Imaginary Kählerian Killing spinors I

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Abstract. We describe and to some extent characterize a new family of Kähler spin manifolds admitting non-trivial imaginary Kählerian Killing spinors.

1 Introduction

Let $(\widetilde{M}^{2n}, g, J)$ a Kähler manifold of real dimension $2n$ and with Kähler-form $\widetilde{\Omega}$ defined by $\widetilde{\Omega}(X,Y) := g(J(X),Y)$ for all vectors $X, Y \in T\widetilde{M}$. We denote by $p_+ : TM \longrightarrow T^{1,0}M$, $X \mapsto \frac{1}{2}(X - iJ(X))$ and $p_- : TM \longrightarrow T^{0,1}M$, $X \mapsto \frac{1}{2}(X + iJ(X))$ the projection maps. In case $\widetilde{M}^{2n}$ is spin, we denote its complex spinor bundle by $\Sigma\widetilde{M}$.

Definition 1.1 *Let $(\widetilde{M}^{2n}, g, J)$ a spin Kähler manifold and $\alpha \in \mathbb{C}$. A pair (ψ, ϕ) of sections of $\Sigma\widetilde{M}$ is called an* α-Kählerian Killing spinor *if and only if it satisfies, for every $X \in \Gamma(T\widetilde{M})$,*

$$\left| \begin{array}{ll} \widetilde{\nabla}_X \psi & = -\alpha p_-(X) \cdot \phi \\ \widetilde{\nabla}_X \phi & = -\alpha p_+(X) \cdot \psi. \end{array} \right.$$

An α-Kählerian Killing spinor is said to be real *(resp.* imaginary*) if and only if $\alpha \in \mathbb{R}$ (resp. $\alpha \in i\mathbb{R}^*$).*

If $\alpha = 0$, then an α-Kählerian Killing spinor is nothing but a pair of parallel spinors. The classification of Kähler spin manifolds (resp. spin manifolds) admitting real non-parallel Kählerian Killing (resp. parallel) spinors has been established by A. Moroianu in [12] (resp. by McK. Wang in [14]).

In this paper, we describe and partially classify those Kähler spin manifolds carrying non-trivial imaginary Kählerian Killing spinors. Note first that there is no restriction in assuming $\alpha = i$: obviously, changing (ψ, ϕ) into $(\psi, -\phi)$ changes α into $-\alpha$; moreover, (ψ, ϕ) is an α-Kählerian Killing spinor on $(\widetilde{M}^{2n}, g, J)$ if and only if it is an $\frac{\alpha}{\lambda}$-Kählerian Killing spinor on $(\widetilde{M}^{2n}, \lambda^2 g, J)$ for any constant $\lambda > 0$.

K.-D. Kirchberg, who introduced this equation (see [9] for references), showed that, if a non-zero i-Kählerian Killing spinor (ψ, ϕ) exists on $(\widetilde{M}^{2n}, g, J)$, then necessarily the complex dimension n of $\widetilde{M}$ is odd, the manifold $(\widetilde{M}^{2n}, g)$ is Einstein with scalar curvature $-4n(n+1)$, the pair (ψ, ϕ) vanishes nowhere and satisfies $\widetilde{\Omega} \cdot \psi = -i\psi$ as well as $\widetilde{\Omega} \cdot \phi = i\phi$, see [9] and Proposition 2.1 below for further properties. Moreover, he proved in the case $n = 3$ that the holomorphic sectional curvature must be constant [9, Thm. 16], in particular only the complex hyperbolic space $\mathbb{C}\mathrm{H}^3$ occurs as simply-connected complete $(\widetilde{M}^6, g, J)$ with non-trivial i-Kählerian Killing spinors.

We extend Kirchberg's results in several ways. First, we study in detail the critical points of the length function $|\psi|$ of ψ. We show that, if the underlying Riemannian manifold $(\widetilde{M}^{2n}, g)$ is connected and complete, then $|\psi|$ has at most one critical value, which then has to be a (global) minimum and that the corresponding set of critical points is a Kähler totally geodesic submanifold (Proposition 2.3).
As a next step, we describe a whole family of examples of Kähler manifolds admitting non-trivial i-Kählerian Killing spinors (Theorem 3.9), including the complex hyperbolic space and some Kähler manifolds with non-constant holomorphic sectional curvature (Corollary 3.13). All arise as so-called doubly-warped products over Sasakian manifolds. A more detailed study of the induced spinor equation on that

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Sasakian manifold allows the complex hyperbolic space to be characterized within the family (Theorem 3.18).

In the last section, we show that doubly-warped products are the only possible Kähler manifolds with non-trivial i-Kählerian Killing spinors as soon as both components of (ψ, ϕ) have the same length and are exchanged through the Clifford multiplication by a (real) vector field (Theorem 4.1). This shows an interesting analogy with H. Baum's classification [3, 4] of complete Riemannian spin manifolds with imaginary Killing spinors.

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2 General integrability conditions

In this section we look for further necessary conditions for the existence of imaginary Kählerian Killing spinors. Consider the vector field V on $\widetilde{M}$ defined by

$$g(V, X) := \Im m(\langle p_+(X) \cdot \psi, \phi \rangle) \tag{1}$$

for every vector X on $\widetilde{M}$. We recall the following

Proposition 2.1 (see [9]) *Let (ψ, ϕ) be an i-Kählerian Killing spinor on $(\widetilde{M}^{2n}, g, J)$ which does not vanish identically. Then the following properties hold:*

i) $\mathrm{grad}(|\psi|^2) = \mathrm{grad}(|\phi|^2) = 2V$.

ii) For all vectors $X, Y \in T\widetilde{M}$,

$$g(\widetilde{\nabla}_X V, Y) = \Re e\left(\langle p_-(X) \cdot \phi, p_-(Y) \cdot \phi \rangle + \langle p_+(X) \cdot \psi, p_+(Y) \cdot \psi \rangle\right).$$

In particular,

$$\mathrm{Hess}(|\psi|^2)(X, Y) = \mathrm{Hess}(|\phi|^2)(X, Y) = 2\Re e\left(\langle p_-(X) \cdot \phi, p_-(Y) \cdot \phi \rangle + \langle p_+(X) \cdot \psi, p_+(Y) \cdot \psi \rangle\right).$$

iii) $\Delta(|\psi|^2) = \Delta(|\phi|^2) = -2(n+1)(|\psi|^2 + |\phi|^2)$, *where* $\Delta := -\mathrm{tr}_g(\mathrm{Hess})$.

iv) The vector field V is holomorphic, i.e., it satisfies: $\widetilde{\nabla}_{J(X)} V = J(\widetilde{\nabla}_X V)$ for every $X \in T\widetilde{M}$. In particular, the vector field $J(V)$ is Killing on $\widetilde{M}$.

v) $\mathrm{grad}(|V|^2) = 2\widetilde{\nabla}_V V$.

Note that, from Proposition 2.1, the identity $\Delta(|\psi|^2 + |\phi|^2) = -4(n+1)(|\psi|^2 + |\phi|^2)$ holds on $\widetilde{M}$, therefore $\widetilde{M}$ cannot be compact.

Next we are interested in the critical points of $|\psi|^2$ (or of $|\phi|^2$, they are the same by Proposition 2.1.*i*)). We need a technical lemma:

Lemma 2.2 *Under the hypotheses of* Proposition 2.1*, one has*

$$\widetilde{\nabla}_X \widetilde{\nabla}_Y V = \widetilde{\nabla}_{\widetilde{\nabla}_X Y} V + \{2g(V, X)Y + g(V, Y)X - g(V, J(Y))J(X) + g(X, Y)V + g(J(X), Y)J(V)\}$$

for all vector fields X, Y on $\widetilde{M}$. Therefore,

$$\mathrm{Hess}(|V|^2)(X, Y) = 2g(\widetilde{\nabla}_X V, \widetilde{\nabla}_Y V) + 2\left(3g(X, V)g(Y, V) + |V|^2 g(X, Y) - g(X, J(V))g(Y, J(V))\right).$$

Proof: Using Proposition 2.1, we compute in a local orthonormal basis $\{e_j\}_{1\leq j\leq 2n}$ of $T\widetilde{M}$:

$$\begin{aligned}
\widetilde{\nabla}_X\widetilde{\nabla}_Y V &= \sum_{j=1}^{2n}\Re e\Big(\langle p_-(\widetilde{\nabla}_X Y)\cdot\phi, p_-(e_j)\cdot\phi\rangle + \langle p_+(\widetilde{\nabla}_X Y)\cdot\psi, p_+(e_j)\cdot\psi\rangle \\
&\qquad + \langle p_-(Y)\cdot\widetilde{\nabla}_X\phi, p_-(e_j)\cdot\phi\rangle + \langle p_-(Y)\cdot\phi, p_-(e_j)\cdot\widetilde{\nabla}_X\phi\rangle \\
&\qquad + \langle p_+(Y)\cdot\widetilde{\nabla}_X\psi, p_+(e_j)\cdot\psi\rangle + \langle p_+(Y)\cdot\psi, p_+(e_j)\cdot\widetilde{\nabla}_X\psi\rangle\Big)e_j \\
&= \sum_{j=1}^{2n}\Re\mathfrak{e}\Big(\langle p_-(\widetilde{\nabla}_X Y)\cdot\phi, p_-(e_j)\cdot\phi\rangle + \langle p_+(\widetilde{\nabla}_X Y)\cdot\psi, p_+(e_j)\cdot\psi\rangle \\
&\qquad - \alpha\langle p_-(Y)\cdot p_+(X)\cdot\psi, p_-(e_j)\cdot\phi\rangle + \alpha\langle p_-(Y)\cdot\phi, p_-(e_j)\cdot p_+(X)\cdot\psi\rangle \\
&\qquad - \alpha\langle p_+(Y)\cdot p_-(X)\cdot\phi, p_+(e_j)\cdot\psi\rangle + \alpha\langle p_+(Y)\cdot\psi, p_+(e_j)\cdot p_-(X)\cdot\phi\rangle\Big)e_j \\
&= \widetilde{\nabla}_{\widetilde{\nabla}_X Y}V \\
&\quad + \sum_{j=1}^{2n}\Im m\Big(\langle p_-(Y)\cdot p_+(X)\cdot\psi, p_-(e_j)\cdot\phi\rangle + \langle p_+(Y)\cdot p_-(X)\cdot\phi, p_+(e_j)\cdot\psi\rangle\Big)e_j \\
&\quad - \sum_{j=1}^{2n}\Im m\Big(\langle p_-(Y)\cdot\phi, p_-(e_j)\cdot p_+(X)\cdot\psi\rangle + \langle p_+(Y)\cdot\psi, p_+(e_j)\cdot p_-(X)\cdot\phi\rangle\Big)e_j.
\end{aligned}$$

We compute the second line of the right-hand side of the preceding equation (the treatment of the third one is analogous). Using $\langle p_+(X)\cdot\psi,\phi\rangle = 2ig(V,p_+(X))$, we obtain

$$\begin{aligned}
\langle p_+(Y)\cdot p_-(X)\cdot\phi, p_+(e_j)\cdot\psi\rangle &= \overline{\langle\psi, p_-(X)\cdot p_+(Y)\cdot p_-(e_j)\cdot\phi\rangle} + 4ig(Y,p_-(e_j))g(V,p_-(X)) \\
&\quad + 4ig(Y,p_-(X))g(V,p_-(e_j)).
\end{aligned}$$

We deduce that, for every $j\in\{1,\dots,2n\}$,

$$\begin{aligned}
\langle p_-(Y)\cdot p_+(X)\cdot\psi, p_-(e_j)\cdot\phi\rangle + \langle p_+(Y)\cdot p_-(X)\cdot\phi, p_+(e_j)\cdot\psi\rangle &= 2\Re e\left(\langle\psi, p_-(X)\cdot p_+(Y)\cdot p_-(e_j)\cdot\phi\rangle\right) \\
&\quad + 4ig(Y,p_-(e_j))g(V,p_-(X)) \\
&\quad + 4ig(Y,p_-(X))g(V,p_-(e_j)).
\end{aligned}$$

The imaginary part of the right-hand side of the last equality is then given for every $j\in\{1,\dots,2n\}$ by

$$\begin{aligned}
4\Re e\left(g(Y,p_-(e_j))g(V,p_-(X)) + g(Y,p_-(X))g(V,p_-(e_j))\right) &= g(V,X)g(Y,e_j) + g(V,J(X))g(J(Y),e_j) \\
&\quad + g(X,Y)g(V,e_j) + g(J(X),Y)g(J(V),e_j).
\end{aligned}$$

This shows that

$$\begin{aligned}
\sum_{j=1}^{2n}\Im m\Big(\langle p_-(Y)\cdot p_+(X)\cdot\psi, p_-(e_j)\cdot\phi\rangle + \langle p_+(Y)\cdot p_-(X)\cdot\phi, p_+(e_j)\cdot\psi\rangle\Big)e_j &= g(V,X)Y \\
&\quad + g(V,J(X))J(Y) \\
&\quad + g(X,Y)V \\
&\quad + g(J(X),Y)J(V).
\end{aligned}$$

Similarly, one shows that

$$\begin{aligned}
\sum_{j=1}^{2n}\Im m\Big(\langle p_-(Y)\cdot\phi, p_-(e_j)\cdot p_+(X)\cdot\psi\rangle + \langle p_+(Y)\cdot\psi, p_+(e_j)\cdot p_-(X)\cdot\phi\rangle\Big)e_j &= -g(V,Y)X \\
&\quad + g(V,J(Y))J(X) \\
&\quad - g(V,X)Y \\
&\quad + g(V,J(X))J(Y).
\end{aligned}$$

Combining the computations above, we obtain

$$\begin{aligned}\widetilde{\nabla}_X\widetilde{\nabla}_Y V &= \widetilde{\nabla}_{\widetilde{\nabla}_X Y}V \\ &\quad + (g(V,X)Y + g(V,J(X))J(Y) + g(X,Y)V + g(J(X),Y)J(V)) \\ &\quad - (-g(V,Y)X + g(V,J(Y))J(X) - g(V,X)Y + g(V,J(X))J(Y)) \\ &= \widetilde{\nabla}_{\widetilde{\nabla}_X Y}V \\ &\quad + (2g(V,X)Y + g(V,Y)X - g(V,J(Y))J(X) + g(X,Y)V + g(J(X),Y)J(V)),\end{aligned}$$

which shows the first identity. We deduce for the Hessian of $|V|^2$ that, for all vector fields X, Y on $\widetilde{M}$,

$$\begin{aligned}\mathrm{Hess}(|V|^2)(X,Y) &= 2g(\widetilde{\nabla}_X\widetilde{\nabla}_V V, Y) \\ &= 2g(\widetilde{\nabla}_{\widetilde{\nabla}_X V}V, Y) + 2\Big(2g(V,X)g(V,Y) + |V|^2 g(X,Y) - 0 + g(X,V)g(V,Y) \\ &\qquad + g(J(X),V)g(J(V),Y)\Big) \\ &= 2g(\widetilde{\nabla}_X V, \widetilde{\nabla}_Y V) + 2\left(3g(X,V)g(Y,V) + |V|^2 g(X,Y) - g(X,J(V))g(Y,J(V))\right),\end{aligned}$$

which is the second identity. This concludes the proof of Lemma 2.2. □

We can now describe more precisely the set of critical values and points of $|\psi|^2$ and $|V|^2$.

Proposition 2.3 *Under the hypotheses of* Proposition 2.1*, assume furthermore $(\widetilde{M}^{2n}, g)$ to be connected and complete. Then the following holds:*

i) The set $\{V = 0\}$ of zeros of V coincides with $\{\widetilde{\nabla}_V V = 0\}$. As a consequence, the zeros of V are the only critical points of the function $|V|^2$ on $\widetilde{M}^{2n}$.

ii) The subset $\{V = 0\}$ is a (possibly empty) connected totally geodesic Kähler submanifold of complex dimension $k < n$ in $(\widetilde{M}^{2n}, g, J)$. Furthermore, for all $x, y \in \{V = 0\}$, every geodesic segment between x and y lies in $\{V = 0\}$.

iii) The function $|\psi|^2$ has at most one critical value on $\widetilde{M}^{2n}$, which is then a global minimum of $|\psi|^2$. Furthermore, the set of critical points of $|\psi|^2$ is a connected totally geodesic Kähler submanifold in $(\widetilde{M}^{2n}, g, J)$.

Proof: The proof relies on simple computations and arguments.
i) Proposition 2.1.*v*) already implies that $\{\widetilde{\nabla}_V V = 0\}$ coincides with the set of critical points of $|V|^2$. Every zero of V is obviously a zero of $\widetilde{\nabla}_V V$, i.e., a critical point of $|V|^2$. Conversely, let $x \in \{\widetilde{\nabla}_V V = 0\}$. Then $0 = g_x(\widetilde{\nabla}_V V, V) = |p_-(V_x)\cdot\phi|^2 + |p_+(V_x)\cdot\psi|^2$, so that $p_-(V_x)\cdot\phi = 0$ and $p_+(V_x)\cdot\psi = 0$, which, in turn, implies $0 = \Im m\left(\langle p_+(V_x)\cdot\psi, \phi\rangle\right) = g(V_x, V_x)$, that is, $V_x = 0$. This shows *i*).
ii) The subset $\{V = 0\}$ - if non-empty - is the fixed-point-set in $\widetilde{M}^{2n}$ of the flow of the holomorphic Killing field $J(V)$, therefore it is a totally geodesic Kähler submanifold of $\widetilde{M}^{2n}$ (see e.g. [10, Sec. II.5]); moreover, it cannot contain any open subset of $\widetilde{M}^{2n}$ since otherwise V would identically vanish as a holomorphic vector field. To show the connectedness of $\{V = 0\}$, it suffices to prove the second part of the statement. Pick any two points x_0, x_1 in $\{V = 0\}$ (or, equivalently, any critical points of $|V|^2$) and any geodesic c in $(\widetilde{M}^{2n}, g)$ with $c(0) = x_0$ and $c(1) = x_1$. Consider the real-valued function $f(t) := |V|^2_{c(t)}$ defined on $\mathbb{R}$. Then, for any $t \in \mathbb{R}$ one has $f'(t) = g(\mathrm{grad}(|V|^2), c'(t)) = 2g(\widetilde{\nabla}_{c'(t)}V, V)$ and

$$f''(t) = \mathrm{Hess}(|V|^2)(c'(t), c'(t)).$$

Lemma 2.2 provides the Hessian of $|V|^2$: for every $X \in T\widetilde{M}$,

$$\mathrm{Hess}(|V|^2)(X,X) = 2|\widetilde{\nabla}_X V|^2 + 2\left(3g(V,X)^2 + |V|^2|X|^2 - g(X,J(V))^2\right).$$

By Cauchy-Schwarz inequality, $|V|^2|X|^2 - g(X,J(V))^2 \geq 0$, so that $\mathrm{Hess}(|V|^2)(X,X) \geq 0$ for all X, in particular f is convex. This in turn implies that, if $f'(0) = f'(1) = 0$, then necessarily f vanishes on $[0,1]$. This proves *ii*).
iii) Set, for any $t \in \mathbb{R}$, $h(t) := |\psi|^2_{c(t)}$ where c is an arbitrary geodesic on $(\widetilde{M}^{2n}, g)$. We show again that

h is convex. As before $h''(t) = \mathrm{Hess}(|\psi|^2)(c'(t), c'(t)) \geq 0$ for every $t \in \mathbb{R}$, where $\mathrm{Hess}(|\psi|^2)(X, X) = 2(|p_-(X) \cdot \phi|^2 + |p_+(X) \cdot \psi|^2) \geq 0$ for every $X \in T\widetilde{M}$ (Proposition 2.1). We already know that, if $V = \frac{1}{2}\mathrm{grad}(|\psi|^2)$ vanishes at two different points of c, then it vanishes on any geodesic segment joining the two points, therefore $|\psi|^2$ is constant on it. This proves that $|\psi|^2$ has at most one critical value. Since h is convex this critical value is necessarily a minimum. The last part of the statement is a straightforward consequence of *ii*) since $\mathrm{grad}(|\psi|^2) = 2V$ by Proposition 2.1. This shows *iii*) and concludes the proof. □

3 Doubly warped products with imaginary Kählerian Killing spinors

In this section, we describe the so-called doubly-warped products carrying non-zero imaginary Kählerian Killing spinors. Doubly warped products were introduced in the spinorial context by Patrick Baier in his master thesis [1] to compute the Dirac spectrum of the complex hyperbolic space, using its representation as a doubly-warped product over an odd-dimensional sphere.

First we recall general formulas on warped products.

Lemma 3.1 *Let $(\widetilde{M} := M \times I, \widetilde{g} := g_t \oplus \beta dt^2)$ be a warped product, where $I \subset \mathbb{R}$ is an open interval, g_t is a smooth 1-parameter family of Riemannian metrics on M and $\beta \in C^\infty(M \times I, \mathbb{R}_+^\times)$. Denote by $\widetilde{M} \xrightarrow{\pi_1} M$ the first projection. Then, for all $X, Y \in \Gamma(\pi_1^* TM)$,*

$$\begin{aligned}
\widetilde{\nabla}_{\frac{\partial}{\partial t}} \frac{\partial}{\partial t} &= -\frac{1}{2}\mathrm{grad}_{g_t}(\beta(t,\cdot)) + \frac{1}{2\beta}\frac{\partial \beta}{\partial t}\frac{\partial}{\partial t} \\
\widetilde{\nabla}_{\frac{\partial}{\partial t}} X &= \frac{\partial X}{\partial t} + \frac{1}{2} g_t^{-1}\frac{\partial g_t}{\partial t}(X,\cdot) + \frac{1}{2\beta}\frac{\partial \beta}{\partial x}(X)\frac{\partial}{\partial t} \\
\widetilde{\nabla}_X \frac{\partial}{\partial t} &= \frac{1}{2} g_t^{-1}\frac{\partial g_t}{\partial t}(X,\cdot) + \frac{1}{2\beta}\frac{\partial \beta}{\partial x}(X)\frac{\partial}{\partial t} \\
\widetilde{\nabla}_X Y &= \nabla^M_X Y - \frac{1}{2\beta}\frac{\partial g_t}{\partial t}(X,Y)\frac{\partial}{\partial t},
\end{aligned}$$

where $\frac{\partial X}{\partial t} = [\frac{\partial}{\partial t}, X]$ and ∇^M (resp. $\widetilde{\nabla}$) is the Levi-Civita covariant derivative of (M, g_t) (resp. of $(\widetilde{M}, \widetilde{g})$).

Proof: straightforward consequence of the Koszul identity. □

From now on we restrict ourselves to the following particular case: the manifold M will be equipped with a *Riemannian flow*.

Definition 3.2

i) A Riemannian flow *is a triple $(M, \widehat{g}, \widehat{\xi})$, where M is a smooth manifold and $\widehat{\xi}$ is a smooth unit vector field whose flow is isometric on the orthogonal distribution, i.e., $\widehat{g}(\widehat{\nabla}^M_Z \widehat{\xi}, Z') = -\widehat{g}(Z, \widehat{\nabla}^M_{Z'}\widehat{\xi})$ for all $Z, Z' \in \widehat{\xi}^\perp$, where $\widehat{\nabla}^M$ denotes the Levi-Civita covariant derivative of $(M, \widehat{g})$.*

ii) A Riemannian flow $(M, \widehat{g}, \widehat{\xi})$ is called minimal *if and only if $\widehat{\nabla}^M_{\widehat{\xi}}\widehat{\xi} = 0$, that is, if $\widehat{\xi}$ is actually a Killing vector field on M.*

Let $(M, \widehat{g}, \widehat{\xi})$ be a minimal Riemannian flow. Let $\widehat{h}$ denote the endomorphism-field of $\widehat{\xi}^\perp$ defined by $\widehat{h}(Z) := \widehat{\nabla}^M_Z \widehat{\xi}$ for every $Z \in \widehat{\xi}^\perp$. Let $\widehat{\nabla}$ be the covariant derivative on $\widehat{\xi}^\perp$ defined for all $Z \in \Gamma(\widehat{\xi}^\perp)$ by $\widehat{\nabla}_X Z := \begin{cases} [\widehat{\xi}, Z]^{\widehat{\xi}^\perp} & \text{if } X = \widehat{\xi} \\ (\widehat{\nabla}^M_X Z)^{\widehat{\xi}^\perp} & \text{if } X \perp \widehat{\xi} \end{cases}$. Alternatively, $\widehat{\nabla}$ can be described by the following formulas: for all $Z, Z' \in \Gamma(\widehat{\xi}^\perp)$,

$$\widehat{\nabla}^M_{\widehat{\xi}} Z = \widehat{\nabla}_{\widehat{\xi}} Z + \widehat{h}(Z) \qquad \text{and} \qquad \widehat{\nabla}^M_Z Z' = \widehat{\nabla}_Z Z' - \widehat{g}(\widehat{h}(Z), Z')\widehat{\xi}.$$

It is important to notice that, if $(M, \widehat{g}, \widehat{\xi})$ is a (minimal) Riemannian flow and $g := r^2(s^2 \widehat{g}_{\widehat{\xi}} \oplus \widehat{g}_{\widehat{\xi}^\perp})$ for some constants $r, s > 0$, then $(M, g, \xi := \frac{1}{rs}\widehat{\xi})$ is a (minimal) Riemannian flow with corresponding objects given by

$$h = \frac{s}{r}\widehat{h} \qquad \text{and} \qquad \nabla = \widehat{\nabla}. \tag{2}$$

In this language, a *Sasakian* manifold is a minimal Riemannian flow $(M,\widehat{g},\widehat{\xi})$ such that $\widehat{h}$ is a transversal Kähler structure, that is, $\widehat{h}^2 = -\mathrm{Id}_{\widehat{\xi}^\perp}$ and $\widehat{\nabla}\widehat{h} = 0$. Further on in the text we shall need for normalization purposes so-called $\mathcal{D}$-*homothetic deformations* of a Sasakian structure: a $\mathcal{D}$-homothetic deformation of $(M,\widehat{g},\widehat{\xi})$ is $(M,\lambda^2(\lambda^2\widehat{g}_{\widehat{\xi}}\oplus\widehat{g}_{\widehat{\xi}^\perp}),\frac{1}{\lambda^2}\widehat{\xi})$ for some $\lambda\in\mathbb{R}_+^\times$. The identities (2) imply that $(M,\lambda^2(\lambda^2\widehat{g}_{\widehat{\xi}}\oplus\widehat{g}_{\widehat{\xi}^\perp}),\frac{1}{\lambda^2}\widehat{\xi})$ is Sasakian as soon as $(M,\widehat{g},\widehat{\xi})$ is Sasakian.

We can now make the concept of doubly-warped product precise:

Definition 3.3 *A* doubly-warped product *is a warped product of the form*

$$(\widetilde{M},\widetilde{g}) := (M\times I,\rho(t)^2(\sigma(t)^2\widehat{g}_{\widehat{\xi}}\oplus\widehat{g}_{\widehat{\xi}^\perp})\oplus dt^2),$$

where I is an open interval, $(M,\widehat{g},\widehat{\xi})$ is a minimal Riemannian flow, $\rho,\sigma : I\longrightarrow\mathbb{R}_+^\times$ are smooth functions and $\widehat{g}_{\widehat{\xi}} := \widehat{g}_{|_{\mathbb{R}\widehat{\xi}\oplus\mathbb{R}\widehat{\xi}}}$, $\widehat{g}_{\widehat{\xi}^\perp} := \widehat{g}_{|_{\widehat{\xi}^\perp\oplus\widehat{\xi}^\perp}}$.

As for warped products, it can be easily proved that a doubly-warped product $(\widetilde{M},\widetilde{g})$ is complete as soon as $I=\mathbb{R}$ and $(M,\widehat{g})$ is complete.

It is easy to check that, setting $g_t := \rho(t)^2(\sigma(t)^2\widehat{g}_{\widehat{\xi}}\oplus\widehat{g}_{\widehat{\xi}^\perp})$, one has $\frac{\partial g_t}{\partial t} = 2\frac{\rho'}{\rho}g_t + \frac{2\sigma'}{\sigma}g_t(\pi_{\widehat{\xi}^\perp},\cdot)$ and the unit vector field providing the Riemannian flow on (M,g_t) is $\xi = \frac{1}{\rho\sigma}\widehat{\xi}$. In particular, the formulas in Lemma 3.1 simplify:

$$\begin{aligned}
\widetilde{\nabla}_{\frac{\partial}{\partial t}}\frac{\partial}{\partial t} &= 0\\
\widetilde{\nabla}_{\frac{\partial}{\partial t}}\xi &= 0\\
\widetilde{\nabla}_{\frac{\partial}{\partial t}}Z &= \frac{\partial Z}{\partial t}+\frac{\rho'}{\rho}Z\\
\widetilde{\nabla}_{\xi}\frac{\partial}{\partial t} &= \frac{(\rho\sigma)'}{\rho\sigma}\xi\\
\widetilde{\nabla}_{\xi}\xi &= -\frac{(\rho\sigma)'}{\rho\sigma}\frac{\partial}{\partial t}\\
\widetilde{\nabla}_{\xi}Z &= \nabla_\xi Z + h(Z)\\
\widetilde{\nabla}_{Z}\frac{\partial}{\partial t} &= \frac{\rho'}{\rho}Z\\
\widetilde{\nabla}_{Z}\xi &= h(Z)\\
\widetilde{\nabla}_{Z}Z' &= \nabla_Z Z' - g_t(h(Z),Z')\xi - \frac{\rho'}{\rho}g_t(Z,Z')\frac{\partial}{\partial t},
\end{aligned}$$

where we have denoted the corresponding objects on (M,g_t,ξ) without the hat "$\widehat{\ }$".

Next we look at a possible construction of Kähler structures on doubly-warped products.

Lemma 3.4 *Let $(\widetilde{M},\widetilde{g}) := (M\times I,\rho(t)^2(\sigma(t)^2\widehat{g}_{\widehat{\xi}}\oplus\widehat{g}_{\widehat{\xi}^\perp})\oplus dt^2)$ be a doubly-warped product. Assume the existence of a transversal Kähler structure J on $(M,\widehat{g},\widehat{\xi})$ and define the almost complex structure $\widetilde{J}$ on $\widetilde{M}$ by $\widetilde{J}(\xi) := \frac{\partial}{\partial t}$, $\widetilde{J}(\frac{\partial}{\partial t}) := -\xi$ and $\widetilde{J}(Z) := J(Z)$ for all $Z\in\{\xi,\frac{\partial}{\partial t}\}^\perp$. Then $(\widetilde{M}^{2n},\widetilde{g},\widetilde{J})$ is Kähler if and only if $\widehat{h} = -\frac{\rho'}{\sigma}J$ on $\{\xi,\frac{\partial}{\partial t}\}^\perp$ (in particular $\frac{\rho'}{\sigma}$ must be constant).*

Proof: Using the identities above we write down the condition $\widetilde{\nabla}\widetilde{J}=0$. Denote by h and ∇ the objects corresponding to g_t on M. Note first that, by definition and (2), one has $\nabla J = 0$ on $\{\xi,\frac{\partial}{\partial t}\}^\perp$ and $\widetilde{J}_{|_{\{\xi,\frac{\partial}{\partial t}\}^\perp}} = J$, which does not depend on t. Hence we obtain, for all $Z,Z'\in\Gamma(\widehat{\xi}^\perp)$:

$$\begin{aligned}
\widetilde{\nabla}_{\frac{\partial}{\partial t}}(\widetilde{J}(\frac{\partial}{\partial t})) - \widetilde{J}(\widetilde{\nabla}_{\frac{\partial}{\partial t}}\frac{\partial}{\partial t}) &= 0\\
\widetilde{\nabla}_{\frac{\partial}{\partial t}}(\widetilde{J}(\xi)) - \widetilde{J}(\widetilde{\nabla}_{\frac{\partial}{\partial t}}\xi) &= 0\\
\widetilde{\nabla}_{\frac{\partial}{\partial t}}(\widetilde{J}(Z)) - \widetilde{J}(\widetilde{\nabla}_{\frac{\partial}{\partial t}}Z) &= \frac{\partial(\widetilde{J}(Z))}{\partial t} - \widetilde{J}(\frac{\partial Z}{\partial t}) = \frac{\partial(J(Z))}{\partial t} - J(\frac{\partial Z}{\partial t}) = 0
\end{aligned}$$

$$
\begin{aligned}
\widetilde{\nabla}_\xi(\widetilde{J}(\frac{\partial}{\partial t})) - \widetilde{J}(\widetilde{\nabla}_\xi \frac{\partial}{\partial t}) &= 0 \\
\widetilde{\nabla}_\xi(\widetilde{J}(\xi)) - \widetilde{J}(\widetilde{\nabla}_\xi \xi) &= 0 \\
\widetilde{\nabla}_\xi(\widetilde{J}(Z)) - \widetilde{J}(\widetilde{\nabla}_\xi Z) &= h \circ J(Z) - J \circ h(Z) \\
\widetilde{\nabla}_Z(\widetilde{J}(\frac{\partial}{\partial t})) - \widetilde{J}(\widetilde{\nabla}_Z \frac{\partial}{\partial t}) &= -h(Z) - \frac{\rho'}{\rho} J(Z) \\
\widetilde{\nabla}_Z(\widetilde{J}(\xi)) - \widetilde{J}(\widetilde{\nabla}_Z \xi) &= \frac{\rho'}{\rho} Z - J \circ h(Z) \\
\widetilde{\nabla}_Z(\widetilde{J}(Z')) - \widetilde{J}(\widetilde{\nabla}_Z Z') &= -g_t(h(Z), J(Z'))\xi - \frac{\rho'}{\rho} g_t(Z, J(Z'))\frac{\partial}{\partial t} + g_t(h(Z), Z')\frac{\partial}{\partial t} - \frac{\rho'}{\rho} g_t(Z, Z')\xi.
\end{aligned}
$$

Therefore, $\widetilde{\nabla}\widetilde{J} = 0$ implies $h = -\frac{\rho'}{\rho} J$ on $\xi^\perp$ which, in turn, implies $h \circ J = J \circ h$. Moreover, (2) implies that $h = \frac{\sigma}{\rho}\widehat{h}$, which yields $\widehat{h} = -\frac{\rho'}{\sigma} J$. The reverse implication is obvious. □

Remarks 3.5

1. With the assumptions of Lemma 3.4, the function ρ' vanishes either identically or nowhere on the interval I. In the former case the vanishing of $\widehat{h}$ is equivalent to M being locally the Riemannian product of an interval with a Kähler manifold; in the latter one, we may assume, up to changing σ into $|\frac{\rho'}{\sigma}|\sigma$ (and $\widehat{g}$ into $(\frac{\sigma}{\rho'})^2 \widehat{g}_{\widehat{\xi}} \oplus \widehat{g}_{\widehat{\xi}^\perp}$), that $\widehat{h} = -\varepsilon J$ and $\rho' = \varepsilon\sigma$ with $\varepsilon \in \{\pm 1\}$.

2. Given a Kähler doubly warped product $(\widetilde{M}, \widetilde{g}, \widetilde{J})$ as in Lemma 3.4 and a real constant C, the map $(x,t) \mapsto (x, \pm t + C)$ provides a holomorphic isometry $(\widetilde{M}, \widetilde{g}, \widetilde{J}) \longrightarrow (\widetilde{M}', \widetilde{g}', \widetilde{J}')$, where $(\widetilde{M}', \widetilde{g}') := (M \times (C \pm I), g_{\pm t + C} \oplus dt^2)$ and $\widetilde{J}'$ is the corresponding complex structure (again as in Lemma 3.4). If furthermore M is spin, then this isometry preserves the corresponding spin structures. Thus, in the case where $\rho' \neq 0$, we may assume that $\varepsilon = 1$, i.e., that $\widehat{h} = -J$ and $\rho' = \sigma$.

Now we examine the correspondence of spinors. Let the underlying manifold M of some minimal Riemannian flow (M, g, ξ) be spin and, in case M is the total space of a Riemannian submersion with $\mathbb{S}^1$-fibres over a spin manifold N, let M carry the spin structure induced by that of N. Let ΣM denote the spinor bundle of (M, g) and "$\underset{M}{\cdot}$" its Clifford multiplication. Let the doubly warped product $\widetilde{M}$ carry the product spin structure (with Clifford multiplication denoted by "$\cdot$"). Then the transversal covariant derivative ∇ induces a covariant derivative - also denoted by ∇ - on ΣM, which is related to the spinorial Levi-Civita covariant derivative ∇^M on ΣM via (see e.g. [7, eq. (2.4.7)] or [8, Sec. 4])

$$
\nabla^M_\xi \varphi = \nabla_\xi \varphi + \frac{1}{4} \sum_{j=1}^{2n-2} e_j \underset{M}{\cdot} h(e_j) \underset{M}{\cdot} \varphi \qquad \text{and} \qquad \nabla^M_Z \varphi = \nabla_Z \varphi + \frac{1}{2} \xi \underset{M}{\cdot} h(Z) \underset{M}{\cdot} \varphi
$$

for every $\varphi \in \Gamma(\Sigma M)$, where $\{e_j\}_{1 \leq j \leq 2n-2}$ is a local orthonormal basis of $\xi^\perp \subset TM$.

Lemma 3.6 *Let a minimal Riemannian flow $(M, \widehat{g}, \widehat{\xi})$ carry a transversal Kähler structure J such that the doubly-warped product $(\widetilde{M}, \widetilde{g}, \widetilde{J})$ is Kähler, where $\widetilde{J}$ is the almost-complex structure induced by J as in* Lemma 3.4. *Assume furthermore M to be spin. Let $\widetilde{M}$ carry the induced spin structure. Then the following identities hold for all $\varphi \in \Gamma(\Sigma\widetilde{M})$ and $Z \in \{\xi, \frac{\partial}{\partial t}\}^\perp$:*

$$
\begin{aligned}
\widetilde{\nabla}_{\frac{\partial}{\partial t}} \varphi &= \frac{\partial \varphi}{\partial t} \\
\widetilde{\nabla}_\xi \varphi &= \nabla_\xi \varphi - \frac{\rho'}{2\rho} \widetilde{\Omega} \cdot \varphi - \frac{\sigma'}{2\sigma} \xi \cdot \frac{\partial}{\partial t} \cdot \varphi \\
\widetilde{\nabla}_Z \varphi &= \nabla_Z \varphi - \frac{\rho'}{2\rho} (\xi \cdot J(Z) + Z \cdot \frac{\partial}{\partial t}) \cdot \varphi,
\end{aligned}
$$

where $\widetilde{\Omega}$ denotes the Kähler form of $(\widetilde{M}, \widetilde{g}, \widetilde{J})$.

Proof: Let $(e_1,\dots,e_{2n-2},e_{2n-1}:=\xi,e_{2n}:=\frac{\partial}{\partial t})$ be a local positively-oriented orthonormal basis of $T\widetilde{M}$ and $(\psi_\alpha)_\alpha$ the corresponding spinorial frame. It can be assumed that $e_j=\rho^{-1}\widehat{e_j}$ with $\widehat{g}(\widehat{e_j},\widehat{e_k})=\delta_{jk}$ and $\frac{\partial\widehat{e_j}}{\partial t}=0$ (extend some $\widehat{g}$-orthonormal basis independently of time). Split $\varphi=\sum_\alpha c_\alpha\psi_\alpha$, then

$$\begin{aligned}
\widetilde{\nabla}_{\frac{\partial}{\partial t}}\varphi &= \frac{1}{4}\sum_\alpha c_\alpha\sum_{j,k=1}^{2n}\widetilde{g}(\widetilde{\nabla}_{\frac{\partial}{\partial t}}e_j,e_k)e_j\cdot e_k\cdot\psi_\alpha+\underbrace{\sum_\alpha\frac{\partial c_\alpha}{\partial t}\psi_\alpha}_{=:\frac{\partial\varphi}{\partial t}}\\
&= \frac{\partial\varphi}{\partial t}+\frac{1}{4}\sum_\alpha c_\alpha\sum_{j,k=1}^{2n-2}\widetilde{g}(\widetilde{\nabla}_{\frac{\partial}{\partial t}}e_j,e_k)e_j\cdot e_k\cdot\psi_\alpha\\
&= \frac{\partial\varphi}{\partial t}+\frac{1}{4}\sum_\alpha c_\alpha\sum_{j,k=1}^{2n-2}\{g_t(\frac{\partial e_j}{\partial t},e_k)+\frac{\rho'}{\rho}\delta_{jk}\}e_j\cdot e_k\cdot\psi_\alpha\\
&= \frac{\partial\varphi}{\partial t},
\end{aligned}$$

where we have used $\widetilde{\nabla}_{\frac{\partial}{\partial t}}\frac{\partial}{\partial t}=\widetilde{\nabla}_{\frac{\partial}{\partial t}}\xi=0$ and $\frac{\partial e_j}{\partial t}=-\frac{\rho'}{\rho}e_j$ by the above choice of e_j. On the other hand, the Weingarten endomorphism field of (M,g_t) in $\widetilde{M}$ is given by $A(\xi):=-\widetilde{\nabla}_\xi\frac{\partial}{\partial t}=-\frac{(\rho\sigma)'}{\rho\sigma}\xi$ and $A(Z):=-\widetilde{\nabla}_Z\frac{\partial}{\partial t}=-\frac{\rho'}{\rho}Z$ for all $Z\in\{\xi,\frac{\partial}{\partial t}\}^\perp$, so that the Gauss-Weingarten formula implies

$$\begin{aligned}
\widetilde{\nabla}_\xi\varphi &= \nabla^M_\xi\varphi+\frac{1}{2}A(\xi)\cdot\frac{\partial}{\partial t}\cdot\varphi\\
&= \nabla_\xi\varphi+\frac{1}{4}\sum_{j=1}^{2n-2}e_j\underset{M}{\cdot}h(e_j)\underset{M}{\cdot}\varphi-\frac{(\rho\sigma)'}{2\rho\sigma}\xi\cdot\frac{\partial}{\partial t}\cdot\varphi\\
&= \nabla_\xi\varphi-\frac{\rho'}{4\rho}\sum_{j=1}^{2n-2}e_j\cdot J(e_j)\cdot\varphi-\frac{(\rho\sigma)'}{2\rho\sigma}\xi\cdot\frac{\partial}{\partial t}\cdot\varphi\\
&= \nabla_\xi\varphi-\frac{\rho'}{2\rho}\Omega\cdot\varphi-\frac{(\rho\sigma)'}{2\rho\sigma}\xi\cdot\frac{\partial}{\partial t}\cdot\varphi,
\end{aligned}$$

where Ω is the 2-form associated to J on $\{\xi,\frac{\partial}{\partial t}\}^\perp$, i.e., $\Omega(Z,Z')=g_t(J(Z),Z')$ for all $Z,Z'\in\{\xi,\frac{\partial}{\partial t}\}^\perp$. Since $\widetilde{\Omega}=\Omega+\xi\wedge\frac{\partial}{\partial t}$, we deduce that

$$\begin{aligned}
\widetilde{\nabla}_\xi\varphi &= \nabla_\xi\varphi-\frac{\rho'}{2\rho}\widetilde{\Omega}\cdot\varphi+(\frac{\rho'}{2\rho}-\frac{(\rho\sigma)'}{2\rho\sigma})\xi\cdot\frac{\partial}{\partial t}\cdot\varphi\\
&= \nabla_\xi\varphi-\frac{\rho'}{2\rho}\widetilde{\Omega}\cdot\varphi-\frac{\sigma'}{2\sigma}\xi\cdot\frac{\partial}{\partial t}\cdot\varphi.
\end{aligned}$$

For any $Z\in\{\xi,\frac{\partial}{\partial t}\}^\perp$, one has

$$\begin{aligned}
\widetilde{\nabla}_Z\varphi &= \nabla^M_Z\varphi+\frac{1}{2}A(Z)\cdot\frac{\partial}{\partial t}\cdot\varphi\\
&= \nabla_Z\varphi+\frac{1}{2}\xi\underset{M}{\cdot}h(Z)\underset{M}{\cdot}\varphi-\frac{\rho'}{2\rho}Z\cdot\frac{\partial}{\partial t}\cdot\varphi\\
&= \nabla_Z\varphi-\frac{\rho'}{2\rho}\xi\cdot J(Z)\cdot\varphi-\frac{\rho'}{2\rho}Z\cdot\frac{\partial}{\partial t}\cdot\varphi,
\end{aligned}$$

which shows the last identity and concludes the proof. □

Later on we shall need to split spinors into different components. Recall that, on any Kähler spin manifold $(\widetilde{M}^{2n},\widetilde{g},\widetilde{J})$, the spinor bundle $\Sigma\widetilde{M}$ of $(\widetilde{M}^{2n},\widetilde{g})$ splits under the Clifford action of the Kähler form $\widetilde{\Omega}$ into

$$\Sigma\widetilde{M}=\bigoplus_{r=0}^{n}\Sigma_r\widetilde{M},$$

where $\Sigma_r\widetilde{M}:=\mathrm{Ker}(\widetilde{\Omega}\cdot-i(2r-n)\mathrm{Id})$. Now if $(\widetilde{M}^{2n},\widetilde{g},\widetilde{J})$ is a doubly-warped product as above, then any $\varphi\in\Sigma_r\widetilde{M}$ (with $r\in\{0,1,\dots,n\}$) can be further split into eigenvectors for the Clifford action of

$\Omega = g(J\cdot,\cdot)$. Namely, since $[\xi \wedge \frac{\partial}{\partial t}, \Omega] = 0$, the automorphism $\xi \cdot \frac{\partial}{\partial t}$ of $\Sigma\widetilde{M}$ leaves $\Sigma_r\widetilde{M}$ invariant; from $(\xi \cdot \frac{\partial}{\partial t})^2 = -1$ one deduces the orthogonal decomposition $\Sigma_r\widetilde{M} = \mathrm{Ker}(\xi \cdot \frac{\partial}{\partial t} + i\mathrm{Id}) \oplus \mathrm{Ker}(\xi \cdot \frac{\partial}{\partial t} - i\mathrm{Id})$. Since both Clifford actions of ξ and $\frac{\partial}{\partial t}$ are ∇-parallel, so is the latter splitting. But, for any $\varphi \in \Sigma_r\widetilde{M}$, one has

$$\begin{aligned} \varphi \in \mathrm{Ker}(\xi \cdot \frac{\partial}{\partial t} \pm i\mathrm{Id}) &\iff \Omega \cdot \varphi = i(2r-n)\varphi \pm i\varphi \\ &\iff \Omega \cdot \varphi = i(2r-n\pm 1)\varphi, \end{aligned}$$

that is, $\Sigma_r\widetilde{M} \cap \mathrm{Ker}(\xi \cdot \frac{\partial}{\partial t} + i\mathrm{Id}) = \Sigma_r M$ and $\Sigma_r\widetilde{M} \cap \mathrm{Ker}(\xi \cdot \frac{\partial}{\partial t} - i\mathrm{Id}) = \Sigma_{r-1}M$, where by definition $\Sigma_r M := \mathrm{Ker}(\Omega\cdot - i(2r-(n-1)\mathrm{Id}))$ for $r \in \{0,1,\ldots,n-1\}$ and $\{0\}$ otherwise. Out of dimensional reasons one actually has

$$\Sigma_r\widetilde{M} = \Sigma_r M \oplus \Sigma_{r-1}M \tag{3}$$

for every $r \in \{0,1,\ldots,n\}$. Beware here that, if r is even, then $\Sigma_r\widetilde{M}$ is a subspace of $\Sigma^+\widetilde{M}$ hence $\Sigma_r\widetilde{M}_{|_M}$ is canonically identified with a subspace of $\Sigma^+\widetilde{M}_{|_M} = \Sigma M$, whereas if r is odd then it is a subspace of $\Sigma_-\widetilde{M}$ and is also identified as a subspace of ΣM, but this time with opposite Clifford multiplication.

Lemma 3.7 *Under the hypotheses of* Lemma 3.6*, let $\varphi \in \Gamma(\Sigma_r\widetilde{M})$ for some $r \in \{0,1\ldots,n\}$ and consider its decomposition $\varphi = \varphi_r + \varphi_{r-1}$ w.r.t.* (3)*. Then the identities of* Lemma 3.6 *read:*

$$\begin{aligned} \widetilde{\nabla}_{\frac{\partial}{\partial t}}\varphi_r &= \frac{\partial \varphi_r}{\partial t} \\ \widetilde{\nabla}_{\frac{\partial}{\partial t}}\varphi_{r-1} &= \frac{\partial \varphi_{r-1}}{\partial t} \\ \widetilde{\nabla}_\xi\varphi_r &= \nabla_\xi\varphi_r + \frac{i}{2}((n-2r)\frac{\rho'}{\rho} + \frac{\sigma'}{\sigma})\varphi_r \\ \widetilde{\nabla}_\xi\varphi_{r-1} &= \nabla_\xi\varphi_{r-1} + \frac{i}{2}((n-2r)\frac{\rho'}{\rho} - \frac{\sigma'}{\sigma})\varphi_{r-1} \\ \widetilde{\nabla}_Z\varphi &= \nabla_Z\varphi_r - \frac{\rho'}{\rho}p_+(Z)\cdot\frac{\partial}{\partial t}\cdot\varphi_{r-1} + \nabla_Z\varphi_{r-1} - \frac{\rho'}{\rho}p_-(Z)\cdot\frac{\partial}{\partial t}\cdot\varphi_r \end{aligned}$$

for all $Z \in \{\xi, \frac{\partial}{\partial t}\}^\perp$, where, as usual, $p_\pm(Z) = \frac{1}{2}(Z \mp iJ(Z))$.

Proof: The first two identities follow from $\widetilde{\nabla}_{\frac{\partial}{\partial t}}(\xi \wedge \frac{\partial}{\partial t}) = 0$ and $\frac{\partial J}{\partial t} = 0$. For the third and fourth ones, note that $\widetilde{\nabla}_\xi(\xi \wedge \frac{\partial}{\partial t}) = 0$, so that

$$\begin{aligned} \widetilde{\nabla}_\xi\varphi_r + \widetilde{\nabla}_\xi\varphi_{r-1} &= \nabla_\xi\varphi_r + \nabla_\xi\varphi_{r-1} - \frac{i\rho'}{2\rho}(2r-n)(\varphi_r+\varphi_{r-1}) - \frac{i\sigma'}{2\sigma}(\varphi_{r-1}-\varphi_r) \\ &= \nabla_\xi\varphi_r + \frac{i}{2}((n-2r)\frac{\rho'}{\rho} + \frac{\sigma'}{\sigma})\varphi_r + \nabla_\xi\varphi_{r-1} + \frac{i}{2}((n-2r)\frac{\rho'}{\rho} - \frac{\sigma'}{\sigma})\varphi_{r-1}, \end{aligned}$$

which is the result. As for the last identity, one does not have $\widetilde{\nabla}_Z(\xi \wedge \frac{\partial}{\partial t}) = 0$, however

$$\begin{aligned} (\xi\cdot J(Z) + Z\cdot\frac{\partial}{\partial t})\cdot\varphi &= (-J(Z)\cdot\frac{\partial}{\partial t}\cdot\xi\cdot\frac{\partial}{\partial t} + Z\cdot\frac{\partial}{\partial t})\cdot\varphi \\ &= -iJ(Z)\cdot\frac{\partial}{\partial t}\cdot(\varphi_{r-1}-\varphi_r) + Z\cdot\frac{\partial}{\partial t}\cdot(\varphi_r+\varphi_{r-1}) \\ &= 2p_+(Z)\cdot\frac{\partial}{\partial t}\cdot\varphi_{r-1} + 2p_-(Z)\cdot\frac{\partial}{\partial t}\cdot\varphi_r \end{aligned}$$

for all $Z \in \{\xi, \frac{\partial}{\partial t}\}^\perp$. This concludes the proof. □

We now have all we need to rewrite the imaginary Kähler Killing spinor equation on doubly warped products.

Lemma 3.8 *Let a spin minimal Riemannian flow $(M^{2n-1}, \widehat{g}, \widehat{\xi})$ carry a transversal Kähler structure J such that the doubly-warped product $(\widetilde{M}, \widetilde{g}, \widetilde{J})$ is Kähler, where $\widetilde{J}$ is the almost-complex structure induced by J as in* Lemma 3.4*. Let $\widetilde{M}$ carry the induced spin structure and assume $n \geq 3$ to be odd. Then a pair*

(ψ,ϕ) is an i-Kählerian Killing spinor on $(\widetilde{M}^{2n},\widetilde{g},\widetilde{J})$ if and only if the following identities are satisfied by the components $\phi=\phi_{\frac{n+1}{2}}+\phi_{\frac{n-1}{2}}$ and $\psi=\psi_{\frac{n-1}{2}}+\psi_{\frac{n-3}{2}}$ w.r.t. (3):

$$\left|\begin{array}{ll}
\frac{\partial\phi_{\frac{n+1}{2}}}{\partial t} & =0\\
\frac{\partial\phi_{\frac{n-1}{2}}}{\partial t} & =-i\frac{\partial}{\partial t}\cdot\psi_{\frac{n-1}{2}}\\
\frac{\partial\psi_{\frac{n-1}{2}}}{\partial t} & =-i\frac{\partial}{\partial t}\cdot\phi_{\frac{n-1}{2}}\\
\frac{\partial\psi_{\frac{n-3}{2}}}{\partial t} & =0\\
\nabla_\xi\phi_{\frac{n+1}{2}} & =\frac{i}{2}(\frac{\rho'}{\rho}-\frac{\sigma'}{\sigma})\phi_{\frac{n+1}{2}}\\
\nabla_\xi\phi_{\frac{n-1}{2}} & =\frac{i}{2}(\frac{\rho'}{\rho}+\frac{\sigma'}{\sigma})\phi_{\frac{n-1}{2}}-\frac{\partial}{\partial t}\cdot\psi_{\frac{n-1}{2}}\\
\nabla_\xi\psi_{\frac{n-1}{2}} & =-\frac{i}{2}(\frac{\rho'}{\rho}+\frac{\sigma'}{\sigma})\psi_{\frac{n-1}{2}}+\frac{\partial}{\partial t}\cdot\phi_{\frac{n-1}{2}}\\
\nabla_\xi\psi_{\frac{n-3}{2}} & =-\frac{i}{2}(\frac{\rho'}{\rho}-\frac{\sigma'}{\sigma})\psi_{\frac{n-3}{2}}\\
\nabla_Z\phi_{\frac{n+1}{2}} & =p_+(Z)\cdot(\frac{\rho'}{\rho}\frac{\partial}{\partial t}\cdot\phi_{\frac{n-1}{2}}-i\psi_{\frac{n-1}{2}})\\
\nabla_Z\phi_{\frac{n-1}{2}} & =\frac{\rho'}{\rho}p_-(Z)\cdot\frac{\partial}{\partial t}\cdot\phi_{\frac{n+1}{2}}-ip_+(Z)\cdot\psi_{\frac{n-3}{2}}\\
\nabla_Z\psi_{\frac{n-1}{2}} & =\frac{\rho'}{\rho}p_+(Z)\cdot\frac{\partial}{\partial t}\cdot\psi_{\frac{n-3}{2}}-ip_-(Z)\cdot\phi_{\frac{n+1}{2}}\\
\nabla_Z\psi_{\frac{n-3}{2}} & =p_-(Z)\cdot(\frac{\rho'}{\rho}\frac{\partial}{\partial t}\cdot\psi_{\frac{n-1}{2}}-i\phi_{\frac{n-1}{2}})
\end{array}\right| \tag{4}$$

for every $Z\in\{\xi,\frac{\partial}{\partial t}\}^\perp$.

Proof: Since $p_+(\frac{\partial}{\partial t})\cdot\psi=\frac{1}{2}(\frac{\partial}{\partial t}+i\xi)\cdot\psi=\frac{1}{2}\frac{\partial}{\partial t}\cdot(1+i\xi\cdot\frac{\partial}{\partial t}\cdot)\psi=\frac{\partial}{\partial t}\cdot\psi_{\frac{n-1}{2}}$ and similarly $p_-(\frac{\partial}{\partial t})\cdot\phi=\frac{\partial}{\partial t}\cdot\phi_{\frac{n-1}{2}}$, the i-Kählerian Killing spinor equation is satisfied by (ψ,ϕ) for $X=\frac{\partial}{\partial t}$ if and only if

$$\begin{aligned}
\frac{\partial\phi_{\frac{n+1}{2}}}{\partial t}+\frac{\partial\phi_{\frac{n-1}{2}}}{\partial t} &= -ip_+(\frac{\partial}{\partial t})\cdot\psi=-i\frac{\partial}{\partial t}\cdot\psi_{\frac{n-1}{2}}\\
\frac{\partial\psi_{\frac{n-1}{2}}}{\partial t}+\frac{\partial\psi_{\frac{n-3}{2}}}{\partial t} &= -ip_-(\frac{\partial}{\partial t})\cdot\phi=-i\frac{\partial}{\partial t}\cdot\phi_{\frac{n-1}{2}},
\end{aligned}$$

which gives the first four identities (use $[\Omega,\frac{\partial}{\partial t}]=0$).
From $p_+(\xi)\cdot\psi=-ip_+(\frac{\partial}{\partial t})\cdot\psi=-i\frac{\partial}{\partial t}\cdot\psi_{\frac{n-1}{2}}$ and $p_-(\xi)\cdot\phi=ip_-(\frac{\partial}{\partial t})\cdot\phi=i\frac{\partial}{\partial t}\cdot\phi_{\frac{n-1}{2}}$ we deduce that the i-Kählerian Killing spinor equation is satisfied by (ψ,ϕ) for $X=\xi$ if and only if

$$\begin{aligned}
\nabla_\xi\phi_{\frac{n+1}{2}}+\frac{i}{2}(-\frac{\rho'}{\rho}+\frac{\sigma'}{\sigma})\phi_{\frac{n+1}{2}} &= 0\\
\nabla_\xi\phi_{\frac{n-1}{2}}-\frac{i}{2}(\frac{\rho'}{\rho}+\frac{\sigma'}{\sigma})\phi_{\frac{n-1}{2}} &= -\frac{\partial}{\partial t}\cdot\psi_{\frac{n-1}{2}}\\
\nabla_\xi\psi_{\frac{n-1}{2}}+\frac{i}{2}(\frac{\rho'}{\rho}+\frac{\sigma'}{\sigma})\psi_{\frac{n-1}{2}} &= \frac{\partial}{\partial t}\cdot\phi_{\frac{n-1}{2}}\\
\nabla_\xi\psi_{\frac{n-3}{2}}+\frac{i}{2}(\frac{\rho'}{\rho}-\frac{\sigma'}{\sigma})\psi_{\frac{n-3}{2}} &= 0,
\end{aligned}$$

which implies the next four equations.
Let $Z\in\{\xi,\frac{\partial}{\partial t}\}^\perp$, then the i-Kählerian Killing spinor equation is satisfied by (ψ,ϕ) for $X=Z$ if and only if

$$\begin{aligned}
-ip_+(Z)\cdot\psi_{\frac{n-1}{2}} &= \nabla_Z\phi_{\frac{n+1}{2}}-\frac{\rho'}{\rho}p_+(Z)\cdot\frac{\partial}{\partial t}\cdot\phi_{\frac{n-1}{2}}\\
-ip_+(Z)\cdot\psi_{\frac{n-3}{2}} &= \nabla_Z\phi_{\frac{n-1}{2}}-\frac{\rho'}{\rho}p_-(Z)\cdot\frac{\partial}{\partial t}\cdot\phi_{\frac{n+1}{2}}\\
-ip_-(Z)\cdot\phi_{\frac{n+1}{2}} &= \nabla_Z\psi_{\frac{n-1}{2}}-\frac{\rho'}{\rho}p_+(Z)\cdot\frac{\partial}{\partial t}\cdot\psi_{\frac{n-3}{2}}\\
-ip_-(Z)\cdot\phi_{\frac{n-1}{2}} &= \nabla_Z\psi_{\frac{n-3}{2}}-\frac{\rho'}{\rho}p_-(Z)\cdot\frac{\partial}{\partial t}\cdot\phi_{\frac{n-1}{2}},
\end{aligned}$$

which concludes the proof. □

Next we want to describe all doubly warped products with non-zero imaginary Kählerian Killing spinors.

Theorem 3.9 *For $n \geq 3$ odd let $(\widetilde{M}^{2n}, \widetilde{g}, \widetilde{J})$ be a Kähler spin doubly warped product as in* Lemma 3.8. *If there exists a non-zero i-Kählerian Killing spinor (ψ, ϕ) on $(\widetilde{M}^{2n}, \widetilde{g}, \widetilde{J})$, then*

- *the minimal Riemannian flow $(M^{2n-1}, \widehat{g}, \widehat{\xi})$ is Sasakian,*
- *up to changing t into $-t$, applying a $\mathcal{D}$-homothety and translating the interval I by a constant, one has either $\rho = e^t$ or $\rho = \sinh$ or $\rho = \cosh$,*
- *the components ψ_r and ϕ_r of (ψ, ϕ) w.r.t.* (3) *satisfy:*

i) In case $\rho = e^t$: Then $\sigma = e^t$ and, setting $\widetilde{\psi}_{\frac{n-3}{2}} := i\frac{\partial}{\partial t} \cdot \psi_{\frac{n-3}{2}}$ and $\varphi_{\frac{n-1}{2}} := e^t(\phi_{\frac{n-1}{2}} + i\frac{\partial}{\partial t} \cdot \psi_{\frac{n-1}{2}})$, one has

$$\left|\begin{array}{ll} \frac{\partial}{\partial t}\phi_{\frac{n+1}{2}} & = 0 \\ \frac{\partial}{\partial t}\widetilde{\psi}_{\frac{n-3}{2}} & = 0 \\ \frac{\partial}{\partial t}\varphi_{\frac{n-1}{2}} & = 0 \\ \widehat{\nabla}_{\widehat{\xi}}\phi_{\frac{n+1}{2}} & = 0 \\ \widehat{\nabla}_{\widehat{\xi}}\widetilde{\psi}_{\frac{n-3}{2}} & = 0 \\ \widehat{\nabla}\varphi_{\frac{n-1}{2}} & = 0 \\ \widehat{\nabla}_Z\phi_{\frac{n+1}{2}} & = (-1)^{\frac{n+1}{2}} p_+(Z)\widehat{\cdot}_M\varphi_{\frac{n-1}{2}} \\ \widehat{\nabla}_Z\widetilde{\psi}_{\frac{n-3}{2}} & = (-1)^{\frac{n+1}{2}} p_-(Z)\widehat{\cdot}_M\varphi_{\frac{n-1}{2}}. \end{array}\right.$$

If furthermore $\varphi_{\frac{n-1}{2}} = 0$, then for $\widehat{\phi}_{\frac{n-1}{2}} := e^{-t}\phi_{\frac{n-1}{2}}$ one has $\frac{\partial}{\partial t}\widehat{\phi}_{\frac{n-1}{2}} = 0$ and

$$\left|\begin{array}{ll} \widehat{\nabla}\phi_{\frac{n+1}{2}} & = 0 \\ \widehat{\nabla}\widetilde{\psi}_{\frac{n-3}{2}} & = 0 \\ \widehat{\nabla}_{\widehat{\xi}}\widehat{\phi}_{\frac{n-1}{2}} & = 0 \\ \widehat{\nabla}_Z\widehat{\phi}_{\frac{n-1}{2}} & = (-1)^{\frac{n+1}{2}}(p_-(Z)\widehat{\cdot}_M\phi_{\frac{n+1}{2}} + p_+(Z)\widehat{\cdot}_M\widetilde{\psi}_{\frac{n-3}{2}}). \end{array}\right.$$

In particular, the manifold $(M^{2n-1}, \widehat{g}, \widehat{\xi})$ admits a non-zero transversally parallel spinor. Conversely, every non-zero transversally parallel spinor $\widehat{\phi}_{\frac{n-1}{2}} \in \Gamma(\Sigma_{\frac{n-1}{2}}M)$ provides a non-zero i-Kählerian Killing spinor by setting $\phi_{\frac{n+1}{2}} := \psi_{\frac{n-3}{2}} := 0$ and $\phi_{\frac{n-1}{2}} := e^t\widehat{\phi}_{\frac{n-1}{2}}$, $\psi_{\frac{n-1}{2}} := -e^t i\frac{\partial}{\partial t}\cdot\widehat{\phi}_{\frac{n-1}{2}}$. Moreover, for any i-Kählerian Killing spinor (ψ, ϕ) on that doubly warped product $(\widetilde{M}^{2n}, \widetilde{g}, \widetilde{J})$, the component $\phi_{\frac{n-1}{2}}$ is transversally parallel on $(M, \widehat{g}, \widehat{\xi})$ if and only if $i\frac{\partial}{\partial t}\cdot\psi = -\phi$.

ii) In case $\rho = \sinh$: One has $\sigma = \cosh$ on $I = \mathbb{R}_+^\times$ and there is a one-to-one correspondence between the space of i-Kählerian Killing spinors on $(\widetilde{M}^{2n}, \widetilde{g}, \widetilde{J})$ and that of sections $(\varphi_{\frac{n+1}{2}}, \varphi_{\frac{n-1}{2}}, \widetilde{\varphi}_{\frac{n-1}{2}}, \widetilde{\varphi}_{\frac{n-3}{2}})$ of $\Sigma_{\frac{n+1}{2}}M \oplus \Sigma_{\frac{n-1}{2}}M \oplus \Sigma_{\frac{n-1}{2}}M \oplus \Sigma_{\frac{n-3}{2}}M \longrightarrow M$ satisfying

$$\left|\begin{array}{lll} \widehat{\nabla}_{\widehat{\xi}}\overset{(\sim)}{\varphi}_r & = & \frac{(-1)^r}{2}(n-2r)\widehat{\xi}\widehat{\cdot}_M\overset{(\sim)}{\varphi}_r \\ \widehat{\nabla}_{\widehat{\xi}}\overset{(\sim)}{\varphi}_{r-1} & = & \frac{-(-1)^r}{2}(n-2r)\widehat{\xi}\widehat{\cdot}_M\overset{(\sim)}{\varphi}_{r-1} \\ \widehat{\nabla}_Z\overset{(\sim)}{\varphi}_r & = & (-1)^r p_+(Z)\widehat{\cdot}_M\overset{(\sim)}{\varphi}_{r-1} \\ \widehat{\nabla}_Z\overset{(\sim)}{\varphi}_{r-1} & = & (-1)^r p_-(Z)\widehat{\cdot}_M\overset{(\sim)}{\varphi}_r \end{array}\right. \tag{5}$$

on $(M^{2n-1}, \widehat{g}, \widehat{\xi})$, for every $Z \in \widehat{\xi}^\perp$ (this means that $(\varphi_{\frac{n+1}{2}}, \varphi_{\frac{n-1}{2}})$ must satisfy (5) *for $r = \frac{n+1}{2}$ and $(\widetilde{\varphi}_{\frac{n-1}{2}}, \widetilde{\varphi}_{\frac{n-3}{2}})$ must satisfy* (5) *for $r = \frac{n-1}{2}$).*

iii) In case $\rho = \cosh$: One has $\sigma = \sinh$ on $I = \mathbb{R}_+^\times$ and there is a one-to-one correspondence between the space of i-Kählerian Killing spinors on $(\widetilde{M}^{2n}, \widetilde{g}, \widetilde{J})$ and that of sections $(\varphi_{\frac{n+1}{2}}, \varphi_{\frac{n-1}{2}}, \widetilde{\varphi}_{\frac{n-1}{2}}, \widetilde{\varphi}_{\frac{n-3}{2}})$

of $\Sigma_{\frac{n+1}{2}}M \oplus \Sigma_{\frac{n-1}{2}}M \oplus \Sigma_{\frac{n-1}{2}}M \oplus \Sigma_{\frac{n-3}{2}}M \longrightarrow M$ *satisfying*

$$\left|\begin{array}{lcl} \widehat{\nabla}_{\widehat{\xi}} \overset{(\sim)}{\varphi}_r & = & -\frac{(-1)^r}{2}(n-2r)\widehat{\xi}\,\widehat{\cdot}_M \overset{(\sim)}{\varphi}_r \\ \widehat{\nabla}_{\widehat{\xi}} \overset{(\sim)}{\varphi}_{r-1} & = & \frac{(-1)^r}{2}(n-2r)\widehat{\xi}\,\widehat{\cdot}_M \overset{(\sim)}{\varphi}_{r-1} \\ \widehat{\nabla}_Z \overset{(\sim)}{\varphi}_r & = & (-1)^{\frac{n+1}{2}}p_+(Z)\,\widehat{\cdot}_M \overset{(\sim)}{\varphi}_{r-1} \\ \widehat{\nabla}_Z \overset{(\sim)}{\varphi}_{r-1} & = & (-1)^{\frac{n-1}{2}}p_-(Z)\,\widehat{\cdot}_M \overset{(\sim)}{\varphi}_r \end{array}\right. \tag{6}$$

on $(M^{2n-1},\widehat{g},\widehat{\xi})$*, for every* $Z \in \widehat{\xi}^{\perp}$ *(this means that* $(\varphi_{\frac{n+1}{2}},\varphi_{\frac{n-1}{2}})$ *must satisfy* (6) *for* $r=\frac{n+1}{2}$ *and* $(\widetilde{\varphi}_{\frac{n-1}{2}},\widetilde{\varphi}_{\frac{n-3}{2}})$ *must satisfy* (6) *for* $r=\frac{n-1}{2}$*).*

Proof: We first show $\rho''=\rho$ on I. In order to express all equations of (4) in an intrinsic way, we have to compare all objects on (M,g_t,ξ) with the corresponding ones on $(M,\widehat{g},\widehat{\xi})$. Recall that $g_t=\rho(t)^2(\sigma(t)^2\widehat{g}_{\widehat{\xi}}\oplus\widehat{g}_{\widehat{\xi}^{\perp}})$ and $\xi=\frac{1}{\rho\sigma}\widehat{\xi}$. As for (2), it is elementary to check the following relations:

$$\widehat{\nabla}=\nabla,\quad \xi\cdot=\widehat{\xi}\,\widehat{\cdot},\quad \xi\underset{M}{\cdot}=\widehat{\xi}\,\widehat{\underset{M}{\cdot}},\quad Z\cdot=\rho Z\,\widehat{\cdot},\quad Z\underset{M}{\cdot}=\rho Z\,\widehat{\underset{M}{\cdot}},$$

for all $Z\in\xi^{\perp}$. Applying $\frac{\partial}{\partial t}$ onto

$$\left|\begin{array}{ll} \widehat{\nabla}_Z\phi_{\frac{n+1}{2}} & = p_+(Z)\,\widehat{\cdot}\,(\rho'\frac{\partial}{\partial t}\cdot\phi_{\frac{n-1}{2}}-i\rho\psi_{\frac{n-1}{2}}) \\ \widehat{\nabla}_Z\psi_{\frac{n-3}{2}} & = p_-(Z)\,\widehat{\cdot}\,(\rho'\frac{\partial}{\partial t}\cdot\psi_{\frac{n-1}{2}}-i\rho\phi_{\frac{n-1}{2}}) \end{array}\right.$$

and using $\frac{\partial\phi_{\frac{n+1}{2}}}{\partial t}=\frac{\partial\psi_{\frac{n-3}{2}}}{\partial t}=0$, one obtains

$$\begin{aligned} 0 &= p_+(Z)\,\widehat{\cdot}\,(\rho''\frac{\partial}{\partial t}\cdot\phi_{\frac{n-1}{2}}+\rho'\frac{\partial}{\partial t}\cdot\frac{\partial\phi_{\frac{n-1}{2}}}{\partial t}-i\rho'\psi_{\frac{n-1}{2}}-i\rho\frac{\partial\psi_{\frac{n-1}{2}}}{\partial t}) \\ &= p_+(Z)\,\widehat{\cdot}\,(\rho''\frac{\partial}{\partial t}\cdot\phi_{\frac{n-1}{2}}+\rho'\frac{\partial}{\partial t}\cdot(-i\frac{\partial}{\partial t}\cdot\psi_{\frac{n-1}{2}})-i\rho'\psi_{\frac{n-1}{2}}-i\rho(-i\frac{\partial}{\partial t}\cdot\phi_{\frac{n-1}{2}})) \\ &= (\rho''-\rho)p_+(Z)\,\widehat{\cdot}\,\frac{\partial}{\partial t}\cdot\phi_{\frac{n-1}{2}} \end{aligned}$$

and analogously $(\rho''-\rho)p_-(Z)\,\widehat{\cdot}\,\frac{\partial}{\partial t}\cdot\psi_{\frac{n-1}{2}}=0$ for all $Z\in\widehat{\xi}^{\perp}$. Fix a local $\widehat{g}$-orthonormal basis $(e_j)_{1\le j\le 2n-2}$ of $\widehat{\xi}^{\perp}$. Putting $Z=e_j$, Clifford-multiplying by e_j and summing over j gives $(\rho''-\rho)\phi_{\frac{n-1}{2}}=(\rho''-\rho)\psi_{\frac{n-1}{2}}=0$. On the other hand, both equations involving $\frac{\partial\phi_{\frac{n-1}{2}}}{\partial t}$ and $\frac{\partial\psi_{\frac{n-1}{2}}}{\partial t}$ provide the existence of smooth sections $A^{\pm}_{\frac{n-1}{2}}$ of $\Sigma_{\frac{n-1}{2}}M$ (independent of t) such that $\phi_{\frac{n-1}{2}}=e^tA^+_{\frac{n-1}{2}}+e^{-t}A^-_{\frac{n-1}{2}}$ and $\psi_{\frac{n-1}{2}}=-e^ti\frac{\partial}{\partial t}\cdot A^+_{\frac{n-1}{2}}+e^{-t}i\frac{\partial}{\partial t}\cdot A^-_{\frac{n-1}{2}}$. We deduce that $(\rho''-\rho)A^+_{\frac{n-1}{2}}=(\rho''-\rho)A^-_{\frac{n-1}{2}}=0$. If both $A^+_{\frac{n-1}{2}}$ and $A^-_{\frac{n-1}{2}}$ vanished identically on M, then so would $\phi_{\frac{n-1}{2}}$ and $\psi_{\frac{n-1}{2}}$ and the identities involving $\widehat{\nabla}_Z\phi_{\frac{n-1}{2}}$ and $\widehat{\nabla}_Z\psi_{\frac{n-1}{2}}$ would provide (after contracting with the Clifford multiplication just as above) $\phi_{\frac{n+1}{2}}=\psi_{\frac{n-3}{2}}=0$, so that $(\psi,\phi)=0$, which is a contradiction. Therefore $\rho''-\rho=0$ on I.
It follows in particular that $\rho'=0$ on I cannot hold, so we may assume that $\widehat{h}=-J$ (hence $(M^{2n-1},\widehat{g},\widehat{\xi})$ is Sasakian) and $\rho'=\sigma$ (see Remarks 3.5). Furthermore, in the case where the constant $(\rho')^2-\rho^2$ does not vanish, up to replacing ρ by $\frac{\rho}{\sqrt{|(\rho'^2)-\rho^2|}}$ (which is equivalent to performing a $\mathcal{D}$-homothetic deformation of the Sasakian structure), we may assume that $(\rho'^2)-\rho^2=1$ or -1 on I. Next we rewrite the equations from Lemma 3.8 considering the new sections $\varphi_{\frac{n+1}{2}},\varphi_{\frac{n-1}{2}},\widetilde{\varphi}_{\frac{n-1}{2}},\widetilde{\varphi}_{\frac{n-3}{2}}$ defined by

$$\left|\begin{array}{ll} \varphi_{\frac{n+1}{2}} & := \phi_{\frac{n+1}{2}} \\ \varphi_{\frac{n-1}{2}} & := \rho'\phi_{\frac{n-1}{2}}+i\rho\frac{\partial}{\partial t}\cdot\psi_{\frac{n-1}{2}} \\ \widetilde{\varphi}_{\frac{n-1}{2}} & := i\rho\frac{\partial}{\partial t}\cdot\phi_{\frac{n-1}{2}}+\rho'\psi_{\frac{n-1}{2}} \\ \widetilde{\varphi}_{\frac{n-3}{2}} & := \psi_{\frac{n-3}{2}}. \end{array}\right.$$

Note that the linear transformation $(\phi_{\frac{n+1}{2}},\phi_{\frac{n-1}{2}},\psi_{\frac{n-1}{2}},\psi_{\frac{n-3}{2}})\mapsto(\varphi_{\frac{n+1}{2}},\varphi_{\frac{n-1}{2}},\widetilde{\varphi}_{\frac{n-1}{2}},\widetilde{\varphi}_{\frac{n-3}{2}})$ is invertible if and only if $(\rho')^2-\rho^2\neq 0$. From (4) we have, for all $Z\in\widehat{\xi}^{\perp}$:

$$\frac{\partial}{\partial t}\varphi_{\frac{n+1}{2}} = 0$$

$$
\begin{aligned}
\frac{\partial}{\partial t}\varphi_{\frac{n-1}{2}} &= 0\\
\frac{\partial}{\partial t}\widetilde{\varphi}_{\frac{n-1}{2}} &= 0\\
\frac{\partial}{\partial t}\widetilde{\varphi}_{\frac{n-3}{2}} &= 0\\
\widehat{\nabla}_{\widehat{\xi}}\varphi_{\frac{n+1}{2}} &= \frac{(-1)^{\frac{n+1}{2}}}{2}(n-2(\frac{n+1}{2}))((\rho')^2-\rho^2)\widehat{\xi}\,\widehat{\cdot}_M\varphi_{\frac{n+1}{2}}\\
\widehat{\nabla}_{\widehat{\xi}}\varphi_{\frac{n-1}{2}} &= -\frac{(-1)^{\frac{n+1}{2}}}{2}(n-2(\frac{n+1}{2}))((\rho')^2-\rho^2)\widehat{\xi}\,\widehat{\cdot}_M\varphi_{\frac{n-1}{2}}\\
\widehat{\nabla}_{\widehat{\xi}}\widetilde{\varphi}_{\frac{n-1}{2}} &= \frac{(-1)^{\frac{n-1}{2}}}{2}(n-2(\frac{n-1}{2}))((\rho')^2-\rho^2)\widehat{\xi}\,\widehat{\cdot}_M\widetilde{\varphi}_{\frac{n-1}{2}}\\
\widehat{\nabla}_{\widehat{\xi}}\widetilde{\varphi}_{\frac{n-3}{2}} &= -\frac{(-1)^{\frac{n-1}{2}}}{2}(n-2(\frac{n-1}{2}))((\rho')^2-\rho^2)\widehat{\xi}\,\widehat{\cdot}_M\widetilde{\varphi}_{\frac{n-3}{2}}\\
\widehat{\nabla}_Z\varphi_{\frac{n+1}{2}} &= (-1)^{\frac{n+1}{2}}p_+(Z)\,\widehat{\cdot}_M\varphi_{\frac{n-1}{2}}\\
\widehat{\nabla}_Z\varphi_{\frac{n-1}{2}} &= (-1)^{\frac{n+1}{2}}((\rho')^2-\rho^2)p_-(Z)\,\widehat{\cdot}_M\varphi_{\frac{n+1}{2}}\\
\widehat{\nabla}_Z\widetilde{\varphi}_{\frac{n-1}{2}} &= (-1)^{\frac{n-1}{2}}((\rho')^2-\rho^2)p_+(Z)\,\widehat{\cdot}_M\widetilde{\varphi}_{\frac{n-3}{2}}\\
\widehat{\nabla}_Z\widetilde{\varphi}_{\frac{n-3}{2}} &= (-1)^{\frac{n-1}{2}}p_-(Z)\,\widehat{\cdot}_M\widetilde{\varphi}_{\frac{n-1}{2}}.
\end{aligned}
$$

If $(\rho')^2-\rho^2\neq 0$ on I, then the required equations directly follow from the above ones. Moreover, since in that case the correspondence $(\phi_{\frac{n+1}{2}},\phi_{\frac{n-1}{2}},\psi_{\frac{n-1}{2}},\psi_{\frac{n-3}{2}})\mapsto(\varphi_{\frac{n+1}{2}},\varphi_{\frac{n-1}{2}},\widetilde{\varphi}_{\frac{n-1}{2}},\widetilde{\varphi}_{\frac{n-3}{2}})$ is bijective, the "If" in the assumptions is actually an "if and only if". If now $(\rho')^2-\rho^2=0$, then $\rho'=\pm\rho$ on I; since we have assumed $\rho'>0$ (up to changing t into $-t$), we only have to consider $\rho'=\rho$, hence $\rho=Ce^t$ for some positive constant C. Since translating t provides a holomorphic isometry (again see Remarks 3.5), one may assume that $C=1$, i.e., $\rho=e^t$. In that case, one has $\widehat{\nabla}\varphi_{\frac{n-1}{2}}=0$ on M, hence $\varphi_{\frac{n-1}{2}}$ vanishes either identically or nowhere on M (and on $\widetilde{M}$ since it is constant in t). If $\varphi_{\frac{n-1}{2}}\neq 0$, then all right members in the equations listed just above vanish except

$$
\left|\begin{aligned}
\widehat{\nabla}_Z\phi_{\frac{n+1}{2}} &= (-1)^{\frac{n+1}{2}}p_+(Z)\,\widehat{\cdot}_M\varphi_{\frac{n-1}{2}}\\
\widehat{\nabla}_Z\widetilde{\varphi}_{\frac{n-3}{2}} &= (-1)^{\frac{n-1}{2}}p_-(Z)\,\widehat{\cdot}_M\widetilde{\varphi}_{\frac{n-1}{2}},
\end{aligned}\right.
$$

which together with $\widetilde{\varphi}_{\frac{n-1}{2}}=i\frac{\partial}{\partial t}\cdot\varphi_{\frac{n-1}{2}}$ gives the result. If $\varphi_{\frac{n-1}{2}}=0$ on M, then coming back to the equations from Lemma 3.8, one has $\widehat{\nabla}\phi_{\frac{n+1}{2}}=\widehat{\nabla}\psi_{\frac{n-3}{2}}=0$ and $\phi_{\frac{n-1}{2}}$ satisfies the required equations. □

Remark 3.10 In Theorem 3.9.*i*) not every i-Kählerian Killing spinor on $\widetilde{M}$ must come from a transversally parallel spinor on M. For instance, consider the complex hyperbolic space $\mathbb{C}\mathrm{H}^n$ (for n odd) endowed with its Fubini-Study metric of constant holomorphic sectional curvature -4 and its canonical spin structure. Then $\mathbb{C}\mathrm{H}^n$ (possibly with a suitable submanifold removed) can be viewed as a doubly warped product in several ways. For example, $\mathbb{C}\mathrm{H}^n$ is a doubly-warped product over the Heisenberg group M, which admits a $\binom{n-1}{\frac{n-1}{2}}$-dimensional space of transversally parallel spinors lying pointwise in $\Sigma_{\frac{n-1}{2}}M$ (see below). However, $\mathbb{C}\mathrm{H}^n$ carries a $2\binom{n}{\frac{n+1}{2}}$-dimensional space of i-Kählerian Killing spinors [9, Sec. 3]. Therefore there exists at least one non-zero Kählerian Killing spinor on $\mathbb{C}\mathrm{H}^n$ which does not come from any transversally parallel spinor on M.

As an example for Theorem 3.9.*i*), any Heisenberg manifold of dimension $4k+1$ (with $k\geq 1$) has a spin structure for which the corresponding spinor bundle is trivialized by transversally parallel spinors. This follows from three facts: every Heisenberg manifold is an $\mathbb{S}^1$-bundle with totally geodesic fibres over a flat torus; every $\mathbb{S}^1$-bundle over a manifold carrying parallel spinors carries transversally parallel spinors for the induced spin structure, see e.g. [6, Prop. 3.6]; the whole spinor bundle of any flat torus endowed

with its so-called trivial spin structure is trivialized by parallel spinors. Note that, as a consequence of Lemma 3.12 below, the doubly warped product arising from a $(2n-1)$-dimensional Heisenberg manifold M choosing $\rho = \sigma = e^t$ has constant holomorphic sectional curvature -4, therefore it is holomorphically isometric to $\mathbb{C}\mathrm{H}^n$ as soon as it is simply-connected and complete.
Examples for Theorem 3.9.*i*) with non-constant holomorphic sectional curvature can be constructed out of the following lemma:

Lemma 3.11 *For each integer $n \equiv 1$ (4), let (N^{2n-2}, g_N, J) be any simply-connected closed Hodge hyperkähler manifold. Then there exists an $\mathbb{S}^1$-bundle M over N carrying an $\mathbb{S}^1$-invariant metric $\widehat{g}$ for which $(M^{2n-1}, \widehat{g}, \widehat{\xi})$ is Sasakian and for which there exists a parallel spinor lying pointwise in $\Sigma_{\frac{n-1}{2}} M$.*

Proof: Recall first that every hyperkähler manifold is spin (this follows from the structure group $\mathrm{Sp}(\frac{n-1}{2})$ being simply-connected). McK. Wang's classification [14] of manifolds with parallel spinors provides the existence of exactly $\frac{n-1}{2}+1$ linearly independent parallel spinors on N, one of which lies pointwise in $\Sigma_{\frac{n-1}{2}} N$ if and only if $\frac{n-1}{2}$ is even [14, (ii) p.61]. Now, for any Hodge Kähler manifold (N, g, J) ("Hodge" meaning that its Kähler class is proportional to an integral class), there exists an $\mathbb{S}^1$-bundle $M \xrightarrow{\pi} N$ carrying an $\mathbb{S}^1$-invariant metric $\widehat{g}$ for which $(M^{2n-1}, \widehat{g}, \widehat{\xi})$ is Sasakian with $\widehat{h} = -J$, see [13, Prop. 2] (as usual $\widehat{\xi}$ denotes the fundamental vector field of the $\mathbb{S}^1$-action). By [6, Prop. 3.6], the lift of the non-zero parallel spinor in $\Sigma_{\frac{n-1}{2}} N$ to M gives a non-zero transversal parallel spinor on $(M^{2n-1}, \widehat{g}, \widehat{\xi})$ provided the spin structure on M is induced by the one on $\pi^*(TN)$ and the trivial covering of $\mathbb{S}^1$; because of $\widehat{h} = -J$, this spinor lies pointwise in $\Sigma_{\frac{n-1}{2}} M$. □

Kodaira's embedding theorem states that a closed Kähler manifold is Hodge if and only if it is projective, i.e., if and only if it can be holomorphically embedded in some complex projective space. Therefore projective hyperkähler manifolds of complex dimension $4k$ (with $k \geq 1$) provide examples for N in Lemma 3.11. For instance, simply connected hyperkähler manifolds can be constructed as the Hilbert scheme of a K3-surface (cf. [5]). Indeed, let X be a K3-surface, then the Hilbert scheme $\mathrm{Hilb}^{2k}(X)$, which is the blow-up along the diagonal of the $2k$-th symmetric product of X, is a compact, simply-connected hyperkähler manifold of complex dimension $4k$. If X is projective, e.g. a quartic, then $\mathrm{Hilb}^{2k}(X)$ is projective too and thus has an integer Kähler class.

In order to decide whether the doubly warped product we construct is the complex hyperbolic space or not, the transversal holomorphic curvature of $(M, \widehat{g}, \widehat{\xi})$ and the holomorphic sectional curvature of $(\widetilde{M}^{2n}, \widetilde{g}, \widetilde{J})$ have to be compared:

Lemma 3.12 *Let $(\widetilde{M}^{2n}, \widetilde{g}, \widetilde{J})$ be a Kähler doubly warped product as in* Lemma 3.4 *with $\rho'' = \rho$, $\sigma = \rho'$ and $\widehat{h} = -J$. Then the holomorphic sectional curvature $\widetilde{K}_{\mathrm{hol}}(Z)$ of $(\widetilde{M}, \widetilde{g}, J)$ and the transversal holomorphic sectional curvature $\widehat{K}_{\mathrm{hol}}(Z)$ of $(M, \widehat{g}, \widehat{\xi})$ are related by*

$$\widetilde{K}_{\mathrm{hol}}(Z) = \frac{1}{\rho^2}\Big(\widehat{K}_{\mathrm{hol}}(Z) - 4(\rho')^2\Big),$$

for all $Z \in \{\widehat{\xi}, \frac{\partial}{\partial t}\}^{\perp} \setminus \{0\}$. In particular, the doubly warped product $(\widetilde{M}^{2n}, \widetilde{g}, \widetilde{J})$ has constant holomorphic sectional curvature -4 if and only if the transversal holomorphic sectional curvature of $(M, \widehat{g}, \widehat{\xi})$ is constant equal to $4((\rho')^2 - \rho^2)$.

Proof: Recall that $\widetilde{K}_{\mathrm{hol}}(Z)$ and $\widehat{K}_{\mathrm{hol}}(Z)$ are defined by

$$\widetilde{K}_{\mathrm{hol}}(Z) := \frac{\widetilde{g}(\widetilde{R}(Z, JZ)Z, JZ)}{\widetilde{g}(Z,Z)^2} \qquad \text{and} \qquad \widehat{K}_{\mathrm{hol}}(Z) := \frac{\widehat{g}(\widehat{R}(Z, JZ)Z, JZ)}{\widehat{g}(Z,Z)^2},$$

where $\widetilde{R}_{X,Y} := \widetilde{\nabla}_{[X,Y]} - [\widetilde{\nabla}_X, \widetilde{\nabla}_Y]$ and $\widehat{R}_{Z,Z'} := \widehat{\nabla}_{[Z,Z']} - [\widehat{\nabla}_Z, \widehat{\nabla}_{Z'}]$ are the curvature tensors associated to $\widetilde{\nabla}$ and $\widehat{\nabla}$ on $T\widetilde{M}$ and $\widehat{\xi}^{\perp}$ respectively. The following identities can be deduced from the formulas in Lemma 3.1, taking into account $\rho' = \sigma$ and $\rho'' = \rho$:

$$\begin{aligned} \widetilde{g}(\widetilde{R}_{\xi, \frac{\partial}{\partial t}}\xi, \frac{\partial}{\partial t}) &= -\frac{(\rho\sigma)''}{\rho\sigma} = -4 \\ \widetilde{g}(\widetilde{R}(Z, JZ)Z, JZ) &= \widetilde{g}(\widehat{R}(Z, JZ)Z, JZ) - 4(\frac{\rho'}{\rho})^2 \widetilde{g}(Z,Z)^2, \end{aligned}$$

for every $Z \in \{\widehat{\xi}, \frac{\partial}{\partial t}\}^\perp \setminus \{0\}$. Using $\widetilde{g}(Z,\cdot) = \rho^2 \widehat{g}(Z,\cdot)$, we obtain

$$\begin{aligned}\widetilde{K}_{\mathrm{hol}}(Z) &= \frac{\widetilde{g}(\widehat{R}(Z,JZ)Z,JZ)}{\widetilde{g}(Z,Z)^2} - 4(\frac{\rho'}{\rho})^2 \\ &= \frac{1}{\rho^2}\frac{\widehat{g}(\widehat{R}(Z,JZ)Z,JZ)}{\widehat{g}(Z,Z)^2} - 4(\frac{\rho'}{\rho})^2,\end{aligned}$$

which gives the first statement. Since by the computation above $\widetilde{K}_{\mathrm{hol}}(\xi) = -4$ (independently of $\widehat{g}$), the second follows from the first (note that $(\rho')^2 - \rho^2$ is constant by the assumption $\rho'' = \rho$). □

As a consequence of Theorem 3.9.*i*), Lemma 3.11 and Lemma 3.12, we obtain:

Corollary 3.13 *For an integer $n \equiv 1$ (4), let (N^{2n-2}, g_N, J) be any simply-connected closed Hodge hyperkähler manifold. Let $(M^{2n-1}, \widehat{g}, \widehat{\xi})$ be constructed from N as in* Lemma 3.11 *and $(\widetilde{M}^{2n}, \widetilde{g}, \widetilde{J})$ be the Kähler spin doubly warped product constructed from M as in* Lemma 3.6 *with $\rho = \sigma = e^t$. Then $(\widetilde{M}^{2n}, \widetilde{g}, \widetilde{J})$ carries a non-zero i-Kählerian Killing spinor but has non-constant holomorphic sectional curvature.*

Proof: The existence of a non-zero i-Kählerian Killing spinor follows from Theorem 3.9.*i*) and Lemma 3.11. In case $\rho = \sigma = e^t$, Lemma 3.12 implies that the holomorphic sectional curvature of the doubly warped product $(\widetilde{M}^{2n}, \widetilde{g}, J)$ is -4 if and only if the transversal holomorphic sectional curvature of $(M, \widehat{g}, \widehat{\xi})$ vanishes, that is, if and only if its transversal curvature vanishes (see e.g. [11, Prop. 7.1 p.166]). Now for any $\mathbb{S}^1$-bundle as in Lemma 3.11, the transversal (holomorphic) sectional curvature of M and the (holomorphic) sectional curvature of N coincide. Since simply-connected closed hyperkähler manifolds cannot be flat, the Kähler manifold $(\widetilde{M}^{2n}, \widetilde{g}, \widetilde{J})$ cannot have constant holomorphic sectional curvature. □

Corollary 3.13 provides the first family of examples of Kähler spin manifolds of non-constant holomorphic sectional curvature carrying non-zero imaginary Kählerian Killing spinors.

The two other subcases $(\rho')^2 - \rho^2 = 1$ and $(\rho')^2 - \rho^2 = -1$ are geometrically more simple to describe. We do it in separate lemmas.

Lemma 3.14 *Let (M^{2n-1}, g, ξ) be a Sasakian spin manifold with $h = -J$ and fix $r \in \{0, 1, \ldots, n\}$. Then a section (ψ_r, ψ_{r-1}) of $\Sigma_r M \oplus \Sigma_{r-1} M$ satisfies* (5) *if and only if $\psi := \psi_r + \psi_{r-1}$ is a $\frac{(-1)^r}{2}$-Killing spinor on (M, g).*

Proof: Let Ω be the 2-form associated to J on $\xi^\perp$, i.e., $\Omega(Z, Z') = g(J(Z), Z')$ for all $Z, Z' \perp \xi$. Using $\Omega \underset{M}{\cdot} \psi_r = (-1)^{r+1}(2r - n + 1)\xi \underset{M}{\cdot} \psi_r$ (for all r) we have on the one hand

$$\begin{aligned}\nabla_\xi \psi &= \nabla^M_\xi \psi + \frac{1}{2}\Omega \underset{M}{\cdot} \psi \\ &= \nabla^M_\xi \psi - \frac{(-1)^r}{2}\xi \underset{M}{\cdot} \psi + \frac{(-1)^r}{2}\xi \underset{M}{\cdot} \psi + \frac{1}{2}\Omega \underset{M}{\cdot} \psi \\ &= \nabla^M_\xi \psi - \frac{(-1)^r}{2}\xi \underset{M}{\cdot} \psi + \frac{(-1)^r}{2}\xi \underset{M}{\cdot} \psi - \frac{(-1)^r}{2}(2r - n + 1)\xi \underset{M}{\cdot} \psi_r \\ &\quad + \frac{(-1)^r}{2}(2(r-1) - n + 1)\xi \underset{M}{\cdot} \psi_{r-1} \\ &= \nabla^M_\xi \psi - \frac{(-1)^r}{2}\xi \underset{M}{\cdot} \psi + \frac{(-1)^r}{2}(n - 2r)\xi \underset{M}{\cdot} \psi_r + \frac{(-1)^r}{2}(2(r-1) - n + 2)\xi \underset{M}{\cdot} \psi_{r-1},\end{aligned}$$

which implies

$$\left|\begin{array}{ll} \nabla_\xi \psi_r & = (\nabla^M_\xi \psi - \frac{(-1)^r}{2}\xi \underset{M}{\cdot} \psi)_r + \frac{(-1)^r}{2}(n - 2r)\xi \underset{M}{\cdot} \psi_r \\ \nabla_\xi \psi_{r-1} & = (\nabla^M_\xi \psi - \frac{(-1)^r}{2}\xi \underset{M}{\cdot} \psi)_{r-1} - \frac{(-1)^r}{2}(n - 2r)\xi \underset{M}{\cdot} \psi_{r-1}. \end{array}\right. \tag{7}$$

On the other hand, for every $Z \in \xi^\perp$ one has,

$$\nabla_Z \psi = \nabla^M_Z \psi - \frac{1}{2}\xi \underset{M}{\cdot} h(Z) \underset{M}{\cdot} \psi$$

$$
\begin{aligned}
&= \nabla^M_Z\psi - \frac{(-1)^r}{2} Z \mathbin{\dot{}}_M \psi + \frac{(-1)^r}{2} Z \mathbin{\dot{}}_M \psi - \frac{1}{2} J(Z) \mathbin{\dot{}}_M \xi \mathbin{\dot{}}_M \psi \\
&= \nabla^M_Z\psi - \frac{(-1)^r}{2} Z \mathbin{\dot{}}_M \psi + \frac{(-1)^r}{2} Z \mathbin{\dot{}}_M \psi - \frac{1}{2} J(Z) \mathbin{\dot{}}_M \{(-1)^{r+1} i\psi_r + (-1)^r i\psi_{r-1}\} \\
&= \nabla^M_Z\psi - \frac{(-1)^r}{2} Z \mathbin{\dot{}}_M \psi + \frac{(-1)^r}{2} (Z + iJ(Z)) \cdot \psi_r + \frac{(-1)^r}{2} (Z - iJ(Z)) \cdot \psi_{r-1},
\end{aligned}
$$

which implies

$$
\left|
\begin{array}{ll}
\nabla_Z\psi_r & = (\nabla^M_Z\psi - \frac{(-1)^r}{2} Z \mathbin{\dot{}}_M \psi)_r + (-1)^r p_+(Z) \cdot \psi_{r-1} \\
\nabla_Z\psi_{r-1} & = (\nabla^M_Z\psi - \frac{(-1)^r}{2} Z \mathbin{\dot{}}_M \psi)_{r-1} + (-1)^r p_-(Z) \cdot \psi_r.
\end{array}
\right. \tag{8}
$$

Therefore the pair (ψ_r, ψ_{r-1}) satisfies (5) if and only if $\psi := \psi_r + \psi_{r-1}$ satisfies $\nabla^M_X\psi = \frac{(-1)^r}{2} X \mathbin{\dot{}}_M \psi$ for all $X \in TM$, that is, if and only if ψ is a $\frac{(-1)^r}{2}$-Killing spinor on (M, g). □

The case $(\rho')^2 - \rho^2 = -1$ is analogous to the case $(\rho')^2 - \rho^2 = 1$ up to a Lorentzian detour. We call (9) the following system of equations:

$$
\left|
\begin{array}{ll}
\nabla_\xi\psi_r & = -\frac{(-1)^r}{2}(n-2r)\xi \mathbin{\dot{}}_M \psi_r \\
\nabla_\xi\psi_{r-1} & = \frac{(-1)^r}{2}(n-2r)\xi \mathbin{\dot{}}_M \psi_{r-1} \\
\nabla_Z\psi_r & = (-1)^r \epsilon p_+(Z) \mathbin{\dot{}}_M \psi_{r-1} \\
\nabla_Z\psi_{r-1} & = -(-1)^r \epsilon p_-(Z) \mathbin{\dot{}}_M \psi_r
\end{array}
\right. \tag{9}
$$

for all $Z, Z' \in \xi^\perp$, where $\epsilon \in \{\pm 1\}$.

Lemma 3.15 *Let (M^{2n-1}, g, ξ) be a Sasakian spin manifold with $h = -J$ and fix $r \in \{0, 1, \ldots, n\}$ as well as $\epsilon \in \{\pm 1\}$. Then a section (ψ_r, ψ_{r-1}) of $\Sigma_r M \oplus \Sigma_{r-1} M$ satisfies (9) if and only if $\psi := \psi_r + i\epsilon\psi_{r-1}$ is a $\frac{(-1)^{r+1} i}{2}$-Killing spinor on the Lorentzian manifold $(M, -g_\xi \oplus g_{\xi^\perp})$.*

Proof: First, there exists the analog of Riemannian flow in the Lorentzian context. A Lorentzian flow is given by a triple $(M, \widehat{g}, \widehat{\xi})$, where $(M, \widehat{g})$ is a Lorentzian manifold and $\widehat{\xi}$ a smooth tangent vector field on M with $\widehat{g}(\widehat{\xi}, \widehat{\xi}) = -1$ and $\widehat{g}(\widehat{\nabla}^M_Z \widehat{\xi}, Z') = -\widehat{g}(\widehat{\nabla}^M_{Z'} \widehat{\xi}, Z)$ for all $Z, Z' \in \widehat{\xi}^\perp$. Note that $(M, \widehat{g})$ is necessarily time-oriented because of the existence of $\widehat{\xi}$. Setting $\widehat{\nabla}_X Z := \left| \begin{array}{ll} [\widehat{\xi}, Z]^{\widehat{\xi}^\perp} & \text{if } X = \widehat{\xi} \\ (\widehat{\nabla}^M_X Z)^{\widehat{\xi}^\perp} & \text{if } X \perp \widehat{\xi} \end{array} \right.$ for all $Z \in \Gamma(\widehat{\xi}^\perp)$ and $\widehat{h} := \widehat{\nabla}^M \widehat{\xi}$, one obtains a metric connection $\widehat{\nabla}$ and a skew-symmetric endomorphism-field $\widehat{h}$ on $\widehat{\xi}^\perp$ such that

$$
\left|
\begin{array}{ll}
\widehat{\nabla}^M_{\widehat{\xi}} Z & = \widehat{\nabla}_{\widehat{\xi}} Z + \widehat{h}(Z) + \widehat{g}(\widehat{\nabla}^M_{\widehat{\xi}} \widehat{\xi}, Z)\widehat{\xi} \\
\widehat{\nabla}^M_Z Z' & = \widehat{\nabla}_Z Z' + \widehat{g}(\widehat{h}(Z), Z')\widehat{\xi}
\end{array}
\right.
$$

for all $Z, Z' \in \Gamma(\widehat{\xi}^\perp)$. Moreover, in case M is spin, the corresponding Gauss-type formula for spinors reads

$$
\left|
\begin{array}{ll}
\widehat{\nabla}_{\widehat{\xi}}\varphi & = \widehat{\nabla}^M_{\widehat{\xi}}\varphi - \frac{1}{2}\widehat{\Omega} \mathbin{\dot{}}_M \varphi + \frac{1}{2}\widehat{\xi} \mathbin{\dot{}}_M \widehat{\nabla}^M_{\widehat{\xi}} \widehat{\xi} \mathbin{\dot{}}_M \varphi \\
\widehat{\nabla}_Z\varphi & = \widehat{\nabla}^M_Z\varphi + \frac{1}{2}\widehat{\xi} \mathbin{\dot{}}_M \widehat{h}(Z) \mathbin{\dot{}}_M \varphi
\end{array}
\right.
$$

for all $\varphi \in \Gamma(\Sigma M)$ and $Z \in \widehat{\xi}^\perp$, where $\widehat{\Omega}(Z, Z') := \widehat{g}(\widehat{h}(Z), Z')$. In case $(M, \widehat{g}, \widehat{\xi})$ is Lorentzian Sasakian, i.e., if furthermore $\widehat{\nabla}^M_{\widehat{\xi}} \widehat{\xi} = 0$, $\widehat{h}^2 = -\mathrm{Id}$ and $\widehat{\nabla}\widehat{h} = 0$, then we still have the $\widehat{\nabla}$-parallel decomposition $\Sigma M = \oplus_{r=0}^{n-1} \Sigma_r M$ with $\Sigma_r M := \mathrm{Ker}(\widehat{\Omega} \mathbin{\dot{}}_M - i(2r - (n-1)\mathrm{Id}))$. This time one has $\widehat{\xi} \mathbin{\dot{}}_M \varphi_r = (-1)^{r+1}\varphi_r$ for all $\varphi_r \in \Sigma_r M$.
Assume now $(M, \widehat{g}, \widehat{\xi})$ to be Lorentzian Sasakian and pick a section $\psi = \psi_r + \psi_{r-1}$ of $\Sigma_r M \oplus \Sigma_{r-1} M$, then the formulas above imply

$$
\begin{aligned}
\widehat{\nabla}_{\widehat{\xi}}\psi &= \widehat{\nabla}^M_{\widehat{\xi}}\psi - \frac{1}{2}\widehat{\Omega} \mathbin{\dot{}}_M \psi \\
&= \widehat{\nabla}^M_{\widehat{\xi}}\psi - \frac{(-1)^{r+1} i}{2}\widehat{\xi} \mathbin{\dot{}}_M \psi + \frac{(-1)^{r+1} i}{2}\widehat{\xi} \mathbin{\dot{}}_M \psi - \frac{i}{2}\big((2r - (n-1))\psi_r + (2(r-1) - (n-1))\psi_{r-1}\big)
\end{aligned}
$$

$$
\begin{aligned}
&= \widehat{\nabla}^M_{\widehat{\xi}}\psi - \frac{(-1)^{r+1}i}{2}\widehat{\xi}\,\widehat{\cdot}_M\psi + \frac{(-1)^{r+1}i}{2}\widehat{\xi}\,\widehat{\cdot}_M\psi \\
&\quad + \frac{(-1)^r i}{2}(2r-(n-1))\widehat{\xi}\,\widehat{\cdot}_M\psi_r - \frac{(-1)^r i}{2}(2(r-1)-(n-1))\widehat{\xi}\,\widehat{\cdot}_M\psi_{r-1} \\
&= \widehat{\nabla}^M_{\widehat{\xi}}\psi - \frac{(-1)^{r+1}i}{2}\widehat{\xi}\,\widehat{\cdot}_M\psi + \frac{(-1)^{r+1}i}{2}(n-2r)\widehat{\xi}\,\widehat{\cdot}_M\psi_r - \frac{(-1)^{r+1}i}{2}(n-2r)\widehat{\xi}\,\widehat{\cdot}_M\psi_{r-1},
\end{aligned}
$$

that is,

$$
\left|\begin{array}{ll}
\widehat{\nabla}_{\widehat{\xi}}\psi_r & = \big(\widehat{\nabla}^M_{\widehat{\xi}}\psi - \frac{(-1)^{r+1}i}{2}\widehat{\xi}\,\widehat{\cdot}_M\psi\big)_r + \frac{(-1)^{r+1}i}{2}(n-2r)\widehat{\xi}\,\widehat{\cdot}_M\psi_r \\
\widehat{\nabla}_{\widehat{\xi}}\psi_{r-1} & = \big(\widehat{\nabla}^M_{\widehat{\xi}}\psi - \frac{(-1)^{r+1}i}{2}\widehat{\xi}\,\widehat{\cdot}_M\psi\big)_{r-1} - \frac{(-1)^{r+1}i}{2}(n-2r)\widehat{\xi}\,\widehat{\cdot}_M\psi_{r-1}.
\end{array}\right.
$$

This is still valid for $r=0$ or $r=n$ (setting $\psi_{-1}:=\psi_n:=0$). Similarly, for all $Z\in\widehat{\xi}^\perp$,

$$
\begin{aligned}
\widehat{\nabla}_Z\psi &= \widehat{\nabla}^M_Z\psi + \frac{1}{2}\widehat{\xi}\,\widehat{\cdot}_M\widehat{h}(Z)\,\widehat{\cdot}_M\psi \\
&= \widehat{\nabla}^M_Z\psi - \frac{(-1)^{r+1}i}{2}Z\,\widehat{\cdot}_M\psi + \frac{(-1)^{r+1}i}{2}Z\,\widehat{\cdot}_M\psi - \frac{(-1)^{r+1}}{2}\widehat{h}(Z)\,\widehat{\cdot}_M\psi_r + \frac{(-1)^{r+1}}{2}\widehat{h}(Z)\,\widehat{\cdot}_M\psi_{r-1} \\
&= \widehat{\nabla}^M_Z\psi - \frac{(-1)^{r+1}i}{2}Z\,\widehat{\cdot}_M\psi + (-1)^{r+1}ip_-(Z)\,\widehat{\cdot}_M\psi_r + (-1)^{r+1}ip_+(Z)\,\widehat{\cdot}_M\psi_{r-1},
\end{aligned}
$$

that is,

$$
\left|\begin{array}{ll}
\widehat{\nabla}_Z\psi_r & = \big(\widehat{\nabla}^M_Z\psi - \frac{(-1)^{r+1}i}{2}Z\,\widehat{\cdot}_M\psi\big)_r + (-1)^{r+1}ip_+(Z)\,\widehat{\cdot}_M\psi_{r-1} \\
\widehat{\nabla}_Z\psi_{r-1} & = \big(\widehat{\nabla}^M_Z\psi - \frac{(-1)^{r+1}i}{2}Z\,\widehat{\cdot}_M\psi\big)_{r-1} + (-1)^{r+1}ip_-(Z)\,\widehat{\cdot}_M\psi_r.
\end{array}\right.
$$

If one changes the Lorentzian metric $\widehat{g}$ into $g:=-\widehat{g}_{\widehat{\xi}}\oplus\widehat{g}_{\widehat{\xi}^\perp}$, then one obtains a smooth Riemannian metric g on M and the triple $(M,g,\xi:=\widehat{\xi})$ is a Riemannian flow with

$$
\left|\begin{array}{ll}
\nabla^M_\xi\xi & = -\widehat{\nabla}^M_{\widehat{\xi}}\widehat{\xi} \\
h & = -\widehat{h} \\
\nabla & = \widehat{\nabla}.
\end{array}\right.
$$

Moreover, the Clifford multiplications are related by

$$
\left|\begin{array}{ll}
\xi\,\dot{}_M & = i\widehat{\xi}\,\widehat{\cdot}_M \\
Z\,\dot{}_M & = Z\,\widehat{\cdot}_M,
\end{array}\right.
$$

for all $Z\in\xi^\perp=\widehat{\xi}^\perp$. Therefore the equations above become on (M,g,ξ)

$$
\left|\begin{array}{ll}
\nabla_\xi\psi_r & = \big(\widehat{\nabla}^M_{\widehat{\xi}}\psi - \frac{(-1)^{r+1}i}{2}\widehat{\xi}\,\widehat{\cdot}_M\psi\big)_r - \frac{(-1)^r}{2}(n-2r)\xi\,\dot{}_M\psi_r \\
\nabla_\xi\psi_{r-1} & = \big(\widehat{\nabla}^M_{\widehat{\xi}}\psi - \frac{(-1)^{r+1}i}{2}\widehat{\xi}\,\widehat{\cdot}_M\psi\big)_{r-1} + \frac{(-1)^r}{2}(n-2r)\xi\,\dot{}_M\psi_{r-1} \\
\nabla_Z\psi_r & = \big(\widehat{\nabla}^M_Z\psi - \frac{(-1)^{r+1}i}{2}Z\,\widehat{\cdot}_M\psi\big)_r + (-1)^{r+1}ip_+(Z)\,\dot{}_M\psi_{r-1} \\
\nabla_Z\psi_{r-1} & = \big(\widehat{\nabla}^M_Z\psi - \frac{(-1)^{r+1}i}{2}Z\,\widehat{\cdot}_M\psi\big)_{r-1} + (-1)^{r+1}ip_-(Z)\,\dot{}_M\psi_r.
\end{array}\right.
$$

Therefore, $\psi_r - i\epsilon\psi_{r-1}$ satisfies (9) if and only if ψ is a $\frac{(-1)^{r+1}i}{2}$-Killing spinor on $(M,\widehat{g},\widehat{\xi})$. □

Round spheres provide examples of spin Sasakian manifolds where (5) is fulfilled for the right r.

Lemma 3.16 *For any odd $n\geq 3$, the $(2n-1)$-dimensional round sphere M with its canonical Sasakian and spin structures admits a $2\begin{pmatrix} n \\ \frac{n+1}{2}\end{pmatrix}$-dimensional space of sections of $\Sigma_{\frac{n+1}{2}}M\oplus\Sigma_{\frac{n-1}{2}}M\oplus\Sigma_{\frac{n-1}{2}}M\oplus\Sigma_{\frac{n-3}{2}}M$ satisfying (5).*

Proof: Consider the standard embedding $\mathbb{S}^{2n-1}\subset\mathbb{C}^n$, with unit normal $\nu_x=x$ and hence Weingarten-endomorphism field $A=-\mathrm{Id}_{TM}$. Set $\xi:=-i\nu$. It is well-known that $(\mathbb{S}^{2n-1},g,\xi)$ is a Sasakian spin

manifold with $h=-J$ on $\xi^\perp\subset TM$, where J is the standard complex structure induced from $\mathbb{C}^n$. Let $\psi\in\Sigma_r\mathbb{C}^n$ with $r\in\{0,1\ldots,n\}$ (i.e., $\widetilde{\Omega}\cdot\psi=i(2r-n)\psi$ where $\widetilde{\Omega}$ is the standard Kähler form of $\mathbb{C}^n$). If r is even then $\psi\in\Sigma^+\mathbb{C}^n$. In that case the spinorial Gauss formula reads

$$\nabla^M_X\varphi=\nabla^{\mathbb{C}^n}_X\varphi-\frac{1}{2}A(X)\underset{M}{\cdot}\varphi$$

so that the restriction of ψ on $\mathbb{S}^{2n-1}$ satisfies $\nabla^M_X\psi=\frac{1}{2}X\underset{M}{\cdot}\psi$, i.e., is a $\frac{1}{2}$-Killing spinor. If r is odd, then $\psi\in\Sigma_-\mathbb{C}^n$. The spinorial Gauss formula for a section $\varphi\in\Sigma^-\mathbb{C}^n_{|_{\mathbb{S}^{2n-1}}}$, which can be identified with $\Sigma\mathbb{S}^{2n-1}$ provided we change the sign of the Clifford multiplication, reads then

$$\nabla^M_X\varphi=\nabla^{\mathbb{C}^n}_X\varphi+\frac{1}{2}A(X)\underset{M}{\cdot}\varphi$$

for every $X\in TM$. We deduce that $\nabla^M_X\psi=-\frac{1}{2}X\underset{M}{\cdot}\psi$ for every $X\in TM$, that is, the restriction of ψ to $\mathbb{S}^{2n-1}$ is a $-\frac{1}{2}$-Killing spinor. To sum up, the restriction of a constant section $\psi\in\Sigma_r\mathbb{C}^n$ to $M:=\mathbb{S}^{2n-1}$ is a $\frac{(-1)^r}{2}$-Killing spinor on M. Decompose such a ψ into $\psi=\psi_r+\psi_{r-1}$, see (3). From Lemma 3.14 and $\mathrm{rk}_{\mathbb{C}}(\Sigma_r\mathbb{C}^n)=\left(\begin{array}{c}n\\r\end{array}\right)$ we conclude. □

The analog of $\mathbb{S}^{2n-1}$ in the Lorentzian context is the Anti-deSitter spacetime $\mathbb{H}^{2n-1}$, that can be defined by

$$\mathbb{H}^{2n-1}:=\{z\in\mathbb{C}^n\mid -|z_0|^2+\sum_{j=1}^{n-1}|z_j|^2=-1\}.$$

Lemma 3.17 *For any odd $n\geq 3$, the $(2n-1)$-dimensional Anti-deSitter spacetime $M:=\mathbb{H}^{2n-1}$ with its induced Lorentzian Sasakian structure (with $\widehat{\xi}_x=ix$ and $\widehat{h}=J$) and induced spin structure admits an $\left(\begin{array}{c}n\\r\end{array}\right)$-dimensional space of $\frac{(-1)^{r+1}i}{2}$-Killing spinors lying pointwise in $\Sigma_rM\oplus\Sigma_{r-1}M$. In particular, if one considers the (Riemannian) Sasakian metric given by $-\widehat{g}_{\widehat{\xi}}\oplus\widehat{g}_{\widehat{\xi}^\perp}$, where $\widehat{g}$ is the canonical Lorentzian metric of sectional curvature -1, then $\mathbb{H}^{2n-1}$ admits a $2\left(\begin{array}{c}n\\\frac{n+1}{2}\end{array}\right)$-dimensional space of sections of $\Sigma_{\frac{n+1}{2}}M\oplus\Sigma_{\frac{n-1}{2}}M\oplus\Sigma_{\frac{n-1}{2}}M\oplus\Sigma_{\frac{n-3}{2}}M$ satisfying* (6).

Proof: First recall that M is a Lorentzian Sasakian manifold and simultaneously an $\mathbb{S}^1$-bundle with totally geodesic fibres over $\mathbb{C}\mathrm{H}^{n-1}$. Just as for the sphere, one can restrict spinors from $\mathbb{C}^n$ onto M so that the following Gauss-Weingarten-formula holds for all $\psi\in C^\infty(\mathbb{C}^n,\Sigma_{2n})$ and all $X\in TM$:

$$\begin{aligned}\nabla^M_X\psi &= -\frac{A(X)}{2}\cdot\nu\cdot\psi\\ &= \left|\begin{array}{ll}\frac{iA(X)}{2}\underset{M}{\cdot}\psi & \text{if } \psi(x)\in\Sigma^+_{2n}\ \forall x\\ -\frac{iA(X)}{2}\underset{M}{\cdot}\psi & \text{if } \psi(x)\in\Sigma^-_{2n}\ \forall x,\end{array}\right.\end{aligned}$$

where $A(X):=\widetilde{\nabla}_X\nu$ is the Weingarten endormorphism of M in $\mathbb{C}^n$. Moreover, there still exists a $\widetilde{\nabla}$-parallel splitting $\Sigma_{2n}=\oplus_{r=0}^n\Sigma_{2n,r}$ where $\Sigma_{2n,r}:=\mathrm{Ker}(\widetilde{\Omega}\cdot-i(2r-n)\mathrm{Id})$ (with dimension $\left(\begin{array}{c}n\\r\end{array}\right)$) and $\widetilde{\Omega}$ is the Kähler form associated to the standard complex structure J on $\widetilde{M}$. Choosing $\nu_x:=-x$ as unit normal on M, one has $A=-\mathrm{Id}_{TM}$, so that the restriction of any constant section of $\mathbb{C}^n\times\Sigma_{2n,r}$ onto M provides a $\frac{(-1)^{r+1}i}{2}$-Killing spinor. Since again $\Sigma_r\widetilde{M}_{|M}=\Sigma_rM\oplus\Sigma_{r-1}M$, the first statement follows. The second statement is a consequence of the first one together with Lemma 3.15. □

The doubly warped product of Theorem 3.9.*ii*) corresponding to $M=\mathbb{S}^{2n-1}$ is the complement of a point in the complex hyperbolic space $\mathbb{C}\mathrm{H}^n$ with its canonical Fubini-Study metric of constant holomorphic sectional curvature -4 (compare with [1, Satz 5.1]). Therefore we obtain a new description of the imaginary Kählerian Killing spinors on $\mathbb{C}\mathrm{H}^n$ after the explicit one by K.-D. Kirchberg [9, Sec. 3]. Actually $\mathbb{C}\mathrm{H}^n$ is essentially the only example occurring in Theorem 3.9.*ii*):

Theorem 3.18 *For $n \geq 3$ odd let $(\widetilde{M}^{2n}, \widetilde{g}, \widetilde{J})$ be a Kähler doubly warped product as in* Lemma 3.6 *with $(M^{2n-1}, \widehat{g}, \widehat{\xi})$ complete, Sasakian, simply-connected, spin, $I = \mathbb{R}_+^\times$, $\rho = \sinh$ and $\sigma = \cosh$. Let $\widetilde{M}$ carry the induced spin structure and assume $(\widetilde{M}^{2n}, \widetilde{g}, \widetilde{J})$ admits a non-zero i-Kählerian Killing spinor (ψ, ϕ). Then $(\widetilde{M}^{2n}, \widetilde{g}, \widetilde{J})$ is holomorphically isometric to $\mathbb{C}\mathrm{H}^n \setminus \{x\}$ for some $x \in \mathbb{C}\mathrm{H}^n$.*

Proof: It suffices to show that $(M^{2n-1}, \widehat{g}, \widehat{\xi})$ is $\mathbb{S}^{2n-1}$ with its standard Sasakian structure. By assumption and Lemma 3.14, the section $\varphi_{\frac{n+1}{2}} + \varphi_{\frac{n-1}{2}}$ is a $\frac{(-1)^{\frac{n+1}{2}}}{2}$-Killing spinor on $(M^{2n-1}, \widehat{g}, \widehat{\xi})$ lying pointwise in $\Sigma_{\frac{n+1}{2}} M \oplus \Sigma_{\frac{n-1}{2}} M$ and the section $\widetilde{\varphi}_{\frac{n-1}{2}} + \widetilde{\varphi}_{\frac{n-3}{2}}$ is a $-\frac{(-1)^{\frac{n+1}{2}}}{2}$-Killing spinor on $(M^{2n-1}, \widehat{g}, \widehat{\xi})$ lying pointwise in $\Sigma_{\frac{n-1}{2}} M \oplus \Sigma_{\frac{n-3}{2}} M$. At least one of them does not vanish. Now C. Bär's classification (see in particular [2, Thm. 3]) implies that either $M = \mathbb{S}^{2n-1}$ or M is a compact Einstein-Sasakian manifold with exactly one non-zero $\frac{1}{2}$- and one non-zero $-\frac{1}{2}$-Killing spinor. Moreover, each Killing spinor induces a parallel spinor on the Riemannian cone $\overline{M}$ over M [2]. But coming back to McK. Wang's classification of simply-connected complete Riemannian spin manifolds with parallel spinors, it turns out that, in the latter case, the reduced holonomy of $\overline{M}$ is SU_n (where n is its complex dimension) and the parallel spinors lie in $\Sigma_0 \overline{M}$ and $\Sigma_n \overline{M}$ (see [14, (i) p.61]), in particular not in $\Sigma_{\frac{n+1}{2}} \overline{M}$. Thus only $\mathbb{S}^{2n-1}$ occurs. □

In case $M = \mathbb{H}^{2n-1}$ is equipped with its associated Riemannian Sasakian structure, the corresponding doubly warped product with $\rho = \cosh$ and $\sigma = \sinh$ has again constant holomorphic sectional curvature -4 by Lemma 3.12. It is actually the complement in $\mathbb{C}\mathrm{H}^n$ of some submanifold. We conjecture that, up to covering, $\mathbb{H}^{2n-1}$ is the only Lorentzian Sasakian manifold having non-zero imaginary Killing spinors lying pointwise in the "middle" eigenspaces $\Sigma_r M$ (with $r \in \{\frac{n-3}{2}, \ldots, \frac{n+1}{2}\}$) of the Clifford action of the transversal Kähler form. If this happens, then only the complex hyperbolic space can occur as (simply-connected complete) example of doubly warped product in Theorem 3.9.*iii*).

4 Classification in a particular case

In this section, we show that the structure of a doubly warped product can be recovered from the length function of a non-zero imaginary Kählerian Killing spinor satisfying certain supplementary assumption on the Kähler manifold $\widetilde{M}$. The following result can be seen as analogous to H. Baum's one [3] about imaginary Killing spinors of so-called *type I*. Recall for the next theorem that V was defined by (1).

Theorem 4.1 *Let $(\widetilde{M}^{2n}, g, J)$ be a connected complete Kähler spin manifold carrying a non-zero i-Kählerian Killing spinor (ψ, ϕ). Assume $|\psi| = |\phi|$ and the existence of a real vector field W on $\widetilde{M}$ together with a non-identically vanishing continuous function $\mu : \widetilde{M} \longrightarrow \mathbb{C}$ such that $W \cdot \psi = \mu\phi$. Then the vector field V has no zero, the Kähler manifold $(\widetilde{M}^{2n}, g, J)$ is a doubly warped product as in* Theorem 3.9.*i*) *and (ψ, ϕ) comes from a transversally parallel spinor on $(M, \widehat{g}, \widehat{\xi})$.*

Proof: We construct a holomorphic isometry between $(\widetilde{M}^{2n}, g, J)$ and some doubly warped product. This isometry is provided by the flow of some vector field associated to the Kählerian Killing spinor (compare with the case of imaginary Killing spinors [3]).
First note that, if $|\psi| = |\phi|$, then both ψ and ϕ have no zero on $\widetilde{M}$. Because of $|W| \cdot |\psi| = |W \cdot \psi| = |\mu| \cdot |\phi|$, this already implies $|W| = |\mu|$ on $\widetilde{M}$. Fix a neighbourhood U of a point x with $\mu(x) \neq 0$ for all $x \in U$. It follows from the definition of V that

$$\mu = 2i \frac{g(p_+(W), V)}{|\phi|^2} \tag{10}$$

on U, in particular $W(x) \neq 0$ and $V(x) \neq 0$ for all $x \in U$. Now Cauchy-Schwarz inequality with $X = V$ in (1) gives $|V| \leq |\psi| \cdot |\phi|$ on $\widetilde{M}$. With (10) we deduce that

$$\begin{aligned} |\mu|^2 &= \frac{|V|^2}{|\phi|^4} \Big(g(W, \frac{V}{|V|})^2 + g(W, \frac{J(V)}{|V|})^2 \Big) \\ &\leq \frac{|V|^2 |W|^2}{|\phi|^4} \\ &\leq |W|^2 \end{aligned}$$

on U, which together with $|\mu| = |W|$ provides $|V| = |\phi|^2$. By the equality case in Cauchy-Schwarz inequality, we obtain $V \cdot \psi = i|V|\phi$ and $V \cdot \phi = i|V|\psi$ on U. This identity holds on $\widetilde{M}$ because of the analyticity of all objects involved (by definition, ψ is anti-holomorphic and ϕ is holomorphic). This in turn implies $|V| = |\phi|^2$ on $\widetilde{M}$, in particular $\{V = 0\} = \varnothing$ and $\frac{V}{|V|} \cdot \psi = i\phi$ as well as $\frac{V}{|V|} \cdot \phi = i\psi$ on $\widetilde{M}$.
Next we look at the level hypersurfaces $M_r := \{x \in \widetilde{M},\ |\phi(x)| = r\}$ (with $r \in \mathbb{R}_+^\times$) which, if non-empty, are smooth because of $\{V = 0\} = \varnothing$ and Proposition 2.1. A unit normal to M_r is given by $\nu := \frac{V}{|V|}$ and the associated Weingarten endomorphism field is

$$\begin{aligned} A(X) &:= -\widetilde{\nabla}_X \nu \\ &= -\frac{1}{|V|}\left(\widetilde{\nabla}_X V - g(\widetilde{\nabla}_X V, \frac{V}{|V|})\frac{V}{|V|}\right) \end{aligned}$$

for every $X \in \nu^\perp$. Setting $\xi := -J(\nu)$ (note that the vector field ξ is pointwise tangent to M_r), using $\nu \cdot \psi = i\phi$ and Proposition 2.1.*ii*), we compute, for all $X, Y \in \nu^\perp$,

$$\begin{aligned} g(A(X),Y) &= -\frac{1}{|V|} g(\widetilde{\nabla}_X V, Y) \\ &= -\frac{1}{|V|}\Re e\left(\langle p_-(X)\cdot\phi, p_-(Y)\cdot\phi\rangle + \langle p_+(X)\cdot\psi, p_+(Y)\cdot\psi\rangle\right) \\ &= -\frac{1}{|V|}\Re e\left(\langle p_-(X)\cdot\nu\cdot\psi, p_-(Y)\cdot\nu\cdot\psi\rangle + \langle p_+(X)\cdot\psi, p_+(Y)\cdot\psi\rangle\right) \\ &= -\frac{1}{|V|}\Re e\Big(-\langle p_-(X)\cdot\nu\cdot\psi, \nu\cdot p_-(Y)\cdot\psi\rangle - 2\overline{g(\nu,p_-(Y))}\langle p_-(X)\cdot\nu\cdot\psi,\psi\rangle \\ &\qquad + \langle p_+(X)\cdot\psi, p_+(Y)\cdot\psi\rangle\Big) \\ &= -\frac{1}{|V|}\Re e\Big(\langle \nu\cdot p_-(X)\cdot\psi, \nu\cdot p_-(Y)\cdot\psi\rangle + 2g(\nu,p_-(X))\langle\psi,\nu\cdot p_-(Y)\cdot\psi\rangle \\ &\qquad - 2\overline{g(\nu,p_-(Y))}\langle p_-(X)\cdot\nu\cdot\psi,\psi\rangle + \langle p_+(X)\cdot\psi, p_+(Y)\cdot\psi\rangle\Big) \\ &= -\frac{1}{|V|}\Re e\Big(\langle X\cdot\psi, Y\cdot\psi\rangle + ig(\nu,J(X))\langle\psi,\nu\cdot p_-(Y)\cdot\psi\rangle + ig(\nu,J(Y))\langle p_-(X)\cdot\nu\cdot\psi,\psi\rangle\Big) \\ &= -\frac{1}{|V|}\Big(|\psi|^2 g(X,Y) + g(\nu,J(X))\Re e(\underbrace{\langle\phi, p_-(Y)\cdot\psi\rangle}_{0}) - g(\nu,J(Y))\Re e(\langle p_-(X)\cdot\phi,\psi\rangle)\Big) \\ &= -\frac{1}{|V|}\Big(|\psi|^2 g(X,Y) + g(\nu,J(Y))g(J(X),V)\Big) \\ &= -(g(X,Y) + g(\xi,X)g(\xi,Y)), \end{aligned}$$

that is, $A = -\mathrm{Id}_{TM_r} - \xi^\flat \otimes \xi$. In particular, the Gauß-Weingarten formula for the inclusion $M_r \subset \widetilde{M}$ reads $\widetilde{\nabla}_X Y = \nabla^{M_r}_X Y - (g(X,Y) + g(\xi,X)g(\xi,Y))\nu$ for all vector fields X, Y tangent to M_r.
We begin with the reconstruction of the doubly warped product structure of Theorem 3.9.*i*). From $A(\xi) = -2\xi$, we deduce that $A(J(V)) = -2J(V)$, hence $\widetilde{\nabla}_{J(V)}\nu = 2J(V)$. Proposition 2.1.*ii*) gives

$$J(V)(|V|) = \frac{g(\widetilde{\nabla}_V V, J(V))}{|V|} = \frac{1}{|V|}\Re e\left(\langle p_-(V)\cdot\phi, p_-(J(V))\cdot\phi\rangle + \langle p_+(V)\cdot\psi, p_+(J(V))\cdot\psi\rangle\right) = 0.$$

Therefore $\widetilde{\nabla}_{J(V)}V = 2|V|J(V)$, that is, $\widetilde{\nabla}_V V = 2|V|V$ using $\widetilde{\nabla}_{J(X)}V = J(\widetilde{\nabla}_X V)$ for all X. This implies for the commutator of ξ and ν (which we need later for the identification of the metric and of the Sasakian structure)

$$\begin{aligned} [\xi,\nu] &= -[J(\nu),\nu] \\ &= -[\frac{J(V)}{|V|},\frac{V}{|V|}] \\ &= -\frac{1}{|V|}\underbrace{J(V)(\frac{1}{|V|})}_{0}V - \frac{1}{|V|}[\frac{J(V)}{|V|},V] \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{|V|}V(\frac{1}{|V|})J(V) - \frac{1}{|V|^2}[J(V),V] \\
&= -\frac{g(\widetilde{\nabla}_V V,V)}{|V|^3}J(\frac{V}{|V|}) - \frac{1}{|V|^2}J(\underbrace{[V,V]}_{0}) \\
&= 2\xi. \qquad (11)
\end{aligned}
$$

We show now that each (non-empty) $(M_r, g_{|_{M_r}}, \xi_{|_{M_r}})$ is Sasakian. For every $X \in TM_r$, one has

$$
\begin{aligned}
\widetilde{\nabla}_X\xi &= -\widetilde{\nabla}_X(J(\nu)) \\
&= -J(\widetilde{\nabla}_X\nu) \\
&= J(A(X)) \\
&= -J(X) - g(\xi,X)\nu,
\end{aligned}
$$

so that $\widetilde{\nabla}_\xi\xi = -2\nu$, from which $\nabla^{M_r}_\xi\xi = 0$ follows and, for every $Z \in \{\xi,\nu\}^\perp$, the identity $\widetilde{\nabla}_Z\xi = -J(Z)$ implies $\nabla^{M_r}_Z\xi = -J(Z)$. In particular, $\xi_{|_{M_r}}$ defines a minimal Riemannian flow on $(M_r, g_{|_{M_r}})$ and $h = -J$ is an almost Hermitian structure on $\xi^\perp \subset TM_r$. It remains to show that h - or, equivalently, J - is transversally parallel on $\xi^\perp$. Recall that, from the definition of the transversal covariant derivative ∇ one has, for all sections Z, Z' of $\xi^\perp$,

$$
\begin{aligned}
\nabla_\xi Z &= \nabla^{M_r}_\xi Z - h(Z) \\
&= \widetilde{\nabla}_\xi Z - g(A(\xi),Z)\nu + J(Z) \\
&= \widetilde{\nabla}_\xi Z + J(Z)
\end{aligned}
$$

and

$$
\begin{aligned}
\nabla_Z Z' &= \nabla^{M_r}_Z Z' + g(h(Z),Z')\xi \\
&= \widetilde{\nabla}_Z Z' - g(A(Z),Z')\nu - g(J(Z),Z')\xi \\
&= \widetilde{\nabla}_Z Z' + g(Z,Z')\nu - g(J(Z),Z')\xi,
\end{aligned}
$$

from which one deduces that

$$
\begin{aligned}
(\nabla_\xi J)(Z) &= \nabla_\xi(J(Z)) - J(\nabla_\xi Z) \\
&= \widetilde{\nabla}_\xi(J(Z)) - Z - J(\widetilde{\nabla}_\xi Z) + Z \\
&= 0
\end{aligned}
$$

and

$$
\begin{aligned}
(\nabla_Z J)(Z') &= \nabla_Z(J(Z')) - J(\nabla_Z Z') \\
&= \widetilde{\nabla}_Z(J(Z')) + g(Z,J(Z'))\nu - g(J(Z),J(Z'))\xi \\
&\quad -J(\widetilde{\nabla}_Z Z') + g(Z,Z')\xi + g(J(Z),Z')\nu \\
&= 0,
\end{aligned}
$$

i.e., $\nabla J = 0$, which proves that $(M_r, g_{|_{M_r}}, \xi_{|_{M_r}})$ is Sasakian.

We come to the holomorphic isometry. Denote $M := M_1$, $\widehat{g} := g_{|_M}$ and $\widehat{\xi} := \xi_{|_M}$. Up to rescaling (ψ,ϕ) by a positive constant (this does not influence both conditions on (ψ,ϕ)), we may assume that $M \neq \varnothing$. Let F^ν_t be the flow of ν on $\widetilde{M}$. The vector field ν is complete since ν is bounded and $(\widetilde{M}, g)$ is complete. Consider the map

$$
\begin{aligned}
F : M \times \mathbb{R} &\longrightarrow \widetilde{M} \\
(x,t) &\longmapsto F^\nu_t(x).
\end{aligned}
$$

We first show that F is a diffeomorphism. If $F^\nu_t(x) = F^\nu_{t'}(x')$ for some $t,t' \in \mathbb{R}$ and $x,x' \in M$, then x and x' lie on the same integral curve of ν. Let now c be any integral curve of ν on $\widetilde{M}$ with $c(0) \in M$ and set $f(t) := |V|_{c(t)}$ (note that f *a priori* depends on the curve and in particular on the chosen starting

point). Then f is smooth with first derivative given by $f'(t) = \frac{g(\widetilde{\nabla}_V V, V)}{|V|^2}(c(t)) = 2|V|_{c(t)} = 2f(t)$ for all t, so that $f = f(0)e^{2t} = e^{2t}$. This has several consequences. On the one hand, f is injective, so that c meets M at most once, hence $x = x'$ and $t = t'$, which proves the injectivity of F. On the other hand, f does *a posteriori* not depend on the chosen starting point on M, in particular F_t^ν preserves the foliation by the level hypersurfaces M_r of $|\phi|$ and hence the orthogonal splitting $TM_r \oplus \mathbb{R}\nu$. Together with the surjectivity of $f : \mathbb{R} \to \mathbb{R}_+^\times$, we obtain that of F and the pointwise invertibility of the differential of F. Therefore F is a diffeomorphism.
Next we determine the metric F^*g. The map F sends $\frac{\partial}{\partial t}$ onto ν, so that obviously $F^*g(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}) = 1$. The preceding considerations also yield $F^*g(\frac{\partial}{\partial t}, X) = 0$ for all $t \in \mathbb{R}$ and $X \in TM$. Since

$$\frac{\partial}{\partial s}(F_s^\nu)_*\xi_{|s=t} = (F_t^\nu)_*\frac{\partial}{\partial s}(F_s^\nu)_*\xi_{|s=0} = (F_t^\nu)_*[\xi, \nu] \overset{(11)}{=} 2(F_t^\nu)_*\xi,$$

we have

$$(F_t^\nu)_*\xi = e^{2t}\xi \tag{12}$$

for every $t \in \mathbb{R}$. Moreover, the Lie derivative of g in direction of ν is given for all $X, Y \in \nu^\perp$ by

$$\begin{aligned}(\mathcal{L}_\nu g)(X,Y) &= g(\widetilde{\nabla}_X\nu, Y) + g(\widetilde{\nabla}_Y\nu, X)\\ &= -2g(A(X), Y)\\ &= 2(g(X,Y) + g(\xi, X)g(\xi, Y)),\end{aligned}$$

that is, $(\mathcal{L}_\nu g)_{|_{\nu^\perp}} = 2(g + \xi^\flat \otimes \xi^\flat)$. The identity $\frac{\partial}{\partial s}(F_s^\nu)^*g_{|s=t} = (F_t^\nu)^*\mathcal{L}_\nu g$ provides, for any $X, Y \in TM$ and $t \in \mathbb{R}$

$$\begin{aligned}\frac{\partial}{\partial s}((F_s^\nu)^*g(X,Y))_{|s=t} &= (\frac{\partial}{\partial s}(F_s^\nu)^*g_{|s=t})(X,Y)\\ &= \{(F_t^\nu)^*\mathcal{L}_\nu g\}(X,Y)\\ &= \mathcal{L}_\nu g((F_t^\nu)_*X, (F_t^\nu)_*Y) \circ F_t^\nu\\ &= 2\Big(g((F_t^\nu)_*X, (F_t^\nu)_*Y) + g(\xi, (F_t^\nu)_*X)g(\xi, (F_t^\nu)_*Y)\Big) \circ F_t^\nu\\ &= 2\Big((F_t^\nu)^*g(X,Y) + (F_t^\nu)^*g((F_{-t}^\nu)_*\xi, X)(F_t^\nu)^*g((F_{-t}^\nu)_*\xi, Y)\Big)\\ &\overset{(12)}{=} 2\Big((F_t^\nu)^*g(X,Y) + e^{-4t}(F_t^\nu)^*g(\xi, X)(F_t^\nu)^*g(\xi, Y)\Big).\end{aligned} \tag{13}$$

Since $(F_t^\nu)^*g(\xi,\xi) = g((F_t^\nu)_*\xi, (F_t^\nu)_*\xi) \circ F_t^\nu \overset{(12)}{=} (e^{4t}g(\xi,\xi)) \circ F_t^\nu = e^{4t}$, we deduce from (13) that, for $X = \xi$,

$$\frac{\partial}{\partial s}((F_s^\nu)^*g(\xi, Y))_{|s=t} = 4(F_t^\nu)^*g(\xi, Y),$$

from which $(F_t^\nu)^*g(\xi, Y) = e^{4t}g(\xi, Y)$ follows. In particular, $(F_t^\nu)^*g(\xi, Y) = 0$ for every $Y \in \{\xi, \nu\}^\perp$. For $X, Y \in \{\xi, \nu\}^\perp$, the identity (13) becomes

$$\frac{\partial}{\partial s}((F_s^\nu)^*g(X,Y))_{|s=t} = 2(F_t^\nu)^*g(X,Y),$$

which implies $(F_t^\nu)^*g(X,Y) = e^{2t}g(X,Y)$. To sum up, the pull-back metric on $M \times \mathbb{R}$ is given by

$$F^*g = e^{2t}(e^{2t}\widehat{g}_{\widehat{\xi}} \oplus \widehat{g}_{\widehat{\xi}^\perp}) \oplus dt^2,$$

where $\widehat{g}_{\widehat{\xi}} = \widehat{\xi}^\flat \otimes \widehat{\xi}^\flat = \widehat{g}(\widehat{\xi}, \cdot) \otimes \widehat{g}(\widehat{\xi}, \cdot)$ and, as in the beginning of this section, $\widehat{g}_{\widehat{\xi}^\perp}$ denotes the restriction of $\widehat{g}$ onto the subspace $\{\widehat{\xi}, \frac{\partial}{\partial t}\}^\perp \subset TM$. Hence the map F provides an isometry with the doubly warped product of Theorem 3.9.*i*). This isometry pulls the spin structure of $\widetilde{M}$ back onto the product spin structure of $M \times \mathbb{R}$, where M carries the spin structure induced by its embedding in $\widetilde{M}$. It remains to show that F identifies the complex structures. This follows from the definition of the complex structure on the doubly warped product $M \times \mathbb{R}$ (see Lemma 3.4), from $(F_t^\nu)_*\nu = \nu$, $(F_t^\nu)_*(e^{-2t}\widehat{\xi}) = \xi$ and from $[J(Z), \nu] = \widetilde{\nabla}_{J(Z)}\nu - \widetilde{\nabla}_\nu J(Z) = -A(J(Z)) - J(\widetilde{\nabla}_\nu Z) = J(Z) - J(\widetilde{\nabla}_\nu Z) = J([Z, \nu])$ for every section Z of $\{\xi, \nu\}^\perp$ (use the computation of A above).
Last but not the least, the identity $\nu \cdot \psi = i\phi$ implies that ϕ (or, equivalently, ψ) is transversally parallel on $(M, \widehat{g}, \widehat{\xi})$ by Theorem 3.9.*i*). This concludes the proof of Theorem 4.1. □

It is important to note that only the condition $W \cdot \psi = \mu\phi$ for some real vector field W is restrictive, since by [9, Thm. 11] the identity $|\psi| = |\phi|$ can always be assumed.

We conjecture that the examples of Section 3 describe all Kähler spin manifolds admitting non-trivial imaginary Kählerian Killing spinors. This will be the object of a forthcoming paper.

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