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Abstract

We derive a general large deviation principle for a canonical sequence of probability measures, having its origins in random matrix theory, on unbounded sets K of $\mathbb{C}$ with weakly admissible external fields Q and very general measures ν on K. For this we use logarithmic potential theory in $\mathbb{R}^n$, $n \geq 2$, and a standard contraction principle in large deviation theory which we apply from the two-dimensional sphere in $\mathbb{R}^3$ to the complex plane $\mathbb{C}$.

This paper is dedicated to our good friend and colleague, Ed Saff.

Keywords: Large deviation principle, logarithmic potential theory

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1 Introduction and main results

Let K be a closed subset of the complex plane $\mathbb{C}$ and ν a measure on K. For $k = 1, 2, ...,$ we will be concerned with the following ensemble of probability measures on K^{k+1}:

$$\frac{1}{Z_k}|VDM(z_0, ..., z_k)|^{2\beta} \exp\Big(- 2k[Q(z_0) + \cdots + Q(z_k)]\Big) d\nu(z_0)...d\nu(z_k).$$

Here

- Z_k is a normalization constant;
- $VDM(z_0, ..., z_k) = \prod_{0\le i<j\le k}(z_j - z_i)$ is the usual Vandermonde determinant;
- $Q : K \to (-\infty, +\infty]$ is a lower semicontinuous function; and
- $\beta > 0$.

These probability measures occur in random matrix theory as the joint probability of eigenvalues and also in the theory of Coulomb gases, where $z_0, ..., z_k$ are the positions of particles. They have been extensively studied but generally only when ν is Lebesgue measure (cf., [1] or [18]).

We will deal with the global behavior as $k \to \infty$. In particular, we study the almost sure convergence of the empirical measure of a random point $\frac{1}{k+1}\sum_{i=0}^{k} \delta_{z_i}$ to the equilibrium measure given by the unique minimizer of the weighted energy functional; i.e.,

$$\inf\{I^Q(\mu) : \ \mu \in \mathcal{M}(K)\}$$

where $\mathcal{M}(K)$ are the probability measures on K and

$$I^Q(\mu) = \int_K \int_K \log \frac{1}{|z - t|w(z)w(t)} d\mu(z)d\mu(t),$$

with $w(z) = \exp(-Q(z))$. We will also establish a large deviation principle (LDP).

Ben Arous and Guionnet [2], building on work of Voiculescu, first proved a large deviation principle for the Gaussian Unitary Ensemble. This was subsequently extended to general unitary invariant ensembles. Hiai and Petz [18] extended these methods to the complex plane and *strongly admissible* (see Definition 3.1) continuous weights Q. In these settings, ν was taken to be Lebesgue measure.

More recently, the case of *weakly admissible weights* (see Definition 3.1) on unbounded subsets of the plane was studied in [15] and the existence of a unique minimizer of the weighted energy functional (which in this case may not have compact support) was established. In [14] a large deviation principle was established for Q weakly admissible, continuous on $\mathbb{R}$ or $\mathbb{C}$ and ν the Lebesgue measure. Such weights occur in certain ensembles (see [14], the Cauchy ensemble) and in certain vector energy problems (see [15]).

In this paper we will systematically develop the case when Q is lower semicontinuous, weakly admissible and ν is more general than Lebesgue measure. We will use the methods

of [9] which first of all give the almost sure convergence of the empirical measure of a random point, and, subsequently, we obtain a large deviation principle.

The paper is organized as follows: in the next section, we give some basic results on logarithmic potential theory in $\mathbb{R}^n$ valid for $n \geq 2$. Using the results in $\mathbb{R}^3$ together with inverse stereographic projection from a two-dimensional sphere $\mathbf{S}$ to the complex plane, in section 3 we readily extend some classical potential-theoretic results valid for compact subsets of $\mathbb{C}$ to closed, unbounded sets with weakly admissible weights.

In sections 4-7, we return to the setting of compact sets K in $\mathbb{R}^n$ and admissible weights (see Definition 2.2; such weights need only be lower semicontinuous). Corollary 4.12 establishes the almost sure convergence of the empirical measure of a random point to the equilibrium measure in this setting, for appropriate measures ν.

Our next goal is to show that two functionals $\underline{J}$ and $\overline{J}$ on the space $\mathcal{M}(K)$ of probability measures coincide. These functionals are defined as asymptotic $L^2(\nu)-$averages of Vandermonde determinants with respect to a *Bernstein-Markov* measure ν on K. As in previous work (cf., [9] and [8]), weighted versions of these functionals are of essential use (Theorem 6.6). This equality immediately yields a large deviation principle in this $\mathbb{R}^n$ setting, Theorem 7.1, in which the rate function is given in terms of the weighted energy functional independent of the Bernstein-Markov measure ν.

In section 8, we deal with compact subsets of the sphere in $\mathbb{R}^3$ and measures of infinite mass, again establishing a LDP (Theorem 8.6). Measures of infinite mass arise as the push-forward of measures on unbounded subsets of the plane under stereographic projection.

Our ultimate goal, achieved in sections 9 and 10, is to utilize the $\mathbb{R}^n$ result to prove the analogous equality of the appropriate $J-$functionals for probability measures on closed, unbounded sets in $\mathbb{C}$ allowing weakly admissible weights and very general measures of infinite mass (Theorem 9.4). Then, via a contraction principle, we obtain an LDP (Theorem 10.2):

Theorem 1.1. *Let $K \subset \mathbb{C}$ be closed, and let Q be a weakly admissible weight on K. Assume (K, ν, Q) satisfies the weighted Bernstein-Markov property (9.1). If ν has finite mass, assume that (K, ν) satisfies a strong Bernstein-Markov property while if ν has infinite mass in a neighborhood of infinity, assume that (9.4) and (9.5) are satisfied for some function $\epsilon(z)$. Define a sequence $\{\sigma_k\}$ of probability measures on $\mathcal{M}(K)$ by*

$$\sigma_k(G) = \frac{1}{Z_k} \int_{\widetilde{G}_k} |VDM(z_0, ..., z_k)|^2 \prod_{i=0}^{k} e^{-2kQ(z_i)} \prod_{i=0}^{k} d\nu(z_i)$$

where $\widetilde{G}_k = \{(z_0, ..., z_k) \in K^{k+1} : \frac{1}{k+1} \sum \delta_{z_i} \in G\}$. Then $\{\sigma_k\}$ satisfies a **large deviation principle** *with speed k^2 and good rate function $\mathcal{I} := \mathcal{I}_{K,Q}$ where, for $\mu \in \mathcal{M}(K)$,*

$$\mathcal{I}(\mu) = I^Q(\mu) - I^Q(\mu_{K,Q}).$$

In section 11 we extend this result to the case of general β (Theorem 11.2). Our results include the LDP for a number of ensembles occurring in the literature (see Remark 10.3)

and also the results of Hardy [15] for Lebesgue measure in $\mathbb{R}$ or $\mathbb{C}$ (see the discussion after Theorem 11.2). The idea of using inverse stereographic projection and working in $\mathbb{R}^3$ to obtain an LDP for unbounded sets in $\mathbb{C}$ comes from this work.

2 Logarithmic Potential Theory in $\mathbb{R}^n$

Let $K \subset \mathbb{R}^n$ be compact and let $\mathcal{M}(K)$ be the set of probability measures on K endowed with the topology of weak convergence from duality with continuous functions. We consider the logarithmic energy minimization problem:

$$\inf_{\mu \in \mathcal{M}(K)} I(\mu)$$

where

$$I(\mu) := \int_K \int_K \log \frac{1}{|x-y|} d\mu(x) d\mu(y)$$

is the logarithmic energy of μ. We will say that K is *log-polar* if $I(\mu) = \infty$ for all $\mu \in \mathcal{M}(K)$. It is known that any compact set of positive Hausdorff dimension is non log-polar [11]. For a Borel set $E \subset \mathbb{R}^n$ we will say E is log-polar if every compact subset of E is log-polar. We write

$$U^\mu(x) := \int_K \log \frac{1}{|x-y|} d\mu(y)$$

for the logarithmic potential of μ. It is locally integrable and superharmonic in all of $\mathbb{R}^n$.

We gather known results about logarithmic potentials in $\mathbb{R}^n$ in the next theorem.

Theorem 2.1. *The following results, whose precise statements can be found in [21] for logarithmic potentials in $\mathbb{C} = \mathbb{R}^2$, hold true for logarithmic potentials in $\mathbb{R}^n$, $n \geq 2$:*

1. *for $\mu = \mu_1 - \mu_2$ a signed measure with compact support and total mass zero, with μ_1 and μ_2 of finite energies, $I(\mu)$ is nonnegative and is zero if and only if $\mu_1 = \mu_2$.*
2. *principle of descent and lower envelope theorem (with "q.e." in the latter replaced by "off of a log-polar set");*
3. *maximum principle;*
4. *continuity principle.*

Proof. The version of Item 1. in $\mathbb{C}$ is [21, Lemma 1.8]. In $\mathbb{R}^n$, it follows from [12, Theorem 2.5]. An extension of item 1. in case of unbounded support and whenever $I(\mu)$ is well-defined is given in [19], see Example 3.3. One checks that the proofs of the principle of descent and lower envelope theorem in $\mathbb{C}$, Theorems I.6.8. and I.6.9. of [21], are valid in $\mathbb{R}^n$. Items 3. and 4. are Theorems 5.2 and 5.1 of [17]. A maximum principle restricted to the two dimensional sphere also follows as a particular case of [10, Theorem 5]. □

We will need to work in a weighted setting. We caution the reader that, unlike the setting of compact sets in $\mathbb{R}^n$ where we have a single notion of admissibility for a weight function, when we work on unbounded sets in $\mathbb{C}$ in the next section we will have several different notions.

Definition 2.2. Given a compact set $K \subset \mathbb{R}^n$ which is not log-polar, let Q be a lower semicontinuous function on K with $\{x \in K : Q(x) < \infty\}$ not log-polar. We call such Q *admissible* and write $Q \in \mathcal{A}(K)$. We define $w(x) := e^{-Q(x)}$.

We refer to either Q or w as the *weight*; in [21] this terminology is reserved for w.

We consider now the weighted logarithmic energy minimization problem:

$$\inf I^Q(\mu), \quad \mu \in \mathcal{M}(K),$$

where

$$I^Q(\mu) := \int_K \int_K \log \frac{1}{|x-y|w(x)w(y)} d\mu(x) d\mu(y) = I(\mu) + 2\int_K Q(x) d\mu(x).$$

Following the arguments on pp. 27-33 in [21], we have the following.

Theorem 2.3. *For $K \subset \mathbb{R}^n$ compact and not log-polar, and $Q \in \mathcal{A}(K)$,*

1. *$V_w := \inf_{\mu \in \mathcal{M}(K)} I^Q(\mu)$ is finite;*
2. *there exists a unique weighted equilibrium measure $\mu_{K,Q} \in \mathcal{M}(K)$ with $I^Q(\mu_{K,Q}) = V_w$ and the logarithmic energy $I(\mu_{K,Q})$ is finite;*
3. *the support $S_w :=$supp$(\mu_{K,Q})$ is contained in $\{x \in K : Q(x) < \infty\}$ and S_w is not log-polar;*
4. *Let $F_w := V_w - \int_K Q(x) d\mu_{K,Q}(x)$ denote the (finite) Robin constant. Then*

$$U^{\mu_{K,Q}}(x) + Q(x) \geq F_w \text{ on } K \setminus P \text{ where } P \text{ is log-polar (possibly empty);}$$
$$U^{\mu_{K,Q}}(x) + Q(x) \leq F_w \text{ for all } x \in S_w.$$

Remark 2.4. In the proof of the Frostman-type property 4. in [21], one simply replaces "q.e." – off of a set of positive logarithmic capacity in $\mathbb{C}$ – by "off of a log-polar set" as the essential property used is the existence of a measure of finite logarithmic energy on a compact subset of a set of positive logarithmic capacity in $\mathbb{C}$. We should mention that, in the unweighted case, the existence portion of 2. and property 4. can be found in [17], Theorems 5.4 and 5.8.

Next we discretize: for $k \geq 2$, let the k-th weighted diameter $\delta_k^Q(K)$ be defined by

$$\delta_k^Q(K) := \sup_{x_1,...,x_k \in K} |VDM_k^Q(x_1, ..., x_k)|^{2/k(k-1)},$$

where $|VDM_k^Q(x_1, ..., x_k)|$ denotes the weighted Vandermonde:

$$|VDM_k^Q(x_1, ..., x_k)| := \prod_{i<j} |x_i - x_j| w(x_i) w(x_j) = \prod_{i<j} |x_i - x_j| \prod_{j=1}^k w(x_j)^{k-1}$$
$$=: |VDM_k(x_1, ..., x_k)| \cdot \prod_{j=1}^k w(x_j)^{k-1}. \tag{2.1}$$

By the uppersemicontinuity of $(x_1, ..., x_k) \to \prod_{i<j} |x_i - x_j| w(x_i) w(x_j)$ on K^k the supremum is attained; we call any collection of k points of K at which the maximum is attained *weighted Fekete points* of order k for K, Q. Following the proofs of Propositions 3.1–3.3 of [9, Section 3] we may derive similar results in $\mathbb{R}^n$.

Theorem 2.5. *Given $K \subset \mathbb{R}^n$ compact and not log-polar, and $Q \in \mathcal{A}(K)$,*

1. *if $\{\mu_k = \frac{1}{k} \sum_{j=1}^k \delta_{x_j^{(k)}}\} \subset \mathcal{M}(K)$ converge weakly to $\mu \in \mathcal{M}(K)$, then*

$$\limsup_{k\to\infty} |VDM_k^Q(x_1^{(k)}, ..., x_k^{(k)})|^{2/k(k-1)} \le \exp(-I^Q(\mu)); \tag{2.2}$$

2. *we have*

$$\delta^Q(K) := \lim_{k\to\infty} \delta_k^Q(K) = \exp(-V_w);$$

3. *if $\{x_j^{(k)}\}_{j=1,...,k;\ k=2,3,...} \subset K$ and*

$$\lim_{k\to\infty} |VDM_k^Q(x_1^{(k)}, ..., x_k^{(k)})|^{2/k(k-1)} = \exp(-V_w)$$

then

$$\mu_k = \frac{1}{k} \sum_{j=1}^k \delta_{x_j^{(k)}} \to \mu_{K,Q} \text{ weakly.}$$

3 Weighted potential theory on unbounded sets in $\mathbb{C}$

We use the previous results in $\mathbb{R}^3$ and the inverse stereographic projection from the two-dimensional sphere to $\mathbb{C}$ to extend classical results concerning potential theory on compact subsets of $\mathbb{C}$ to unbounded closed sets with weakly admissible weights. Some of these results already appeared in the literature, see, e.g., [16, 23].

Thus let $K \subset \mathbb{C}$ be closed and unbounded. We consider three types of admissibility for weight functions on K.

Definition 3.1. Let Q be a lower semicontinuous function on K with $\{z \in K : Q(z) < \infty\}$ a nonpolar subset of $\mathbb{C}$ (equivalently a non log-polar subset of $\mathbb{R}^2$). We say Q is

1. *weakly admissible* if there exists $M \in (-\infty, \infty)$ such that

$$\liminf_{z\in K,\ |z|\to\infty} \big(Q(z) - \log|z|\big) = M. \tag{3.1}$$

2. *admissible* if $\liminf_{z\in K,\ |z|\to+\infty} \big(Q(z) - \log|z|\big) = +\infty$.

3. *strongly admissible* if for some $\epsilon > 0$, there exists $R > 0$ with $Q(z) > (1+\epsilon)\log|z|$ for $z \in K$ and $|z| > R$.

Examples of weakly admissible weights arise from logarithmic potentials: if μ is a probability measure on $\mathbb{C}$ such that U^μ is continuous, then $Q = -U^\mu$ is weakly admissible on $K = \mathbb{C}$.

We assume now that Q is weakly admissible. We consider the inverse stereographic projection $T : \mathbb{C} \cup \{\infty\} \to \mathbf{S}$ where $\mathbf{S}$ is the sphere in $\mathbb{R}^3$ centered in $(0, 0, 1/2)$ of radius $1/2$. It is defined by

$$T(z) = \left(\frac{\mathrm{Re}\,(z)}{1+|z|^2}, \frac{\mathrm{Im}\,(z)}{1+|z|^2}, \frac{|z|^2}{1+|z|^2} \right), \quad z \in \mathbb{C} \tag{3.2}$$

and $T(\infty) = P_0$, where $P_0 = (0, 0, 1)$ denotes the "north pole" of $\mathbf{S}$. The map T is a homeomorphism with

$$|T(z) - T(u)| = \frac{|z-u|}{\sqrt{1+|z|^2}\sqrt{1+|u|^2}}, \quad z, u \in \mathbb{C}, \tag{3.3}$$

where $|\cdot|$ denotes the Euclidean distance.

For ν a positive Borel measure supported on K, not necessarily finite, we denote by $T_*\nu$ its push-forward by T, that is, the measure on $T(K)$ such that

$$\int_{T(K)} f(x) dT_*\nu(x) = \int_K f(T(z)) d\nu(z),$$

for any Borel function f on $T(K)$. Lemma 2.1 in [15] shows that the map

$$T_* : \ \mathcal{M}(K) \to \mathcal{M}(T(K)),$$

is a homeomorphism from $\mathcal{M}(K)$ to the subset of $\mathcal{M}(T(K))$ of measures which put no mass at the north pole P_0 of $\mathbf{S}$. Here, $\mathcal{M}(K)$ and $\mathcal{M}(T(K))$ are endowed with the topology of weak convergence. This is the topology coming from duality with bounded, continuous functions. On K, it suffices to consider bounded, continuous functions $f : K \to \mathbb{C}$ such that $\lim_{|z|\to\infty} f(z)$ exists. This follows from the correspondence of $\mathcal{M}(K)$ with the measures in $\mathcal{M}(T(K))$ putting no mass at P_0.

When the support of a measure $\mu \in \mathcal{M}(K)$ is unbounded, its potential

$$U^\mu(z) = \int \log \frac{1}{|z-t|} d\mu(t), \qquad z \in \mathbb{C}$$

is not always well-defined. However, the following lemma holds true.

Lemma 3.2. *If there exists a $z_0 \in \mathbb{C}$ with $U^\mu(z_0) > -\infty$ then*

$$\int \log(1+|t|) d\mu(t) < \infty, \tag{3.4}$$

which implies that $U^\mu(z)$ is well-defined as a function on $\mathbb{C}$ with values in $(-\infty, \infty]$. Moreover, $U^\mu(z)$ is then superharmonic and

$$-U^\mu(z) \le \log(1+|z|) + \int \log(1+|t|) d\mu(t).$$

Also,

$$\int \log(1+|t|)d\mu(t) < \infty \quad \Longleftrightarrow \quad -\infty < I(\mu). \tag{3.5}$$

Proof. If $1+2|z_0| \le |t|$ then $1+|t| \le 2(|t|-|z_0|) \le 2|t-z_0|$, hence

$$\int_{1+2|z_0|\le|t|} \log(1+|t|)d\mu(t) \le \log 2 + \int_{1+2|z_0|\le|t|} \log|z_0-t|d\mu(t) < \infty. \tag{3.6}$$

Conversely, if (3.4) holds then $-\infty < U^\mu(z)$ and $-\infty < I(\mu)$ since $|z-t| \le (1+|z|)(1+|t|)$. Under assumption (3.4), the potential $U^\mu(z)$ is superharmonic. This follows e.g. from the fact that

$$U^\mu(z) = \int \log\frac{1+|t|}{|z-t|}d\mu(t) - \int \log(1+|t|)d\mu(t),$$

and the first integral on the right-hand side is superharmonic with respect to z, see [20, Theorem 2.4.8]. The direct implication in (3.5) was noted above. Conversely, if $-\infty < I(\mu)$ then $U^\mu(z)$ cannot be constant, equal to $-\infty$, for all z, so the inequality on the left of (3.5), which is (3.4), follows from (3.6). □

Logarithmic potentials on $\mathbb{C}$ and on the sphere $\mathbf{S}$ correspond by the relation

$$U^\mu(z) = U^{T_*\mu}(Tz) - \frac{1}{2}\log(1+|z|^2) - \frac{1}{2}\int \log(1+|t|^2)d\mu(t), \qquad z \in \mathbb{C}. \tag{3.7}$$

The weighted logarithmic energy of a measure $\mu \in \mathcal{M}(K)$ is defined as

$$I^Q(\mu) = \int_K \int_K \log\frac{1}{|z-t|w(z)w(t)}d\mu(z)d\mu(t) = I(\mu) + 2\int_K Q d\mu, \tag{3.8}$$

where $w = e^{-Q}$. The double integral is always well-defined. Indeed it follows from the upper semicontinuity of w and (3.1) that the integrand is bounded below. On the contrary, the second expression has a meaning only if $I(\mu) > -\infty$ which is not necessarily true. Another equivalent way to define $I^Q(\mu)$, which is always valid, is by using the map T as was done in [16]. Here, one identifies $I^Q(\mu)$ with the weighted logarithmic energy of the measure $T_*\mu \in \mathcal{M}(T(K))$,

$$I^{\widetilde{Q}}(T_*\mu) = \int_{T(K)} \int_{T(K)} \log\frac{1}{|x-y|}dT_*\mu(x)dT_*\mu(y) + 2\int_{T(K)} \widetilde{Q}(x)dT_*\mu(x), \tag{3.9}$$

where

$$\widetilde{Q}(T(z)) = Q(z) - \frac{1}{2}\log(1+|z|^2). \tag{3.10}$$

To define $\widetilde{Q}$ on the whole of $T(K)$ we set $\widetilde{Q}(P_0) = M$, so that $\widetilde{Q}$ becomes lower semicontinuous, and we get a correspondence between *weakly admissible weights on the closed set* K *in* $\mathbb{C}$ and *admissible weights on the compact set* $T(K) \subset \mathbf{S}$ *in* $\mathbb{R}^3$.

Lemma 3.3. *A closed subset $K \subset \mathbb{C}$ is polar if and only if $T(K) \subset \mathbf{S}$ is log-polar.*

Proof. We have

$$\int_{T(K)} \int_{T(K)} \log \frac{1}{|x-y|} dT_*\mu(x) dT_*\mu(y) = \int_K \int_K \log \frac{1}{|z-t|} d\mu(z) d\mu(t) + \int_K \log(1+|t|^2) d\mu(t).$$

Recall that a closed subset $K \subset \mathbb{C}$ is polar if $K_r = K \cap B(0,r)$ is polar for all $r > 0$. Thus, if K is nonpolar, there exists $r > 0$ with K_r nonpolar, that is, there is a measure μ_r supported on K_r of finite energy. By the above equality, $T_*\mu_r$ is a measure on $T(K_r) \subset T(K)$ of finite energy, so $T(K)$ is not log-polar. Conversely, if $K \subset \mathbb{C}$ is polar, for any finite measure μ of compact support in K we have $I(\mu) = \infty$ in $\mathbb{C}$ (cf. [20]) and thus $I(T_*\mu) = \infty$ in $\mathbf{S}$. Since any measure on $T(K)$ charging the north pole P_0 has infinite energy, it follows that $T(K)$ is log-polar. □

Theorem 2.3 asserts the existence and uniqueness of a weighted energy minimizing measure on a non log-polar compact subset of $\mathbb{R}^n$ with an admissible weight. Obviously, this minimizing measure does not charge any point of the set, in particular the north pole P_0 if it belongs to the set. Hence the above correspondence implies the following.

Theorem 3.4. *Let K be a nonpolar closed subset of $\mathbb{C}$ and Q a weakly admissible weight on K. Then,*

1. *$V_w := \inf_{\mu \in \mathcal{M}(K)} I^Q(\mu)$ is finite;*
2. *there exists a unique weighted equilibrium measure $\mu_{K,Q} \in \mathcal{M}(K)$ with $I^Q(\mu_{K,Q}) = V_w$ and the logarithmic energy $I(\mu_{K,Q})$ is finite (hence $-\Delta U^{\mu_{K,Q}} = 2\pi\mu_{K,Q}$);*
3. *the support $S_w :=$supp$(\mu_{K,Q})$ is contained in $\{x \in K : Q(z) < \infty\}$ and S_w is not polar;*
4. *Let $F_w := V_w - \int_K Q(z) d\mu_{K,Q}(z)$ denote the (finite) Robin constant. Then*

$$U^{\mu_{K,Q}}(z) + Q(z) \geq F_w \text{ on } K \setminus P \text{ where } P \text{ is polar (possibly empty);}$$
$$U^{\mu_{K,Q}}(z) + Q(z) \leq F_w \text{ for all } z \in S_w.$$

Proof. The above assertions correspond by the map T to the similar assertions from Theorem 2.3 applied with a non log-polar compact subset of the sphere $\mathbf{S}$. Note that

$$T_*\mu_{K,Q} = \mu_{T(K),\widetilde{Q}}, \qquad V_w = V_{\widetilde{w}}, \qquad F_w = F_{\widetilde{w}} - \frac{1}{2}\int_K \log(1+|t|^2) d\mu_{K,Q}(t),$$

where we have set $\widetilde{w} := e^{-\widetilde{Q}}$. The fact that $I(\mu_{K,Q}) < \infty$ follows from

$$I(\mu_{K,Q}) = I^Q(\mu_{K,Q}) - 2\int_K Q d\mu_{K,Q} = I^{\widetilde{Q}}(T_*\mu_{K,Q}) - 2\int_K Q d\mu_{K,Q},$$

where we know that $I^{\widetilde{Q}}(T_*\mu_{K,Q}) = V_{\widetilde{w}}$ is finite and Q is bounded below. If S_w is compact, it is clear that the other inequality $-\infty < I(\mu_{K,Q})$ is satisfied. If S_w is not compact, we may use

$$\iint \log|z-t| d\mu_{K,Q}(z) d\mu_{K,Q}(t) \leq 2\int \log(1+|t|) d\mu_{K,Q}(t),$$

so to verify $-\infty < I(\mu_{K,Q})$ it suffices to show that

$$\int_K \log(1+|t|)d\mu_{K,Q}(t) < \infty,$$

which holds true by Lemma 3.2 since the equilibrium potential satisfies $U^{\mu_{K,Q}} > -\infty$. □

In particular, if μ is a probability measure on $\mathbb{C}$ such that U^μ is continuous, taking $Q = -U^\mu$ on $K = \mathbb{C}$ we have $\mu = \mu_{K,Q}$ so that, in general, $\mu_{K,Q}$ need not have compact support. As specific examples, if $K = \mathbb{C}$ and $Q(z) = \frac{1}{2}\log(1+|z|^2)$, then $d\mu_{K,Q} = \pi^{-1}(1+|z|^2)^{-2}dm(z)$ where dm is Lebesgue measure, cf., Example 1.4 of [15]. If $K = \mathbb{R}$ and $Q(x) = \frac{1}{2}\log(1+x^2)$, then $d\mu_{K,Q} = \pi^{-1}(1+x^2)^{-1}dx$, cf., Example 1.3 of [15]. We mention that in [16], existence and uniqueness of a minimizing measure was proven in the more general context of weakly admissible vector equilibrium problems.

Let $L(\mathbb{C})$ be the set of all subharmonic functions u on $\mathbb{C}$ with the property that

$$u(z) - \log|z| \text{ is bounded above as } |z| \to \infty.$$

We will need the following version of the domination principle, see [7, Corollary A.2].

Proposition 3.5. *Let $u, v \in L(\mathbb{C})$ with $u(z)-v(z)$ bounded above as $|z| \to \infty$ and suppose $I(\Delta v) < \infty$. If $u \le v$ a.e.-Δv, then $u \le v$ on $\mathbb{C}$.*

Here, Δv need not have compact support.

We can now state a weighted version of the Bernstein-Walsh lemma with a weakly admissible weight (see [21, Theorem III.2.1] for the case of an admissible weight). This will be used in section 8 to get a version for appropriate polynomials on the sphere (Theorem 8.1). We let $\mathcal{P}_k(\mathbb{C})$ denote the complex-valued polynomials of a complex variable of degree at most k.

Theorem 3.6. *Let K be a closed nonpolar subset of $\mathbb{C}$ and Q a weakly admissible weight on K. If $p_k \in \mathcal{P}_k(\mathbb{C})$ and*

$$|p_k(z)e^{-kQ(z)}| \le M \qquad \text{for q.e. } z \in S_w,$$

then

$$|p_k(z)| \le M\exp(k(-U^{\mu_{K,Q}}(z) + F_w)), \qquad z \in \mathbb{C},$$

and

$$|p_k(z)e^{-kQ(z)}| \le M, \qquad \text{for q.e. } z \in K.$$

Proof. The function $g := \log(|p_k|/M)/k$ belongs to $L(\mathbb{C})$ and

$$g(z) \le Q(z) \le -U^{\mu_{K,Q}}(z) + F_w \quad \text{for q.e. } z \in S_w.$$

By Lemma 3.2, $-U^{\mu_{K,Q}} + F_w$ also belongs to $L(\mathbb{C})$ and $-\Delta U^{\mu_{K,Q}} = 2\pi\mu_{K,Q}$ is of finite energy. Hence, by the above domination principle,

$$g(z) \le -U^{\mu_{K,Q}}(z) + F_w, \qquad z \in \mathbb{C},$$

which, together with the first inequality in item 4. of Theorem 3.4, proves our contention. □

We proceed with properties of the weighted Vandermonde. We have the relation

$$|VDM_k^Q(z_1,...,z_k)| = |VDM_k^{\widetilde{Q}}(T(z_1),...,T(z_k))|,$$

from which it follows that the assertions of Theorem 2.5 about the Vandermonde can be carried over to $\mathbb{C}$. Since the result may be of interest on its own, we state it as a theorem.

Theorem 3.7. *Let K be a closed nonpolar subset of $\mathbb{C}$ and Q a weakly admissible weight on K. The k-th weighted diameters $\delta_k^Q(K)$, $k \geq 2$, are finite and*

1. *if $\{\mu_k = \frac{1}{k}\sum_{j=1}^k \delta_{x_j^{(k)}}\} \subset \mathcal{M}(K)$ converge weakly to $\mu \in \mathcal{M}(K)$, then*

$$\limsup_{k\to\infty} |VDM_k^Q(x_1^{(k)},...,x_k^{(k)})|^{2/k(k-1)} \leq \exp(-I^Q(\mu)); \tag{3.11}$$

2. *we have $\delta^Q(K) := \lim_{k\to\infty} \delta_k^Q(K) = \exp(-V_w)$;*

3. *if $\{x_j^{(k)}\}_{j=1,...,k;\ k=2,3,...} \subset K$ and*

$$\lim_{k\to\infty} |VDM_k^Q(x_1^{(k)},...,x_k^{(k)})|^{2/k(k-1)} = \exp(-V_w)$$

then

$$\mu_k = \frac{1}{k}\sum_{j=1}^k \delta_{x_j^{(k)}} \to \mu_{K,Q} \text{ weakly.}$$

Remark 3.8. The Frostman-type result in 4. of Theorem 3.4 and 2. of Theorem 3.7 (as well as 3. in the special case of arrays of weighted Fekete points) have also been proved in [3].

4 Bernstein-Markov properties in $\mathbb{R}^n$

In sections 4-8, we return to the setting of compact sets in $\mathbb{R}^n$. In particular, admissible weights will be in the sense of Definition 2.2. For $k = 1, 2, ...$, let $\mathcal{P}_k = \mathcal{P}_k^{(n)}$ denote the *real* polynomials in n real variables $x = (x_1, ..., x_n)$ of degree at most k and $\mathcal{P}_k(\mathbb{C})$ denote the *complex* holomorphic polynomials in n complex variables $z = (z_1, ..., z_n)$ of degree at most k. Given a compact set $K \subset \mathbb{C}^n$ and a positive measure ν on K, we say that (K, ν) satisfies the Bernstein-Markov property (or ν is a Bernstein-Markov measure for K) if for all $p_k \in \mathcal{P}_k(\mathbb{C})$,

$$||p_k||_K := \sup_{z\in K} |p_k(z)| \leq M_k ||p_k||_{L^2(\nu)} \text{ with } \limsup_{k\to\infty} M_k^{1/k} = 1.$$

It was shown in [8] that any compact set in $\mathbb{C}^n$ admits a Bernstein-Markov measure for complex holomorphic polynomials; indeed, the following stronger statement is true.

Proposition 4.1 ([8]). *Let $K \subset \mathbb{R}^n$. There exists a measure $\nu \in \mathcal{M}(K)$ such that for all complex-valued polynomials p of degree at most k in the (real) coordinates $x = (x_1, ..., x_n)$ we have*

$$||p||_K \leq M_k ||p||_{L^2(\nu)}$$

where $\limsup_{k\to\infty} M_k^{1/k} = 1$.

For a compact set $K \subset \mathbb{R}^n$ and a positive measure ν on K, we will say that (K, ν) satisfies the Bernstein-Markov property if for all $p_k \in \mathcal{P}_k$,

$$||p_k||_K := \sup_{x\in K} |p_k(x)| \leq M_k ||p_k||_{L^2(\nu)} \text{ with } \limsup_{k\to\infty} M_k^{1/k} = 1.$$

More generally, for $K \subset \mathbb{R}^n$ compact, $Q \in \mathcal{A}(K)$, and ν a measure on K, we say that the triple (K, ν, Q) satisfies the weighted Bernstein-Markov property if for all $p_k \in \mathcal{P}_k$,

$$||e^{-kQ} p_k||_K \leq M_k ||e^{-kQ} p_k||_{L^2(\nu)} \text{ with } \limsup_{k\to\infty} M_k^{1/k} = 1.$$

We have the analogous notion of weighted Bernstein-Markov property for $p_k \in \mathcal{P}_k(\mathbb{C})$ if $K \subset \mathbb{C}^n$ and $Q \in \mathcal{A}(K)$.

Remark 4.2. These properties can be stated with L^p norms for any $0 < p < \infty$. The proof of Theorem 3.4.3 in [22] in $\mathbb{C}$ that if (K, ν) satisfies an (weighted) L^p-Bernstein-Markov property for $\mathcal{P}_k(\mathbb{C})$ for some $0 < p < \infty$ then (K, ν) satisfies an (weighted) L^p-Bernstein-Markov property for all $0 < p < \infty$ just uses Hölder's inequality and remains valid in our setting.

Now another very important observation: Theorem 3.2 of [4] works – indeed, is even stated – in $\mathbb{R}^n$ for any $n \geq 2$:

Theorem 4.3. *Given $K \subset \mathbb{R}^n$ compact, and Q a continuous weight, if ν is a finite measure on K such that (K, ν) satisfies a Bernstein-Markov property, then the triple (K, ν, Q) satisfies a weighted Bernstein-Markov property.*

Definition 4.4. Given $K \subset \mathbb{R}^n$ compact, a finite measure ν on K is called a *strong Bernstein-Markov measure* for K if for any continuous weight Q on K, the triple (K, ν, Q) satisfies a weighted Bernstein-Markov property. We have the analogous notion if $K \subset \mathbb{C}^n$ using $p_k \in \mathcal{P}_k(\mathbb{C})$.

Remark 4.5. Combining Proposition 4.1 and Theorem 4.3 we see that *any compact set K in $\mathbb{R}^n$ admits a strong Bernstein-Markov measure; and any Bernstein-Markov measure on K is automatically a strong Bernstein-Markov measure for K*. This latter equivalence is not necessarily true in the complex setting. For $K \subset \mathbb{C}$, there are well-known sufficient mass-density conditions on a measure ν on K so that (K, ν) satisfies a Bernstein-Markov property for $\mathcal{P}_k(\mathbb{C})$ [22]. In particular, Lebesgue measure on an interval or Lebesgue planar measure on a compact set in $\mathbb{C}$ having C^1 boundary satisfy the Bernstein-Markov property.

We remark that if $\mathbb{C} \setminus K$ is regular for the Dirichlet problem, the condition that (K, ν) satisfies a Bernstein-Markov property for $\mathcal{P}_k(\mathbb{C})$ is equivalent to the condition that ν be a regular measure; i.e., $\nu \in \mathbf{Reg}$ in the terminology of [22]. We refer to this book for more details.

Furthermore, for every compact set K in $\mathbb{R}^n$ there exist discrete measures which satisfy the (strong) Bernstein-Markov property [8]. If one considers $K \subset \mathbb{R}^n \subset \mathbb{C}^n$, there are sufficient mass-density conditions on a measure ν on K so that (K, ν) satisfies a Bernstein-Markov property for polynomials on $\mathbb{C}^n$ and hence on $\mathbb{R}^n$. For more on this, cf., [6] and [8].

Remark 4.6. Let $K \subset \mathbb{R}^n$ be compact and not log-polar and let $v \in \mathcal{A}(K)$. If α is a finite measure on K such that (K, α, v) satisfies a weighted Bernstein-Markov property, then

1. $(K, c\alpha, v)$ satisfies a weighted Bernstein-Markov property for any $0 < c < \infty$ and
2. $(K, \alpha + \beta, v)$ satisfies a weighted Bernstein-Markov property for any finite measure β on K.

The importance of a (weighted) Bernstein-Markov property is the following consequence on the asymptotic behavior of the (weighted) L^2 *normalization constants* defined by

$$Z_k^Q = Z_k^Q(K, \nu) := \int_{K^k} |VDM_k^Q(\mathbf{X_k})|^2 d\nu(\mathbf{X_k}), \tag{4.1}$$

where $\mathbf{X_k} := (x_1, ..., x_k) \in K^k$ and ν is a finite positive measure on K.

Remark 4.7. The quantity Z_k^Q appears as the normalization constant in the law of eigenvalues of random matrix models. It is also referred to as the partition function in the theory of Coulomb gases. See Section 11 for more details on the link between these notions.

Proposition 4.8. *Given $K \subset \mathbb{R}^n$ compact and not log-polar, $Q \in \mathcal{A}(K)$, and ν a finite measure on K such that (K, ν, Q) satisfies a weighted Bernstein-Markov property, we have*

$$\lim_{k\to\infty} (Z_k^Q)^{1/k(k-1)} = \exp(-V_w) = \delta^Q(K).$$

Proof. We clearly have $\limsup_{k\to\infty} (Z_k^Q)^{1/k(k-1)} \leq \exp(-V_w)$ from 2. of Theorem 2.5. For the reverse inequality with $\liminf$, note that

$$|VDM_k(x_1, ..., x_k)|^2 = \prod_{i<j} |x_i - x_j|^2 = \prod_{i<j} \Big(\sum_{l=1}^n (x_{i,l} - x_{j,l})^2\Big)$$

(where we write $x_i = (x_{i,1}, ..., x_{i,n})$) is a polynomial of degree $k(k-1)$ in the nk real coordinates $\{x_{i,l}\}_{l=1,...,n;\ i=1,...,k}$. Now if $\mathbf{F_k} = (f_1, ..., f_k)$ is a set of weighted Fekete points of order k for K, Q, then

$$p(x_1) := |VDM_k(x_1, f_2, ..., f_k)|^2 \prod_{j=2}^k e^{-2(k-1)Q(f_j)}$$

is a (nonnegative) polynomial of degree $2(k-1)$ in (the coordinates of) x_1. By definition of weighted Fekete points, for any $x_1 \in K$,

$$p(x_1)e^{-2(k-1)Q(x_1)} \leq \max_{x\in K} p(x)e^{-2(k-1)Q(x)} = p(f_1)e^{-2(k-1)Q(f_1)}$$

since this right-hand-side is precisely $|VDM_k^Q(\mathbf{F_k})|^2$. By the weighted Bernstein-Markov property using L^1 norm instead of L^2 (see Remark 4.2),

$$|VDM_k^Q(\mathbf{F_k})|^2 \leq M_{2(k-1)} \int_K |VDM_k(x_1, f_2, ..., f_k)|^2 \cdot e^{-2(k-1)Q(x_1)} \cdot \prod_{j=2}^{k} e^{-2(k-1)Q(f_j)} d\nu(x_1).$$

Now for each fixed $x_1 \in K$, we consider

$$p_2(x_2) := |VDM_k(x_1, x_2, f_3..., f_k)|^2 \cdot e^{-2(k-1)Q(x_1)} \cdot \prod_{j=3}^{k} e^{-2(k-1)Q(f_j)}$$

which is a (nonnegative) polynomial of degree $2(k-1)$ in (the coordinates of) x_2. Then

$$p_2(f_2)e^{-2(k-1)Q(f_2)} \leq \max_{x\in K} p_2(x)e^{-2(k-1)Q(x)}.$$

The left-hand-side is

$$|VDM_k(x_1, f_2, f_3..., f_k)|^2 \cdot e^{-2(k-1)Q(x_1)} \cdot \prod_{j=2}^{k} e^{-2(k-1)Q(f_j)}.$$

The right-hand-side, by the weighted Bernstein-Markov property, is bounded above by

$$M_{2(k-1)} \int_K |VDM_k(x_1, x_2, f_3, ..., f_k)|^2 \cdot e^{-2(k-1)Q(x_1)} \cdot \prod_{j=3}^{k} e^{-2(k-1)Q(f_j)} e^{-2(k-1)Q(x_2)} d\nu(x_2).$$

Plugging these into our first estimate, we have

$$\begin{aligned}
|VDM_k^Q(\mathbf{F_k})|^2 &\leq M_{2(k-1)} \int_K |VDM_k(x_1, f_2, ..., f_k)|^2 \cdot e^{-2(k-1)Q(x_1)} \cdot \prod_{j=2}^{k} e^{-2(k-1)Q(f_j)} d\nu(x_1) \\
&\leq M_{2(k-1)} \int_K \Big[M_{2(k-1)} \int_K |VDM_k(x_1, x_2, f_3, ..., f_k)|^2 \cdot e^{-2(k-1)Q(x_1)} \cdot \\
&\qquad \prod_{j=3}^{k} e^{-2(k-1)Q(f_j)} e^{-2(k-1)Q(x_2)} d\nu(x_2) \Big] d\nu(x_1) \\
&= (M_{2(k-1)})^2 \int_{K^2} |VDM_k(x_1, x_2, f_3, ..., f_k)|^2 \prod_{j=3}^{k} e^{-2(k-1)Q(f_j)} \prod_{j=1}^{2} e^{-2(k-1)Q(x_j)} d\nu(x_2) d\nu(x_1).
\end{aligned}$$

Continuing the process and using $M_{2(k-1)}^{1/2k} \to 1$ gives the result.

□

Given *any* $Q \in \mathcal{A}(K)$, we can always find a finite measure satisfying the important conclusion of Proposition 4.8.

Proposition 4.9. *Let $K \subset \mathbb{R}^n$ be compact and not log-polar and let $Q \in \mathcal{A}(K)$. Then there exists a finite measure μ on K such that*

$$\lim_{k\to\infty} (Z_k^Q(K,\mu))^{1/k(k-1)} = \exp(-V_w) = \delta^Q(K). \tag{4.2}$$

We can even construct μ so that, in addition, μ is a (strong) Bernstein-Markov measure for K.

Proof. Consider the weighted equilibrium measure $\mu_{K,Q}$. We have $V_w = I(\mu_{K,Q}) < \infty$. By Lusin's continuity theorem, for every integer $m > 1$, there exists a compact subset K_m of K such that $\mu_{K,Q}(K \setminus K_m) \leq 1/m$ and Q (considered as a function on K_m only) is continuous on K_m. We may assume that each K_m is not log-polar and that the sets K_m are increasing as m tends to infinity. Let $\mu_m \in \mathcal{M}(K_m)$ be a (strong) Bernstein-Markov measure for K_m. We claim that $\mu = \sum_{m=1}^{\infty} \frac{1}{2^m}\mu_m$ satisfies (4.2).

Since $\mu_m \in \mathcal{M}(K_m)$ is a (strong) Bernstein-Markov measure for K_m and $Q_m := Q|_{K_m}$ is continuous, it also follows from 1. of Remark 4.6 that $(K_m, Q_m, \frac{1}{2^m}\mu_m)$ satisfies a weighted Bernstein-Markov property.

Since $Z_k^Q(K,\mu) \leq \max_{\mathbf{x}\in K^k} |VDM^Q(\mathbf{x})|^2 \mu(K)^k$, we have

$$\limsup_{k\to\infty} (Z_k^Q(K,\mu))^{1/k(k-1)} \leq \delta^Q(K).$$

To show

$$\liminf_{k\to\infty} (Z_k^Q(K,\mu))^{1/k(k-1)} \geq \delta^Q(K),$$

let $\lambda_m := \mu_{K,Q}(K_m)$ so that $\lambda_m \uparrow 1$. Letting $\theta_m := \frac{1}{\lambda_m}\mu_{K,Q}|_{K_m} \in \mathcal{M}(K_m)$, we have

$$I^{Q_m}(\theta_m) \geq I^{Q_m}(\mu_{K_m,Q_m}).$$

Since $(K_m, Q_m, \frac{1}{2^m}\mu_m)$ satisfies a weighted Bernstein-Markov property,

$$\exp(-I^{Q_m}(\mu_{K_m,Q_m})) = \lim_{k\to\infty} (Z_k^{Q_m}(K_m, \frac{1}{2^m}\mu_m))^{1/k(k-1)}.$$

Clearly

$$Z_k^Q(K,\mu) \geq Z_k^{Q_m}(K_m, \mu|_{K_m}) \geq Z_k^{Q_m}(K_m, \frac{1}{2^m}\mu_m).$$

Thus

$$\begin{aligned}\liminf_{k\to\infty} (Z_k^Q(K,\mu))^{1/k(k-1)} &\geq \liminf_{k\to\infty} (Z_k^{Q_m}(K_m, \frac{1}{2^m}\mu_m))^{1/k(k-1)} \\ &= \exp(-I^{Q_m}(\mu_{K_m,Q_m})) \geq \exp(-I^{Q_m}(\theta_m)).\end{aligned}$$

By monotone convergence we have

$$\lim_{m\to\infty} I^{Q_m}(\theta_m) = I^Q(\mu_{K,Q})$$

so that

$$\liminf_{k\to\infty}(Z_k^Q(K,\mu))^{1/k(k-1)} \geq \exp -I^Q(\mu_{K,Q}) = \delta^Q(K),$$

as desired.

For the second part, let ν be a (strong) Bernstein-Markov measure for K and define

$$\mu := \sum_{m=1}^{\infty} 2^{-m}\mu_m + \nu.$$

The fact that μ is a (strong) Bernstein-Markov measure for K follows from the fact that ν is a (strong) Bernstein-Markov measure for K and 2. of Remark 4.6. Finally, μ satisfies (4.2) from the previous part applied to $\sum_{m=1}^{\infty} 2^{-m}\mu_m$ and the obvious inequality $Z_k^Q(K,\mu) \geq Z_k^Q(K,\sum_{m=1}^{\infty} 2^{-m}\mu_m)$. □

Example 4.10. If μ is a (strong) Bernstein-Markov measure for K and the set of points of discontinuity of $Q \in \mathcal{A}(K)$ is of μ measure zero, then (4.2) holds for μ. As a simple but illustrative example, let $K = [-1,1] \subset \mathbb{R}$ and take

$$Q(x) = 0 \text{ at all } x \in [-1,1] \setminus \{0\};\ Q(0) = -1.$$

It is easy to see that Lebesgue measure $d\mu$ on $[-1,1]$ satisfies (4.2) but (K,Q,μ) does not satisfy the weighted Bernstein-Markov property. On the other hand, $(K,Q,\mu+\delta_0)$ does satisfy the weighted Bernstein-Markov property.

For $Q \in \mathcal{A}(K)$ and ν a finite measure on K, we define a probability measure $Prob_k$ on K^k: for a Borel set $A \subset K^k$,

$$Prob_k(A) := \frac{1}{Z_k^Q} \cdot \int_A |VDM_k^Q(\mathbf{X_k})|^2 d\nu(\mathbf{X_k}). \tag{4.3}$$

Directly from Proposition 4.9 and (4.3) we obtain the following estimate.

Corollary 4.11. *Let $Q \in \mathcal{A}(K)$ and ν a finite measure on K satisfying*

$$\lim_{k\to\infty}(Z_k^Q(K,\nu))^{1/k(k-1)} = \exp(-V_w) = \delta^Q(K).$$

Given $\eta > 0$, define

$$A_{k,\eta} := \{\mathbf{X_k} \in K^k : |VDM_k^Q(\mathbf{X_k})|^2 \geq (\delta^Q(K) - \eta)^{k(k-1)}\}. \tag{4.4}$$

Then there exists $k^* = k^*(\eta)$ such that for all $k > k^*$,

$$Prob_k(K^k \setminus A_{k,\eta}) \leq \Big(1 - \frac{\eta}{2\exp(-V_w)}\Big)^{k(k-1)} \nu(K^k).$$

We get the induced product probability measure $\mathbf{P}$ on the space of arrays on K,

$$\chi := \{X = \{\mathbf{X_k} \in K^k\}_{k\geq 1}\},$$

namely,

$$(\chi, \mathbf{P}) := \prod_{k=1}^{\infty} (K^k, Prob_k).$$

As an immediate consequence of the Borel-Cantelli lemma and 3. of Theorem 2.5, we obtain:

Corollary 4.12. *Let $Q \in \mathcal{A}(K)$ and ν a finite measure on K satisfying*

$$\lim_{k\to\infty} (Z_k^Q(K,\nu))^{1/k(k-1)} = \exp(-V_w) = \delta^Q(K).$$

For $\mathbf{P}$-a.e. array $X = \{x_j^{(k)}\}_{j=1,\ldots,k;\ k=2,3,\ldots} \in \chi$,

$$\frac{1}{k}\sum_{j=1}^{k} \delta_{x_j^{(k)}} \to \mu_{K,Q} \text{ weakly as } k \to \infty.$$

5 Approximation of probability measures

For the proof of a large deviation principle (LDP) in $\mathbb{R}^n$, as in [9], we will need to approach general measures in $\mathcal{M}(K)$ by weighted equilibrium measures. For that, we consider equilibrium problems with weights that are the negatives of potentials. We first verify that the natural candidate solution to such a problem is, indeed, the true solution.

Lemma 5.1. *Let $\mu \in \mathcal{M}(K)$, $K \subset \mathbb{R}^n$ compact, $I(\mu) < \infty$. Consider the possibly non-admissible weight $u := -U^\mu$ on K. The weighted minimal energy on K is obtained with the measure μ, that is*

$$\forall \nu \in \mathcal{M}(K), \quad I(\mu) + 2\int u d\mu \leq I(\nu) + 2\int u d\nu,$$

with equality if and only if $\nu = \mu$.

Proof. We may assume that $I(\nu) < \infty$. The inequality may be rewritten as

$$0 \leq I(\nu) - 2I(\mu,\nu) + I(\mu) = I(\nu - \mu),$$

which is true, and, moreover, the energy $I(\nu - \mu)$ can vanish only when $\nu = \mu$, see item 1. of Theorem 2.1. □

The following two approximation results are analogous to [21, Lemma I.6.10].

Lemma 5.2. *Let $K \subset \mathbb{R}^n$ be compact and non log-polar and let $\mu \in \mathcal{M}(K)$. Let $Q \in \mathcal{A}(K)$ be finite μ-almost everywhere. There exist an increasing sequence of compact sets K_m in K and a sequence of measures $\mu_m \in \mathcal{M}(K_m)$ satisfying*

1. *the measures μ_m tend weakly to μ, as $m \to \infty$;*
2. *the functions $Q_{|K_m} \in C(K_m)$ and $\int Q d\mu_m$ tend to $\int Q d\mu$ as $m \to \infty$;*
3. *the energies $I(\mu_m)$ tend to $I(\mu)$ as $m \to \infty$.*

Proof. By Lusin's continuity theorem, for every integer $m \geq 1$, there exists a compact subset K_m of K such that $\mu(K \setminus K_m) \leq 1/m$ and Q (considered as a function on K_m only) is continuous on K_m. We may assume that K_m is increasing as m tends to infinity. Then, the measures $\widetilde{\mu}_m := \mu_{|K_m}$ are increasing and tend weakly to μ. Since Q is bounded below on K, the monotone convergence theorem tells us that

$$\int Q d\widetilde{\mu}_m = \int Q_{|K_m} d\mu \to \int Q d\mu, \quad \text{as } m \to \infty.$$

Denoting as usual by $\log^+$ and $\log^-$ the positive and negative parts of the log function, we have, as $m \to \infty$,

$$\chi_m(z,t) \log^+ |z-t| \uparrow \log^+ |z-t| \quad \text{and} \quad \chi_m(z,t) \log^- |z-t| \uparrow \log^- |z-t|,$$

$(\mu \times \mu)$-almost everywhere on $K \times K$ where $\chi_m(z,t)$ is the characteristic function of $K_m \times K_m$ and we agree that the left-hand sides vanish when $z = t \notin K_m$. By monotone convergence, we obtain

$$I(\widetilde{\mu}_m) \to I(\mu), \quad \text{as } m \to \infty.$$

Finally, defining $\mu_m := \widetilde{\mu}_m / \mu(K_m)$ gives the result. □

Corollary 5.3. *Let $K \subset \mathbb{R}^n$ be compact and non log-polar and let $\mu \in \mathcal{M}(K)$ with $I(\mu) < \infty$. Let K_m be the sequence of increasing compact sets in K and μ_m the sequence of measures in $\mathcal{M}(K_m)$ given by Lemma 5.2 with $Q = U^\mu$. There exist a sequence of continuous functions Q_m on K such that*

1. *the measures μ_m tend weakly to μ and the energies $I(\mu_m)$ tend to $I(\mu)$, as $m \to \infty$;*
2. *the measures μ_m are equal to the weighted equilibrium measures μ_{K,Q_m}.*

Proof. First, note that $U^\mu \in \mathcal{A}(K)$ and is finite μ-almost everywhere since $I(\mu) < \infty$, so that Lemma 5.2 applies with $Q = U^\mu$. Now we define

$$Q_m := -U^{\mu_m}|_K = -\mu(K_m)^{-1} U^{\widetilde{\mu}_m}|_K.$$

Since $Q = U^\mu$ is continuous on K_m, it follows that Q_m is continuous on K_m. By the continuity principle for logarithmic potentials (4. of Theorem 2.1), $-U^{\mu_m}$ is continuous on $\mathbb{R}^n$ and hence Q_m is continuous on K. Items 1. and 2. follow from Lemmas 5.2 and 5.1 respectively. □

6 The J^Q functionals on $\mathbb{R}^n$

In this section, we introduce and establish the main properties of the weighted L^2 functionals $\overline{J}^Q, \underline{J}^Q$ as well as the relation with the weighted energy I^Q. Our goal is to establish an LDP in the next section.

Fix a compact set K in $\mathbb{R}^n$, a measure ν in $\mathcal{M}(K)$ and $Q \in \mathcal{A}(K)$. We recall that $\mathcal{M}(K)$ endowed with the weak topology is a Polish space, i.e., a separable complete metrizable space. Given $G \subset \mathcal{M}(K)$, for each $k = 1, 2, ...$ we let

$$\widetilde{G}_k := \{\mathbf{a} = (a_1, ..., a_k) \in K^k,\ \frac{1}{k}\sum_{j=1}^{k} \delta_{a_j} \in G\}, \tag{6.1}$$

and set

$$J_k^Q(G) := \Big[\int_{\widetilde{G}_k} |VDM_k^Q(\mathbf{a})|^2 d\nu(\mathbf{a})\Big]^{1/k(k-1)}. \tag{6.2}$$

Definition 6.1. For $\mu \in \mathcal{M}(K)$ we define

$$\overline{J}^Q(\mu) := \inf_{G \ni \mu} \overline{J}^Q(G) \text{ where } \overline{J}^Q(G) := \limsup_{k\to\infty} J_k^Q(G);$$

$$\underline{J}^Q(\mu) := \inf_{G \ni \mu} \underline{J}^Q(G) \text{ where } \underline{J}^Q(G) := \liminf_{k\to\infty} J_k^Q(G);$$

Here the infima are taken over all neighborhoods G of the measure μ in $\mathcal{M}(K)$. Note that, a priori, $\overline{J}^Q, \underline{J}^Q$ depend on ν. For the unweighted case $Q = 0$, we simply write $\overline{J}$ and $\underline{J}$.

Lemma 6.2. *The functionals $\underline{J}(\mu)$, $\overline{J}(\mu)$, $\underline{J}^Q(\mu)$, $\overline{J}^Q(\mu)$, are upper semicontinuous on $\mathcal{M}(K)$ in the weak topology.*

Proof. The proof is similar to the one of [7, Lemma 3.1]. □

Lemma 6.3. *The following properties hold (and with the $\underline{J}, \underline{J}^Q$ functionals as well):*

1. $\overline{J}^Q(\mu) \le e^{-I^Q(\mu)}$ *for* $Q \in \mathcal{A}(K)$;
2. $\overline{J}^Q(\mu) \le \overline{J}(\mu) \cdot e^{-2\int_K Q d\mu}$ *for* $Q \in \mathcal{A}(K)$;
3. $\overline{J}^Q(\mu) = \overline{J}(\mu) \cdot e^{-2\int_K Q d\mu}$ *for* Q **continuous**.

Proof. Item 1. follows from

$$J_k^Q(G) \le \sup_{\mathbf{a}\in\widetilde{G}_k} |VDM_k^Q(\mathbf{a})|^{2/k(k-1)},$$

and the upper bound (2.2) on the limit of the Vandermonde. We prove item 2. and item 3. simultaneously. We first observe that if $\mu \in \mathcal{M}(K)$ and Q is continuous on K, given $\epsilon > 0$, there exists a neighborhood $G \subset \mathcal{M}(K)$ of μ with

$$\Big| \int_K Q\Big(d\mu - \frac{1}{k}\sum_{j=1}^{k} \delta_{a_j}\Big)\Big| \leq \epsilon \quad \text{for } \mathbf{a} \in \widetilde{G}_k$$

for k sufficiently large. Thus we have

$$-\epsilon - \int_K Q d\mu \leq -\frac{1}{k}\sum_{j=1}^{k} Q(a_j) \leq \epsilon - \int_K Q d\mu. \tag{6.3}$$

Note that for $Q \in \mathcal{A}(K)$, hence lower semicontinuous, we only have the second inequality. Since

$$|VDM_k^Q(\mathbf{a})| = |VDM_k(\mathbf{a})| \cdot \prod_{j=1}^{k} e^{-(k-1)Q(a_j)},$$

we deduce from (6.3) that

$$|VDM_k(\mathbf{a})| e^{-k(k-1)(\epsilon + \int_K Q d\mu)} \leq |VDM_k^Q(\mathbf{a})| \leq |VDM_k(\mathbf{a})| e^{k(k-1)(\epsilon - \int_K Q d\mu)}.$$

Now we take the square, integrate over $\mathbf{a} \in \widetilde{G}_k$ and take a $k(k-1)$-th root of each side to get

$$J_k(G) e^{-2(\epsilon + \int_K Q d\mu)} \leq J_k^Q(G) \leq J_k(G) e^{2(\epsilon - \int_K Q d\mu)}.$$

Precisely, given $\epsilon > 0$, these inequalities are valid for G a sufficiently small neighborhood of μ. Hence we get, upon taking $\limsup_{k\to\infty}$, the infimum over $G \ni \mu$, and noting that $\epsilon > 0$ is arbitrary,

$$\overline{J}(\mu) = \overline{J}^Q(\mu) \cdot e^{2\int_K Q d\mu}$$

as desired. If Q is only lower semicontinuous, we still have the upper bounds in the above, which gives item 2. □

From item 1. in Lemma 6.3, we know that for $Q \in \mathcal{A}(K)$

$$\log \underline{J}^Q(\mu) \leq \log \overline{J}^Q(\mu) \leq -I^Q(\mu). \tag{6.4}$$

In the remainder of this section, we show that when the measure ν satisfies a Bernstein-Markov property, equalities hold in (6.4).

We first consider the unweighted functionals $\underline{J}$ and $\overline{J}$ and the case of an equilibrium measure $\mu = \mu_{K,v}$ where $v \in \mathcal{A}(K)$

Lemma 6.4. *Let K be non log-polar, $v \in \mathcal{A}(K)$, and let $\nu \in \mathcal{M}(K)$ such that (K, ν, v) satisfy a weighted Bernstein-Markov property. Then,*

$$\log \overline{J}(\mu_{K,v}) = \log \underline{J}(\mu_{K,v}) = -I(\mu_{K,v}). \tag{6.5}$$

Proof. To prove (6.5), we first verify the following.

Claim: Fix a neighborhood G of $\mu_{K,v}$. For $\eta > 0$, define

$$A_{k,\eta} := \{\mathbf{Z_k} \in K^k : |VDM_k^v(\mathbf{Z_k})|^2 \geq (\delta^v(K) - \eta)^{k(k-1)}\}.$$

Given a sequence $\{\eta_j\}$ with $\eta_j \downarrow 0$, there exists a j_0 and a k_0 such that

$$\forall j \geq j_0, \quad \forall k \geq k_0, \quad A_{k,\eta_j} \subset \widetilde{G}_k. \tag{6.6}$$

We prove (6.6) by contradiction: if false, there are sequences $\{k_l\}$ and $\{j_l\}$ tending to infinity such that for all l sufficiently large we can find a point $\mathbf{Z_{k_l}} = (z_1, ..., z_{k_l})$ with $\mathbf{Z_{k_l}} \in A_{k_l,\eta_{j_l}} \setminus \widetilde{G}_{k_l}$. But

$$\mu^l := \frac{1}{k_l} \sum_{i=1}^{k_l} \delta_{z_i} \notin G$$

for l sufficiently large is a contradiction with item 3. of Theorem 2.5 since $\mathbf{Z_{k_l}} \in A_{k_l,\eta_{j_l}}$ and $\eta_{j_l} \to 0$ imply $\mu^l \to \mu_{K,v}$ weakly. This proves the claim.

Fix a neighborhood G of $\mu_{K,v}$ and a sequence $\{\eta_j\}$ with $\eta_j \downarrow 0$. For $j \geq j_0$, choose $k = k_j$ large enough so that the inclusion in (6.6) holds true as well as

$$Prob_{k_j}(K^{k_j} \setminus A_{k_j,\eta_j}) \leq \Big(1 - \frac{\eta_j}{2\delta^v(K)}\Big)^{k_j(k_j-1)}, \tag{6.7}$$

and

$$\Big(1 - \frac{\eta_j}{2\delta^v(K)}\Big)^{k_j(k_j-1)} \to 0 \quad \text{as} \quad j \to \infty, \tag{6.8}$$

which is possible (for (6.7) we make use of Corollary 4.11). In view of (6.6), the definition of $Prob_{k_j}$, and (6.7), we have

$$\begin{aligned} \frac{1}{Z_{k_j}^v} \int_{\widetilde{G}_{k_j}} |VDM_{k_j}^v(\mathbf{Z_{k_j}})|^2 d\nu(\mathbf{Z_{k_j}}) &\geq \frac{1}{Z_{k_j}^v} \int_{A_{k_j,\eta_j}} |VDM_{k_j}^v(\mathbf{Z_{k_j}})|^2 d\nu(\mathbf{Z_{k_j}}) \\ &\geq 1 - \Big(1 - \frac{\eta_j}{2\delta^v(K)}\Big)^{k_j(k_j-1)}. \end{aligned} \tag{6.9}$$

Note that, because of (6.8), the lower bound in (6.9) tends to 1 as $j \to \infty$. Then, since (K, ν, v) satisfy a weighted Bernstein-Markov property, we derive, with the asymptotics of $Z_{k_j}^v$ given in Proposition 4.8, that

$$\liminf_{j\to\infty} \frac{1}{k_j(k_j-1)} \log \int_{\widetilde{G}_{k_j}} |VDM_{k_j}^v(\mathbf{Z_{k_j}})|^2 d\nu(\mathbf{Z_{k_j}}) \geq \log \delta^v(K).$$

Given any sequence of positive integers $\{k\}$ we can find a subsequence $\{k_j\}$ as above corresponding to some $\eta_j \downarrow 0$; hence

$$\liminf_{k\to\infty} \frac{1}{k(k-1)} \log \int_{\widetilde{G}_k} |VDM_k^v(\mathbf{Z}_k)|^2 d\nu(\mathbf{Z}_k) \geq \log \delta^v(K).$$

It follows that

$$\log \underline{J}^v(G) \geq \log \delta^v(K).$$

Taking the infimum over all neighborhoods G of $\mu_{K,v}$ we obtain

$$\log \underline{J}^v(\mu_{K,v}) \geq \log \delta^v(K).$$

Using item 2. of Lemma 6.3 with $\mu = \mu_{K,v}$, we get

$$\log \underline{J}(\mu_{K,v}) \geq -I(\mu_{k,v}),$$

and with the unweighted version of item 1., we obtain (6.5). □

Remark 6.5. We observe that the proof only used the property

$$\lim_{k\to\infty} (Z_k^v(K,\nu))^{1/k(k-1)} = \delta^v(K).$$

Theorem 6.6. *Let K be a non log-polar compact subset of $\mathbb{R}^n$ and let $\nu \in \mathcal{M}(K)$ satisfy the (strong) Bernstein-Markov property.*
(i) For any $\mu \in \mathcal{M}(K)$,

$$\log \overline{J}(\mu) = \log \underline{J}(\mu) = -I(\mu). \tag{6.10}$$

(ii) Let $Q \in \mathcal{A}(K)$. Then

$$\overline{J}^Q(\mu) = \overline{J}(\mu) \cdot e^{-2\int_K Q d\mu}, \tag{6.11}$$

(and with the $\underline{J}, \underline{J}^Q$ functionals as well) so that,

$$\log \overline{J}^Q(\mu) = \log \underline{J}^Q(\mu) = -I^Q(\mu). \tag{6.12}$$

Proof. We first prove (i). The upper bound

$$\log \overline{J}(\mu) \leq -I(\mu) \tag{6.13}$$

is the unweighted version of (6.4). For the lower bound $-I(\mu) \leq \log \underline{J}(\mu)$ we first assume that $I(\mu) < \infty$. Using Corollary 5.3, there exists a sequence of (continuous) functions Q_m defined on K and measures $\mu_m = \mu_{K,Q_m}$ tending weakly to μ such that,

$$\lim_{m\to\infty} I(\mu_m) = I(\mu). \tag{6.14}$$

Thus we can apply Lemma 6.4 to conclude

$$\log \overline{J}(\mu_m) = \log \underline{J}(\mu_m) = -I(\mu_m),$$

and from (6.14) along with the uppersemicontinuity of the functional $\mu \to \underline{J}(\mu)$, we derive

$$\lim_{m\to\infty} \log \underline{J}(\mu_m) = -I(\mu) \leq \log \underline{J}(\mu).$$

Together with (6.13) we get

$$\log \underline{J}(\mu) = \log \overline{J}(\mu) = -I(\mu).$$

If $\mu \in \mathcal{M}(K)$ satisfies $I(\mu) = \infty$, Item 1. of Lemma 6.3 shows that $\overline{J}(\mu) = 0$.

We next proceed with assertion (ii). For (6.11), it is sufficient to prove the inequality

$$\overline{J}^Q(\mu) \geq \overline{J}(\mu) \cdot e^{-2\int_K Q d\mu}. \tag{6.15}$$

We first assume that Q is finite μ-almost everywhere and $I(\mu) < \infty$ so that Lemma 5.2 can be applied on K. Let K_m be the sequence of compact subsets of K and μ_m be the sequence of measures in $\mathcal{M}(K_m)$ given by that lemma. By the upper semicontinuity of the functional $\overline{J}^Q$,

$$\overline{J}^Q(\mu) \geq \limsup_{m\to\infty} \overline{J}^Q(\mu_m).$$

Also, by item 3. of Lemma 6.3 and (6.10), since $Q|_{K_m}$ is continuous,

$$\overline{J}^Q(\mu_m) = \overline{J}(\mu_m) e^{-2\int Q d\mu_m} = e^{-I(\mu_m) - 2\int Q d\mu_m}.$$

Hence, (6.15) follows from items 2. and 3. of Lemma 5.2. When $I(\mu) = \infty$, both sides of (6.15) equal 0, since $\overline{J}(\mu) = e^{-I(\mu)}$ and, by definition, $0 \leq \overline{J}^Q(\mu)$. If $\mu(\{Q = \infty\}) > 0$, this is true as well because $\overline{J}(\mu) > -\infty$ while the exponential in the right-hand side vanishes.

Finally, (6.12) follows from (6.10) and (6.11). □

Remark 6.7. We note that the Bernstein-Markov property of the measure ν has only been applied with the sequence of continuous weights Q_m that appear when approaching μ with Corollary 5.3.

From now on, we simply use the notation J, J^Q, without the overline or underline. It follows from (6.10) and (6.12) that these functionals are independent of the measure ν; i.e., we have shown: *if $\nu \in \mathcal{M}(K)$ is* **any** *(strong) Bernstein-Markov measure, for any $Q \in \mathcal{A}(K)$, and for any $\mu \in \mathcal{M}(K)$ we have*

$$\log J^Q(\mu) = -I^Q(\mu). \tag{6.16}$$

7 Large Deviation Principle in $\mathbb{R}^n$

Fix a non log-polar compact set K in $\mathbb{R}^n$, a measure ν on K and $Q \in \mathcal{A}(K)$. Define $j_k : K^k \to \mathcal{M}(K)$ via

$$j_k(x_1, ..., x_k) = \frac{1}{k} \sum_{j=1}^{k} \delta_{x_j}. \tag{7.1}$$

The push-forward $\sigma_k := (j_k)_*(Prob_k)$ (see (4.3) for the definition of $Prob_k$) is a probability measure on $\mathcal{M}(K)$: for a Borel set $G \subset \mathcal{M}(K)$,

$$\sigma_k(G) = \frac{1}{Z_k^Q} \int_{\widetilde{G}_k} |VDM_k^Q(x_1, ..., x_k)|^2 d\nu(x_1) \cdots d\nu(x_k), \tag{7.2}$$

recall (4.1), (4.3) and (6.1); here, Z_k^Q depends on K, Q and ν.

Theorem 7.1. *Assume ν is a (strong) Bernstein-Markov measure on K, $Q \in \mathcal{A}(K)$, and ν satisfies*

$$\lim_{k\to\infty} (Z_k^Q(K,\nu))^{1/k(k-1)} = \exp(-V_w) = \delta^Q(K). \tag{7.3}$$

The sequence $\{\sigma_k = (j_k)_*(Prob_k)\}$ of probability measures on $\mathcal{M}(K)$ satisfies a **large deviation principle** *with speed k^2 and good rate function $\mathcal{I} := \mathcal{I}_{K,Q}$ where, for $\mu \in \mathcal{M}(K)$,*

$$\mathcal{I}(\mu) := \log J^Q(\mu_{K,Q}) - \log J^Q(\mu) = I^Q(\mu) - I^Q(\mu_{K,Q}).$$

This means that $\mathcal{I} : \mathcal{M}(K) \to [0, \infty]$ is a lower semicontinuous mapping such that the sublevel sets $\{\mu \in \mathcal{M}(K) : \mathcal{I}(\mu) \le \alpha\}$ are compact in the weak topology on $\mathcal{M}(K)$ for all $\alpha \ge 0$ ($\mathcal{I}$ is "good") satisfying (7.4) and (7.5):

Definition 7.2. The sequence $\{\mu_k\}$ of probability measures on $\mathcal{M}(K)$ satisfies a **large deviation principle** (LDP) with good rate function $\mathcal{I}$ and speed k^2 if for all measurable sets $\Gamma \subset \mathcal{M}(K)$,

$$-\inf_{\mu\in\Gamma^0} \mathcal{I}(\mu) \le \liminf_{k\to\infty} \frac{1}{k^2} \log \mu_k(\Gamma) \text{ and} \tag{7.4}$$

$$\limsup_{k\to\infty} \frac{1}{k^2} \log \mu_k(\Gamma) \le -\inf_{\mu\in\bar{\Gamma}} \mathcal{I}(\mu). \tag{7.5}$$

In the setting of $\mathcal{M}(K)$, to prove a LDP it suffices to work with a base for the weak topology. The following is a special case of a basic general existence result for a LDP given in Theorem 4.1.11 in [13].

Proposition 7.3. *Let $\{\sigma_\epsilon\}$ be a family of probability measures on $\mathcal{M}(K)$. Let $\mathcal{B}$ be a base for the topology of $\mathcal{M}(K)$. For $\mu \in \mathcal{M}(K)$ let*

$$\mathcal{I}(\mu) := -\inf_{\{G\in\mathcal{B}:\mu\in G\}} \big(\liminf_{\epsilon\to 0} \epsilon \log \sigma_\epsilon(G)\big).$$

Suppose for all $\mu \in \mathcal{M}(K)$,

$$\mathcal{I}(\mu) = -\inf_{\{G\in\mathcal{B}:\mu\in G\}} \big(\limsup_{\epsilon\to 0} \epsilon \log \sigma_\epsilon(G)\big).$$

Then $\{\sigma_\epsilon\}$ satisfies a LDP with rate function $\mathcal{I}(\mu)$ and speed $1/\epsilon$.

We give our proof of Theorem 7.1 using Theorem 6.6.

Proof. As a base $\mathcal{B}$ for the topology of $\mathcal{M}(K)$, we simply take all open sets. For $\{\sigma_\epsilon\}$, we take the sequence of probability measures $\{\sigma_k\}$ on $\mathcal{M}(K)$ and we take $\epsilon = k^{-2}$. For $G \in \mathcal{B}$,

$$\frac{1}{k^2} \log \sigma_k(G) = \frac{k-1}{k} \log J_k^Q(G) - \frac{1}{k^2} \log Z_k^Q$$

using (6.2) and (7.2). From (6.16), and the fact that (7.3) holds,

$$\lim_{k\to\infty} \frac{1}{k^2} \log Z_k^Q = \log \delta^Q(K) = \log J^Q(\mu_{K,Q});$$

and by Theorem 6.6,

$$\inf_{G\ni\mu} \limsup_{k\to\infty} \log J_k^Q(G) = \inf_{G\ni\mu} \liminf_{k\to\infty} \log J_k^Q(G) = \log J^Q(\mu).$$

Thus by Proposition 7.3 and Theorem 6.6, $\{\sigma_k\}$ satisfies an LDP with rate function

$$\mathcal{I}(\mu) := \log J^Q(\mu_{K,Q}) - \log J^Q(\mu) = I^Q(\mu) - I^Q(\mu_{K,Q})$$

and speed k^2. This rate function is good since $\mathcal{M}(K)$ is compact. □

Remark 7.4. Note that the rate function is independent of the (strong) Bernstein-Markov measure ν satisfying (7.3).

8 Measures ν of infinite mass on $K \subset \mathbf{S} \subset \mathbb{R}^3$

In this section, we restrict to the setting of compact subsets of the two-dimensional sphere $\mathbf{S}$ in $\mathbb{R}^3$, of center $(0, 0, 1/2)$ and radius $1/2$. Then in the following sections, we use stereographic projection from $\mathbf{S}$ to the complex plane to derive a large deviation principle on unbounded subsets of $\mathbb{C}$. Now typically, on the complex plane, one would like to consider locally finite measures with infinite mass like, e.g., the Lebesgue measure. We use the stereographic projection T defined in (3.2) which sends the north pole $P_0 = (0, 0, 1)$ of $\mathbf{S}$ to the point at infinity in $\mathbb{C}$. On the sphere $\mathbf{S}$ we are thus led to consider positive measures ν, locally finite in $\mathbf{S} \setminus P_0$, such that

$$\nu(V_{P_0}) = \infty, \quad \text{for all neighborhoods } V_{P_0} \text{ of } P_0. \tag{8.1}$$

The goal of this section is to extend the results from the previous sections to such measures.

Fix a compact subset K of $\mathbf{S}$ containing P_0. To ensure the finiteness of the different quantities defined previously, some condition should be satisfied linking the measure ν and the increase of the weights Q near P_0. We assume that

$$\exists a > 0, \quad \int_K \epsilon(x)^a d\nu(x) < \infty, \tag{8.2}$$

where $\epsilon(x)$ is some nonnegative continuous function that tends to 0 as x tends to P_0, and that

$$Q(x) \geq -\log \epsilon(x), \quad \text{as } x \to P_0. \tag{8.3}$$

This implies in particular that $Q(P_0) = \infty$. We next state a weighted Bernstein-Walsh lemma on the sphere.

Theorem 8.1. *Let K be a closed non log-polar subset of $\mathbf{S}$ and $Q \in \mathcal{A}(K)$. Let*

$$p_k(x) = \prod_{j=1}^{k} |x - x_j|, \qquad x \in \mathbf{S}, \tag{8.4}$$

where $x_1, \ldots, x_k \in \mathbf{S}$ and assume

$$|p_k(x)e^{-kQ(x)}| \leq M \quad \text{for } x \in S_w \setminus P \text{ where } P \text{ is log-polar (possibly empty).}$$

Then

$$|p_k(x)| \leq M \exp(k(-U^{\mu_{K,Q}}(x) + F_w)), \qquad x \in \mathbf{S},$$

and

$$|p_k(z)e^{-kQ(z)}| \leq M \quad \text{for } x \in K \setminus \widetilde{P} \text{ where } \widetilde{P} \text{ is log-polar (possibly empty).}$$

Proof. Using the stereographic map T defined in (3.2), this theorem is a translation of Theorem 3.6 on $\mathbb{C}$. □

We will also need a lemma related to where the L^p norm of a weighted "polynomial" lives, see [21, III, Theorem 6.1], [5, Theorem 6.1] for polynomials on $\mathbb{C}$. For $w = e^{-Q}$, we set

$$S_w^* = \{x \in K,\ U^{\mu_{K,Q}}(x) + Q(x) \leq F_w\}.$$

Note that, as $U^{\mu_{K,Q}}(x) + Q(x)$ is lower semicontinuous, S_w^* is a closed subset of $\mathbf{S}$ which, moreover, does not contain P_0. Indeed, $U^{\mu_{K,Q}}(x)$ is bounded below while $Q(x)$ tends to infinity as x tends to P_0. Moreover, from Theorem 3.4, $S_w \subset S_w^*$.

Lemma 8.2. *Let $p > 0$, K a non log-polar compact subset of $\mathbf{S}$ containing P_0, $Q \in \mathcal{A}(K)$ and ν a positive measure on K satisfying (8.1)–(8.3). We assume that (K, ν, Q) satisfies the weighted Bernstein-Markov property. Let $N \subset K$ be a closed neighborhood of S_w^*. Then, there exists a constant $c > 0$ independent of k and p such that, for all expressions p_k of the form (8.4),*

$$\int_K |p_k e^{-kQ}|^p d\nu \leq (1 + \mathcal{O}(e^{-ck})) \int_N |p_k e^{-kQ}|^p d\nu.$$

Proof. We normalize p_k so that $\|p_k e^{-kQ}\|_{S_w^*} = 1$. It is sufficient to show that there exists a constant $c > 0$ such that for k large,

$$\int_{K \setminus N} |p_k e^{-kQ}|^p d\nu \leq e^{-ck}, \tag{8.5}$$

and that, for every $\epsilon > 0$ and k large,

$$\int_K |p_k e^{-kQ}|^p d\nu \geq e^{-\epsilon k}.$$

For the second inequality, we use the $L^{p/2}$–Bernstein-Markov property (recall Remark 4.2) which gives

$$\int_K |p_k e^{-kQ}|^p d\nu \geq M_{2k}^{-p/2} \|p_k e^{-kQ}\|_K^p \geq M_{2k}^{-p/2} \|p_k e^{-kQ}\|_{S_w^*}^p = M_{2k}^{-p/2} \geq e^{-\epsilon k},$$

for k large, where we notice that p_k^2 is a real polynomial of degree $2k$.

For the first inequality, we use Theorem 8.1 and the fact that $S_w \subset S_w^*$. This implies that, for $x \in K$,

$$|e^{-kQ(z)} p_k(x)| \leq \|e^{-kQ} p_k\|_{S_w} e^{-k(U^{\mu_{K,Q}}+Q-F_w)} \leq e^{-k(U^{\mu_K,Q}+Q-F_w)}.$$

Since $U^{\mu_{K,Q}}$ is bounded below on K, there exists a constant b_0 such that

$$-U^{\mu_{K,Q}}(x) - Q(x) + F_w \leq \log \epsilon(x) + b_0, \quad x \in K,$$

and, as $\epsilon(x)$ tends to 0 as x tends to P_0, there exists a neighborhood V_{P_0} of P_0 such that $e^{b_0}\epsilon(x)^{1/2} < 1$ for $x \in V_{P_0}$. On the other hand, since N is a closed neighborhood of S_w^* and $-U^{\mu_{K,Q}} - Q$ is upper semicontinuous, there exists a constant $b_1 > 0$ such that

$$-U^{\mu_{K,Q}}(x) - Q(x) + F_w \leq -b_1 < 0, \quad x \in K \setminus N.$$

From this we deduce that

$$\begin{aligned}
\int_{K\setminus N} |p_k e^{-kQ}|^p d\nu &= \int_{V_{P_0}} |p_k e^{-kQ}|^p d\nu + \int_{K\setminus(N\cup V_{P_0})} |p_k e^{-kQ}|^p d\nu \\
&\leq \int_{V_{P_0}} e^{kpb_0}\epsilon(x)^{kp} d\nu + e^{-kpb_1}\nu(K \setminus V_{P_0}) \\
&\leq \|e^{b_0}\epsilon(x)^{1/2}\|_{V_{P_0}}^{kp} \int_K \epsilon(x)^a d\nu + e^{-kpb_1}\nu(K \setminus V_{P_0}),
\end{aligned}$$

for k large, which implies (8.5). □

We are now in a position to prove an extended version of Proposition 4.8.

Proposition 8.3. *Let K be a non log-polar compact subset of $\mathbf{S}$ containing P_0, $Q \in \mathcal{A}(K)$ and ν a positive measure on K satisfying (8.1)–(8.3). We assume that (K, ν, Q) satisfies a weighted Bernstein-Markov property. Then the L^2 normalization constants Z_k^Q defined in (4.1) are finite and*

$$\lim_{k\to\infty} (Z_k^Q)^{1/k(k-1)} = \delta^Q(K).$$

Proof. In view of (8.2) and (8.3), it is clear that, for k large, the integral defining Z_k^Q is finite. The lower bound,

$$\delta^Q(K) \leq \liminf_{k\to\infty} (Z_k^Q)^{1/k(k-1)},$$

is proved as in the proof of Proposition 4.8 by making use of the weighted Bernstein-Markov property. For the upper bound,

$$\limsup_{k\to\infty}(Z_k^Q)^{1/k(k-1)} \le \delta^Q(K), \tag{8.6}$$

we first note that the expression $|VDM_k^Q(\mathbf{X_k})|^2$ is, in each variable, of the form $e^{-2(k-1)Q}|q|^2$ with $|q|$ as in (8.4) for $k-1$. Hence, by using Lemma 8.2 with $p=2$ for each of the k variables, and with $N \subset K$ a closed neighborhood of S_w^* as in Lemma 8.2 with $\nu(N) < \infty$ and $P_0 \notin N$, we get

$$\begin{aligned} Z_k^Q = \int_{K^k} |VDM_k^Q(\mathbf{X_k})|^2 d\nu(\mathbf{X_k}) &\le (1+\mathcal{O}(e^{-c(k-1)}))^k \int_{N^k} |VDM_k^Q(\mathbf{X_k})|^2 d\nu(\mathbf{X_k}) \\ &\le (1+\mathcal{O}(e^{-c(k-1)}))^k (\delta_k^Q(K))^{k(k-1)} \nu(N)^k, \end{aligned}$$

which implies (8.6) by taking the $k(k-1)$-th root and letting k go to infinity. □

The next goal is to generalize Corollary 4.11.

Corollary 8.4. *We assume that the conditions (8.1)–(8.3) are satisfied and that (K,ν,Q) satisfies the weighted Bernstein-Markov property. Then, with the notation of Corollary 4.11, there exist a constant $c>0$ and $k^*=k^*(\eta)$ such that for all $k>k^*$,*

$$Prob_k(K^k \setminus A_{k,\eta}) \le \Big(1-\frac{\eta}{2\delta^Q(K)}\Big)^{k(k-1)} \nu(N)^k + \mathcal{O}(e^{-ck}), \tag{8.7}$$

where $N \subset K$ is a closed neighborhood of S_w^* as in Lemma 8.2 with $\nu(N)<\infty$ and $P_0 \notin N$.

Proof. We set $B_{k,\eta} := K^k \setminus A_{k,\eta}$ and decompose the integral in

$$Prob_k(K^k \setminus A_{k,\eta}) = \frac{1}{Z_k^Q} \int_{B_{k,\eta}} |VDM_k^Q(\mathbf{X_k})|^2 d\nu(\mathbf{X_k})$$

as a sum of two integrals over $B_{k,\eta} \cap N^k$ and $B_{k,\eta} \cap (K^k \setminus N^k)$. Recalling the definition of the set $A_{k,\eta}$, the first term is less than

$$\Big(1-\frac{\eta}{2\delta^Q(K)}\Big)^{k(k-1)} \nu(N)^k,$$

for k large. The second term is less than

$$\frac{1}{Z_k^Q} \int_{K^k\setminus N^k} |VDM_k^Q(\mathbf{X_k})|^2 d\nu(\mathbf{X_k}) \le \sum_{j=1}^k \frac{1}{Z_k^Q} \int_{U_j} |VDM_k^Q(\mathbf{X_k})|^2 d\nu(\mathbf{X_k}),$$

where $U_j = K \times \cdots \times (K \setminus N) \times \cdots K$ and the subset $K \setminus N$ is in j-th position. As already observed in the previous proof, the expression $|VDM_k^Q(\mathbf{X_k})|^2$ is, in each variable, of the

form $e^{-2(k-1)Q}|q|^2$ with $|q|$ as in (8.4) for $k-1$. Hence, applying Lemma 8.2 with the j-th variable to the integral over U_j, we get the upper bound

$$\mathcal{O}(e^{-c(k-1)})\sum_{j=1}^{k}\frac{1}{Z_k^Q}\int_{V_j}|VDM_k^Q(\mathbf{X_k})|^2 d\nu(\mathbf{X_k}),$$

where $V_j = K\times\cdots\times N\times\cdots K$. Replacing N with K we finally get the upper bound $\mathcal{O}(ke^{-c(k-1)})$, which implies (8.7) with a different c. □

The last result that needs to be extended is the first item of Lemma 6.3, namely, that for $Q\in\mathcal{A}(K)$, and ν satisfying (8.1)–(8.3),

$$\overline{J}^Q(\mu)\le e^{-I^Q(\mu)}. \tag{8.8}$$

With the notation of Section 6, we remark that fixing $a>0$ as in (8.2), we can write

$$\int_{\widetilde{G}_k}|VDM_k^Q(\mathbf{a})|^2 d\nu(\mathbf{a}) = \int_{\widetilde{G}_k}|VDM_k^{Q_k}(\mathbf{a})|^2 d\widetilde{\nu}(\mathbf{a}),$$

where

$$Q_k(x) = Q(x) - \frac{a}{2(k-1)}Q_+(x), \qquad \widetilde{\nu}(x) = e^{-aQ_+(x)}\nu(x),$$

and $Q_+ = \max(Q,0)$. Observe that $\{Q_k\}_k$ is an increasing sequence of admissible weights that converges pointwise to Q as k tends to infinity. Also, in view of (8.2) and (8.3), $\widetilde{\nu}$ is a finite measure. Since

$$\int_{\widetilde{G}_k}|VDM_k^{Q_k}(\mathbf{a})|^2 d\widetilde{\nu}(\mathbf{a}) \le \int_{\widetilde{G}_k}|VDM_k^{Q_{k_0}}(\mathbf{a})|^2 d\widetilde{\nu}(\mathbf{a}), \quad k\ge k_0,$$

we have

$$\int_{\widetilde{G}_k}|VDM_k^{Q}(\mathbf{a})|^2 d\nu(\mathbf{a}) \le \int_{\widetilde{G}_k}|VDM_k^{Q_{k_0}}(\mathbf{a})|^2 d\widetilde{\nu}(\mathbf{a}), \quad k\ge k_0.$$

By letting k go to infinity in this inequality and taking the infima over all neighborhoods G of a measure μ in $\mathcal{M}(K)$, we obtain

$$J_\nu^Q(\mu)\le J_{\widetilde{\nu}}^{Q_{k_0}}(\mu),$$

where, here, the subscript denotes the measure with respect to which the Vandermonde is integrated. Since $\widetilde{\nu}$ is of finite mass, we derive from item 1. of Lemma 6.3 that

$$J_\nu^Q(\mu)\le e^{-I^{Q_{k_0}}(\mu)}.$$

Letting k_0 go to infinity, and making use of

$$\int Q_{k_0}d\mu\to\int Q d\mu, \quad \text{as } k_0\to\infty,$$

which follows from the monotone convergence theorem, we obtain (8.8).

From the results above and the proofs of the previous sections, one may check that Theorem 6.6 extends to the measures ν considered in this section. For future reference, we state this as a theorem.

Theorem 8.5. *Let K be a non log-polar compact subset of the sphere $\mathbf{S}$ in $\mathbb{R}^3$, containing P_0, $Q \in \mathcal{A}(K)$ and ν a positive measure on K satisfying (8.1)–(8.3). Assume ν satisfies a (strong) Bernstein-Markov property. Then,*

$$\log \overline{J}^Q(\mu) = \log \underline{J}^Q(\mu) = -I^Q(\mu).$$

Also, the large deviation principle asserted in Theorem 7.1 extends.

Theorem 8.6. *Let K be a compact subset of the sphere $\mathbf{S}$ in $\mathbb{R}^3$, containing P_0, $Q \in \mathcal{A}(K)$ and ν a positive measure on K satisfying (8.1)–(8.3) and (7.3). Then, the large deviation principle from Theorem 7.1 holds.*

Remark 8.7. In Theorems 8.5 and 8.6, the conclusion is valid for any ν satisfying a (strong) Bernstein-Markov property and (8.1)–(8.3) (with any appropriate function $\epsilon(x)$). Moreover, the rate function in Theorem 8.6 is independent of ν.

9 The J^Q functionals on unbounded sets in $\mathbb{C}$

We return to the case of unbounded sets in $\mathbb{C}$; our goal is to use Theorems 8.5 and 8.6 to derive their versions in our current setting. In the sequel we will need the Bernstein-Markov property on $\mathbb{C}$. For K a closed subset of $\mathbb{C}$, ν a positive measure on K, locally finite but possibly of infinite mass in a neighborhood of infinity, and Q a weakly admissible weight as in (3.1), we say that (K, ν, Q) satisfies the Bernstein-Markov property if

$$\forall p_k \in \mathcal{P}_k(\mathbb{C}), \quad \|e^{-kQ} p_k\|_K \leq M_k \|e^{-kQ} p_k\|_{L^2(\nu)}, \quad \text{with } \limsup_{k\to\infty} M_k^{1/k} = 1. \tag{9.1}$$

As in Section 4, if (9.1) holds true for any continuous weakly admissible weight Q, we will say that ν satisfies a strong Bernstein-Markov property. Note that the polynomials p_k in (9.1) are polynomials with respect to the *complex variable* z.

Example 9.1. For Q a continuous admissible weight on $\mathbb{R}$, the linear Lebesgue measure $d\lambda$ provides an example of a measure with unbounded support satisfying (9.1). Indeed, from Theorem 3.6, the sup norm of $e^{-kQ}p_k$ is attained on S_w which is compact. Hence, it suffices to prove (9.1) on S_w or any compact set containing S_w, for instance a finite interval I (see Remark 4.5). For Q a continuous admissible weight on $\mathbb{C}$, similar reasoning shows that planar Lebesgue measure dm on $\mathbb{C}$ satisfies (9.1) as well (in this case one considers the restriction of dm to a closed disk).

Next, let Q be an admissible weight on $K = \{x \in \mathbb{R} : x \geq 0\}$ which is continuous except $Q(0) = +\infty$. In this case, property (c) of Theorem I.1.3 [21] shows that S_w, the support of $\mu_{K,Q}$, will be compact and disjoint from the origin. Thus linear Lebesgue measure similarly satisfies (9.1). Specific examples are Laguerre weights $Q(x) = \lambda x - s \log x$ with $\lambda, s > 0$ which occur in the Wishart ensemble (see [18], section 5.5); and $Q(x) = c(\log x)^2$ with $c \geq 0$, occurring in the Stieltjes-Wigert ensemble (see [24] and [14]).

Remark 9.2. For future use we observe that, if ν has infinite mass in a neighborhood of infinity, then the Bernstein-Markov property (9.1) is automatically satisfied if we restrict, for each k, to polynomials p_k of *exact degree* k and the weight Q satisfies a condition slightly stronger than weak admissibility (3.1), namely

$$\lim_{z\in K,\ |z|\to\infty}(Q(z)-\log|z|)=M<\infty. \tag{9.2}$$

Indeed, for p_k a monic polynomial of degree k, $|e^{-kQ(z)}p_k(z)|$ behaves like the constant $e^{-kM}>0$ as $z\to\infty$, so that its sup norm on K is finite while its $L^2(\nu)$-norm is infinite. Hence we can take $M_k=1$ for each $k\geq 0$.

Next, we define the (weighted) L^2 normalization constants for a closed subset K of $\mathbb{C}$, Q weakly admissible and ν a positive measure on K,

$$Z_k^Q(K,\nu):=\int_{K^k}|VDM_k^Q(\mathbf{Z_k})|^2 d\nu(\mathbf{Z_k}), \tag{9.3}$$

where $\mathbf{Z_k}:=(z_1,...,z_k)\in K^k$. Then we have the correspondence

$$Z_k^Q(K,\nu)=Z_k^{\widetilde{Q}}(T(K),T_*\nu)$$

where $\widetilde{Q}$ is defined in (3.10). To ensure the finiteness of $Z_k^Q(K,\nu)$ in case ν has infinite mass in a neighborhood of infinity, like, e.g., Lebesgue measure, we assume that the weight Q and the measure ν satisfy conditions that correspond via the inverse of T to the conditions (8.2) and (8.3) on the sphere, namely,

$$\exists a>0,\quad \int_K \epsilon(z)^a d\nu(z)<\infty, \tag{9.4}$$

and

$$Q(z)-\log|z|\geq -\log\epsilon(z),\quad \text{as } z\to\infty, \tag{9.5}$$

where $\epsilon(z)$ is some nonnegative continuous function that tends to 0 as z tends to ∞. Note that the weight Q is then admissible in the sense of [21] or 2. of Definition 3.1, that is

$$Q(z)-\log|z|\to\infty,\quad \text{as } z\to\infty. \tag{9.6}$$

Using the inequality

$$|z_i-z_j|\leq(1+|z_i|)(1+|z_j|),$$

one may also check directly that the $Z_k^Q(K,\nu)$, k large, are, indeed, finite.

In the typical example where $K=\mathbb{R}$ or $K=\mathbb{C}$ and ν is Lebesgue measure, $\epsilon(z)$ can be chosen as $|z|^{-\epsilon}$, $\epsilon>0$, and (9.5) becomes the following *strong admissibility* condition (recall 3. of Definition 3.1):

$$Q(z)-\log|z|\geq\epsilon\log|z|,\quad \text{as } z\to\infty.$$

Our next result is a version of Propositions 4.8 and 8.3 on the $k(k-1)$-th root asymptotic behavior of the L^2 normalization constants for K a closed subset of $\mathbb{C}$.

Proposition 9.3. *Let K be a nonpolar closed subset of $\mathbb{C}$ and ν a positive measure on K. Let Q be a weight on K which is weakly admissible if ν has finite mass and such that (9.4) and (9.5) are satisfied for some function $\epsilon(z)$ if ν has infinite mass in a neighborhood of infinity. We assume that (K, ν, Q) satisfies the weighted Bernstein-Markov property (9.1). Then,*

$$\lim_{k\to\infty} (Z_k^Q)^{1/k(k-1)} = \delta^Q(K).$$

Proof. Via the inverse map of T, the statement is essentially a simple translation of Propositions 4.8 and 8.3. The only observation to be made is that, for the proof of Proposition 4.8 on the sphere, it suffices that a weighted Bernstein-Markov property is satisfied with respect to polynomials p of the particular form

$$p(x) = \prod_{j=1}^{k} |x - T(z_j)|^2, \quad x \in \mathbf{S}, \tag{9.7}$$

and that this Bernstein-Markov property corresponds to (9.1) via T^{-1}. Also, in the proof of Proposition 8.3, it is sufficient to use a version of Lemma 8.2 which only assumes the Bernstein-Markov property for polynomials of the form (9.7) (and thus only holds for such polynomials). □

Weighted J-functionals $\underline{J}^Q(\mu)$ and $\overline{J}^Q(\mu)$ can be defined on the closed subset K of $\mathbb{C}$, with respect to a positive measure ν in K, as was done on compact subsets of $\mathbb{R}^n$, see Definition 6.1. Then,

$$\underline{J}^Q(\mu) = \underline{J}^{\widetilde{Q}}(T_*\mu), \quad \overline{J}^Q(\mu) = \overline{J}^{\widetilde{Q}}(T_*\mu),$$

where the J-functionals on the right-hand sides involve integrals with respect to the measure $T_*\nu$. From this correspondence, and Theorems 6.6 and 8.5, we derive the following.

Theorem 9.4. *With the hypotheses of Proposition 9.3 and assuming that ν satisfies a strong Bernstein-Markov property on K, we have*

$$\log \underline{J}^Q(\mu) = \log \overline{J}^Q(\mu) = -I^Q(\mu).$$

Proof. The statement is a translation of Theorems 6.6 and 8.5 on the sphere. Again, we observe that the strong Bernstein-Markov property for polynomials of the form (9.7) is sufficient for their proofs. We also use the equality of the weighted logarithmic energies (3.8) and (3.9). □

The conclusion is valid for all ν satisfying the hypotheses; in particular, the functional J^Q $(= \underline{J}^Q = \overline{J}^Q)$ $= e^{-I^Q}$ for any such ν.

10 Large deviation principle for unbounded sets in $\mathbb{C}$

A large deviation principle in the spirit of Theorem 7.1 for compact subsets of $\mathbb{C}^m,\ m \geq 1$ has been obtained in [8] using the methods of this paper (see also [9]). Yattselev [25] has proved an LDP associated to a specific type of weight on $\mathbb{C}$; he uses Lebesgue measure on $\mathbb{C}$. The large deviation principle for strongly admissible weights Q on all of $\mathbb{C}$ with Lebesgue measure can be found in the book of Hiai and Petz [18]. There they extend the method of Ben Arous and Guionnet [2]. Here, we will utilize the results from the previous sections to establish a LDP in the setting of a closed set K in $\mathbb{C}$, not necessarily bounded, with a weakly admissible weight Q and an appropriate Bernstein-Markov measure. The proof is based on a standard contraction principle in LDP theory:

Theorem 10.1 ([13, Theorem 4.2.1])**.** *If $\{P_n\}$ is a sequence of probability measures on a Polish space X satisfying an LDP with speed $\{a_n\}$ and rate function $\mathcal{I}$, Y is another Polish space and $f : X \to Y$ is a continuous map, then $\{Q_n := f_* P_n\}$ satisfies an LDP on Y with the same speed and with rate function*

$$\mathcal{J}(y) := \inf\{\mathcal{I}(x) : x \in X, f(x) = y\}. \tag{10.1}$$

For K a closed, possibly unbounded, subset of $\mathbb{C}$, ν a locally finite measure on K, and Q a weakly admissible weight on K, we define the measure $Prob_k$ on K^k as in (4.3) for K in $\mathbb{R}^n$ and $j_k : K^k \to \mathcal{M}(K)$ as in (7.1).

The statement of the large deviation principle is as follows.

Theorem 10.2. *Assume (K, ν, Q) satisfies the weighted Bernstein-Markov property (9.1). If ν has finite mass, we also assume that (K, ν) satisfies a strong Bernstein-Markov property while if ν has infinite mass in a neighborhood of infinity, we assume that (9.4) and (9.5) are satisfied for some function $\epsilon(z)$. Then the sequence $\{\sigma_k = (j_k)_*(Prob_k)\}$ of probability measures on $\mathcal{M}(K)$ satisfies a* **large deviation principle** *with speed k^2 and good rate function $\mathcal{I} := \mathcal{I}_{K,Q}$ where, for $\mu \in \mathcal{M}(K)$,*

$$\mathcal{I}(\mu) := \log J^Q(\mu_{K,Q}) - \log J^Q(\mu) = I^Q(\mu) - I^Q(\mu_{K,Q}). \tag{10.2}$$

We emphasize again that, as in Theorem 8.6, the rate function is independent of the measure ν.

Proof. We apply Theorem 10.1 to the homeomorphism $f = (T^{-1})_*$: thus to prove an LDP in the setting of a closed set K in $\mathbb{C}$, not necessarily bounded, with a weakly admissible weight Q, it suffices, via this contraction principle, to use an LDP in the setting of a compact set $T(K)$ in $\mathbf{S} \subset \mathbb{R}^3$ with the admissible weight $\widetilde{Q}$. This we have from Theorems 7.1 and 8.6. In case ν is of infinite mass in a neighborhood of infinity, we observe that the strong Bernstein-Markov property of (K, ν) is not needed. Indeed, the corresponding Bernstein-Markov property on $T(K)$ is only needed for polynomials that are Vandermonde expressions, hence of maximal degree, and for the weights Q_m appearing in Corollary 5.3. These weights are of the form $-U^{\mu_m}$ with $\mu_m \in \mathcal{M}(T(K))$ and the corresponding weights

on K satisfy (9.2) with $M = -U^{\mu_m}(P_0) < \infty$, so that Remark 9.2 applies. Finally, the rate function $\mathcal{I}(\mu)$ is good because $I^Q(\mu) = I^{\widetilde{Q}}(T_*\mu)$, the energy $I^{\widetilde{Q}}$ is lower semicontinuous on the compact set $\mathcal{M}(\mathbf{S})$, and T_* is a homeomorphism. □

Remark 10.3. In particular (see Example 9.1), we have a large deviation principle on $K = \{x \in \mathbb{R} : x \geq 0\}$ with Lebesgue measure for the Laguerre weights as well as for the weights occurring in the Stieltjes-Wigert ensembles.

11 Applications: β ensembles

Let K be a closed subset of $\mathbb{C}$, ν a positive measure on K, and Q a weakly admissible weight on K. Classical models in random matrix theory involve probability distributions on K^k of the form

$$\frac{1}{\widehat{Z}^Q_{\beta,k}} \prod_{1\leq i<j\leq k} |z_i - z_j|^{2\beta} \prod_{i=1}^{k} e^{-2kQ(z_i)} d\nu(z_i), \tag{11.1}$$

where $\beta > 0$ and the normalization constant $\widehat{Z}^Q_{\beta,k}$ is

$$\widehat{Z}^Q_{\beta,k} = \int_{K^k} \prod_{1\leq i<j\leq k} |z_i - z_j|^{2\beta} \prod_{i=1}^{k} e^{-2kQ(z_i)} d\nu(z_i). \tag{11.2}$$

(We caution the reader that in [15] and [1] the 2β is replaced by β). The probability distribution (11.1) and normalization constant $\widehat{Z}^Q_{\beta,k}$ differ from the distribution in (4.3) and the L^2 normalization constant Z^Q_k, defined in (9.3), by the exponent β and an additional factor $\prod_i e^{-2Q(x_i)}$ in its integrand. One may check that all results from the previous sections remain true, with appropriate modifications, when we consider (11.1) and (11.2). Actually, writing the products in (11.1) and (11.2) as the square of a weighted Vandermonde to the power β, the main modification consists in replacing the weight Q with the weight Q/β and to use the Bernstein-Markov property in L^β instead of L^1 as was done in Section 4. To be precise, because of the factor k, instead of $k-1$, in the exponential factors of (11.1) and (11.2), the Bernstein-Markov property to be satisfied for a given weight Q here is

$$\forall p_k \in \mathcal{P}_k(\mathbb{C}), \quad \|e^{-(k+1)Q} p_k\|_K \leq M_k \|e^{-(k+1)Q} p_k\|_{L^2(\nu)}, \quad \text{with } \limsup_{k\to\infty} M_k^{1/k} = 1. \tag{11.3}$$

This property is slightly weaker than (9.1) as it concerns only polynomials in $\mathcal{P}_k(\mathbb{C})$ instead of $\mathcal{P}_{k+1}(\mathbb{C})$, but it will make a minor difference in the assumptions of the large deviation principle because Remark 9.2 no longer applies.

When ν has infinite mass in a neighborhood of infinity, the conditions (9.4) and (9.5) become

$$\exists a > 0, \quad \int_K \epsilon(z)^a d\nu(z) < \infty, \tag{11.4}$$

and

$$Q(z) - \beta \log |z| \geq -\log \epsilon(z), \quad \text{as } z \to \infty, \tag{11.5}$$

where $\epsilon(z)$ is some nonnegative continuous function that tends to 0 as z tends to ∞.

Based on the above remarks, one may check that we have the following analogue of Proposition 9.3 concerning the asymptotics of $\widehat{Z}^Q_{\beta,k}$.

Proposition 11.1. *Let K be a nonpolar closed subset of $\mathbb{C}$ and ν a positive measure on K. Let Q be a weight on K such that Q/β is weakly admissible if ν has finite mass and such that (11.4) and (11.5) are satisfied for some function $\epsilon(z)$ if ν has infinite mass in a neighborhood of infinity. We assume that $(K, \nu, Q/\beta)$ satisfies the weighted Bernstein-Markov property (11.3). Then*

$$\lim_{k\to\infty} (\widehat{Z}^Q_{\beta,k})^{1/k(k-1)} = (\delta^{Q/\beta}(K))^\beta.$$

The following large deviation principle, an analogue of Theorem 10.2, also holds true.

Theorem 11.2. *Let Q/β, $\beta > 0$, be a weakly admissible weight on K such that $(K, \nu, Q/\beta)$ satisfies the weighted Bernstein-Markov property (11.3). We assume that (K, ν) satisfies a strong Bernstein-Markov property and, in addition, if ν has infinite mass in a neighborhood of infinity, we assume that (11.4) and (11.5) are satisfied for some function $\epsilon(z)$. Then the sequence of probability measures $\widetilde{\sigma}_k$ on $\mathcal{M}(K)$, defined so that for a Borel set $G \subset \mathcal{M}(K)$,*

$$\widetilde{\sigma}_k(G) := \frac{1}{\widehat{Z}^Q_{\beta,k}} \int_{\widetilde{G}_k} \prod_{1\leq i<j\leq k} |z_i - z_j|^{2\beta} \prod_{i=1}^k e^{-2kQ(z_i)} d\nu(z_i),$$

satisfies a **large deviation principle** *with speed k^2 and good rate function $\mathcal{I}^\beta_{K,Q}$ defined by*

$$\mathcal{I}^\beta_{K,Q}(\mu) := I^Q_\beta(\mu) - I^Q_\beta(\mu_{K,Q/\beta}),$$

where

$$I^Q_\beta(\mu) = \int_K \int_K \log \frac{1}{|z-t|^\beta} d\mu(z) d\mu(t) + 2\int_K Q(z) d\mu(z) = \beta I^{Q/\beta}(\mu), \qquad \mu \in \mathcal{M}(K).$$

Proof. One checks that all arguments in the proof of Theorem 10.2 go through when considering the probability distribution (11.1) instead of the one in (4.3). In particular, this entails verifying the analogue of Theorems 6.6 and 9.4, namely that, for appropriate assumptions on ν and Q, one has

$$\log \overline{J}^Q_\beta(\mu) = \log \underline{J}^Q_\beta(\mu) = -I^Q_\beta(\mu),$$

where the functionals $\overline{J}^Q_\beta$ and $\underline{J}^Q_\beta$ are derived from

$$J^Q_{\beta,k}(G) := \Big[\int_{\widetilde{G}_k} |VDM^{Q/\beta}_k(\mathbf{a})|^{2\beta} d\nu(\mathbf{a}) \Big]^{1/k(k-1)}, \qquad G \subset \mathcal{M}(K),$$

in the same way as in Definition 6.1. Since Remark 9.2 does not apply for the Bernstein-Markov property (11.3), the strong Bernstein-Markov property is needed even if ν has infinite mass. □

As an example, we take $K = \mathbb{R}$, $d\nu = d\lambda =$ Lebesgue measure on $\mathbb{R}$, and Q a continuous weight such that there exists $\beta' > \beta$ with Q/β' weakly admissible:

$$\exists M > -\infty, \qquad \liminf_{|z|\to\infty,\ z\in K} (Q(z) - \beta' \log|z|) = M. \tag{11.6}$$

Note that this implies that Q/β is admissible. Also, (11.4) and (11.5) hold true with $\epsilon(z) = |z|^{\beta-\beta'}$. The triple $(\mathbb{R}, d\lambda, Q/\beta)$ satisfies the weighted Bernstein-Markov property since Q/β is admissible, cf., Example 9.1. The measure $d\lambda$ likely also satisfies a strong Bernstein-Markov property, but, as already mentioned in the proof of Theorem 10.2, for an LDP it is sufficient that this property is satisfied for weights which correspond via T to continuous weights on the sphere $\mathbf{S}$ of the form $-U^{\mu_m}$ where $\mu_m \in \mathcal{M}(T(\mathbb{R}))$ are the measures from Corollary 5.3. Moreover, the proof of Lemma 5.2 shows that the supports of the measures μ_m can be chosen to avoid a neighborhood (depending on m) of the north pole P_0 so that the push-backward measures $\nu_m = T_*^{-1}\mu_m$ have compact supports in $\mathbb{C}$. Using the relations (3.7) and (3.10), what is then needed is that $d\lambda$ satisfies the Bernstein-Markov property for continuous weights of the form

$$Q_m(z) = -U^{\nu_m}(z) - \frac{1}{2}\int \log(1+|t|^2) d\nu_m(t),$$

where $\nu_m \in \mathcal{M}(\mathbb{R})$ has compact support. These weights are weakly admissible. Hence, by Theorem 3.6 and the continuity of Q_m, the corresponding weighted polynomials attain their sup norm on S_{w_m} (where $w_m = e^{-Q_m}$), which is equal to the support of ν_m, see Lemma 5.1 (or more precisely its analogue in $\mathbb{C}$). Consequently, we need that $d\lambda$ satisfies the Bernstein-Markov property for continuous weights on a compact set, which we know holds true (cf., the discussion in Example 9.1). Thus we conclude that the large deviation principle asserted in Theorem 11.2 applies on the real line for $d\nu = d\lambda$ and any continuous weight Q satisfying (11.6). We note that this includes the large deviation principle for the law of the spectral measure of Gaussian Wigner matrices ([2], [1, Theorem 2.6.1]) as well as the refined version for weakly confining potentials given in [15].

When $K = \mathbb{C}$, $d\nu = dm =$ planar Lebesgue measure, and Q is a continuous weight on $\mathbb{C}$ satisfying the growth condition (11.6), assumptions (11.4) and (11.5) still hold true with $\epsilon(z) = |z|^{\beta-\beta'}$. Moreover, the measure dm satisfies the required Bernstein-Markov properties. Hence, Theorem 11.2 applies when $K = \mathbb{C}$, $d\nu = dm$, and Q is a continuous weight satisfying (11.6).

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