moreover W^{ε_n} converges to W locally in $\mathscr{S}_2$ and there exists a subsequence of ε_{n_k} such that $W^{\varepsilon_{n_k}}(\omega)$ converges locally uniformly for almost surely all ω.

Denote $\mathfrak{L}(\omega)$ the set of times Y_t spend on the boundary. Let F_0 be a tubular neighbourhood of ∂M. On $\{Y_t \in F_0\}$, we write

$$W_t = W_t^T + f(t)\, \nu_{Y_t}, \tag{5.2}$$

where ν_{Y_t} is its component along ν_{Y_t} and W_t^T its orthogonal complement.

The proposition below is a local result. We prove the following two ways of removing the normal part from the damped parallel translation are equivalent. (1) During an excursion (l_α, r_α), evolve (W_t) with the continuous damped parallel translation equation, then remove the normal part at the touching down time r_α; (2) at the beginning of every excursion remove the normal part of W_t and then evolve (W_t) with the continuous damped parallel translation equation during an excursion. This equivalence is due to the fact that every beginning of excursion is the right limit of ends of excursions and every end of excursion is the left limit of beginning of excursions. Notice the integral with respect to the local time is well explained by the approximation by W_t^ε and later by the approximation by W_t^a, but is absent of the description here.

Proposition 5.1. *Let S_1, S_2 be stopping times and $t \in [S_1(\omega), S_2(\omega)]$. Let $\zeta = \inf\{t > 0 : Y_t \in \partial M\}$. We assume the following conditions.*

1. *The Ricci curvature and the shape operator are bounded on E_δ.*
2. *$Y_t(\omega) \in F_0$ whenever $t \in [S_1(\omega), S_2(\omega)]$.*

Then for almost surely all ω, $W_t = W_t^T + f(t)\, \nu_{Y_t}$ where $f(t,\omega)$ is a right continuous real-valued process vanishing on $[S_1(\omega), S_2(\omega)] \cap \mathfrak{L}(\omega)$. Furthermore,

$$f(t) = \begin{cases} r_t & \text{if} \quad t < \zeta \\ r_t - r_{\alpha_t} & \text{if} \quad t \geq \zeta, \end{cases} \tag{5.3}$$

where $\alpha_t(\omega) = \sup\{s \leq t,\ Y_s(\omega) \in \partial M\} = \sup\left([S_1(\omega), t \wedge S_2(\omega)) \cap \mathfrak{L}(\omega)\right)$, and

$$r_t = \langle W_{S_1}, \nu(Y_{S_1})\rangle - \frac{1}{2}\int_{S_1}^t \mathrm{Ric}(W_s, \nu_{Y_s})\, ds + \int_{S_1}^t \langle W_s, D\nu_{Y_s}\rangle, \tag{5.4}$$

on $[S_1(\omega), S_2(\omega)]$. *Furthermore,*

$$\begin{aligned} DW_t^T = &-\frac{1}{2}\mathrm{Ric}^\sharp(W_t)^T\,dt - \mathcal{S}(W_t^T)\,dL_t - \langle W_t, D\nu_{Y_t}\rangle\nu_{Y_t} \\ &- \langle W_t, \nu_{Y_t}\rangle D\nu_{Y_t} - \frac{1}{2}\sum_{k=2}^m \langle W_t, \nabla_{\sigma_k(Y_t)}\nu(Y_t)\rangle \nabla_{\sigma_k(Y_t)}\nu dt. \end{aligned} \tag{5.5}$$

Conversely, if a right-continuous $L(T_{Y_0}M, T_{Y_t}M)$*-valued process* W_t' *satisfies (5.2-5.5), then it satisfies* (C.7).

This is a local result, hence can be reduced to the half plane model, which was dealt with in [21]. The proof we give below will be used for our approximation result (Theorem 5.6 and Corollary 5.7).

Proof. Denote $f_t = f(t) = \langle W_t, \nu_{Y_t}\rangle$. The formulas below in the proof are interpreted and obtained as following: we first prove the corresponding identity for W_t^ε and then take $\varepsilon \to 0$. Firstly we compute the stochastic differential of f_t:

$$df_t = -\frac{1}{2}\langle \mathrm{Ric}^\sharp(W_t), \nu_{Y_t}\rangle dt + \langle W_t, D\nu_{Y_t}\rangle - \chi_{\{t\in\mathscr{R}(\omega)\}}\langle W_{t-}, \nu_{Y_t}\rangle,$$

for which we used the fact that $\mathcal{S}(W_t^T) = \mathcal{S}(W_t)$ is orthogonal to ν_{Y_t}. Then from $W_t^T = W_t - f_t\nu_{Y_t}$,

$$DW_t^T = DW_t - df_t\nu_{Y_t} - f_t D\nu_{Y_t} - \frac{1}{2}D[f_\cdot, \nu_{Y_\cdot}]_t$$

where the covariant square bracket $D[f_\cdot, \nu_{Y_\cdot}]_t$ is the martingale bracket including the jump part. The jump part of the bracket disappears since ν_{Y_t} is a sample continuous process. Thus

$$DW_t^T = -\frac{1}{2}(\mathrm{Ric}^\sharp(W_t))^T dt - \mathcal{S}(W_t^T)\,dL_t - \langle W_t, D\nu_{Y_t}\rangle\nu_{Y_t} - f_t D\nu_{Y_t} - \frac{1}{2}D\langle f_\cdot, \nu_{Y_\cdot}\rangle_t,$$

where $D\langle f_\cdot, \nu_{Y_\cdot}\rangle_t$ is the continuous part of the martingale bracket. The martingale part of $\nu(y_t)$ is $\sum_{k=2}^m \nabla_{\sigma_k(Y_t)}\nu\, dB_t^k$; while the martingale part of $\langle W_t, D\nu_{Y_t}\rangle$ is

$$\sum_{k=2}^m \langle W_t, \nabla_{\sigma_k(Y_t)}\nu(Y_t)\rangle dB_t^k.$$

This means that

$$D\langle f_\cdot, \nu_{Y_\cdot}\rangle_t = \sum_{k=2}^m \langle W_t, \nabla_{\sigma_k(Y_t)}\nu(Y_t)\rangle \nabla_{\sigma_k(Y_t)}\nu dt,$$

concluding (5.5).

For $t < \zeta$, (5.3) clearly holds. We proves it holds also for $t > \zeta$. If $t \in \mathscr{R}(\omega) \equiv \{r_\alpha(\omega)\}$, $f_t = 0$ by the definition. This agrees with (5.3): $\alpha_t = t$ and $r_t - r_{\alpha_t} = 0$.

On $[S_1, S_2]$ the process R_t is equivalent in law to a reflected Brownian motion. So for every $t \in \mathfrak{L}(\omega)\backslash\mathscr{R}(\omega)$, there exists an increasing sequence $(t_n)_{n\in\mathbf{N}}$ of elements of $\mathscr{R}(\omega)$ converging to t. For all $n \in \mathbf{N}$ we have $f(t_n) = 0$ and

$$f(t) = f(t_n) + \int_{t_n}^t df(s) = 0 + \int_{t_n}^t \langle DW_s, \nu_{Y_s}\rangle + \int_{t_n}^t \langle W_s, D\nu_{Y_s}\rangle.$$

Notice that this formula makes sense by choosing a continuous version of the integral $\int_{S_1}^\cdot \langle W_s, D\nu_{Y_s}\rangle$ and by remarking that (W_t) is the pathwise solution to equation (C.7).

So we have

$$
\begin{aligned}
f(t)^2 &= \int_{t_n}^t 2f(s)\,df(s) + \int_{t_n}^t df(s)\,df(s) \\
&= \int_{t_n}^t 2f(s)\left(\langle DW_s, \nu_{Y_s}\rangle + \langle W_s, D\nu_{Y_s}\rangle\right) + \int_{t_n}^t \mathrm{trace}\langle W_s, \nabla.\nu\rangle\langle W_s, \nabla.\nu\rangle \\
&\quad + \sum_{s\in]t_n,t]\cap\mathscr{R}(\omega)} \langle W_s, \nu_{Y_s}\rangle^2 \\
&= \int_{t_n}^t 2f(s)\left(\left\langle -\frac{1}{2}\mathrm{Ric}^\sharp(W_s)ds - \mathcal{S}(W_s)dL_s, \nu_{Y_s}\right\rangle + \langle W_s, D\nu_{Y_s}\rangle\right) \\
&\quad + \int_{t_n}^t \mathrm{trace}\langle W_s, \nabla.\nu\rangle\langle W_s, \nabla.\nu\rangle - \sum_{s\in]t_n,t]\cap\mathscr{R}(\omega)} f(s)^2.
\end{aligned}
$$

Notice that the last term combines the jump term from $\langle DW_s.\nu_{Y_s}\rangle$ and from $\langle W_s, \nu_{Y_s}\rangle^2$. It is the sum:

$$
-2\sum_{s\in]t_n,t]\cap\mathscr{R}(\omega)} f(s)\langle W_s, \nu_{Y_s}\rangle + \sum_{s\in]t_n,t]\cap\mathscr{R}(\omega)} \langle W_s, \nu_{Y_s}\rangle^2.
$$

Since the jumps are all non-positive and $\langle \mathcal{S}(W_s), \nu_{Y_s}\rangle = 0$, we get

$$
\begin{aligned}
f(t)^2 \le \int_{t_n}^t 2f(s)\left(-\frac{1}{2}\mathrm{Ric}^\sharp(W_s\nu_{Y_s})ds + \langle W_s, D\nu_{Y_s}\rangle\right) \\
+ \int_{t_n}^t \mathrm{trace}\langle W_s, \nabla.\nu\rangle\langle W_s, \nabla.\nu\rangle.
\end{aligned}
$$

But W_s is pathwise bounded in compact intervals, and

$$
\int_u^t f(s)\langle W_s, D\nu_{Y_s}\rangle, \quad \int_u^t \mathrm{trace}\langle W_s, \nabla.\nu\rangle\langle W_s, \nabla.\nu\rangle
$$

are continuous in u. So the right hand side converges to 0 as $n \to \infty$.

This implies that $f(t) = 0$ for all $t \in \mathfrak{L}(\omega)$. In particular for all $t > \xi$, $f(\alpha_t) = 0$ and the second equality of (5.3) is valid.

Conversely let W'_t be a right-continuous process satisfying the conditions of Proposition 5.1. Clearly W'_t satisfies (C.7) when $Y_t \in M^0$. On the other hand $f(t)$ vanishes on left hand sides of excursion, it is right continuous, and all right hand times of excursions are limits of decreasing sequences of left hand times of excursions, again by Lemma C.1. So it also vanishes on $\mathscr{R}$, and consequently $W'_t = W_t$.

□

We can now state the representation theorem for the heat equation on differential 1-forms, c.f.to (1.2), with the absolute boundary conditions $\phi(\nu) = 0$ and $d\phi(\nu) = 0$.

Theorem 5.2. *Suppose that the tubular neighbourhood of ∂M has positive radius, the curvature $-\frac{1}{2}\mathrm{Ric}^\# - \mathcal{S}$ is bounded from below on M^0. Then if ϕ_t is a solution to the heat equation on 1-forms with the absolute boundary conditions, then $\phi_t = \mathbf{E}\phi(W_t)$.*

Proof. It is clear that the reflected Brownian motion (Y_t) is globally defined. Since $\phi_x(v)$ is linear in v, $\frac{1}{2}\sum_{k=1}^m \nabla_{\sigma_k}\sigma_k + \sigma_0 = 0$. Let ϕ be a C^2 differentiate 1-form. We see that

$$\begin{aligned}\phi(W_t) =&\phi(W_0) + \sum_{k=1}^m \int_0^t (\nabla_{\sigma_k(Y_s)}\phi)(W_{s-})dB_s^k + \frac{1}{2}\int_0^t \Delta^1\phi(W_{s-})ds \\ &+ \int_0^t \nabla_{\nu(Y_{s-})}\phi(W_{s-})dL_s - \int_0^t \phi\left(\mathcal{S}(W_{s-})\right)dL_s \\ &- \sum_{s\in\mathscr{R}(\omega)\cap[0,t]} \left(\phi\left(\nu_{Y_s}\right)\langle W_s, \nu_{Y_s}\rangle - \phi\left(\nu_{Y_{s-}}\right)\langle W_{s-}, \nu_{Y_s}\rangle\right).\end{aligned}$$

This equation should be interpreted by first applying W_t^ε. For each ε fixed, $\nu_{Y_s-} = \nu_{Y_s}$, hence the jump terms vanish. Then taking $\varepsilon \to 0$.

We have again used Weitzenböck formula $\Delta^1\phi = \mathrm{trace}\nabla^2\phi - \phi(\mathrm{Ric}^\#)$. By Palais's formula for two vector fields ν and V:

$$d\phi(\nu, V) = L_v(\phi(\nu)) - L_\nu(\phi(V)) - \phi([\nu, V]) = (\nabla_\nu\phi)(V) - \phi(\nabla_V\nu). \tag{5.6}$$

Since $\phi(\nu) = 0$ and $d\phi(\nu,\cdot) = 0$ on the boundary, $(\nabla_\nu)\phi(\cdot) - \phi(\mathcal{S}(\cdot)) = 0$. Since L_s increases only on the boundary the terms in the second line vanishes. Since $\phi(\nu) = 0$ and Y_s is continuous, $\phi(\nu_{Y_{s-}}) = 0$ at the ends of an excursion, the last line vanishes. If $Q_t\phi$ is the solution to the heat equation on 1-forms with absolute boundary conditions and initial value ϕ, on a neighbourhood of the boundary,

$$\phi(W_t) =Q_t\phi(W_0) + \sum_{k=1}^m \int_0^t (\nabla_{\sigma_k(Y_s)}Q_{t-s}\phi)(W_{s-})dB_s^k. \tag{5.7}$$

Note that ϕ is bounded and $\mathbf{E}\sup_{s\le t}|W_s|^2$ is finite, c.f. Proposition 5.1, we take expectations of both sides of (5.7) to obtain $Q_t\phi(W_0) = \phi(W_t)$. □

Let $T > 0$. If $F(t,x)$ is a real valued function on $[0,T]\times M$, we denote by $dF(t,x)$ its differential in the second variable and $\nabla F(t,x)$ the corresponding gradient.

Theorem 5.3. *Let (W_t) be the solution of (C.7). If $F : [0,T]\times M \to \mathbf{R}$ is a $C^{1,2}$ function such that $F(t,Y_t)$ is a continuous local martingale, then so is $dF(t,Y_t)(W_t)$.*

Remark 5.4. The statements in Theorem 5.3, also in Corollary 5.5 and Theorem 5.2, are valid with W_t replaced by W_t^ε. But they are more powerful (and more intrinsic) with W_t, for the reason that $|W_t|$ is expected to be smaller than $|W_t^\varepsilon|$.

Proof. It is clear that, on $\{Y_t \in M^0\}$, $d(\langle\nabla F(t,Y_t), W_t\rangle)$ is the differential of a local martingale, hence we only need to prove the result on $\{Y_t \in F_0\}$. We write the Itô formula for $F(t,Y_t)$, the Itô differential $d(F(t,Y_t))$ satisfies the following identity:

$$\begin{aligned}d(F(t,Y_t)) =&\langle\nabla F(t,Y_t), \sigma(Y_t)\,dB_t\rangle \\ &+ \left(\partial_t + \frac{1}{2}\Delta\right)F(t,Y_t)\,dt + \langle dF(t,Y_t), \nu_{Y_t}\rangle\,dL_t.\end{aligned} \tag{5.8}$$

By the local martingale property of $F(t,Y_t)$ the last two terms vanishes and

$$\begin{aligned}&\left(\partial_t + \frac{1}{2}\Delta\right)F(t,y) = 0,\ (t,y)\in[0,T]\times M^0, \\ &\nu_y \in \ker dF(t,y),\ (t,y)\in[0,T]\times\partial M.\end{aligned} \tag{5.9}$$

Since W_t has finite variation on the set $\{Y_t \notin \partial M\}$ there is no covariation term between $dF(t, Y_t)$ and W_t. Writing an Itô formula for $\langle dF(t, Y_t), W_t\rangle$ yields

$$\begin{aligned} d\left\langle \nabla F(t, Y_t), W_t\right\rangle = \nabla dF(t, Y_t)(\sigma(Y_t)dB_t, W_t) + \nabla dF(t, Y_t)(\nu_{Y_t}, W_t)\, dL_t \\ + \left(\partial_t + \frac{1}{2}\operatorname{trace}\nabla^2\right) dF(t, Y_t)(W_t)\, dt \\ - \frac{1}{2}\left\langle \nabla F(t, Y_t), \mathrm{Ric}^\sharp(W_t)\right\rangle dt - \left\langle \nabla F(t, Y_t)\mathcal{S}(W_t)\right\rangle\, dL_t. \end{aligned}$$

where in the last term we used (5.9). We note that $\Delta^1 = \operatorname{trace}\nabla^2 - \frac{1}{2}\mathrm{Ric}^\sharp$ and $\Delta^1 d = d\Delta$. This together with (5.9), $\left(\partial_t + \frac{1}{2}\Delta^1\right) dF(t, y) = 0$, yields

$$\begin{aligned} d\langle \nabla F(t, Y_t), W_t\rangle =& \nabla dF(t, Y_t)(\sigma(Y_t)dB_t, W_t) \\ &+ \nabla dF(t, Y_t)(\nu_{Y_t}, W_{t-})\, dL_t - \left\langle \nabla F(t, Y_t), \mathcal{S}(W_t)\right\rangle\, dL_t. \end{aligned}$$

Now for $y \in \partial M$ and $w \in T_yM$, since $\nu(y) \in \ker dF(t, y) \in T_y\partial M$ we have

$$\begin{aligned} -\langle \nabla F(t, y), \mathcal{S}(w)\rangle &= \langle \nabla F(t, y), \nabla_w \nu\rangle \\ &= -\langle \nabla_w dF(t, y), \nu_y\rangle = -\nabla dF(t, y)(\nu_y, w). \end{aligned}$$

For the second equality we used the fact that $\nu(y) \in \ker dF(t, y)$. Putting all the calculations together we finally get

$$d\langle dF(t, Y_t), W_t\rangle = \nabla dF(t, Y_t)(\sigma(Y_t)dB_t, W_t),$$

which proves that $\langle \nabla F(t, Y_t), W_t\rangle$ is a continuous local martingale.

□

Applying this theorem to $F(t, y) = \mathbf{E}[f(Y_{T-t}(y))]$ where $(Y(y))$ is reflected Brownian motion started at $y \in M$, f is a smooth function on M with $df(\nu) = 0$ on the boundary, (under this condition F is $C^{1,2}$, see e.g. [29]), we immediately get the following Bismut type formula:

Corollary 5.5. *Assume that M is compact. Let $f : M \to \mathbf{R}$ a smooth bounded function with $\langle df, \nu\rangle = 0$ on the boundary and $T > 0$. Let Q_t be the semi-group associated to the reflected Brownian motion on M. Let $y \in M$, $v \in T_yM$ and (Y_t) a reflected Brownian motion started at $y \in M$, constructed as in Theorem 3.4. Then*

$$d(Q_T f)(v) = \frac{1}{T}\mathbf{E}\left[f(Y_T)\int_0^T \langle W_s(v), \sigma(Y_s)dB_s\rangle\right].$$

Let T and a be positive numbers. Recall that the damped parallel translation along a sample continuous stochastic process (Y_t^a) is the solution to the stochastic covariant differential equation with initial value $W_0^a = \mathrm{Id}_{T_{Y_0^a}M}$,

$$DW_t^a = \left(\nabla_{W_t^a} A^a - \frac{1}{2}\mathrm{Ric}^\sharp(W_t^a)\right) dt. \tag{5.10}$$

Theorem 5.6. *Let M be a compact Riemannian manifold. Let*

$$A^a(x) = \nabla \ln\tanh\left(\frac{R(x)}{a}\right).$$

Let $(Y_t^a, t \in [0,T])$ and $(Y_t, t \in [0,T])$ be the stochastic processes defined in Theorem 3.4. Let W_t^a denote the damped parallel translation along Y_t^a. Then for all $p \in [1,\infty)$ and for any C^2 differential 1-form ϕ vanishing on the normal bundle $\nu(\partial M)$,

$$\lim_{a\to 0} \sup_{s\le t} \mathbf{E}\left[|\phi(W_s^a) - \phi(W_s)|^p\right] = 0.$$

For a non-compact manifold, we have the following result, see Section B.

Corollary 5.7. *Let M be a Riemannian manifold, not necessarily compact. Then for any C^2 differential 1-form ϕ such that $\phi(\nu) = 0$ in ∂M, $\phi(W^a)$ converges to $\phi(W)$ in UCP topology.*

6. Damped parallel translation as a derivative flow

In this section we M is a smooth compact manifold with boundary and prove that the damped parallel translation W_t along reflected Brownian motion has the property of a weak derivative flow.

Let $\varphi : [0,1] \to M^o$ be a C^1 map. Here again for $a > 0$, $A^a(x) = \nabla \ln \tanh\left(\frac{R(x)}{a}\right)$. We will built a family of Brownian flows with drift A^a starting at $\varphi(u)$ whose derivative with respect to u is locally uniformly bounded for a.s. ω. Let us return to (3.4).

$$dY_t^a = \sum_{k=1}^m \sigma_k(Y_t^a) \circ dB_t^k + \sigma_0(Y_t^a)dt + A^a(Y_t^a)\,dt.$$

Let us consider its solution flow Ψ^a. Let $Y_t^a(u) = \Psi^a(\varphi(u))$. Let us use a short hand for the above equation :

$$\begin{cases} dY_t^a(0) &= \sigma(Y_t^a(0))\,dB_t + A^a(Y_t(0))\,dt \\ Y_0^a(0) &= \varphi(0). \end{cases} \tag{6.1}$$

For $u \in (0,1]$, let $Y_t^a(u)$ solve

$$\begin{cases} dY_t^a(u) &= /\!\!/_{0,u}(Y_t^a(\cdot))\,(\sigma(Y_t^a(0))\,dB_t) + A^a(Y_t(u))\,dt \\ Y_0^a(u) &= \varphi(u) \end{cases} \tag{6.2}$$

where $/\!\!/_{0,u}(Y_t^a(\cdot))$ denotes parallel translation along the C^1 path $u \mapsto Y_t^a(u)$. The existence of a solution should follow from an iteration method. A proof is given in M. Arnaudon, K. A. Coulibaly and A. Thalmaier [4], where an approximation procedure with iterated parallel couplings is used to obtain a Cauchy sequence in H_2. The advantage is that at each step and each value of u we have a diffusion with the same generator $\frac{1}{2}\Delta + A^a$, and as the mesh goes to 0 that all problems with cut locus disappear. The solution curves $u \mapsto Y_t^a(u)$ are almost surely differentiable and that their derivatives $\partial_u Y_t^a(u)$ is locally uniformly bounded for almost surely all ω and

$$\partial_u Y_t^a(u) = W_t^{a,u}(\dot\varphi(u)). \tag{6.3}$$

where $W_t^{a,u}(\dot\varphi(u)) = W_t(Y^a(u))$ is the damped parallel translation along $Y^a(u)$. This is, to our knowledge, the only known construction for $\partial_u Y_t^a(u)$ a.s. locally uniformly bounded. Our aim is to obtain a similar property for reflected Brownian motion. For this we will let $a \to 0$ in (6.3) and obtain a limiting identity in a weak sense. However we believe that our construction indeed yields (6.3) for $a = 0$ in a strong sense.

Proposition 6.1. *The two parameter family of stochastic processes $\{Y_t^a(u),\ 0 \le u \le 1,\ a \in (0,1]\}$ on $[0,T]$ is tight.*

Proof. We will use the Kolmogorov criterion. For t_1, t_2, u_1, u_2 satisfying $0 \le t_1 < t_2 \le T$ and $0 \le u_1 < u_2 \le 1$ and $p \ge 1$,

$$\begin{aligned}
&\mathbf{E}\left[\rho^p\left(Y^a_{t_1}(u_1), Y^a_{t_2}(u_2)\right)\right] \\
&\le 2^{p-1}\left(\mathbf{E}\left[\rho^p\left(Y^a_{t_1}(u_1), Y^a_{t_2}(u_1)\right)\right] + \mathbf{E}\left[\rho^p\left(Y^a_{t_2}(u_1), Y^a_{t_2}(u_2)\right)\right]\right) \\
&\le 2^{p-1}\left(\mathbf{E}\left[\rho^p\left(Y^a_{t_1}(u_1), Y^a_{t_2}(u_1)\right)\right] + \mathbf{E}\left[\left(\int_{u_1}^{u_2} |W^a_{t_2}(u)| \cdot |\dot\varphi(u)|\, du\right)^p\right]\right) \\
&\le 2^{p-1}\left(\mathbf{E}\left[\rho^p\left(Y^a_{t_1}(u_1), Y^a_{t_2}(u_1)\right)\right] + (u_2-u_1)^{p-1}\|\dot\varphi\|_\infty \int_{u_1}^{u_2} \mathbf{E}\left[|W^a_{t_2}(u)|^p\right]\, du\right) \\
&\le 2^{p-1}\left(\mathbf{E}\left[\rho^p\left(Y^a_{t_1}(u_1), Y^a_{t_2}(u_1)\right)\right] + (u_2-u_1)^{p}\|\dot\varphi\|_\infty \sup_{u\in[0,1],\ t\in[0,T]} \mathbf{E}\left[|W^a_t(u)|^p\right]\right) \\
&\le 2^{p-1}\left(\mathbf{E}\left[\rho^p\left(Y^a_{t_1}(u_1), Y^a_{t_2}(u_1)\right)\right] + C'(p,T)(u_2-u_1)^{p}\|\dot\varphi\|_\infty\right),
\end{aligned}$$

where $C'(p,T)$ is a constant. We used an estimate on $|w^{a,u}_t$ given in (7.13) below. Here and several time in the sequel, we use the equality in law of the processes $(Y^a(u), W^a(u))$, for each fixed u, and (Y^a, W^a). The latter process was constructed in Sections 3 and 5.

For the first term on the right hand side we again use the fact that u_1 is fixed and use estimates for Y^a, from Theorem 3.4. Since M is compact we can replace the distance $\rho(x,y)$ on M by the equivalent distance $\|H(x) - H(y)\|$ where $H : M \to \mathbf{R}^d$ is an embedding. We can also assume that $H = (\imath, R)$ is an extension of the construction in (3.17) around the boundary. In particular we can assume that the image of ∂M by H is included in $\{R = 0\}$, the drift of $\tilde\imath(Y^a_t(u_1))$ is bounded, the drift of $\left(R(Y^a_t(u_1)) - R(Y^a_{t_1}(u_1))\right)^4$ is bounded in $\{R \ge \delta_0\}$ and that it is positive and bounded above by $4\left(R(Y^a_t(u_1)) - R(Y^a_{t_1}(u_1))\right)^2 + b$ in $\{0 < R < \delta_0\}$ where $b > 0$ is independent of a (this is a consequence of (3.6)). This implies,by a standard calculation, that for some constant $C' > 0$,

$$\mathbf{E}\left[\left\|H(Y^a_{t_2}(u_1)) - H(Y^a_{t_1}(u_1))\right\|^4\right] \le C'|t_2 - t_1|^2. \tag{6.4}$$

Finally, for some positive constant C,

$$\mathbf{E}\left[\rho^4\left(Y^a_{t_1}(u_1), Y^a_{t_2}(u_2)\right)\right] \le 8C|t_2 - t_1|^2 + 8C'(4,T)(u_2-u_1)^4\|\dot\varphi\|_\infty. \tag{6.5}$$

This concludes the required tightness. □

With this result at hand we construct our limiting process.

Theorem 6.2. *There is a family of continuous stochastic process $Y_t(u)$ indexed by $0 \le u \le 1$ with the following properties:*

(1) *for every $u \in [0,1]$, $Y_t(u)$ is a reflected Brownian motion on M started at $\varphi(u)$;*
(2) *for every $p \in [1,\infty)$ there exists a number $C'(p,T)$ s.t. for all $0 \le u_1, u_2 \le 1$,*

$$\sup_{0\le t\le T} \mathbf{E}\left[\rho^p(Y_t(u_1), Y_t(u_2))\right] \le C'(p,T)\|\dot\varphi\|_\infty (u_2-u_1)^p. \tag{6.6}$$

Proof. By Proposition 6.1, there exists a sequence $a_k \to 0$ such that the two parameter family of stochastic processes $Y^{a_k}_\cdot(u)$ converges in law whose limit we denote by $Y_\cdot(u)$. Since u is fixed and the convergence considered in weak convergence, we are allowed to use another construction of $Y^a_t(u)$ and part (2) in Theorem 3.4, in which the convergenceis stronger. In the computation for tightness we take $t_1 = t_2 = t$, Proposition 6.1, then take $k \to \infty$ to obtain (2). □

For each $u \in [0,1]$ fixed, the stochastic processes $Y_t^{a_k}(u)$ converges in law to $Y_t(u)$. So the damped parallel translations $W_t^{a_k}(u)$, as stochastic processes on $[0,T]$, converge in law to $W_t(u)$ in the following sense: if ϕ is a C^2 differential 1-form such that $\phi|\partial M(\nu) = 0$, then $\phi(W_t^{a_k,u})$ converges in law to $\phi(W_t(u))$. This is due to the fact that W^a is a functional of Y^a, W is a functional of Y, so we can apply Corollary 5.7.

Unfortunately this argument does not allow us to prove the convergence of $W_t^{a_k,u}$ converges to $W_t(u)$, which would yield $\partial_u Y_t(u) = W_t(u)\dot{\varphi}(u)$. However the following theorem asserts this equality in a weak sense.

Theorem 6.3. *Let M be compact. For all $f \in C^2(M)$ satisfying $\langle df, \nu\rangle = 0$ on the bundary,*

$$\mathbf{E}\left[f(Y_t(u_2)) - f(Y_t(u_1)) - \int_{u_1}^{u_2} \langle df(Y_t(u)), W_t(u)\dot{\varphi}(u)\rangle\, du\right] = 0. \tag{6.7}$$

Proof. By (6.3),

$$f(Y_t^{a_k}(u_2)) - f(Y_t^{a_k}(u_1)) - \int_{u_1}^{u_2} \langle df(Y_t^{a_k}(u)), W_t^{a_k}(u)\dot{\varphi}(u)\rangle\, du = 0.$$

Since all the terms are integrable we can take the expectation, So

$$\mathbf{E}\left[f(Y_t^{a_k}(u_2))\right] - \mathbf{E}\left[f(Y_t^{a_k}(u_1))\right] - \int_{u_1}^{u_2} \mathbf{E}\left[\langle df(Y_t^{a_k}(u)), W_t^{a_k}(u)\dot{\varphi}(u)\rangle\right] du = 0.$$

Now by Theorem 3.4 and Corollary 5.7 and dominated convergence theorem:

$$\mathbf{E}\left[f(Y_t(u_2))\right] - \mathbf{E}\left[f(Y_t(u_1))\right] - \int_{u_1}^{u_2} \mathbf{E}\left[\langle df(Y_t^{a_k}(u)), W_t(u)\dot{\varphi}(u)\rangle\right] du = 0.$$

Finally we use Fubini-Tonelli Theorem to obtain (6.7). □

7. Proof of Theorem 5.6

We first reduce the proof of Theorem 5.6 to the class of C^2 differential 1-forms ϕ vanishing in a neighbourhood of the boundary, this is the content of Section 7.1. We then prove the convergence of the tangential part of the parallel transport in the topology of UCP, followed by the convergence of its normal part f_t^a. By the latter we mean that for all smooth $\phi : M \to \mathbf{R}_+$ vanishing in a neighbourhood of ∂M, $\phi(Y_t^a)f_a(t) \to \phi(Y_t)f(t)$ in the UCP topology. See Sections 7.4 and 7.5.

We describe briefly the strategy and the main difficulties. Thanks to the convergence of parallel transports established in Proposition 4.1 we only need to prove that $w_t^a \to w_t$, more precisely that, writing

$$w_t^a = w_t^{a,T} + f_a(t)n_t^a \tag{7.1}$$

with $n_t^a = (/\!/_t^a)^{-1}\nu(Y_t^a)$ and $w_t^{a,T}$ orthogonal to n_t^a, $w_t^{a,T} \to w_t^T$ and for any C^2 map $\phi : M \to \mathbf{R}$vanishing in a neighbourhood of ∂M, $\phi(Y_t^a)f_a(t)n_t^a \to \phi(Y_t)f(t)n_t$.

Firstly, the integral equation for $(f_a, w^{a,T})$ has the following form

$$\begin{pmatrix} f_a(t) \\ w_t^{a,T} \end{pmatrix} = \begin{pmatrix} f_a(0) \\ w_0^{a,T} \end{pmatrix} + \int_0^t dM_s^a \begin{pmatrix} f_a(s) \\ w_s^{a,T} \end{pmatrix} + \begin{pmatrix} f_a(0)e^{-\tilde{C}_a(t)} \\ 0 \end{pmatrix}, \tag{7.2}$$

where (M_t^a) is a matrix valued process for the following form

$$M_t^a = \begin{pmatrix} 0 & \tilde{u}_t^a \\ v_t^{a,N} & v_t^{a,T} \end{pmatrix} \tag{7.3}$$

whose components are to be specified later. Also,

$$\begin{pmatrix} f(t) \\ w_t^T \end{pmatrix} = \begin{pmatrix} f(0) \\ w_0^T \end{pmatrix} + \int_0^t dM_s \begin{pmatrix} f(s) \\ w_s^T \end{pmatrix} + \begin{pmatrix} f(0)e^{-\tilde{C}(t)} \\ 0 \end{pmatrix} \tag{7.4}$$

where M_t is of the following form:

$$M_t = \begin{pmatrix} 0 & \tilde{u}_t \\ v_t^N & v_t^T \end{pmatrix}. \tag{7.5}$$

If (V_t) is a vector valued stochastic process, denote

$$\|V\|_{\mathscr{S}_p([0,T])} = \mathbf{E}\left(\sup_{0\le s\le T} \mathbf{E}|V_s|^p\right).$$

We will see that the components of M^a converge to the corresponding components of M in $\mathscr{H}_p([0,T])$ for all $p \ge 1$, $T > 0$, with the exception $\tilde{u}^a$ which contains $e^{-\tilde{C}_a(t)}$ and v_t^T which contains local time of the distance to boundary. The main difficulty is the convergence of $f_a(0)e^{-\tilde{C}_a(t)}$ to $f(0)e^{-\tilde{C}(t)}$. The convergence is only in $L^p(dt \times \mathbb{P})$, see Corollary 7.10. Also the convergence of L_t^a to L_t: it is in $\mathscr{S}_p([0,T])$ but not in $\mathscr{H}_p([0,T])$ and will require several integrations by parts. We note in the last term in equation (7.2), the tangential and the normal part decouples. The matrix (7.3) is in the lower triangular form. It is therefore possible to split the proof into the convergence of $w^{a,T}$ to w^T and the convergence of $f_a(t)$ to $f(t)$. This procedure is essential for our proof to work.

7.1. Localisation.

Lemma 7.1. *Let S_1 and S_2 be stopping times such that for a sufficiently small, $Y_t \in E_0$ and $Y_t^a \in E_0$ on $\{\omega : S_1(\omega) \le t \le S_2(\omega)\}$.*

1. *If Theorem 5.6 holds for the class of C^2 differential 1-forms ϕ with $\phi(\nu)$ vanishing in a neighbourhood of the boundary, then it holds for all C^2 1-form ϕ such that $\phi(\nu) = 0$ on ∂M.*
2. *It is sufficient to prove that for $t \in [S_1, S_2]$, $\phi(W^a)$ converges to $\phi(W)$ in UCP.*
3. *If $\phi(W^a)$ converges to $\phi(W)$ in the UCP topology and M is compact, then*

$$\lim_{a\to 0} \mathbf{E} \sup_{s\le t} |\phi(W^a) - \phi(W)|^p \to 0.$$

Without loss of generality we will assume that $S_1 = 0$ and let $S_2 = S$.

Proof. Let ϕ be a C^2 differential 1-form such that $\phi(\nu) = 0$ on ∂M. Then there exists a family of C^2 differential 1-forms ϕ^ε such that $\langle \phi^\varepsilon, \nu\rangle = 0$ in a neighbourhood of ∂M and $\sup_{x\in M} \|\phi^\varepsilon(x) - \phi(x)\| < \varepsilon$: choose for instance $\phi^\varepsilon(u) = \phi(u) - \langle u, \nu\rangle f^\varepsilon(\pi(u))\phi(\nu)$ where f^ε is a smooth function on M satisfying

- $f^\varepsilon = 0$ on $\{R \ge \alpha\}$,
- $f^\varepsilon = 1$ on $\{R \le \alpha/2\}$,
- $0 \le f \le 1$ on M,

and $\alpha \in (0, \delta_0)$ is chosen in such a way that $|\phi(\nu)| < \varepsilon$ on $\{R < \alpha\}$.

By the assumption, $\phi^\varepsilon(W^a) \to \phi^\varepsilon(W)$ in $\mathscr{S}_p([0,T])$. On the other hand

$$\begin{aligned} |\phi(W_t^a) - \phi(W_t)| &\le |\phi(W_t^a) - \phi^\varepsilon(W_t^a)| + |\phi^\varepsilon(W_t^a) - \phi^\varepsilon(W_t)| + |\phi^\varepsilon(W_t) - \phi(W_t)| \\ &\le \varepsilon|W_t^a| + |\phi^\varepsilon(W_t^a) - \phi^\varepsilon(W_t)| + \varepsilon|W_t|. \end{aligned}$$

If a is sufficiently small then $\|\phi^\varepsilon(W^a) - \phi^\varepsilon(W)\|_{\mathscr{S}_p([0,T])} < \varepsilon$ by Theorem 5.6. Using lemma 7.5 we get

$$\|\phi(W^a) - \phi(W)\|_{\mathscr{S}_p([0,T])} < \varepsilon(1 + 2C)$$

for a sufficiently small. Taking $\varepsilon \to 0$ we obtain $\|\phi(W^a) - \phi(W)\|_{\mathscr{S}_p([0,T])} \to 0$.

(2) We note that Y_t^a converges to Y_t in UCP topology and inside M^0 the coefficients for W_t^a converge smoothly and uniformly to the coefficients of the equation for W_t. If for any $t \in [S_1, S_2]$ where S_1, S_2 are stopping times such that for all $t \in [S_1, S_2]$ and a sufficiently small, $Y_t \in E_0$ and $Y_t^a \in E_0$, then $\phi(W^a)$ converges to $\phi(W)$ in UCP.

(3) By Lemma 7.8 below, $\int_0^t \frac{2}{a \sinh\left(\frac{2R(Y_s^a)}{a}\right)} ds$ converges to L_t in $\mathscr{S}_p$ for all $p \in [1, \infty)$.

Let $t > 0$. Since M is compact, by corollary B.5 and lemma 7.5, if $\phi(W^a)$ converges to $\phi(W)$ in UCP topology,

$$\lim_{a \to 0} \mathbf{E}\left(\sup_{s \le t} |\phi(W_s^a) - \phi(W_s)|^p\right) = 0.$$

□

7.2. Preliminary Computations. Let $a > 0$. The damped parallel translation along Y_t^a satisfies the following equations:

$$DW_t^a = \nabla_{W_t^a} A^a \, dt - \frac{1}{2} \mathrm{Ric}^\sharp_{Y_t^a}(W_t^a) \, dt.$$

In the tubular neighbourhood F_0 on which the approximating SDEs were constructed, take $y \in M^0$ and $w \in T_y M$. Then

$$\begin{aligned} \nabla_w A^a =& \nabla_w \nabla \ln \tanh\left(\frac{R}{a}\right) = \nabla_w \left(\frac{2\nabla R}{a \sinh\left(\frac{2R}{a}\right)}\right) \\ =& -\frac{4}{a^2} \frac{\cosh}{\sinh^2}\left(\frac{2R}{a}\right) \langle w, \nu_y \rangle \nu_y - \frac{2}{a \sinh\left(\frac{2R}{a}\right)} \mathcal{S}(w), \end{aligned}$$

Let us define $R_t^a = R(Y_t^a)$, $R_t = R(Y_t)$,

$$c_a(t) = \frac{4}{a^2} \frac{\cosh}{\sinh^2}\left(\frac{2R_t^a}{a}\right), \tag{7.6}$$

and

$$f_a(t) = \langle w_t^a, n_t^a \rangle = \langle W_t^a, \nu(Y_t^a) \rangle. \tag{7.7}$$

We also denote $W_t^{a,T}$ the tangential part of W_t^a:

$$W_t^{a,T} = W_t^a - f_a(t) \nu(Y_t^a). \tag{7.8}$$

Definition 7.2. Let

$$L_t^a = \int_0^t \frac{2}{a \sinh\left(\frac{2R(Y_s^a)}{a}\right)} ds. \tag{7.9}$$

Below $/\!\!/_t^{-1}$ is shorthand for $/\!\!/_t^{-1}(Y_\cdot^a)$. For $x \in M$, denote $\|\nabla \nu(x)\|^2 = \sum_k \|\nabla_{\sigma_k} \nu\|_x^2$. The latter is the Hilbert-Schmidt norm of the linear operator $\nabla \nu : T_x M \to T_x M$. In the following formulas we should consider the integrals are in Itô form. Hence the equation for $W_t^{a,T}$ should be interpreted as for $/\!\!/_t^{-1} W_t^{a,T}$.

Lemma 7.3. *In the tubular neighbourhood the following formulae hold.*

$$DW_t^a = -\frac{1}{2} \mathrm{Ric}^\sharp(W_t^a) \, dt - c_a(t) f_a(t) \nu(Y_t^a) \, dt - \mathcal{S}(W_t^a) \, dL_t^a,$$

$$d/\!\!/_t^{-1} \nu(Y_t^a) dt = /\!\!/_t^{-1} \sum_k \nabla_{\sigma_k} \nu(Y_t^a) dB_t^k + \frac{1}{2} /\!\!/_t^{-1} \operatorname{trace} \nabla^2 \nu(Y_t^a) dt,$$

Finally the stochastic differential of the tangential part of W_t^a has the following tangential and normal decomposition

$$\begin{aligned}
DW_t^{a,T} =& -\frac{1}{2}\left(\mathrm{Ric}^\sharp(W_t^{a,T})\right)^T dt - \frac{1}{2}f_a(t)\left(\mathrm{Ric}^\sharp(\nu(Y_t^a))\right)^T dt - \mathcal{S}(W_t^a)\, dL_t^a \\
&- f_a(t)\nabla_{\sigma_k}\nu(Y_t^a)dB_t^k - \frac{1}{2}f_a(t)\,\mathrm{trace}\,\nabla^2\nu(Y_t^a)dt \\
&- \frac{1}{2}\langle W_t^{a,T}, \nabla_{\sigma_k}\nu(Y_t^a)\rangle\nabla_{\sigma_k}\nu(Y_t^a)\, dt \\
&- \sum_k \langle W_t^{a,T}, \nabla_{\sigma_k}\nu(Y_t^a)\rangle\nu(Y_t^a)dB_t^k - \langle W_t^{a,T}, \mathrm{trace}\,\nabla^2\nu(Y_t^a)\rangle\nu(Y_t^a)dt \\
&+ f_a(t)\|\nabla\nu(Y_t^a)\|^2\nu(Y_t^a)dt,
\end{aligned}$$

$$\begin{aligned}
df_a(t) = -c_a(t)f_a(t)dt - \frac{1}{2}\mathrm{Ric}(W_t^a, \nu(Y_t^a))dt + \sum_k \langle W_t^{a,T}, \nabla_{\sigma_k}\nu(Y_t^a)\rangle\, dB_t^k \\
+\frac{1}{2}\langle W_t^{a,T}, \mathrm{trace}\,\nabla^2\nu(Y_t^a)\rangle dt - \frac{1}{2}f_a(t)\|\nabla\nu(Y_t^a)\|^2.
\end{aligned}$$

Proof. The first formula is clear, the second is straight forward after applying Itô's formula to the equation for (Y_t^a):

$$(/\!\!/_t^a)^{-1}D\nu(Y_t^a) = (/\!\!/_t^a)^{-1}\left(\langle\nabla\nu, \circ dY_t^a\rangle + \frac{1}{2}\,\mathrm{trace}\,\nabla^2\nu(Y_t^a)\,dt\right).$$

Since $\langle\nabla\nu, A^a\rangle = 0$, this yields

$$\begin{aligned}
(/\!\!/_t^a)^{-1}D\nu(Y_t^a) =& (/\!\!/_t^a)^{-1}\left(\sum_k \langle\nabla\nu, \sigma_k(Y_t^a)\rangle\, dB_t^k + \frac{1}{2}\,\mathrm{trace}\,\nabla^2\nu(Y_t^a)\,dt\right) \\
&+ (/\!\!/_t^a)^{-1}\nabla_{A^a(Y_t^a)}\nu dt.
\end{aligned}$$

Note that $\nu = \nabla R$. If γ_t is the geodesic from x to $\pi(x)$ then $\dot\gamma(t) = \nabla R(\gamma(t))$ and hence $\nabla_\nu\nu = 0$ and the second formula follows. We note also that

$$\begin{aligned}
\langle W_t^a, \circ D(\nu(Y_t^a))\rangle &= \sum_k \langle W_t^a, \nabla_{\sigma_k}\nu(Y_t^a)\rangle dB_t^k + \langle W_t^a, \frac{1}{2}\,\mathrm{trace}\,\nabla^2\nu(Y_t^a)\rangle dt \\
&= \sum_k \langle W_t^a, \nabla_{\sigma_k}\nu(Y_t^a)\rangle dB_t^k - \frac{1}{2}f_t\|\nabla\nu(Y_t^a)\|^2 \\
&+ \frac{1}{2}\langle W_t^{a,T}, \mathrm{trace}\,\nabla^2\nu(Y_t^a)\rangle dt.
\end{aligned}$$

Note that the left hand side is in Stratonovich form and the right hand side in Itô form. We work on the third equation:

$$\begin{aligned}
df_t =& \langle DW_t^a, \nu(Y_t^a)\rangle + \langle W_t^a, D(\nu(Y_t^a))\rangle \\
=& -c_a(t)f_a(t)dt - \frac{1}{2}\mathrm{Ric}(W_t^a, \nu(Y_t^a))dt + \sum_k \langle W_t^a, \nabla_{\sigma_k}\nu(Y_t^a)\rangle\, dB_t^k \\
&+ \frac{1}{2}\langle W_t^a, \mathrm{trace}\,\nabla^2\nu(Y_t^a)\rangle dt.
\end{aligned}$$

All stochastic integrals in the above formula are in Itô form. The required identity follows from the observation below:

$$\langle \sum_k \nabla^2 \nu(\sigma_k, \sigma_k), \nu \rangle = -\sum_k \langle \nabla_{\sigma_k} \nu, \nabla_{\sigma_k} \nu \rangle = -\|\nabla \nu\|^2.$$

Next we compute the tangential part of the damped parallel translation.

$$\begin{aligned} DW_t^{a,T} &= DW_t^a - D(f_t \nu(Y_t^a)) \\ &= -\frac{1}{2}\mathrm{Ric}^\sharp(W_t^a)\, dt - c_a(t) f_a(t) \nu(Y_t^a)\, dt - \mathcal{S}(W_t^a)\, dL_t^a - D(f_t \nu(Y_t^a)). \end{aligned}$$

For the normal part of the damped parallel transport, we use product rule

$$\begin{aligned} D(f_t \nu(Y_t^a)) =& \nu(Y_t^a) df_t + f_t D(\nu(Y_t^a)) + df_t D(\nu(Y_t^a)) \\ =& \nu(Y_t^a) \left(-c_a(t) f_a(t) dt - \frac{1}{2}\mathrm{Ric}(W_t^a, \nu(Y_t^a)) dt \right) \\ &+ \nu(Y_t^a) \left(\sum_k \langle W_t^a, \nabla_{\sigma_k} \nu(Y_t^a) \rangle\, dB_t^k + \frac{1}{2}\langle W_t^a, \mathrm{trace}\, \nabla^2 \nu(Y_t^a) \rangle dt. \right) \\ &+ f_t \left(\sum_k \nabla_{\sigma_k} \nu(Y_t^a) dB_t^k + \frac{1}{2} /\!/_t^{-1} \mathrm{trace}\, \nabla^2 \nu(Y_t^a) dt \right) \\ &+ \sum_k \langle W_t^a, \nabla_{\sigma_k(Y_t^a)} \nu \rangle \nabla_{\sigma_k(Y_t^a)} \nu\, dt \end{aligned}$$

Since $\langle \nabla_\cdot \nu, \nu \rangle$ vanishes, $\langle \nabla_{W_t^a} A^a, \nu(Y_t^a) \rangle dt = \mathcal{S}(W_t^a) dL_t^a$. Finally, we bring the above formula back to the equation for $W_t^{a,T}$ and observe that the cancellation of the term involving $f_a(t)$.

$$\begin{aligned} DW_t^{a,T} =& -\frac{1}{2}\mathrm{Ric}^\sharp(W_t^a)\, dt - \mathcal{S}(W_t^a)\, dL_t^a + \frac{1}{2}\mathrm{Ric}(W_t^a, \nu(Y_t^a)) \nu(Y_t^a) dt \\ &- \sum_k \langle W_t^a, \nabla_{\sigma_k} \nu(Y_t^a) \rangle \nu(Y_t^a) dB_t^k + \frac{1}{2} f_t \|\nabla \nu\|^2 \nu(Y_t^a) \\ &- \frac{1}{2}\langle W_t^{a,T}, \mathrm{trace}\, \nabla^2 \nu(Y_t^a) \rangle \nu(Y_t^a) dt - f_t \sum_k \nabla_{\sigma_k} \nu(Y_t^a) dB_t^k \\ &- \frac{1}{2} f_t\, \mathrm{trace}\, \nabla^2 \nu(Y_t^a) dt - \sum_k \langle W_t^a, \nabla_{\sigma_k(Y_t^a)} \nu \rangle \nabla_{\sigma_k(Y_t^a)} \nu\, dt. \end{aligned}$$

Following this up and observing that

$$-\frac{1}{2}\mathrm{Ric}^\sharp(W_t^a) + \frac{1}{2}\mathrm{Ric}(W_t^a, \nu(Y_t^a)) \nu(Y_t^a) = -\frac{1}{2}(\mathrm{Ric}^\sharp(W_t^a))^T,$$

we see

$$\begin{aligned} DW_t^{a,T} =& -\frac{1}{2}(\mathrm{Ric}^\sharp(W_t^a))^T\, dt - \mathcal{S}(W_t^a)\, dL_t^a - \sum_k \langle W_t^{a,T}, \nabla_{\sigma_k} \nu(Y_t^a) \rangle dB_t^k \\ &+ \frac{1}{2} f_t \|\nabla \nu\|^2 - \frac{1}{2}\langle W_t^{a,T}, \mathrm{trace}\, \nabla^2 \nu(Y_t^a) \rangle dt - f_t \sum_k \nabla_{\sigma_k} \nu(Y_t^a) dB_t^k \\ &- f_t\, \mathrm{trace}\, \nabla^2 \nu(Y_t^a) dt - \sum_k \langle W_t^a, \nabla_{\sigma_k(Y_t^a)} \nu \rangle \nabla_{\sigma_k(Y_t^a)} \nu\, dt. \end{aligned}$$

This completes the proof.

□

Lemma 7.4. *Let* $n_t^a = (/\!/_t^a)^{-1}\nu(Y_t^a)$. *Then* $\lim_{a\to 0} n^a = n$, *in the topology of semi-martingales.*

Proof. By the definition, (7.15), the stochastic differential $D\nu(Y_t^a)$ is essentially n_t^a:

$$dn_t^a = (/\!/_t^a)^{-1}\left(\langle \nabla\nu, \sigma_k(Y_t^a)\rangle\, dB_t^k + \frac{1}{2}\operatorname{trace}\nabla^2\nu(Y_t^a)\, dt\right). \tag{7.10}$$

By the same computation,

$$dn_t = (/\!/_t)^{-1}\left(\langle \nabla\nu, \sigma_k(Y_t)\rangle\, dB_t^k + \frac{1}{2}\operatorname{trace}\nabla^2\nu(Y_t)\, dt\right). \tag{7.11}$$

We recall that $Y^a \to Y$ and $/\!/^a(Y^a) \to /\!/(Y)$. Let

$$\Delta_{\partial M}\nu := \operatorname{trace}\nabla^2\nu,$$

where the trace is taken in the vector space orthogonal to ν. It follows that

$$\nabla\nu(Y^a) \to \nabla\nu(Y), \quad \sigma(Y^a) \to \sigma(Y), \quad \Delta_{\partial M}\nu(Y^a) \to \Delta_{\partial M}\nu(Y),$$

all in the topology of UCP. This implies that n^a converges to n in the topology of semi-martingales.

□

Let us define

$$\underline{\mathrm{Ric}}(x) = \inf_{v\in T_xM, |v|=1}\{\mathrm{Ric}(v,v)\}\,,\ x\in M \quad \text{and} \quad \underline{\mathrm{Ric}} = \inf_{x\in M}\underline{\mathrm{Ric}}(x).$$

We also define

$$\underline{\mathcal{S}}(x) = \inf_{v\in T_xM, |v|=1, \langle v,\nabla R\rangle = 0}\{\mathcal{S}(v,v)\}\,,\ x\in M \quad \text{and} \quad \underline{\mathcal{S}} = \inf_{x\in M}\underline{\mathcal{S}}(x).$$

Lemma 7.5. *For any* $t>0$ *and* a,

$$|W_t^a|^2 \le |W_0^a|^2\, e^{-\int_0^t \underline{\mathrm{Ric}}(Y_s^a)\,ds - 2\int_0^t \underline{\mathcal{S}}(Y_s^a)dL_s^a}. \tag{7.12}$$

Suppose that $\underline{\mathrm{Ric}}$ *and* $\underline{\mathcal{S}}$ *are bounded. Then* $\sup_a \|W^a\|_{\mathscr{S}_p([0,T])}$ *is finite. Furthermore*

$$\mathbf{E}\sup_{t\le T}|W_t^a|^p \le |W_0^a|^p e^{-p\underline{\mathrm{Ric}}T}C(T, -p\underline{\mathcal{S}}) \tag{7.13}$$

where $C(T,\lambda)$ *is defined in* (3.24)

Proof. We begin with W^a. Firstly,

$$\begin{aligned}|W_t^a|^2 =& |W_0^a|^2 - \int_0^t \langle \mathrm{Ric}^\sharp(W_s^a), W_s^a\rangle\, ds - 2\int_0^t \langle \mathcal{S}(W_s^a), W_s^a\rangle\, dL_s^a \\ &- \int_0^t c_s^a \langle W_s^a, \nu_{Y_s^a}\rangle^2\, ds\end{aligned}$$

where c_s^a is the scalar normal part of W_t^a, see (7.6). It is easy to see that $c_s^a > 0$. So

$$\begin{aligned}|W_t^a|^2 \le& |W_0^a|^2 - \int_0^t \mathrm{Ric}(W_s^a, W_s^a)\, ds - 2\int_0^t \langle \mathcal{S}(W_s^a), W_s^a\rangle\, dL_s^a \\ \le& |W_0^a|^2 - \int_0^t |W_s^a|^2\underline{\mathrm{Ric}}(Y_s^a)\, ds - 2\int_0^t |W_s^a|^2\underline{\mathcal{S}}(Y_s^a)dL_s^a.\end{aligned}$$

This implies (7.12) and (7.13) immediately follows.

□

Lemma 7.6. *We also have*

$$\|W\|^p_{\mathscr{S}_p([0,T])} \leq |W_0|^p e^{-pT\underline{\mathrm{Ric}}} C(T, -p\underline{\mathcal{S}}) \tag{7.14}$$

Proof. Since W^ε converges to W, a similar computation holds for W^ε, the conclusion for (W_t) follows. □

7.3. The Local Time. Let us recall the notation

$$n^a_t = (/\!/^a_t)^{-1}\nu(Y^a_t),\ \mathrm{ric}^a_t = (/\!/^a_t)^{-1}\mathrm{Ric}^\sharp(/\!/^a_t(\cdot)),\ \mathrm{s}^a_t = (/\!/^a_t)^{-1}\mathcal{S}(/\!/^a_t(\cdot)). \tag{7.15}$$

Denote $w^a_t = /\!/^{-1}_t(Y^a)W^a_t$. Then

$$dw^a_t = -c_a(t)f_a(t)n^a_t dt - \mathrm{s}^a_t(w^a_t)dL^a_t - \frac{1}{2}\mathrm{ric}^a_t(w^a_t)\,dt.$$

Recall c_a is a real valued stochastic process defined in (7.6). Let us define a new stochastic process

$$\tilde{c}_a(t) = c_a(t) + \frac{1}{2}\|\nabla\nu(Y^a_t)\|^2_{\mathrm{H.S.}} + \frac{1}{2}\langle \mathrm{ric}^a_t(n^a_t), n^a_t\rangle, \tag{7.16}$$

and also

$$C_a(t) = \int_0^t c_a(s)\,ds, \qquad \tilde{C}_a(s,t) = \int_s^t \tilde{c}_a(s)\,ds. \tag{7.17}$$

In the tubular neighbourhood, the following holds.

Lemma 7.7. *For $0 \leq s < t \leq S$, define*

$$\begin{aligned}\tilde{r}^a_t = &\int_0^t \langle w^{a,T}_s, \nabla_{\sigma_k(Y^a_s)}\nu\rangle dB^k_s - \frac{1}{2}\int_0^t \langle \mathrm{ric}^a_s(w^{a,T}_s), n^a_s\rangle\,ds \\ &+ \frac{1}{2}\int_0^t \langle w^{a,T}_s, (/\!/^a_s)^{-1}\,\mathrm{trace}\,\nabla^2\nu(Y^a_s)\rangle\,ds\end{aligned} \tag{7.18}$$

Then

$$f_a(t) = f_a(0)e^{-\int_0^t \tilde{c}_a(r)dr} + \int_0^t e^{-\int_s^t \tilde{c}_a(r)dr}\,d\tilde{r}^a_s. \tag{7.19}$$

Proof. By Lemma 7.3, the function $f_a(t)$ is a solution to the following equation,

$$\begin{aligned}df_t = -c_a(t)f_a(t)dt - \frac{1}{2}\mathrm{Ric}(W^a_t, \nu(Y^a_t))dt + \sum_k \langle W^{a,T}_t, \nabla_{\sigma_k}\nu(Y^a_t)\rangle\,dB^k_t \\ + \frac{1}{2}\langle W^{a,T}_t, \mathrm{trace}\,\nabla^2\nu(Y^a_t)\rangle dt - \frac{1}{2}f_a(t)\|\nabla\nu(Y^a_t)\|^2.\end{aligned}$$

Split the W^a_t terms into its tangential and normal parts:

$$\begin{aligned}df_t = &- c_a(t)f_a(t)dt - \frac{1}{2}f_a(t)\mathrm{Ric}(\nu(Y^a_t), \nu(Y^a_t))dt - \frac{1}{2}f_a(t)\|\nabla\nu\|^2 dt \\ &- \frac{1}{2}\mathrm{Ric}(W^{a,T}_t, \nu(Y^a_t))dt + \sum_k \langle W^{a,T}_t, \nabla_{\sigma_k(Y^a_t)}\nu\rangle dB^k_t \\ &+ \frac{1}{2}\langle W^{a,T}_t, \mathrm{trace}\,\nabla^2\nu(Y^a_t)\rangle dt.\end{aligned}$$

The required identity follows from the variation of constant method .

□

Lemma 7.8. *Define for $t \in [0, S]$,*

$$L_t^a = \int_0^t \frac{2\,ds}{a \sinh\left(\frac{2R_s^a}{a}\right)}. \tag{7.20}$$

Let $p, q \in [1, \infty]$ and r defined by $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. Let Z_t^a and Z_t be continuous real semimartingales defined on $[0, S]$. Then

$$\begin{aligned} &\left\| \int_0^{\cdot} Z_s^a \cdot L_s^a - \int_0^{\cdot} Z_s dL_s \right\|_{\mathscr{S}_r} \\ &\leq \|Z^a - Z\|_{\mathscr{S}_p} \|L_S^a\|_q + \|L^a - L\|_{\mathscr{S}_p} \left(\|Z\|_{\mathscr{S}_q} + \|Z\|_{\mathscr{H}_q} \right). \end{aligned} \tag{7.21}$$

Proof. We have for $t \in [0, S]$

$$\begin{aligned} &\int_0^t Z_s^a \, dL_s^a - \int_0^t Z_s \, dL_s \\ &= \int_0^t (Z_s^a - Z_s) \, dL_s^a + \int_0^t Z_s \, d(L_s^a - L_s) \\ &= \int_0^t (Z_s^a - Z_s) \, dL_s^a + \int_0^t (L_s^a - L_s) \, dZ_s + Z_t(L_t^a - L_t). \end{aligned}$$

Since L^a is nondecreasing we have

$$\|L_\cdot^a\|_{\mathscr{H}_q([0,t])}^q = \mathbf{E}\left(\sup_{s \leq t} |L_s^a|^q \right) = \|L_\cdot^a\|_{\mathscr{S}_q([0,t])}^q,$$

so we get by (B.2),

$$\left\| \int_0^{\cdot} (Z_s^a - Z_s) \, dL_s^a \right\|_{\mathscr{H}_r([0,t])} \leq \|Z_\cdot^a - Z_\cdot\|_{\mathscr{S}_p([0,t])} \|L_\cdot^a\|_{\mathscr{S}_q([0,t])}^q.$$

Similar estimates holds for the last two terms on the right hand side of the identity. This concludes the proof. □

Let S be a stopping time such that $Y_t \in F_0$ on $\{t < S\}$.

Lemma 7.9. *Let $\alpha_t = \sup_{s \leq t}\{s \leq t : Y_s \in \partial M\}$. Suppose that $t \notin \mathscr{R}(\omega)$. For all $s, t \in [0, S]$ satisfying $s < t$,*

$$\begin{aligned} &\lim_{a \to 0} e^{-\int_s^t c_a(u)\,du} = 1 \quad \textit{if} \quad s > \alpha_t \\ &\lim_{a \to 0} e^{-\int_s^t c_a(u)\,du} = 0 \quad \textit{if} \quad s < \alpha_t. \end{aligned} \tag{7.22}$$

The convergence is in probability. As a consequence, for all $p \geq 1$,

$$\lim_{a \to 0} \mathbf{E}\left[\int_0^S \left| e^{-C_a(t)} - \chi_{\{s > \alpha_t\}} \right|^p dt \right] = 0; \tag{7.23}$$

$$\lim_{a \to 0} \mathbf{E}\left[\int_0^S \left(\int_0^t \left| e^{-\int_s^t c_a(u)\,du} - \chi_{\{s > \alpha_t\}} \right|^p ds \right) dt \right] = 0. \tag{7.24}$$

Proof. From (7.22) it is easy to get (7.23) and (7.24) using the fact that $e^{-\int_s^t c_a(u)\,du}$ and $\chi_{\{s>\alpha_t\}}$ are uniformly bounded and Fubini-Tonelli theorem.

So let us prove (7.22). Write

$$\int_s^t c_a(u)\,du = \int_s^t \frac{2\cosh\left(\frac{2R_u^a}{a}\right)}{a\sinh\left(\frac{2R_u^a}{a}\right)} \times \frac{2}{a\sinh\left(\frac{2R_u^a}{a}\right)}\,du.$$

If $s > \alpha_t$ then there exists $\varepsilon(\omega) > 0$ such that for $u \in [s,t]$, $R_u > \varepsilon(\omega)$. Since $R^a \to R$ in UCP topology, $\sup_{u\in[s,t]} c_a(u)$ converges to 0 in probability, and this implies that $e^{-\int_s^t c_a(u)\,du} \to 1$.

If $s < \alpha_t$ then $L_t - L_s > 0$. Indeed, this would be true if R_t was a reflected Brownian motion. But by Girsanov transform we obtain that the law of R_t is equivalent to the one of a reflected Brownian motion (Lemma C.1). So this is true.

Now we have

$$\begin{aligned}\int_s^t \frac{2\cosh\left(\frac{2R_u^a}{a}\right)}{a\sinh\left(\frac{2R_u^a}{a}\right)} \times \frac{2}{a\sinh\left(\frac{2R_u^a}{a}\right)}\,du &> \int_s^t \frac{2}{a} \times \frac{2}{a\sinh\left(\frac{2R_u^a}{a}\right)}\,du \\ &= \frac{2}{a}(L_t^a - L_s^a).\end{aligned}$$

Since $R^a \to R$ in UCP topology we have that $L^a \to L$ in UCP topology. So for all $a_0 > 0$ the $\liminf$ of the right hand side is larger than $\dfrac{2}{a_0}(L_t - L_s)$. This yields

$$\limsup_{a\to 0} e^{-\int_s^t c_a(u)\,du} < e^{-\frac{2}{a_0}(L_t - L_s)}$$

in probability. Letting $a_0 \to 0$ we get

$$\lim_{a\to 0} e^{-\int_s^t c_a(u)\,du} = 0 \quad \text{in probability.}$$

□

From this result we get the following

Corollary 7.10. *Define*

$$\tilde{c}(t) = -\frac{1}{2}\|\nabla\nu(Y_t)\|^2_{\text{H.S.}} - \frac{1}{2}\langle \text{ric}_t(n_t), n_t\rangle \tag{7.25}$$

where ric_t *is defined in* (C.9)*. For* $s,t \in [0,S]$ *satisfying* $s < t$ *we define*

$$\tilde{C}(s,t) = \begin{cases} \int_s^t \tilde{c}(s)\,ds, & \text{if} \quad s > \alpha_t \\ +\infty, & \text{if} \quad s \le \alpha_t \end{cases} \tag{7.26}$$

Then the following convergence holds in probability for $\tilde{C}_a(s,t)$ *defined in* (7.17)*:*

$$\lim_{a\to 0} e^{-\tilde{C}_a(s,t)} = e^{-\tilde{C}(s,t)}. \tag{7.27}$$

Consequently, for all $p \ge 1$*,*

$$\lim_{a\to 0} \mathbf{E}\left[\int_0^S \left|e^{-\tilde{C}_a(t)} - e^{-\tilde{C}(t)}\right|^p dt\right] = 0, \tag{7.28}$$

$$\lim_{a\to 0} \mathbf{E}\left[\int_0^S \left(\int_0^t \left|e^{-\tilde{C}_a(s,t)} - e^{-\tilde{C}(s,t)}\right|^p ds\right) dt\right] = 0. \tag{7.29}$$

With these notations Equation (5.3) *rewrites as*

$$f(t) = f(0)e^{-\tilde{C}(t)} + \int_0^t e^{-\tilde{C}(s,t)}\, d\tilde{r}_s \tag{7.30}$$

where

$$\begin{aligned}\tilde{r}_t = &\int_0^t \langle w_s^T, \nabla_{\sigma_k(Y_s)}\nu\rangle dB_s^k - \frac{1}{2}\int_0^t \langle \mathrm{ric}_s(w_s^T), n_s\rangle\, ds \\ &+ \frac{1}{2}\int_0^t \langle w_s^T, /\!\!/_s^{-1} \operatorname{trace} \nabla^2\nu(Y_s)\rangle\, ds.\end{aligned} \tag{7.31}$$

Proof. The convergences are obvious. For equation (7.31), we see if $t < \zeta$, $e^{-\tilde{C}(s,t)} \to 0$ for any $s \geq 0$. Hence

$$f(t) = f(0) + \int_0^t d\tilde{r}_s = f(0) + \tilde{r}_t.$$

If $t \geq \zeta$,

$$\int_0^t e^{-\tilde{C}(s,t)}\, d\tilde{r}_s = \int_{\alpha(t)}^t e^{-\tilde{C}(s,t)}\, d\tilde{r}_s$$

and so

$$f(t) = \int_{\alpha(t)}^t d\tilde{r}_s = \tilde{r}(t) - \tilde{r}(\alpha(t)).$$

□

The new expression (7.30) for f_t is the same form as the equation for $f_a(t)$:

$$\begin{aligned} f_a(t) =& f_a(0)e^{-\int_0^t \tilde{c}_a(r)dr} + \int_0^t e^{-\int_s^t \tilde{c}_a(r)dr}\, d\tilde{r}_s^a \\ =& f_a(0)e^{-\tilde{C}_a(t)} + \int_0^t e^{-\tilde{C}_a(s,t)}\, d\tilde{r}_s^a. \end{aligned}$$

We observe also that $c_a(t)$ does not converge to a finite stochastic process, hence we only expect that $f_a(t)$ converges to $f(t)$ in a weak sense. Especially it is only for a set of t of full measure that $f_a(t) \to f(t)$. This will be made precise in part 7.5

7.4. Convergence of the tangential parts. We will see that tangential parts of W_t^a converges in UCP topology, as for normal parts we have to exclude the boundary times. But both of them converge in $L^p([0,T] \times \Omega)$, this will be proved at the very end of the proof. Let us begin with the first convergence.

Lemma 7.11. *As $a \to 0$, $W_\cdot^{a,T} \to W_\cdot^T$ in UCP topology.*

Proof. Since $/\!\!/_t^a \to /\!\!/_t$ in the UCP topology it is sufficient to prove that $w_\cdot^{a,T} \to w_\cdot^T$ in the UCP topology. We recall from Lemma 7.3, the term involving $c_a(t)$ cancels and we have

$$
\begin{aligned}
DW_t^{a,T} = &- \frac{1}{2}\left(\mathrm{Ric}^\sharp(W_t^{a,T})\right)^T dt - \frac{1}{2} f_a(t)\left(\mathrm{Ric}^\sharp(\nu(Y_t^a))\right)^T dt - \mathcal{S}(W_t^a)\, dL_t^a \\
&- f_a(t)\nabla_{\sigma_k}\nu(Y_t^a)dB_t^k - \frac{1}{2} f_t \,\mathrm{trace}\,\nabla^2\nu(Y_t^a)dt \\
&- \frac{1}{2}\langle W_t^{a,T}, \nabla_{\sigma_k}\nu(Y_t^a)\rangle \nabla_{\sigma_k}\nu(Y_t^a)\, dt \\
&- \sum_k \langle W_t^{a,T}, \nabla_{\sigma_k}\nu(Y_t^a)\rangle \nu(Y_t^a)dB_t^k - \langle W_t^{a,T}, \mathrm{trace}\,\nabla^2\nu(Y_t^a)\rangle \nu(Y_t^a)dt \\
&+ f_t \|\nabla\nu(Y_t^a)\|^2 \nu(Y_t^a)dt.
\end{aligned}
$$

Hence

$$
\begin{aligned}
dw_t^{a,T} = &- \frac{1}{2}\mathrm{ric}_t^a(w_t^{a,T})\, dt - \frac{1}{2} f_a(t)\left(\mathrm{Ric}^\sharp(\nu(Y_t^a))\right)^T dt - \mathrm{s}_t^a(w_t^{a,T})\, dL_t^a \\
&- f_a(t)(/\!\!/_t^a)^{-1}\nabla_{\sigma_k}\nu(Y_t^a)dB_t^k - \frac{1}{2} f_a(t)(/\!\!/_t^a)^{-1}\,\mathrm{trace}\,\nabla^2\nu(Y_t^a)dt \\
&- \frac{1}{2}\langle w_t^{a,T}, (/\!\!/_t^a)^{-1}\nabla_{\sigma_k}\nu(Y_t^a)\rangle (/\!\!/_t^a)^{-1}\nabla_{\sigma_k}\nu(Y_t^a)\, dt \\
&- \sum_k \langle w_t^{a,T}, (/\!\!/_t^a)^{-1}\nabla_{\sigma_k}\nu(Y_t^a)\rangle n_t^a\, dB_t^k \\
&- \langle w_t^{a,T}, \mathrm{trace}(/\!\!/_t^a)^{-1}\nabla^2\nu(Y_t^a)\rangle n_t^a\, dt \\
&+ f_a(t)\|\nabla\nu(Y_t^a)\|^2 n_t^a dt.
\end{aligned}
\tag{7.32}
$$

We define the processes $v_t^{a,T}$, $v_t^{a,\vec{n}}$:

$$
\begin{aligned}
v_u^{a,T}(\cdot) = &- \frac{1}{2}\int_0^u \mathrm{ric}_t^a(\cdot)\, dt - \int_0^u \mathrm{s}_t^a(\cdot)\, dL_t^a \\
&- \frac{1}{2}\int_0^u \langle \cdot, (/\!\!/_t^a)^{-1}\nabla_{\sigma_k}\nu(Y_t^a)\rangle (/\!\!/_t^a)^{-1}\nabla_{\sigma_k}\nu(Y_t^a)\, dt \\
&- \sum_k \int_0^u \langle \cdot, (/\!\!/_t^a)^{-1}\nabla_{\sigma_k}\nu(Y_t^a)\rangle n_t^a\, dB_t^k \\
&- \int_0^u \langle \cdot, \mathrm{trace}(/\!\!/_t^a)^{-1}\nabla^2\nu(Y_t^a)\rangle n_t^a\, dt.
\end{aligned}
\tag{7.33}
$$

Also,

$$
\begin{aligned}
v_u^{a,\nu} = &- \int_0^u \frac{1}{2}\left(\mathrm{Ric}^\sharp(\nu(Y_t^a))\right)^T dt - \int_0^u (/\!\!/_t^a)^{-1}\nabla_{\sigma_k}\nu(Y_t^a)dB_t^k \\
&- \frac{1}{2}\int_0^u (/\!\!/_t^a)^{-1}\,\mathrm{trace}\,\nabla^2\nu(Y_t^a)dt + \int_0^u \|\nabla\nu(Y_t^a)\|^2 n_t^a dt.
\end{aligned}
\tag{7.34}
$$

With these notations and the expression for $f_a(t)$ in formula (7.19) we have

$$
dw_t^{a,T} = dv_t^{a,T}(w_t^{a,T}) + f_a(t)dv_t^{a,\nu}
$$

and so

$$
dw_t^{a,T} = dv_t^{a,T}(w_t^{a,T}) + \left(f_a(0)e^{-\tilde{C}_a(t)} + \int_0^t e^{-\int_s^t \tilde{c}_a(u)\,du}\, d\tilde{r}_s^a\right) dv_t^{a,\nu}. \tag{7.35}
$$

We also have

$$dw_t^T = dv_t^T(w_t^T) + \left(f(0)e^{-\tilde{C}(t)} + \int_0^t e^{-\tilde{C}(s,t)}\, d\tilde{r}_s \right) dv_t^\nu, \tag{7.36}$$

(recall that $e^{-\tilde{C}(s,t)} = 0$ if $s < \alpha_t$), where

$$\begin{aligned} v_u^\nu = &-\frac{1}{2}\int_0^u \mathrm{ric}_t(n_t)^T\, dt + \int_0^u \|\nabla\nu(Y_t)\|^2 n_t\, dt - \int_0^u (/\!\!/_t)^{-1}\nabla_{\sigma(Y_t)\, dB_t}\nu \\ &-\frac{1}{2}\int_0^u (/\!\!/_t)^{-1}\Delta^{h,T}\nu(Y_t)\, dt, \end{aligned} \tag{7.37}$$

$$\begin{aligned} v_u^T(\cdot) = &-\frac{1}{2}\int_0^u \mathrm{ric}_t(\cdot)\, dt - \int_0^u \mathrm{s}_t(\cdot)\, dL_t - \int_0^u \langle \cdot, (/\!\!/_t)^{-1}\nabla_{\sigma^T(Y_t)\, dB_t}\nu\rangle n_t \\ &-\frac{1}{2}\int_0^u \langle \cdot, (/\!\!/_t)^{-1}\Delta^h\nu(Y_t)\rangle n_t\, dt \\ &-\sum_{j\geq 2}\int_0^u \langle \cdot, \nabla_{\sigma_j(Y_t)}\nu\rangle (/\!\!/_t)^{-1}\nabla_{\sigma_j(Y_t)}\nu\, dt. \end{aligned} \tag{7.38}$$

We investigate further (7.35)

$$\begin{aligned} dw_t^{a,T} =& dv_t^{a,T}(w_t^{a,T}) + f_a(0)e^{-\tilde{C}_a(t)} dv_t^{a,\nu} \\ &+ \left(\int_0^t e^{-\int_s^t \tilde{c}_a(u)\, du} \langle w_s^{a,T}, \nabla_{\sigma_k(Y_s^a)}\nu\rangle dB_s^k \right) dv_t^{a,\nu} \\ &- \frac{1}{2}\left(\int_0^t e^{-\int_s^t \tilde{c}_a(u)\, du} \langle \mathrm{ric}_s^a(w_s^{a,T}), n_s^a\rangle\, ds \right) dv_t^{a,\nu} \\ &+ \frac{1}{2}\left(\int_0^t e^{-\int_s^t \tilde{c}_a(u)\, du} \langle w_s^{a,T}, (/\!\!/_s^a)^{-1}\, \mathrm{trace}\, \nabla^2\nu(Y_s^a)\rangle\, ds \right) dv_t^{a,\nu}. \end{aligned}$$

From this the required convergence should follow: when a approaches zero, $v_t^{a,\nu}$ approaches v_t^a and $v_t^{a,T}$ approaches v_t^T. If furthermore if $f_a(0) \to f(0)$, then

$$\lim_{a\to 0} f_a(0)e^{-\tilde{C}_a(t)} = f(0)e^{-\tilde{C}(t)}.$$

Hence the components of $w_t^{a,T}$ is the solution to a system of non-Markovian stochastic differential equations whose coefficients converge, and furthermore $v_t^{a,T}(w_t^{a,T})$ converges only in UCP, not in $\mathscr{H}_p([0,T])$. These factors explain why the proof below is length given this simple explanation. To prove that $v_t^{a,T} \to v_t^T$ in UCP topology, c.f. (7.33) and (7.38), we only need to prove that $\int_0^\cdot \mathrm{s}_t^a(\cdot)\, dL_t^a \to \int_0^\cdot \mathrm{s}_t(\cdot)\, dL_t$ in UCP topology. This is a consequence of Lemma 7.8, together with the facts that UCP topology is equivalent to local convergence in $\mathscr{S}_p$ and that the random variables L_S^a are uniformly bounded in L^2.

To make the rest of the proof more transparent let us define

$$\begin{aligned} \tilde{u}_t = &\int_0^t e^{-\tilde{C}(s,t)} \langle \cdot, \nabla_{\sigma^T(Y_s)\, dB_s}\nu\rangle \\ &-\frac{1}{2}\int_0^t e^{-\tilde{C}(s,t)} \langle \mathrm{ric}_s(\cdot), n_s\rangle\, ds + \frac{1}{2}\int_0^t e^{-\tilde{C}(s,t)} \langle \cdot, /\!\!/_s^{-1}\Delta^{h,T}\nu(Y_s)\rangle\, ds; \end{aligned} \tag{7.39}$$

$$\begin{aligned}\tilde{u}_t^a = &\int_0^t e^{-\tilde{C}_a(s,t)}\langle \cdot, \nabla_{\sigma^T(Y_s^a)\,dB_s}\nu\rangle \\ &- \frac{1}{2}\int_0^t e^{-\tilde{C}_a(s,t)}\langle \mathrm{ric}_s^a(\cdot), n_s^a\rangle\, ds + \frac{1}{2}\int_0^t e^{-\tilde{C}_a(s,t)}\langle \cdot, /\!\!/_s^{-1}\Delta^{h,T}\nu(Y_s^a)\rangle\, ds\end{aligned} \tag{7.40}$$

Then, by Lemma 7.3, we may write

$$f_a(t) = f_a(0)e^{-\tilde{C}_a(t)} + \int_0^t d\tilde{u}_s^a(w_s^{a,T}), \tag{7.41}$$

$$f(t) = f(0)e^{-\tilde{C}(t)} + \int_0^t d\tilde{u}_s(w_s^T). \tag{7.42}$$

Take theses back to equations (7.35) and (7.36), we see

$$dw_t^{a,T} = dv_t^{a,T}(w_t^{a,T}) + \left(f_a(0)e^{-\tilde{C}_a(t)} + \int_0^t d\tilde{u}_s^a(w_s^{a,T})\right) dv_t^{a,\nu}.$$

We also have

$$dw_t^T = dv_t^T(w_t^T) + \left(f(0)e^{-\tilde{C}(t)} + \int_0^t d\tilde{u}_s(w_s^T)\right) dv_t^\nu.$$

Let us simply compute the difference of the two matrices:

$$\begin{aligned}&d(w_t^{a,T} - w_t^T) \\ =&d(v_t^{a,T} - v_t^T)(w_t^{a,T}) + dv_t^T(w_t^{a,T} - w_t^T) + f_a(t)\, d(v_t^{a,\nu} - v_t^\nu) \\ &+ dv_t^\nu\left(f_a(0)e^{-\tilde{C}_a(t)} - f(0)e^{-\tilde{C}(t)}\right) \\ &+ dv_t^\nu \int_0^t d(\tilde{u}_s^a - \tilde{u}_s)(w_s^{a,T}) + dv_t^\nu \int_0^t d\tilde{u}_s(w_s^{a,T} - w_s^T).\end{aligned} \tag{7.43}$$

Now we recall that convergence in UCP topology is implied by local convergence in $\mathscr{S}_1$. For a stopping time S' smaller than S we have

$$\left\|\left(\int_0^\cdot dv_t^T(w_t^{a,T} - w_t^T)\right)^{S'}\right\|_{\mathscr{S}_1} \leq \|(v^T)^{S'}\|_{\mathscr{H}_\infty} \cdot \|(w_\cdot^{a,T} - w_\cdot^T)^{S'}\|_{\mathscr{S}_1} \tag{7.44}$$

Since $v_0^T = 0$ and v^T has locally bounded $\mathscr{H}_\infty$ norm we can split the time interval and we only have to make the proof on $[0, S']$ where $S' \leq S$ is a stopping time so that

$$\|(v^T)^{S'}\|_{\mathscr{H}_\infty} < 1. \tag{7.45}$$

Then using an argument analogous to that for (7.44) we see

$$\begin{aligned}&\left\|\left(\int_0^\cdot dv_t^\nu \int_0^t d\tilde{u}_s(w_s^{a,T} - w_s^T)\right)^{S'}\right\|_{\mathscr{S}_1} \\ &\leq \|(v^\nu)^{S'}\|_{\mathscr{H}_\infty} \cdot \left\|\left(\int_0^\cdot d\tilde{u}_t(w_t^{a,T} - w_s^T)\right)^{S'}\right\|_{\mathscr{S}_1} \\ &\leq \|(v^\nu)^{S'}\|_{\mathscr{H}_\infty} \cdot \|(\tilde{u})^{S'}\|_{\mathscr{H}_\infty} \cdot \|(w_\cdot^{a,T} - w_\cdot^T)^{S'}\|_{\mathscr{S}_1}\end{aligned} \tag{7.46}$$

Since v^ν and $\tilde{u}$ have locally bounded $\mathscr{H}_\infty$ norms, with the same argument we can take S' so that

$$\|(v^\nu)^{S'}\|_{\mathscr{H}_\infty} \cdot \|(\tilde{u})^{S'}\|_{\mathscr{H}_\infty} < 1. \tag{7.47}$$

We want to prove that $\|(w_\cdot^{a,T} - w_\cdot^T)^{S'}\|_{\mathscr{S}_1} \to 0$ as $a \to 0$. Using (7.43-7.45), (7.46), (7.47) and Gronwall lemma, it is sufficient to prove that

$$\lim_{a\to 0} \left\| \left(\int_0^\cdot d(v_t^{a,T} - v_t^T)(w_t^{a,T}) \right)^{S'} \right\|_{\mathscr{S}_1} = 0, \tag{7.48}$$

$$\lim_{a\to 0} \left\| \left(\int_0^\cdot dv_t^\nu \left(f_a(0) e^{-\tilde{C}_a(t)} - f(0) e^{-\tilde{C}(t)} \right) \right)^{S'} \right\|_{\mathscr{S}_1} = 0. \tag{7.49}$$

and

$$\lim_{a\to 0} \left\| \left(\int_0^\cdot dv_t^\nu \int_0^t d(\tilde{u}_s^a - \tilde{u}_s)(w_s^{a,T}) \right)^{S'} \right\|_{\mathscr{S}_1} = 0. \tag{7.50}$$

For (7.48) we write

$$\int_0^t d(v_s^{a,T} - v_s^T)(w_s^{a,T}) = w_t^{a,T}(v_t^{a,T} - v_t^T) - \int_0^t v_s^{a,T} - v_s^T) dw_s^{a,T}. \tag{7.51}$$

From (7.32) and Lemma 7.5 we see that the processes $w^{a,T}$ are uniformly bounded in $\mathscr{H}_2$. Since$v_t^{a,T} \to v_t^T$ in UCP topology, $v_t^{a,T} \to v_t^T$ locally in $\mathscr{S}_\infty$. We have

$$\left\| \left(\int_0^\cdot v_s^{a,T} - v_s^T) dw_s^{a,T} \right)^{S'} \right\|_{\mathscr{S}_2} \le \|(v^{a,T} - v^T)^{S'}\|_{\mathscr{S}_\infty} \cdot \|w_s^{a,T}\|_{\mathscr{H}_2}$$

and

$$\begin{aligned} \|w_\cdot^{a,T}(v_\cdot^{a,T} - v_\cdot^T)\|_{\mathscr{S}_2} &\le \|(v^{a,T} - v^T)^{S'}\|_{\mathscr{S}_\infty} \cdot \|w_s^{a,T}\|_{\mathscr{S}_2} \\ &\le 3\|(v^{a,T} - v^T)^{S'}\|_{\mathscr{S}_\infty} \cdot \|w_s^{a,T}\|_{\mathscr{H}_2}. \end{aligned}$$

From this, (7.51) and the fact that $\mathscr{S}_1$ norm is smaller than $\mathscr{S}_2$ norm, we obtain (7.48).

For (7.49) it is sufficient to compute the $\mathscr{H}_2$ norm of

$$\left(\int_0^\cdot dv_t^\nu \left(f_a(0) e^{-\tilde{C}_a(t)} - f(0) e^{-\tilde{C}(t)} \right) \right)^{S'}$$

and to use the dominated convergence theorem.

Finally let us prove (7.50). This can be done by modifying S', using the facts that the processes $W^{a,T}$ have uniformly bounded $\mathscr{S}_2$ norms and $\tilde{u}^a \to \tilde{u}$ in UCP topology. For this last point, use (7.29) in Corollary 7.10 and Corollary B.4. □

7.5. Convergence of the normal parts.

Lemma 7.12. *For all $p \in [1, \infty)$ and $T > 0$,*

$$\mathbf{E}\left[\int_0^T |f_a(t) - f(t)|^p \, dt \right] \to 0. \tag{7.52}$$

Proof. Write

$$\begin{aligned} f_a(t) - f(t) &= (f_a(0) - f(0)) \, e^{-\tilde{C}_a(t)} + f(0) \left(e^{-\tilde{C}_a(t)} - e^{-\tilde{C}(t)} \right) \\ &\quad + \int_0^t \left(e^{-\tilde{C}_a(s,t)} - e^{-\tilde{C}(s,t)} \right) d\tilde{r}_s^a + \int_0^t e^{-\tilde{C}(s,t)} \, d\left(\tilde{r}_s^a - \tilde{r}_s\right). \end{aligned}$$

The first term in the right converges to 0 in $L^p([0,T] \times \mathbb{P})$ due to the positiveness of $\tilde{C}_a(t)$. The second term in the right converges to 0 due to (7.28). For the last term in

the right we use boundedness of $e^{-\tilde{C}(s,t)}$ and the fact that $\tilde{r}_t^a \to \tilde{r}_t$ in $\mathscr{H}_p([0,T])$ due to (7.18) and (7.31) together with (B.2) and Lemmas 7.11 and 7.5 which allow to prove that $w_t^a \to w_t^T$ in $\mathscr{S}_q([0,T])$, $q \in [1,\infty)$.

We are left to prove that

$$\int_0^t \left(e^{-\tilde{C}_a(s,t)} - e^{-\tilde{C}(s,t)}\right) d\tilde{r}_s^a \to 0 \quad \text{in} \quad L^p([0,T]\times\mathbb{P}.$$

Here it is easier to replace S by $T \geq S$ which is deterministic. We have

$$\begin{aligned}
&\mathbf{E}\left[\int_0^T \left|\int_0^s \left(e^{-\tilde{C}_a(u,s)} - e^{-\tilde{C}(u,s)}\right) d\tilde{r}_u^a\right|^p ds\right] \\
&= \int_0^T \mathbf{E}\left[\left|\int_0^s \left(e^{-\tilde{C}_a(u,s)} - e^{-\tilde{C}(u,s)}\right) d\tilde{r}_u^a\right|^p\right] ds \\
&\leq C(p,T)\int_0^T \mathbf{E}\left[\int_0^s \left|e^{-\tilde{C}_a(u,s)} - e^{-\tilde{C}(u,s)}\right|^p du\right] ds.
\end{aligned}$$

The last inequality comes from the fact that the identity map from $\mathscr{S}_p([0,s])$ to $\mathscr{H}_p([0,s])$ is continuous and bounded by $C(p,s)$ satisfying $0 < C(p,s) \leq C(p,T)$. Notice that the fact that $u \mapsto e^{-\tilde{C}_a(u,s)} - e^{-\tilde{C}(u,s)}$ is not adapted is not a problem since in $d\tilde{r}_u^a$ there is no integration with respect to B^1. We conclude with (7.29) which is easily seen to be true with S replaced by T. □

Lemma 7.13. *For all $p \in [1,\infty)$, $T > 0$ and all smooth $\phi : M \to \mathbf{R}_+$ vanishing in a neighbourhood of ∂M, $\phi(Y_t^a)f_a(t) \to \phi(Y_t)f(t)$ in $\mathscr{S}_p([0,T])$.*

Proof. Since ϕ is bounded and the processes $f_a(t)$ are uniformly bounded in $\mathscr{S}_p([0,T])$ independently of a, it is sufficient to prove convergence in UCP topology.

We have

$$\phi(R_t^a)f_a(t) - \phi(R_t)f(t) = \left(\phi(R_t^a) - \phi(R_t)\right) f_a(t) + \phi(R_t)\left(f_a(t) - f(t)\right).$$

Since the processes $f_a(t)$ are uniformly bounded in $\mathscr{S}_p([0,T])$ independently of a and $R_t^a \to R_t$ in $\mathscr{S}_p([0,T])$, the fist term in the right converges to 0 in UCP topology. Let us consider the second term:

$$\begin{aligned}
d\left(\phi(R_t)(f_a(t) - f(t))\right) =& (f_a(t) - f(t))d\phi(R_t) + \phi(R_t)d\left(f_a(t) - f(t)\right) \\
&+ d\phi(R_t)d\left(f_a(t) - f(t)\right).
\end{aligned}$$

The integral of the first term in the right converges to 0 in UCP topology, due to (7.29) and the fact that $\phi(R_t)$ has uniformly bounded absolutely continuous local characteristics.

On the other hand

$$\begin{aligned}
&\phi(R_t)d\left(f_a(t) - f(t)\right) \\
&= -\tilde{c}_a(t)\phi(R_t)(f_a(t) - f(t))\,dt + \phi(R_t)(\tilde{c}(t) - \tilde{c}_a(t))f(t)\,dt \\
&\quad + \phi(R_t)(f_a(t) - f(t))d\tilde{r}_a(t) + \phi(R_t)f(t)d(\tilde{r}_t^a - \tilde{r}_t).
\end{aligned}$$

From subsection 7.4 together with (7.18) and (B.2) we get that $\tilde{r}^a \to \tilde{r}$ in semimartingale topology.

So due to the presence of $\phi(R_t)$ which vanishes in a neighbourhood of ∂M all the terms behave nicely, with the help of (7.52).

Finally the covariance term can be treated with similar methods. □

With this we completed the proof of Theorem 5.6 and close this section.

APPENDIX A.
THE HALF LINE EXAMPLE

On the half line we select a reflected Brownian motion with 'good' sample path properties. To begin with we consider two reflected Brownian motions: the solution to the Skorohod problem associated with a Brownian motion $x + B_t$ and the solution to the Skorohod problem associated with $x + \int_0^t \operatorname{sign}(x + B_s)dB_s$. The first is a stochastic flow, see lemma A.1 below, while the second is not. The solution and the derivative flow to the Skorohod problem for $x+B_t$ is approximated by solutions and derivative flows to a family of SDEs with explicit drifts. Furthermore, the derivative flow to the Skorohod problem is shown to coincide with the damped parallel translation introduced in Appendix C.

Denote $C_0(\mathbf{R};\mathbf{R})$ the space of real valued continuous function with $f(0) = 0$ and $C_0(\mathbf{R}_+;\mathbf{R}_+)$ its subset of non-negative valued functions. To each $x \geq 0$ and $f \in C_0(\mathbf{R},\mathbf{R})$ there exists a function $h \in C_0(\mathbf{R},\mathbf{R}_+)$ such that the function g, given by $g(t) = x + f(t) + h(t)$, satisfies $\int_0^t \chi_{\{0\}} g(s)\dot{h}(s)ds = h(t)$. The pair (g, h) is the solution to the Skorohod problem associated to (x, f) and is denoted by

$$\Phi_\cdot(x, f) = (g, h). \tag{A.1}$$

It is well known that $h(t) = -\inf_{0\leq s\leq t}\{(x + f(s)) \wedge 0\}$. If B_t is a standard real valued Brownian motion, then Skorohod problem defines the pair of stochastic processes $(X_t(x), L_t(x))$, and $L_t(x)$ is called the local time at 0 of $X_t(x)$ and

$$X_t(x) = x + B_t + L_t(x). \tag{A.2}$$

On the other hand, $|x + B_t|$ is also a reflected Brownian motion. In fact, by Tanaka's formula, $|x+B_t| = x+\beta_t+2\ell_t^0(\omega)$ where $\beta_t = \int_0^t \operatorname{sign}(x+B_s)dB_s$ is a Brownian motion and ℓ_t^0 is the local times of $x + B_t$. The local time ℓ_t^0 is also the boundary time, the total time spent by $x + B_t$ on the boundary $\{0\}$ before time t. This was introduced by P. Lévy (1940) as a Borel measurable function $\Omega \times \mathbf{R}_+ \times \mathbf{R} \to \mathbf{R}_+$ such that $\int_0^t f(x + B_s)ds = \int_{\mathbf{R}} f(a)\ell_t^a da$ for all $f \in \mathcal{B}_b(\mathbf{R};\mathbf{R})$ and such that $(t, a) \mapsto \ell_t^a(\omega)$ is continuous for a.s. ω. It is well known that $\ell_t^0 = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\int_0^t \chi_{[0,\varepsilon)}(x+B_s)ds = \lim_{\varepsilon\to 0} \frac{1}{2\varepsilon}\int_0^t \chi_{(-\varepsilon,\varepsilon)}(x+B_s)ds$. It is clear that $(|x + B_t(\omega)|, 2\ell_t^0(\omega))$ is the solution to the Skorohod problem associated with $x + \beta_\cdot(\omega)$, and $|x + B_t|$ is not a stochastic flow.

It turns out that $X_t(x) = x + B_t + L_t(x)$ has many nice properties. Despite that the probability distribution of $X_t(x)$ is that of a reflecting Brownian, on a sample path level it is not at all the reflected path! It is rather, a lifted path, by 'the lower envelope' curve. The lower envelope curve is the unique continuous decreasing curve that is below the given curve (B_t). Let $0 < x < y$. Let $\tau(y) = \inf\{t > 0 : X_t(y) = 0\}$. It is clear that $X_t(y) - X_t(x) = y - x$ until $X_t(x)$ reaches zero and the two stochastic processes coalesce when $X_t(y)$ reaches zero. If we compensate x by $L_t(x)$, the two processes $X_t(x + L_t(x))$ and $X_t(x)$ are equal for all t.

In Lemma A.1 we summarise the sample properties of $X_t(x)$ and discuss differentiability of $X_t(x)$ with respect to x. These properties are elementary and not surprising. It is perhaps more surprising that these elementary properties of $X_t(x)$ are passed to the reflected Brownian motion on a manifold with boundary. We should mention differentiability with respect to the initial value was studied in K. Burdzy [10] and S. Andres [3] for more general domains of the Euclidean space: Lipschitz domains.

For $s < t$ define $\theta_s B = B_{s+\cdot} - B_s$. Let ξ an $\mathcal{F}_s$ measurable random variable and $(X_{s,t}(\xi, \theta_s B), L_{s,t}(\xi, \theta_s B))$ the solution to the Skorohod problem for $(\xi, \theta_s B)$,

$$X_{s,t}(\xi, \theta_s B) = \xi + (\theta_s B)_{t-s} + L_{s,t}(\xi, \theta_s B).$$

Define $L_{s,t}(\xi, \theta_s B) = 0$ for $0 \le t \le s$. For simplicity we also omit B in the flow, and write $X_t(x)$ for $X_t(x, B)$. Let $T_{x,y} = \inf\{t > 0,\ X_t(x) = X_t(y)\}$ be the first time $X_t(x)$ and $X_t(y)$ meets.

Lemma A.1. *The following statements hold pathwise.*

(1) *For all* $0 \le s < t$, $x \in \mathbf{R}$,

$$X_{s,t}(X_s(x, B), \theta_s B) = X_t(x, B), \qquad L_t(x, B) = L_s(x, B) + L_{s,t}(X_s(x, B), \theta_s B).$$

(2) *Let* $0 < x < y$, *then* $X_t(x)$ *and* $X_t(y)$ *coalesce at the finite time* $T(x, y)$. *Furthermore* $T(x, y) = \tau(y)$ *and* $L_{\tau(y)}(x) = y - x$.

(3) *For all* $t \ge 0$ *and* $x > 0$, $X_t(x + L_t(x)) = X_t(x)$.

(4) *For all* $x \ge 0$ *and* $t \ge 0$,

$$\partial_x X_t(x) = \begin{cases} 1, & t < \tau(x) \\ 0, & t > \tau(x) \end{cases}.$$

Proof. For part (1), we observe that,

$$X_{s,t}(X_s(x, B), \theta_s B) = x + B_t + L_s(x, B) + L_{s,t}(X_s(x), \theta_s B).$$

Define $\tilde{L}(r) = L_r(x, B)$ when $r \le s$ and $\tilde{L}(t) = L_s(x, B) + L_{s,t}(X_s(x), \theta_s B)$ for $t > s$. Then $\tilde{L} \in C_0(\mathbf{R}_+, \mathbf{R}_+)$, and $(X_{s,t}(X_s(x, B), \theta_s B), \tilde{L})$ solves the Skorohod problem for (x, B). By the uniqueness of the Skorohod problem, $X_t(x, B) = X_{s,t}(X_s(x, B), \theta_s B)$ and $\tilde{L}(t) = L_t(x, B)$.

Part (2). From the construction of the solution of the Skorohod problem, it is easy to see that $\tau(x) < \tau(y)$ and $X_t(y) - X_t(x) = y - x$ on $\{t < \tau(x)\}$, and $X_{\tau(y)}(y) = X_{\tau(x)}(x)$ on $\{t = \tau(y)\}$. By the flow property, $0 \le X_t(x) \le X_t(y)$ a.s. for all time. In other words, the two curves $\{X_s(x), s \le t\}$ and $\{X_s(y), s \le t\}$ are parallel on $\{t < \tau(x)\}$, until the lower curve hits zero after which the distance between the two curves decreases until $X_t(y)$ reaches zero, upon which point the two curves meet. The accumulated upward lift that $X_t(x)$ receives up to $\tau(y)$ is

$$-\inf_{0 \le s \le \tau(y,\omega)} \{(x - y + y + B_s(\omega)) \wedge 0\} = y - x.$$

This shows that $X_{\tau(y)}(x) = 0$ and together with the flow property we see the coalescence. We completed the proof that $T(x, y) = \tau(y)$ and $L_{\tau(y)}(x) = y - x$.

Part (3). On $\{t < \tau(x)\}$, $X_t(x + L_t(x)) = X_t(x)$ trivially. If $t \ge \tau(x + L_t(x))$,

$$\begin{aligned} X_t(x + L_t(x)) &= x + L_t(x) + B_t(x) - \inf_{0 \le s \le t} ((x + L_t(x) + B_s) \wedge 0) \\ &= x + L_t(x) + B_t(x) - \inf_{0 \le s \le t} ((x + B_s) \wedge 0) - L_t(x) = X_t(x). \end{aligned}$$

If $\tau(x) \le t < \tau(x + L_t(x))$, $X_t(x + L_t(x)) = x + L_t(x) + B_t$ while $X_t(x)$ receives the kick of the size $L_t(x)$: $X_t(x) = x + B_t + L_t(x)$.

Part (4). Take $t < \tau(x)$. Then $t < \tau(x+\varepsilon)$ for $\varepsilon > t - \tau(x)$ and $X_t(x+\varepsilon) = X_t(x) + \varepsilon$, consequently $\partial_x X_t(x) = 1$. Suppose $t > \tau(x)$. Then by part (2), $X_t(x) = X_t(x - \varepsilon)$ for any $\varepsilon < 0$. If $0 < \varepsilon < L_t(x)$, $0 \le X_t(x + \varepsilon) - X_t(x) \le X_t(x + L_t(x)) - X_t(x) = 0$. We used part (3) in the last step. Hence $\partial_x X_t(x) = 0$ for $t > \tau(x)$. This completes the proof. □

A consequence of Lemma A.1 is the following. If we pick up a time $t > \tau(x)$, $X_t(x + L_t(x))$ must reach 0 between $\tau(x)$ and t.

In the following we construct a family of stochastic processes $\{X^a_\cdot(x), a > 0\}$ with the properties stated below illustrating the general construction. [1] For each a, $X^a_\cdot$ is a stochastic flow and $x \mapsto X^a_t(x)$is a diffeomorphisms on its image; [2] they approximate the reflected Brownian motion; [3] their derivatives approximate $\partial_x X_t(x)$.

Let $\phi(x) = \int_0^x e^{-\frac{y^2}{2}} dy$. For $x > 0$ and $a > 0$ let

$$u^a(x) = P(\tau(x) > a) = \sqrt{\frac{2}{\pi}}\phi\left(\frac{x}{\sqrt{a}}\right) = P_a \chi_{(-\infty,x)},$$

where P_t denotes the heat semigroup. Thus $\partial_x u^a = 2p_a$ where p_a is the Gaussian kernel. Formally $u^0(0) = P(\tau(0) > 0) = 0$ and for $x > 0$, $u^0(x) = P(\tau(x) > 0) = 1$, and $\frac{\partial}{\partial x} \ln u^0(x) = \frac{\partial}{\partial x}\chi_{(-\infty,x)} = \delta_0(x)$, the Dirac mass at 0. Note that $\ln u^a$ is a concave function with positive gradient:

$$\partial_x \ln u^a = \frac{1}{\sqrt{a}}(\ln\phi)'(\frac{x}{\sqrt{a}}) = \frac{1}{\sqrt{a}}\frac{e^{-\frac{x^2}{2a}}}{\phi(\frac{x}{\sqrt{a}})} > 0; \tag{A.3}$$

$$\partial_x^2(\ln u^a) = \frac{1}{a}(\ln\phi)''(\frac{x}{\sqrt{a}}) = -\frac{xe^{-\frac{x^2}{2a}}}{a^{\frac{3}{2}}\phi(\frac{x}{\sqrt{a}})} - \frac{e^{-\frac{x^2}{2a}}}{a\phi(\frac{x}{\sqrt{a}})} < 0. \tag{A.4}$$

Proposition A.2. *Let $X^a_t(x)$ be the solution to*

$$X^a_t(x) = x + B_t + \int_0^t \partial_x \ln u^a(X^a_s(x))ds. \tag{A.5}$$

Then $x \mapsto X^a_t(x)$ is an increasing function, $a \mapsto X^a_t(x)$ decrases as a decreases to zero. For every (t, x, ω), $\lim_{a\downarrow 0} X^a_t(x)$ exists. For every $x \geq 0$, the following holds for almost surely all ω: $\lim_{a\downarrow 0} X^a_t(x) = X_t(x)$ for all t.

Proof. That $X^a_t(x)$ increases with x follows from the comparison theorem one dimensional SDEs. We also observe that the drift $\partial_x \ln u^a(x)$ in (A.5) increases with a.

$$\partial_a\partial_x \ln u^a = -\frac{1}{2a^{\frac{3}{2}}}(\ln\phi)'(\frac{x}{\sqrt{a}}) - \frac{x}{2\sqrt{a}a^{\frac{3}{2}}}(\ln\phi)''(\frac{x}{\sqrt{a}}) > 0.$$

For $y > 0$, define

$$F(y) = y(\ln\phi)''(y) + (\ln\phi)'(y).$$

It is clear that $F(y)$ is negative for y sufficiently large. By the comparison theorem, $X^a_t(x)$ increases with a and $\bar{X}_t(x) = \lim_{a\downarrow 0} X^a_t(x)$ exists for every t, x, ω. Consequently

$$A^x_t := \lim_{a\downarrow 0}\int_0^t \partial_x \ln u^a(X^a_s(x))ds$$

exists and

$$\bar{X}_t(x) = x + B_t + A^x_t.$$

Let $f(t, a) = \int_0^t \partial_x \ln u^a(X^a_s(x))ds$, which is positive and increasing with t. ThusA^x_t is non-negative and nondecreasing in t.

Note that $\lim_{a\to 0}\partial_x \ln u^a(x) = 0$ for $x > 0$, but the convergence is not uniform in x. For $x \in (0, \sqrt{a}]$,

$$\partial_x \ln u^a \geq \frac{1}{\sqrt{a}}\frac{e^{-\frac{x^2}{2a}}}{\frac{x}{\sqrt{a}}} \geq \frac{1 - \frac{x^2}{2a}}{x} > \frac{1}{2x}.$$

By comparison with the Bessel square process Bes^2 or standard criterion for diffusion process, for almost surely all ω, $X_t^a(x)$ cannot reach 0. Next we observe that, $X_t(y)$ is a flow, $X_t^a(y) > X_t^a(x)$ whenever $y > x$. Thus $X_t^a(x) > 0$ for all $a > 0$. The limiting process $\bar{X}_t(x)$ has the property:

$$\bar{X}_0(x) = x, \qquad \bar{X}_t(x) \geq 0.$$

Let $S < T$ be random times with $\bar{X}_t(x,\omega) > 0$ for $t \in [S,T]$. Let

$$\delta(\omega) = \inf\{\bar{X}_t(x,\omega),\ t \in [S(\omega), T(\omega)]\} > 0.$$

The function $x \mapsto \partial_x \ln u^a(x)$ decreases,

$$\int_S^T \partial_x \ln u^a(\bar{X}_s(x))ds \leq \int_S^T \partial_x \ln u^a(\delta(\omega))ds.$$

Then, since $X_s^a(x) \geq \bar{X}_s(x)$,

$$\begin{aligned} A_T(\omega) - A_S(\omega) &= \lim_{a\downarrow 0} \int_S^T \partial_x \ln u^a(X_s^a(\omega))ds \\ &\leq \lim_{a\downarrow 0} \int_S^T \partial_x \ln u^a(\bar{X}_s(\omega))ds \\ &\leq \lim_{a\downarrow 0} \int_S^T \partial_x \ln u^a(\delta(\omega))ds = 0. \end{aligned}$$

This implies that

$$\int_0^t \chi_{\{\bar{X}_s(x)>0\}} dA_s^x = 0$$

and $(\bar{X}_t(x), A_t^x)$ solves the Skorohod problem associated to $x + B_t$. □

Lemma A.3. *For all $x > 0$ and $a > 0$, $\partial_x^3 \ln u^a > 0$.*

Proof. It is clearly sufficient to consider the case $a = 1$.

$$A(x) = \partial_x \ln u^1(x) = \phi'(x).$$

We have from (A.4)

$$A'(x) = -xA(x) - A^2(x)$$

and this implies

$$\phi'''(x) = A''(x) = (x^2 - 1)A(x) + 3xA^2(x) + 2A^3(x). \tag{A.6}$$

It is clearly positive when $x \geq 1$. For $0 \leq x < 1$,

$$A(x) = \frac{e^{-x^2/2}}{\int_0^x e^{-y^2/2}\,dy} > \frac{1 - x^2/2}{x} > \frac{1 - x^2}{3x}.$$

Hence

$$\phi'''(x) \geq A(x)\left((x^2 - 1) + 3xA(x)\right) > 0.$$

This completes the proof. □

Since $\ln u_t^a(x)$ is smooth, the derivative flow $V_t^a(x) = \partial_x X_t^a(x)$ exists and satisfies the linear equation $\dot{V}_t^a = (\partial_x^2 \ln u^a)V_t^a$. We prove that V_t^a converges to 1 when $t < \tau(x)$ and converges to 0 when $t > \tau(x)$. In the sequel, by $V_{\tau(x+h)}^a(x)$ we mean $\partial_x X_s^a(x)|_{s=\tau(x+h)}$, and $\tau(x+h)$ is not differentiated.

Theorem A.4. *Let $X_t^a(x)$ be the solution to (A.5). Let $V_t^a(x) = \partial_x X_t^a(x)$. Then the following holds.*

(1) *For all positive a and t, $x \mapsto V_t^a(x)$ is increasing and $t \mapsto V_t^a(x)$ decreases. For any $y > x$,*

$$\lim_{a\to 0} V^a_{\tau(y)}(x) = 0.$$

(2) *For almost all ω the following holds for all $x > 0$ and $t \geq 0$ such that $t \neq \tau(x)$:*

$$\lim_{a\downarrow 0} V_t^a(x) = \partial_x X_t(x).$$

Furthermore,

$$\lim_{a\to 0} \mathbf{E}\int_0^T |V_t^a(x) - \partial_x X_t(x)|\, dt = 0.$$

Proof. We observe that

$$\frac{d}{dt} V_t^a(x) = \partial_x^2(\ln u_t^a)(X_t^a(x))V_t^a(x),$$

and $V_0^a(x) = 1$, leading to the formula,

$$V_t^a(x) = e^{\int_0^t \partial_x^2(\ln u^a)(X_s^a(x))ds}. \tag{A.7}$$

(1) Since $\partial_x^2(\ln u^a) < 0$, $V_t^a(x)$ decreases with t. We differentiate (A.7) to see that

$$\partial_x V_t^a(x) = V_t^a(x)\int_0^t (\partial_x^3 \ln u^a)(X_s^a(x))V_s^a(x)ds.$$

Firstly, letting $V_0^a = 1$. By Lemma A.3, $\partial_x^3 \ln u^a > 0$, so $x \mapsto V_t^a(x)$ is increasing.

Let x, ω be fixed. Let $t \neq \tau(x,\omega)$ be a non-negative number and $h > 0$. There is a number $\theta(\omega) \in [0,1]$ s.t.

$$\frac{X_t^a(x+h) - X_t^a(x)}{h} = \partial_x X_t^a(x+\theta h) \geq \partial_x X_t^a(x).$$

Since $\tau(x+h,\omega) \neq \tau(x,\omega)$ for a.e. ω, for almost all ω we may set $t = \tau(x,\omega)$:

$$0 \leq \partial_x X_t^a(x)|_{t=\tau(x+h)} \leq \frac{X^a_{\tau(x+h)}(x+h) - X^a_{\tau(x+h)}(x)}{h} \leq \frac{X^a_{\tau(x+h)}(x+h)}{h}.$$

Take $h = L_t(x)$. By Proposition A.2

$$\lim_{a\to 0} X^a_{\tau(x+L_t(x))}(x + L_t(x)) = X_{\tau(x+L_t(x))}(x + L_t(x)) = 0.$$

Thus for any $h > 0$,

$$\lim_{a\downarrow 0} V^a_{\tau(x+h)}(x) = 0.$$

(2) Let $x > 0$. By Lemma A.1, $X_t(x + L_t(x)) = X_t(x)$ for all $t \geq 0$. So if $t > \tau(x)$, then $t \geq \tau(x + L_t(x))$. Suppose that $t > \tau(x)$. Since $V_t^a(x)$ decreases with t,

$$0 \leq V_t^a(x) \leq V^a_{\tau(x+L_t(x))}(x).$$

By the conclusion of part (1), the right hand side converges to 0 as $a \to 0$.

If $t < \tau(x)$, $X_t^a(x) > X_t(x) > 0$ by comparison theorem for SDEs. Also $\partial_x^2 \ln u^a < 0$,

$$1 \geq \lim_{a\downarrow 0} \exp\left(\int_0^t \partial_x^2 \ln u^a(X_s^a(x))ds\right) \geq \exp\left(\lim_{a\downarrow 0}\int_0^t \partial_x^2 \ln u^a(X_s(x))ds\right).$$

On the other hand for every y, $\partial_x^2 \ln u^a(y) \to 0$ and $\inf_{s\in[0,t]} X_s(x) > 0$ for $t < \tau(x)$. This concludes that $\lim_{a\downarrow 0} \exp\left(\int_0^t \partial_x^2 \ln u^a(X_s^a(x))ds\right) = 1$. Note that $V_t^a(x)$ is uniformly bounded to conclude the convergence in L^1. □

APPENDIX B. CONVERGENCE IN $\mathscr{S}^p$ AND IN $\mathscr{H}^p$

Let $a_0 > 0$ and let $\{(Y_t^a,\ t < \xi^a), a \in [0, a_0)\}$ be a family of continuous semi-martingales with values in a manifold M. If U is an open domain in M, let $\tau^{U,a}$ denote the exit times:

$$\tau^{U,a} = \inf\{t > 0 : Y_t^a \notin U\}.$$

Definition B.1. (1) We say that Y^a converges to Y^0 in the topology of uniform convergence in probability on compact time sets (UCP) if

(1a) for all relatively compact open domain $U \subset M$,

$$\liminf_{a\to 0} \tau^{U,a} \geq \tau^{U,0},$$

(1b) for all $t > 0$, the following convergence holds in probability:

$$\lim_{a\to 0} \sup_{s\leq t\wedge\tau^{U,a}\wedge\tau^{U,0}} \rho\left(Y_s^a, Y_s^0\right) \overset{(P)}{=} 0$$

(2) Let $p \in [1, \infty)$. We say that Y^a converges to Y^0 locally in $\mathscr{S}_p$ if there exists an increasing sequence of stopping times $(T_n)_{n\geq 1}$ with $\lim_{n\to\infty} T_n = \xi^0$ such that for some $a_1 > 0$ and for all $a < a_1$ and all $n \in \mathbf{N}$, $T_n < \xi^a$ a.s. and

$$\lim_{a\to 0} \mathbf{E}\left[\sup_{t\leq T_n} \rho^p(Y_t^a, Y_0^a)\right] = 0 \tag{B.1}$$

Notice that given (1b), condition (1a) is equivalent to $\liminf_{a\to 0} \xi^a \geq \xi^0$.

Let $\mathbb{D}$ denote the space of real-valued adapted, Cádlág stochastic processes, defined on some filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$ satisfying the usual conditions. We are mainly interested in special semi-martingales from $\mathbb{D}$. Below an element of $\mathbb{D}$ is assumed to be also a special semi-martingale.

For two real valued semi-martingales $X, Y \in \mathbb{D}$ we define the distance functions:

$$r(X, Y) = \sum_{n>0} 2^{-n}\mathbf{E}\left(1 \wedge \sup_{0\leq t\leq n} |X_t - Y_t|\right),$$

$$\hat{r}(X, Y) = \sup_{|H|\leq 1} r\left(\int_0^t H_s d(X_s - Y_s)\right),$$

where the supremum is taken over all predictable processes H bounded by 1. The distance r is compatible with UCP:

$$\sup_{0\leq s\leq t} |X_s^{(n)} - X_s| \to 0 \quad (\text{ in probability })$$

for each $t > 0$ if and only if $r(X^n - X)$ converges to 0. The distance $\hat{r}$ induces the semi-martingale topology on the vector space of semi-martingales.

Define

$$\mathscr{S}^p = \{X \in \mathbb{D} : \|X\|_{\mathscr{S}^p} = \|\sup_t |X_t|\|_{L^p} < \infty\},$$

$$\mathscr{H}^p = \{X \in \mathbb{D} : \|X\|_{\mathscr{H}^p} = \inf\{(\left| |X_0| + [M, M]_\infty)^{\frac{1}{2}} + \int_0^\infty |dA_s|\right|_{L^p} < \infty\}.$$

where the infimum is taken over all semi-martingale decompositions $X = X_0 + M + A$. When the time interval is restricted to a finite time interval $[0, T]$ the notations will be $\mathscr{S}^p([0, T])$ and $\mathscr{H}^p([0, T])$.

A semi-martingale is locally in $\mathscr{S}^p$ and $\mathscr{H}^p$ if there exists a sequence of stopping times T_n increasing to infinity such that $X^{T_n}\chi_{\{T_n>0\}}$ are in these spaces. They are prelocally in these spaces if $X^{T_n-}\chi_{\{T_n>0\}}$ are, where $X^{T-}(t) = X_t\chi_{t\in[0,T)} + X_{T-}\ \chi_{[T,\infty)}(t)$. Let $(X^{(n)})$ and X be semi-martingales. Let $1 \le p < \infty$. If $X^{(n)}$ converges to X is the semi-martingale topology, then there exists a subsequence that converges prelocally in $\mathscr{H}^p$. If $X^{(n)}$ converges to X prelocally in $\mathscr{H}^p$ then it converges in the semi-martingale topology. Extension to $\mathbf{R}^k$-valued processes is done by considering the components.

The following estimate of M. Emery is useful: If Y is a semi-martingale and H a left continuous process with right limit, and $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}$ where $p,q\in[1,\infty]$, then

$$\left\|\int_0^\infty H_s dZ_s\right\|_{\mathscr{H}^r} \le \|H\|_{\mathscr{S}^p}\|Y\|_{\mathscr{H}^q}. \tag{B.2}$$

We review these convergence in the settings that the semi-martingales may have finite life times or take values in a manifold. See [7] for details.

Let $Z_t = Z_0 + M_t + A_t$ be a semi-martingale in $\mathbf{R}^k$ with lifetime ξ and the canonical decomposition of Z_t into starting point, local martingale M_t starting at 0 and a finite variation process A_t starting at 0. Define

$$v(Z)_t = \sum_{i=1}^k \left(|Z_0^i| + \langle M^i, M^i\rangle_t^{1/2} + \int_0^t |dA^i|_s\right), \quad t<\xi. \tag{B.3}$$

Let $T>0$. We say that a family of semi-martingales $Z^{(n)}$ converges to 0 in $\mathscr{S}_p([0,T])$ if $\mathbf{E}\left[\sup_{s\le T}|Z_s^{(n)}|^p\right] \to 0$. It converges to 0 in $\mathscr{H}_p([0,T])$ if $v(Z^{(n)}) \to 0$ in $\mathscr{S}_p([0,T])$.

To define this for a manifold valued stochastic process, we will use an embedding $\Phi : M \to \mathbf{R}^k$. The definition will in fact be independent of this embedding.

Definition B.2. Let (Y_t^a) be a family of semi-martingales indexed by a.

(1) We say that Y^a converges to Y^0 in semi-martingale topology or in SM topology if the semi-martingale norm of $\Phi(Y^a)-\Phi(Y^0)$, $v\left(\Phi(Y^a)-\Phi(Y^0)\right)$, converges to 0 in UCP topology.
(2) Let $p\in[1,\infty)$. We say that Y^a converges to Y^0 locally in $\mathscr{H}_p$ if the processes $v\left(\Phi(Y^a)-\Phi(Y^0)\right)$ converge to 0 locally in $\mathscr{S}_p$.

The convergence in the semi-martingale topology is stronger than convergence in the UCP topology. However it is a remarkable fact that they coincide on the subset of martingales in the manifold. The following characterisations of convergence will be very useful (see [7]).

Proposition B.3.
- *If $Y^a \to Y^0$ as $a\to 0$ in UCP topology then for all $p\in[0,\infty)$ there exists a sequence $a_k\to 0$ such that $Y^{a_k}\to Y^0$ as $k\to\infty$ locally in $\mathscr{S}_p$.*
- *If $Y^a \to Y^0$ as $a\to 0$ in SM topology then for all $p\in[0,\infty)$ there exists a sequence $a_k\to 0$ such that $Y^{a_k}\to Y^0$ as $k\to\infty$ locally in $\mathscr{H}_p$.*
- *If for some $p\in[1,\infty)$ $Y^a\to Y^0$ as $a\to 0$ locally in $\mathscr{S}_p$ then $Y^a\to Y^0$ as $a\to 0$ in UCP topology.*
- *If for some $p\in[1,\infty)$ $Y^a\to Y^0$ as $a\to 0$ locally in $\mathscr{H}_p$ then $Y^a\to Y^0$ as $a\to 0$ in SM topology.*

As a consequence, a standard way to establish UCP or SM convergence given by the following:

Corollary B.4.
- *$Y^a \to Y^0$ as $a \to 0$ in UCP topology if and only if there exists $p \in [1,\infty)$ such that for any $a_k \to 0$ there exists a subsequence a_{k_ℓ} such that $Y^{a_{k_\ell}} \to Y^0$ locally in $\mathscr{S}_p$.*
- *$Y^a \to Y^0$ as $a \to 0$ in SM topology if and only if there exists $p \in [1,\infty)$ such that for any $a_k \to 0$ there exists a subsequence a_{k_ℓ} such that $Y^{a_{k_\ell}} \to Y^0$ locally in $\mathscr{H}_p$.*

For processes which take their values in a compact manifold and which are defined in bounded times, we have the following easy relations.

Corollary B.5. *Assume that M is compact and that all processes are defined on some deterministic time interval $[0,T]$. The following equivalences hold:*
- *$Y^a \to Y^0$ as $a \to 0$ in UCP topology;*
- *$Y^a \to Y^0$ as $a \to 0$ in $\mathscr{S}_p$ for some $p \in [1,\infty)$;*
- *$Y^a \to Y^0$ as $a \to 0$ in $\mathscr{S}_p$ for all $p \in [1,\infty)$.*

Similarly, we have the equivalences
- *$Y^a \to Y^0$ as $a \to 0$ in SM topology;*
- *$Y^a \to Y^0$ as $a \to 0$ in $\mathscr{H}_p$ for some $p \in [1,\infty)$;*
- *$Y^a \to Y^0$ as $a \to 0$ in $\mathscr{H}_p$ for all $p \in [1,\infty)$.*

APPENDIX C.
IKEDA AND WATANABE'S DAMPED PARALLEL TRANSLATION

The parallel transport P_t along a semi-martingale (Z_t) is the semi-martingale with values in $L(T_{Z_0}M, T_{Z_t}M)$ solving the Stratonovich SDE

$$\circ dP_t = \mathfrak{h}_{P_t}(\circ dZ_t), \quad P_0 = \mathrm{Id}_{T_{Z_0}M} \tag{C.1}$$

where $\mathfrak{h}_{P_t}$ denotes horizontal lift to the orthonormal frame bundle. We have identified $\mathbf{R}^d$ with $T_{Z_0}M$. Parallel transports are isometries, a proof for its existence on manifolds with boundary can be found in Ikeda and Watanabe [21]. For simiplitiy we also use the notation $/\!\!/_t(Z)$.

If Z_t is a diffusion process with generator $\mathcal{L} = \frac{1}{2}\Delta + U$, where U is a time dependent vector field, remaining in the interior of M for all time (which happens if M has no boundary or if U is sufficiently strong in a neighbourhood of the boundary), then the parallel transport P_t along Z_t is the diffusion process whose generator on differential 1-forms is $\frac{1}{2}\,\mathrm{trace}\,\nabla^2 + \nabla_U$. If $\Delta^1 = -(d^*d + dd^*)$ is the Hodge Laplacian, $\mathrm{trace}\nabla^2 = \Delta^1 + \mathrm{Ric}$. The damped parallel translation W_t along Z_t is the solution to the equation

$$DW_t = \left(\nabla_{W_t}U - \frac{1}{2}\mathrm{Ric}^\sharp(W_t)\right)dt, \quad W_0 = \mathrm{Id}_{T_{Z_0}M}, \tag{C.2}$$

where the covariant derivative DW_t is defined to be $/\!\!/_t d\left(/\!\!/_t^{-1}W_t\right)$. The process (W_t) is a diffusion process with generator on 1-forms L^W :

$$\mathcal{L}^W\alpha = \frac{1}{2}\Delta^1\alpha + \nabla_U\alpha + \alpha\left(\nabla.U\right). \tag{C.3}$$

The fundamental property of $\mathcal{L}^W$ is its commutation with differentiation:

$$d(\mathcal{L}f) = \mathcal{L}^W(df), \qquad f \in C^\infty(M). \tag{C.4}$$

As a consequence, if $F \in C^{1,2}([0,T]\times M, \mathbf{R})$ is such that $F(t,Z_t)$ is a local martingale, then $dF(t,W_t)$ is also a local martingale, where dF is the differential of F in the second variable. On the other hand, (C.2) together with the fact that P_t is an isometry yield

estimations on the norm of W_t. This allows to estimate the norm of dF. Another fundamental property is that W_t is the derivative of the flow corresponding to parallel couplings of L-diffusions.

We construct a damped parallel transport along Brownian motion in a manifold with boundary. The covariant derivative DW_t has three components: one coming from the behaviour in M^0 (the usual one), one tangential to ∂M absolutely continuous with respect to dL_t and involving the shape operator of ∂M, and the third is normal to ∂M and has jumps. This is similar to what happens for the half line. Concerning the half line case the flow corresponding to parallel coupling of reflected Brownian motion is $\partial_x X_t$, so in this case $W_t = \partial_x X_t$ and the study is complete.

We also define the second fundamental form and shape operator for level sets of the distance function to the boundary. Let $S(r) = \{y \in F_0,\ R(y) = r\}$. Within a tubular neighbourhood F_0 of the boundary, R is smooth around x. Let $r = R(x)$ and $\nu_x = \sigma_1(x) = \nabla R(x)$. For $w \in T_xM$, $w' \in T_xS(r)$ we define $\Pi_x : T_xM \times T_xS(r) \to \mathbf{R}$ and $\mathcal{S}_x : T_xM \to T_xS(r)$ by

$$\Pi(w, w') = \langle \mathcal{S}(w), w' \rangle = -\langle \nabla_w \nu, w' \rangle = -\nabla dR(w, w') \tag{C.5}$$

The bilinear map Π is said to be the second fundamental form of $S(r)$ and $\mathcal{S}_x$ its shape operator or the Weingarten map.

Let $3\delta_0$ be the radius of the tubular neighbourhood of ∂M. If D is a set denote τ_D the exit time of Y from D.

Lemma C.1. *Let U be a relatively compact set of M. Let $R_0 < \delta_0$, $Y_0 \in U$, Y_t the reflected Brownian motion, and*

$$\tau_{2\delta_0} = \inf\{t : R_t = 2\delta_0\}.$$

Then under a probability measure equivalent to P, $\{R_t, t < \tau_{\delta_0} \wedge \tau_U \wedge T\}$ is the solution to a Skorohod problem for a one dimensional Brownian motion on $\mathbf{R}_+$.

Proof. Within E_0, because of Proposition 3.3 (2),

$$R_t = R_0 + B_t^1 + \int_0^t \Delta R(Y_s) ds + L_t.$$

Let Q be the probability measure whose density with respect to P is the exponential martingale of $-\int_0^t \Delta R(Y_s) dB_s^1$. Then under Q, $\tilde{B}_t^1 := B_t^1 + \int_0^t \Delta R(Y_s) ds$ is a Brownian motion. On the other hand, L_t is nondecreasing and dL_t vanishes when $R_t \neq 0$. Since $R_t \geq 0$ we have $(R, L) = \Phi(0, \tilde{B}^1)$, the solution to Skorohod problem. See (A.1). By the uniqueness of the Skorohod problem, under Q, $\{R_t, t < \tau_{2\delta_0} \wedge \tau_U \wedge T\}$ has the law of a one dimensional reflected Brownian motion. □

Let $\mathfrak{L}(\omega) = \{t \geq 0 : Y_t(\omega) \in \partial M\}$ be the set of times that Y_t spends on the boundary. It has Lebesque measure zero for a.s. all ω and its complement

$$(0, \infty) \setminus \mathfrak{L}(\omega) = \cup_\alpha (l_\alpha(\omega), r_\alpha(\omega))$$

is the union of countably many disjoint open intervals, the excursion intervals. Denote the set of right end times of excursions by $\mathscr{R}(\omega)$:

$$\mathscr{R}(\omega) = \cup_\alpha \{r_\alpha(\omega)\}.$$

We are interested in defining a damped parallel translation W_t along Y_t, which agrees with the usual one during an excursion, and pick up a change of direction when exiting the boundary. The normal direction on the boundary is zero: we remove $-\langle W_t, \vec{n}(Y_t) \rangle \vec{n}(Y_t)$

upon the process entering the boundary. We would have liked to define a stochastic processes (w_t), if it were possible, with values in $n \times n$ matrices, satisfying

$$\begin{aligned}(/\!\!/_t)^{-1}W_t =&(/\!\!/_{r_\alpha})^{-1}W_{r_\alpha} - \frac{1}{2}\int_{r_\alpha}^t (/\!\!/_s)^{-1}\mathrm{Ric}^\sharp(/\!\!/_s-)((/\!\!/_s)^{-1}W_s)\,ds \\ &- \int_{r_\alpha}^t (/\!\!/_s)^{-1}\mathcal{S}(/\!\!/_s-)\left((/\!\!/_s)^{-1}W_s^\varepsilon\right)\,dL_s, \quad t \in (r_\alpha, r_{\alpha+1})\end{aligned}$$

$$\begin{aligned}&(/\!\!/_{r_{\alpha+1}})^{-1}W_{r_{\alpha+1}} \\ &= (/\!\!/_{(r_{\alpha+1})-})^{-1}W_{(r_{\alpha+1})-} - \langle W_{(r_{\alpha+1})-}, \nu_{Y_{(r_{\alpha+1})-}}\rangle(/\!\!/_{r_{(\alpha+1)-}})^{-1}\nu_{Y_{(r_{\alpha+1})-}}.\end{aligned}$$

Given (l_α, r_α), for any $\epsilon > 0$ there is an excursion $(l_{\alpha'}, r_{\alpha'})$ such that $0 < l_{\alpha'} - r_\alpha < \epsilon$, and so the heuristic definition given above does not make sense.

We remedy this problem with an approximation adding jumps only on excursions of size greater or equal to ϵ. We consider the set of excursions of lengths greater or equal to a given size $\varepsilon > 0$ and define

$$\mathscr{R}_\varepsilon(\omega) = \{s = r_\alpha(\omega) \in \mathscr{R}(\omega) : r_\alpha(\omega) - l_\alpha(\omega) \geq \varepsilon\},$$

where the excursions are ordered with l_1 the first time Y_t hits the boundary and we consider only $\alpha \in \mathbf{N}$. If $r_\alpha - l_\alpha \geq \varepsilon$,

$$\begin{aligned}(/\!\!/_t)^{-1}W_t^\varepsilon = (/\!\!/_{r_\alpha})^{-1}W_{r_\alpha}^\varepsilon - \frac{1}{2}\int_{r_\alpha}^t (/\!\!/_s)^{-1}\mathrm{Ric}^\sharp(/\!\!/_s-)((/\!\!/_s)^{-1}W_s^\varepsilon)\,ds \\ - \int_{r_\alpha}^t (/\!\!/_s)^{-1}\mathcal{S}(/\!\!/_s-)\left((/\!\!/_s)^{-1}W_s^\varepsilon\right)\,dL_s, \quad t \in (r_\alpha, r_{\alpha+1})\end{aligned}$$

$$\begin{aligned}&(/\!\!/_{r_{\alpha+1}})^{-1}W_{r_{\alpha+1}}^\varepsilon \\ &= (/\!\!/_{(r_{\alpha+1})-})^{-1}W_{(r_{\alpha+1})-}^\varepsilon - \langle W_{(r_{\alpha+1})-}, \nu_{Y_{(r_{\alpha+1})-}}\rangle(/\!\!/_{r_{(\alpha+1)-}})^{-1}\nu_{Y_{(r_{\alpha+1})-}}.\end{aligned}$$

where $(W^{\epsilon,T})$ denotes the tangential part of W_t. This takes into consideration those times slightly before $(l_{\alpha+1}, r_{\alpha+1})$ and is relevant to the integration with respect to L_t.

Since Y_t spends Lebesgue time 0 on the boundary, for integration with respect to a continuous process we could ignore the boundary process. We would like to simply remove the normal part of W_t upon it touches down to the boundary. We are lead to the following alternative description. Let $v \in T_{Y_0}M$ and $t > 0$, for almost surely all ω, the following folds,

$$\begin{aligned}(/\!\!/_t)^{-1}W_t^\varepsilon(v) =&v - \frac{1}{2}\int_0^t (/\!\!/_s)^{-1}\mathrm{Ric}^\sharp(/\!\!/_s-)((/\!\!/_s)^{-1}W_s^\varepsilon v)\,ds \\ &- \int_0^t (/\!\!/_s)^{-1}\mathcal{S}(/\!\!/_s-)\left((/\!\!/_s)^{-1}W_s^\varepsilon\right)\,dL_s, \quad t \notin \mathscr{R}_\varepsilon(\omega) \\ (/\!\!/_t)^{-1}W_t^\varepsilon =&(/\!\!/_{t-})^{-1}W_{t-}^\varepsilon - \sum_{s\leq t, s\in\mathscr{R}_\varepsilon(\omega)} \langle W_{t-}^\varepsilon, \nu_{Y_{t-}}\rangle(/\!\!/_{t-})^{-1}\nu_{Y_{t-}}.\end{aligned}$$

In other words, W_t^ε is continuous at any time t that is not a boundary time, and satisfies the following covariant equation

$$\frac{DW_t}{dt} = -\frac{1}{2}\mathrm{Ric}^\#(W_t) + \mathcal{S}(W_t).$$

If t is the right hand side of an excursion, we remove the normal part of its component. This description will be used in Theorem C.3.

Define

$$\mathcal{T} = \{\alpha_t \in \mathscr{H}_\infty([0,T]) : \alpha_t \in T^*_{Y_t}M, \alpha_t = 0 \text{ on } \{Y_t \notin D\} \text{ for some D}\}. \tag{C.6}$$

where D is relatively compact subset of M. This is the set of bounded semi-martingale with values in the pull back cotangent bundle by Y_t, with the property that there is relatively compact subset D of M such that $\alpha_t = 0$ whenever $Y_t \notin D$. Denote by τ^D the first exit time from D by Y_t.

Definition C.2. The limit process $W_t \in L(T_{Y_0}M, T_{Y_t}M)$, below in Theorem C.3, is said to be a solution to the following equation

$$DW_t = -\frac{1}{2}\mathrm{Ric}^\sharp(W_t)\,dt - \mathcal{S}(W_t)\,dL_t - \chi_{\{t\in\mathscr{R}(\omega)\}}\langle W_{t-}, \nu_{Y_t}\rangle\nu_{Y_t}, \quad W_0 = \mathrm{Id}_{T_{Y_0}M}. \tag{C.7}$$

Theorem C.3. *Let W^ε_t the solution to*

$$DW^\varepsilon_t = -\frac{1}{2}\mathrm{Ric}^\sharp(W^\varepsilon_t)\,dt - \mathcal{S}(W^\varepsilon_t)\,dL_t - \chi_{\{t\in\mathscr{R}_\varepsilon(\omega)\}}\langle W^\varepsilon_{t-}, \nu_{Y_t}\rangle\nu_{Y_t}, \quad W^\varepsilon_0 = \mathrm{Id}_{T_{Y_0}M}. \tag{C.8}$$

There exists an adapted right continuous stochastic process W_t such that $\lim_{\epsilon\to 0} W^\epsilon_t = W_t$ in UCP and in $\mathscr{S}_2$ for M compact. Furthermore for any $\alpha \in \mathcal{T}$,

$$\lim_{\epsilon\to 0}\left(\int_0^{\cdot\wedge\tau_D} \alpha_s(DW^\varepsilon_s)\right) \overset{\mathscr{S}_2}{=} \left(\int_0^{\cdot\wedge\tau_D} \alpha_s(DW_s)\right).$$

The same result but with different formulation can be found in [21, 18, 29]. We give a proof close to [21], which will be used for our approximation result (Theorem 5.6 and Corollary 5.7).

Proof. Since the definition and convergence are local in Y, we can assume that M is compact. Since $Y_t(\omega)$ has a finite number of excursions larger than ε, the process W^ε_t is a well defined right continuous process. We first prove that as $\varepsilon \to 0$, $(W^\varepsilon_t)_{t\in[0,T]}$ converges in $\mathscr{S}_2$ to a process which we will call $(W_t)_{t\in[0,T]}$.

Using the parallel translation process $/\!\!/_t$ along Y_t, we reformulate the equation as an equation in the linear space $L(T_{Y_0}M, T_{Y_0}M)$. Set

$$\mathrm{ric}_t = /\!\!/_t^{-1} \circ \mathrm{Ric}^\sharp_{Y_t} \circ /\!\!/_t, \quad \mathrm{s}_t = /\!\!/_t^{-1} \circ \mathcal{S}_{Y_t} \circ /\!\!/_t, \quad \vec{n}_t = /\!\!/_t^{-1}(\nu_{Y_t}). \tag{C.9}$$

Then W^ϵ_t is a solution to (C.8) if and only if $w^\epsilon_t = (/\!\!/_t)^{-1}W^\varepsilon_t$ satisfies the following equations. For any $r_\alpha \in \mathscr{R}_\varepsilon(\omega)$,

$$\begin{aligned} w^\varepsilon_t =& - w^\varepsilon_{r_\alpha} - \frac{1}{2}\int_{r_\alpha}^t (/\!\!/_s)^{-1}\mathrm{ric}_s(w^\varepsilon_s)\,ds - \int_{r_\alpha}^t \mathrm{s}_s(w^\varepsilon_s)\,dL_s, \quad t \in (r_\alpha, r_{\alpha+1}) \\ w^\varepsilon_{r_{\alpha+1}} =& w^\varepsilon_{(r_{\alpha+1})-} - \langle w^\varepsilon_{(r_{\alpha+1})-}, \vec{n}_{(r_{\alpha+1})-}\rangle\vec{n}_{(r_{\alpha+1}-)}, \end{aligned}$$

This means (w^ε_t) satisfies the following equation:

$$dw^\varepsilon_t = -\frac{1}{2}\mathrm{ric}_t(w^\varepsilon_t)\,dt - \mathrm{s}_t(w^\varepsilon_t)\,dL_t - \chi_{\{t\in\mathscr{R}_\varepsilon(\omega)\}}\langle w^\varepsilon_{t-}, n_t\rangle n_t, \quad w_0 = \mathrm{Id}. \tag{C.10}$$

For $0 < r < t$, denote by $V^\varepsilon_{s,t} \in \mathbb{L}(T_{Y_0}M; T_{Y_0}M)$ the solution to

$$V^\varepsilon_{r,t} = \mathrm{Id} - \frac{1}{2}\int_r^t \mathrm{ric}_s(V^\varepsilon_{r,s})\,ds - \int_r^t \mathrm{s}_s(V^\varepsilon_{r,s})\,dL_s - \sum_{\{s\in\mathscr{R}_\varepsilon(\omega)\cap[r,t]\}} \langle V^\varepsilon_{r,s-}, n_s\rangle n_s. \tag{C.11}$$

It will be used in a variation of constants method. Let $0 < \varepsilon' < \varepsilon$, the difference between w_t^ε and $w_t^{\varepsilon'}$ is given by

$$\begin{aligned} w_t^\varepsilon - w_t^{\varepsilon'} &= \int_0^t \left(-\frac{1}{2}\mathrm{ric}_s(w_s^\varepsilon) + \frac{1}{2}\mathrm{ric}_s(w_s^{\varepsilon'}) \right) ds + \int_0^t \left(-\mathrm{s}_s(w_s^\varepsilon) + \mathrm{s}_s(w_s^{\varepsilon'}) \right) dL_s \\ &\quad - \sum_{\{s \in \mathscr{R}_\varepsilon(\omega) \cap [0,t]\}} \langle w_{s-}^\varepsilon, n_s \rangle n_s + \sum_{\{s \in \mathscr{R}_{\varepsilon'}(\omega) \cap [0,t]\}} \langle w_{s-}^{\varepsilon'}, n_s \rangle n_s \\ &= -\frac{1}{2}\int_0^t \mathrm{ric}_s(w_s^\varepsilon - w_s^{\varepsilon'})\, ds - \int_0^t \mathrm{s}_s(w_s^\varepsilon - w_s^{\varepsilon'})\, dL_s \\ &\quad - \sum_{\{s \in \mathscr{R}_\varepsilon(\omega) \cap [0,t]\}} \langle w_{s-}^\varepsilon - w_{s-}^{\varepsilon'}, n_s \rangle n_s + \sum_{\{s \in (\mathscr{R}_{\varepsilon'}(\omega) \setminus \mathscr{R}_\varepsilon) \cap [0,t]\}} \langle w_{s-}^{\varepsilon'}, n_s \rangle n_s. \end{aligned}$$

Let

$$C_t = \sum_{\{s \in (\mathscr{R}_{\varepsilon'}(\omega) \setminus \mathscr{R}_\varepsilon) \cap [0,t]\}} \langle w_{s-}^{\varepsilon'}, n_s \rangle n_s. \tag{C.12}$$

With the help of (C.11), $w_t^\varepsilon - w_t^{\varepsilon'}$ can be expressed as

$$w_t^\varepsilon - w_t^{\varepsilon'} = \sum_{\{s \in (\mathscr{R}_{\varepsilon'}(\omega) \setminus \mathscr{R}_\varepsilon) \cap [0,t]\}} \langle w_{s-}^{\varepsilon'}, n_s \rangle V_{s,t}^\varepsilon(n_s). \tag{C.13}$$

But the term $-\sum_{\{s \in \mathscr{R}_\varepsilon(\omega) \cap [r,t]\}} \langle V_{r,s-}^\varepsilon, n_s \rangle n_s$ is norm decreasing. It is easy to see that

$$|V_{r,t}^\varepsilon|^2 = |v|^2 - \int_0^t \mathrm{ric}_s(V_{r,s}^\varepsilon) ds + \int_0^t \langle S(V_{r,s}^\varepsilon), V_{r,s}^\varepsilon \rangle dL_r - \chi_{t \in \mathscr{R}(\omega)} \langle V_{r,t-}, n_t \rangle^2.$$

We have the following pathwise estimates:

$$|V_{s,t}^\varepsilon| \le \exp\left(\int_s^t \rho(Y_s)\, ds + \int_s^t |\mathrm{s}_s|\, dL_s \right) \le e^{\int_s^t \rho(Y_s)\, ds} \exp\left(CL_t\right) \tag{C.14}$$

where $C > 0$ is an upper bound for the norm of the shape operator and ρ a lower bound for the Ricci curvature.

On the other hand it is a remarkable but not surprising fact that each term $\langle w_{s-}^{\varepsilon'}, n_s \rangle n_s$ can be written as a stochastic integral over an interval not containing any excursion of size larger than ε. This comes from the fact that the norm part of $w_t^{\varepsilon'}$ is set to zero at the end of each excursion of size at least ε'. Note that $\varepsilon > \varepsilon'$. For $s \in (\mathscr{R}_{\varepsilon'}(\omega) \setminus \mathscr{R}_\varepsilon) \cap [0,t]$, the jump at s is accumulated only during the last excursion. More precisely,

$$\langle w_{s-}^{\varepsilon'}, n_s \rangle n_s = \int_u^s d\langle w_r^{\varepsilon'}, n_r \rangle n_r$$

where u is the last vanishing time of $\langle w_r^{\varepsilon'}, n_r \rangle n_r$ before s. Now since we are outside boundary times, the process inside the integral is a continuous semi-martingale whose drift is absolutely continuous with respect to $ds + dL_s$ with bounded derivative, see (C.10). It is clear that $s - u < \varepsilon$. Consequently the quadratic variation of

$$C_t := \sum_{s \in (\mathscr{R}_{\varepsilon'}(\omega) \setminus \mathscr{R}_\varepsilon) \cap [0,t]} \langle w_{s-}^{\varepsilon'}, n_s \rangle n_s$$

is bounded by $C\lambda(U(\varepsilon) \cap [0,t])$ where λ is the Lebesgue measure, $U(\varepsilon)$ is the set of times not contained in excursions larger than ε, and the drift of C_t is bounded by $C\lambda(U(\varepsilon) \cap [0,t])(t + L_t)$.

We get

$$\mathbf{E}[|\sup_{t\leq T} w_t^\varepsilon - w_t^{\varepsilon'}|^2]^{1/2} \leq C\mathbf{E}\left[\exp\left(C(T+L_T)\right)\right)\lambda(U(\varepsilon)\cap[0,T])(T+L_T)\right].$$

The left hand side goes to 0 as $\varepsilon \to 0$, so for $\varepsilon_n \to 0$, w^{ε_n} is a Cauchy sequence in $\mathscr{S}_2$, it converges to some process w. Clearly w does not depend on the sequence. Letting $W_t = /\!\!/_t w_t$ then W_t^ε converges to W_t in $\mathscr{S}_2$.

Let us now prove that DW^ε converges to DW in the sense given by theorem C.3. Let $(\alpha_t)_{t\in[0,T]}$ be a $\mathscr{H}_2$ semimartingale taking its values in $T^*_{Y_t}M$, bounded by 1. We have for $0<\varepsilon'<\varepsilon$

$$\begin{aligned}&\int_0^t \alpha_s\left(DW_s^\varepsilon\right) - \int_0^t \alpha_s\left(DW_s^{\varepsilon'}\right)\\ &= \alpha_t\left(W_t^\varepsilon - W_t^{\varepsilon'}\right) - \int_0^t D\alpha_s\left(W_s^\varepsilon - W_s^{\varepsilon'}\right).\end{aligned}$$

By (B.2), a results of M. Emery, we see that

$$\begin{aligned}&\mathbf{E}\left[\sup_{t\leq T}\left\|\int_0^t \alpha_s\left(DW_s^\varepsilon\rangle\right) - \int_0^t \alpha_s\left(DW_s^{\varepsilon'}\right)\right\|^2\right]\\ &\leq \|\alpha\|_{\mathscr{S}_\infty([0,T])}\|W^\varepsilon - W^{\varepsilon'}\|_{\mathscr{S}_2([0,T])} + \|W^\varepsilon - W^{\varepsilon'}\|_{\mathscr{S}_2([0,T])}\|\alpha\|_{\mathscr{H}_\infty([0,T])}\end{aligned}$$

From the first part of the proof and the assumption on α, we get that for $\varepsilon_n \to 0$, $\left(\int_0^\cdot \alpha_s\left(DW_s^{\varepsilon_n}\right)\right)_{t\in[0,T]}$ is a Cauchy sequence in $\mathscr{S}_2([0,T])$, so it converges to some process which is linear in α_s. Consequently we denote it by $\left(\left(\int_0^\cdot \alpha_s\left(DW_s\right)\right)_{t\in[0,T]}\right.$. □

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