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Abstract

We propose a matrix analysis approach to analytically provide the cumulative distribution function of the sum of independent Erlang random variables. This reduces to the characterization of the exponential of the involved generator matrix. We propose a particular basis of vectors in which we write the generator matrix. We find, in the new basis, a Jordan-Chevalley decomposition allowing to simplify the calculation of the exponential of the generator matrix. This is a simpler alternative approach to the existing ones in the literature.

Keywords: Erlang random variables, reliability, queueing systems, cumulative distribution function, Jordan-Chevalley decomposition, hypoexponential distribution.

1 Introduction

Many situations in service and manufacturing service systems involve the computation of the sum of independent exponential random variables. Examples include healthcare or production systems with different stages in series, system reliability with exponentially distributed components lifetimes, and wireless mobile systems with cooperative diversity schemes. This summation arises also in the transient analysis of Markovian queueing systems, and in general, semi-Markov processes.

We consider the general case of a hypoexponential distribution defined as the sum of n independent Erlang distributions, for $n \in \mathbb{N}$. An Erlang distribution is defined by two parameters,

a number of i.i.d. exponential stages and a rate per stage. Thus, the general hypoexponential distribution is completely defined by the couples of parameters $(\lambda_i,\, k_i)$ for $i = 1, ..., n$. Each couple $(\lambda_i,\, k_i)$ defines an Erlang distribution ($\lambda_i \in \mathbb{R}$, $k_i \in \mathbb{N}$), and the rates λ_i for $i = 1, ..., n$ are all distinct. We denote by $K_i = k_1 + k_2 + ... + k_i$ for $i = 1, ..., n$ and use the convention $K_0 = k_0 = 0$. The cumulative distribution function (cdf) of the hypoexponential distribution is then given by

$$F(x) = 1 - \boldsymbol{\alpha} e^{xM} \mathbf{1}, \tag{1}$$

for $x \geq 0$, where $\mathbf{1}$ is a column vector of size K_n with ones everywhere, $\boldsymbol{\alpha}$ is a line vector of size K_n and is given by $\boldsymbol{\alpha} = (1, 0, \ldots, 0)$, and $e^{(.)}$ denotes the exponential operator. The generator square matrix M of size $K_n \times K_n$ is defined by the coefficients $m_{i,j}$ for $i, j \in \{1, ..., K_n\}$. We have $m_{j,j} = -\lambda_i$ and $m_{j,j+1} = \lambda_i$, for $K_{i-1} + 1 \leq j \leq K_i$ and $i = 1, ..., n$. All remaining coefficients of M are zero. We thus may write

$$M = \begin{pmatrix} -\lambda_1 & \lambda_1 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\ 0 & -\lambda_1 & \lambda_1 & \ddots & 0 & 0 & 0 & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \ldots & 0 & -\lambda_1 & \lambda_1 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 & 0 & -\lambda_2 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & \ldots & \ldots & \ldots & \ldots & 0 & -\lambda_n & \lambda_n \\ 0 & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & -\lambda_n \end{pmatrix}. \tag{2}$$

Scheuer (1988) provides a formula for $F(.)$ that involves high order derivatives of products of multiple functions. The formula is however hard to compute numerically. Amari and Misra (1997) propose a simplification of Scheuer (1988)'s formula using Laplace transforms and multi-function generalization of the Lebnitz rule for higher order derivatives of products of two functions. For a particular case with constraints on the values of the λ_is, Van Khuong and Kong (2006) provide the probability distribution function by inverting its Fourier transform. Using the Wilk's integral representation of the distribution of the product of independent beta random variables, Favaro and Walker (2010) provide an alternative formula for $F(.)$. We also refer the reader for more details to the review by Nadarajah (2008).

In this paper, we propose an alternative simple approach to analytically derive the cdf of $F(.)$. It

is based on a linear algebraic matrix analysis. The structure of the approach is as follows. We first obtain some particular eigenvectors of the generator matrix M. These are next used to construct a new basis of vectors. The new basis allows to find the Jordan-Chevalley decomposition of M into a sum of two commutative linear operators, a diagonal one and a nilpotent one. The exponential of the matrix M then simply follows by inverting the new basis matrix using the Cayley-Hamilton theorem, which leads to the cdf of $F(.)$.

2 The Result

Lemma 1 provides the eigenvalues of the matrix M, and one eigenvector associated to each eigenvalue.

Lemma 1 *The eigenvalues of M are $-\lambda_i$ for $i = 1, ..., n$. An eigenvector of size K_n associated to $-\lambda_i$ is the column vector u_i, where the coefficients of u_i, denoted by $u_{i,l}$ for $1 \leq l \leq K_n$, are given by*

$$\begin{cases} u_{i,l} = 1 & , \ l = K_{i-1} + 1, \\ u_{i,l} = 0 & , \ l > K_{i-1} + 1, \\ u_{i,l} = \left(\frac{\lambda_{i-1}}{\lambda_{i-1} - \lambda_i}\right)^{K_{i-1} - l + 1} & , \ K_{i-2} + 1 \leq l \leq K_{i-1}, \\ u_{i,l} = \prod_{j=1}^{i-(m+1)} \left(\frac{\lambda_{i-j}}{\lambda_{i-j} - \lambda_i}\right)^{k_{i-j}} \left(\frac{\lambda_m}{\lambda_m - \lambda_i}\right)^{K_m - l + 1} & , \ K_{m-1} + 1 \leq l \leq K_m \text{ and } 0 \leq m < i - 1. \end{cases}$$

Proof. Since M is a triangular matrix, its eigenvalues are its diagonal coefficients, i.e., $-\lambda_i$ for $i = 0, ..., n$. For $0 \leq i = 0, ..., n$, consider the column vector u_i defined with its coefficients $u_{i,l}$ $(1 \leq l \leq K_n)$, where $u_{i,K_{i-1}+1} = 1$, $u_{i,l} = 0$ for $l > K_{i-1} + 1$, $u_{i,l} = \left(\frac{\lambda_{i-1}}{\lambda_{i-1} - \lambda_i}\right)^{K_{i-1} - l + 1}$ for $K_{i-2} + 1 \leq l \leq K_{i-1}$, and $u_{i,l} = \prod_{j=1}^{i-(m+1)} \left(\frac{\lambda_{i-j}}{\lambda_{i-j} - \lambda_i}\right)^{k_{i-j}} \left(\frac{\lambda_m}{\lambda_m - \lambda_i}\right)^{K_m - l + 1}$, for $K_{m-1} + 1 \leq l \leq K_m$ and $0 \leq m < i - 1$. We now define, for $0 \leq i = 0, ..., n$, v_i as $v_i = M u_i$ and we denote its coefficients by $v_{i,l}$, for $1 \leq l \leq K_n$. Consider $1 \leq j \leq n$. For $K_{j-1} + 1 \leq l \leq K_j$ and $l \neq K_n$, we have

$$v_{i,l} = -\lambda_j u_{i,l} + \lambda_j u_{i,l+1}, \tag{3}$$

and $v_{i,K_n} = -\lambda_n u_{i,K_n}$. Since $u_{i,l} = 0$ and $u_{i,K_{i-1}+1} = 1$ for $l > K_{i-1} + 1$, we deduce from Equation (3) that $v_{i,l} = 0$ for $l > K_{i-1} + 1$ and $v_{i,K_{i-1}+1} = -\lambda_i$. For $K_{i-2} + 1 \leq l \leq K_{i-1}$ we

have $u_{i,l} = \left(\frac{\lambda_{i-1}}{\lambda_{i-1}-\lambda_i}\right)^{K_{i-1}-l+1}$, and for $K_{m-1}+1 \leq l \leq K_m$ and $0 \leq m < i-1$ we have $u_{i,l} = \prod_{j=1}^{i-(m+1)} \left(\frac{\lambda_{i-j}}{\lambda_{i-j}-\lambda_i}\right)^{k_{i-j}} \left(\frac{\lambda_m}{\lambda_m-\lambda_i}\right)^{K_m-l+1}$. Equation (3) therefore leads to $v_{i,l} = -\lambda_{i-1}\left(\frac{\lambda_{i-1}}{\lambda_{i-1}-\lambda_i}\right)^{K_{i-1}-l+1} + \lambda_{i-1}\left(\frac{\lambda_{i-1}}{\lambda_{i-1}-\lambda_i}\right)^{K_{i-1}-(l+1)+1} = -\lambda_i\left(\frac{\lambda_{i-1}}{\lambda_{i-1}-\lambda_i}\right)^{K_{i-1}-l+1}$, for $K_{i-2}+1 \leq l \leq K_{i-1}$. We also obtain $v_{i,l} = -\lambda_i \prod_{j=1}^{i-(m+1)} \left(\frac{\lambda_{i-j}}{\lambda_{i-j}-\lambda_i}\right)^{k_{i-j}} \left(\frac{\lambda_m}{\lambda_m-\lambda_i}\right)^{K_m-l+1}$, for $K_{m-1}+1 \leq l \leq K_m$ and $0 \leq m < i-1$. This proves that u is an eigenvector associated to the eigenvalue $-\lambda_i$, for $i = 1, ..., n$, and finishes the proof of the lemma. □

Let us denote by $\mathcal{B}$ the standard basis composed by the family of column vectors e_l, for $1 \leq l \leq K_n$. The coefficients of e_l $(1 \leq l \leq K_n)$ are all zero except the coefficient in line l which is equal to one. Consider now a new family of vectors denoted by $\mathcal{B}'$ and composed by the vectors e'_l for $1 \leq l \leq K_n$, where $e'_{K_{i-1}+1} = u_i$, and $e'_l = e_l$ for $l \neq K_{i-1}+1$ and $i = 1, ...n$. In Theorem 1, we prove that $\mathcal{B}'$ is a basis.

Theorem 1 *The family of vectors B' is a basis.*

Proof. The number of vectors in $\mathcal{B}'$ is equal to the space dimension, K_n. It thus remains to prove that the family of vectors $\mathcal{B}'$ is linearly independent. The proof is done by contradiction. Suppose that the family $\mathcal{B}'$ is linearly dependent. Therefore, there exists some values $a_1, a_2, ..., a_{K_n} \in \mathbb{R}$, non all zero, such that $a_1e'_1 + a_2e'_2 + \cdots a_{K_n}e'_{K_n} = 0$. We have $e'_1 = u_1 = e_1$, $e'_{K_1+1} = u_2 = b_{2,1}e_1 + b_{2,2}e_2 + \cdots + b_{2,K_1+1}e_{K_1+1}$, $\cdots$, $e'_{K_{n-1}+1} = u_n = b_{n,1}e_1 + b_{n,2}e_2 + \cdots + b_{n,K_{n-1}+1}e_{K_{n-1}+1}$, where the coefficients $b_{s,t}$ $(1 \leq s \leq t \leq K_{n-1}+1)$, given by Lemma 1, are all strictly positive. Then

$$
\begin{aligned}
& a_1e'_1 + a_2e'_2 + \cdots a_{K_n}e'_{K_n} \qquad (4)\\
&= a_1(1 + b_{2,1} + b_{3,1} + \cdots b_{n,1})e_1 + a_2(1 + b_{2,2} + b_{3,2} + \cdots + b_{n,2}) \\
&\quad + \cdots + a_{K_{n-1}+1}b_{n,K_{n-1}+1}e_{K_{n-1}+1} + a_{K_{n-1}+2}e_{K_{n-1}+2} + \cdots + a_{K_n}e_{K_n} \\
&= 0
\end{aligned}
$$

Since the coefficients $b_{s,t}$ $(1 \leq s \leq t \leq K_{n-1}+1)$ are all strictly positive and $(e_1, e_2, ..., e_{K_n})$ is the Basis $\mathcal{B}$, we should have $a_1 = a_2 = ... = a_{K_n} = 0$, which is absurd. Thus $\mathcal{B}'$ is a basis. □

Let us now proceed to a change of basis from $\mathcal{B}$ to $\mathcal{B}'$. We want to write M in the new basis $\mathcal{B}'$, which leads to a matrix denoted by M'. We denote by P the new basis matrix allowing to move

from $\mathcal{B}$ to basis $\mathcal{B}'$. This means that P is given by the vectors of the old basis $\mathcal{B}$ but written in the new basis $\mathcal{B}'$. We have $M = PM'P^{-1}$. We next give P and P'. We have

$$P = \begin{pmatrix} I_{k_1} & P_{1,2} & P_{1,3} & \dots & P_{1,n} \\ O_{2,1} & I_{k_2} & P_{2,3} & \dots & P_{2,n} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & P_{n-1,n} \\ O_{n,1} & \dots & \dots & O_{n,n-1} & I_{k_n} \end{pmatrix}, \tag{5}$$

where $P_{i,j}$, for $i, j = 1, \cdots, n$ and $i < j$, is a matrix of size $k_i \times k_j$ with 0 everywhere except in the first column. The coefficient of line k ($1 \leq k \leq k_i$) of the first column of $P_{i,j}$ is

$$\prod_{l=i+1}^{j-1} \left(\frac{\lambda_l}{\lambda_l - \lambda_j}\right)^{k_l} \left(\frac{\lambda_i}{\lambda_i - \lambda_j}\right)^{k_i - k + 1}. \tag{6}$$

The matrix I_{k_i}, for $i = 1, ..., n$, is the identity matrix of size $k_i \times k_i$, and the matrix $O_{i,j}$, for $i, j = 1, \cdots, n$, of size $k_i \times k_j$ has zero coefficients everywhere. We now compute P^{-1} using the Cayley-Hamilton theorem (see for example Gourdon (1994)). Since $(I_{K_n} - P)^n = O_{K_n}$, we deduce using the binomial theorem that $\sum_{k=0}^{n} \binom{n}{k}(-1)^k P^k = O_{K_n}$, where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ for $n \geq 0$ and $0 \leq k \leq n$. Thus $I_{K_n} = -\sum_{k=1}^{n} \binom{n}{k}(-1)^k P^k = P\left(-\sum_{k=1}^{n} \binom{n}{k}(-1)^k P^{k-1}\right)$. The matrix $-\sum_{k=1}^{n} \binom{n}{k}(-1)^k P^{k-1}$ commutes with P. We therefore obtain $P^{-1} = -\sum_{k=1}^{n} \binom{n}{k}(-1)^k P^{k-1}$.

In the basis $\mathcal{B}'$, the generator matrix is

$$M' = \begin{pmatrix} E_1 & O_{1,2} & O_{1,3} & \dots & O_{1,n} \\ O_{2,1} & E_2 & O_{2,3} & \dots & O_{2,n} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & O_{n-1,n} \\ O_{n,1} & \dots & \dots & O_{n,n-1} & E_n \end{pmatrix}, \tag{7}$$

where E_i ($i = 1, \cdots, n$) is a sub-matrix of size $k_i \times k_i$. It is given by

$$E_i = \begin{pmatrix} -\lambda_i & \lambda_i & 0 & \dots & 0 & 0 \\ 0 & -\lambda_i & \lambda_i & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & -\lambda_i & \lambda_i & 0 \\ 0 & 0 & \dots & 0 & -\lambda_i & \lambda_i \\ 0 & 0 & \dots & 0 & 0 & -\lambda_i \end{pmatrix}. \tag{8}$$

For $i = 1, \cdots, n$, E_i may be written as $E_i = D_i + N_i$ with

$$D_i = \begin{pmatrix} -\lambda_i & 0 & 0 & \dots & 0 & 0 \\ 0 & -\lambda_i & 0 & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & -\lambda_i & 0 & 0 \\ 0 & 0 & \dots & 0 & -\lambda_i & 0 \\ 0 & 0 & \dots & 0 & 0 & -\lambda_i \end{pmatrix}, \text{and } N_i = \begin{pmatrix} 0 & \lambda_i & 0 & \dots & 0 & 0 \\ 0 & 0 & \lambda_i & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & 0 & \lambda_i & 0 \\ 0 & 0 & \dots & 0 & 0 & \lambda_i \\ 0 & 0 & \dots & 0 & 0 & 0 \end{pmatrix}. \quad (9)$$

Using the sub-matrices D_i and N_i, we define the matrices D and N as

$$D = \begin{pmatrix} D_1 & O_{1,2} & O_{1,3} & \dots & O_{1,n} \\ O_{2,1} & D_2 & O_{2,3} & \dots & O_{2,n} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & O_{n-1,n} \\ O_{n,1} & \dots & \dots & O_{n,n-1} & D_n \end{pmatrix}, \text{and } N = \begin{pmatrix} N_1 & O_{1,2} & O_{1,3} & \dots & O_{1,n} \\ O_{2,1} & N_2 & O_{2,3} & \dots & O_{2,n} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & O_{n-1,n} \\ O_{n,1} & \dots & \dots & O_{n,n-1} & N_n \end{pmatrix}. \quad (10)$$

Therefore $M' = D + N$. Since $N^{\max(k_1,k_2,\cdots,k_n)} = O_{K_n,K_n}$, the matrix N is nilpotent. Note that D is a diagonal matrix. Then, D commutes with any other matrix, in particular N. Therefore the decomposition of M' into $M' = N + D$ is the unique Jordan-Chevalley decomposition of M' into a summation of two commutative matrices, a nilpotent one and a diagonalisable one. Using the new basis matrix P, we have $M = PM'P^{-1} = P(N + D)P^{-1}$, so, $e^{xM} = Pe^{x(N+D)}P^{-1}$ (see Gourdon (1994)). Since N and D commute, we deduce that $e^{x(N+D)} = e^{xD} \times e^{xN}$. Because xD is a diagonal matrix we have

$$e^{xD} = \begin{pmatrix} e^{xD_1} & O_{1,2} & O_{1,3} & \dots & O_{1,n} \\ O_{2,1} & e^{xD_2} & O_{2,3} & \dots & O_{2,n} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & O_{n-1,n} \\ O_{n,1} & \dots & \dots & O_{n,n-1} & e^{xD_n} \end{pmatrix}, \quad (11)$$

with

$$e^{xD_i} = \begin{pmatrix} e^{-x\lambda_i} & 0 & \dots & 0 \\ 0 & e^{-x\lambda_i} & \ddots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & e^{-x\lambda_i} \end{pmatrix}, \quad (12)$$

for $i = 1, \cdots, n$. Recall that e^{xN} is given by an infinite summation, $e^{xN} = \sum_{j=0}^{+\infty} \frac{(xN)^j}{j!}$. However, since N is nilpotent (for $i = 1, \cdots, n$, we have $N_i^{k_i} = O_{k_i,k_i}$), the evaluation of $\sum_{j=0}^{+\infty} \frac{(xN)^j}{j!}$ is done within a finite number of calculations. We have $e^{xN} = \sum_{j=0}^{\max(k_1,k_2,\cdots,k_n)-1} \frac{(xN)^j}{j!}$. Then,

$$N^j = \begin{pmatrix} N_1^j & O_{1,2} & O_{1,3} & \dots & O_{1,n} \\ O_{2,1} & N_2^j & O_{2,3} & \dots & O_{2,n} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & O_{n-1,n} \\ O_{n,1} & \dots & \dots & O_{n,n-1} & N_n^j \end{pmatrix}, \tag{13}$$

for $j \in \mathbb{N}$, where the matrix N_i^j is of size $k_i \times k_i$ $(i = 1, \cdots, n)$. The N_i^j's coefficients of line s and column t are all zero except if $t - s = j$ and $j = 0, \cdots k_i$ where the coefficients are all equal to λ_i^j. As a consequence

$$e^{xN} = \begin{pmatrix} e^{xN_1} & O_{1,2} & O_{1,3} & \dots & O_{1,n} \\ O_{2,1} & e^{xN_2} & O_{2,3} & \dots & O_{2,n} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & O_{n-1,n} \\ O_{n,1} & \dots & \dots & O_{n,n-1} & e^{xN_n} \end{pmatrix}, \tag{14}$$

where

$$e^{xN_i} = \begin{pmatrix} 1 & x\lambda_i & \frac{x^2}{2}\lambda_i^2 & \dots & \frac{x^{k_i-2}}{(k_i-2)!}\lambda_i^{k_i-2} & \frac{x^{k_i-1}}{(k_i-1)!}\lambda_i^{k_i-1} \\ 0 & 1 & x\lambda_i & \ddots & \frac{x^{k_i-3}}{(k_i-3)!}\lambda_i^{k_i-3} & \frac{x^{k_i-2}}{(k_i-2)!}\lambda_i^{k_i-2} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & 1 & x\lambda_i & \frac{x^2}{2}\lambda_i^2 \\ 0 & 0 & \dots & 0 & 1 & x\lambda_i \\ 0 & 0 & \dots & 0 & 0 & 1 \end{pmatrix}, \tag{15}$$

for $i = 1, \cdots, n$. Finally, we may write

$$e^{x(N+D)} = \begin{pmatrix} e^{xN_1}e^{xD_1} & O_{1,2} & O_{1,3} & \dots & O_{1,n} \\ O_{2,1} & e^{xN_2}e^{xD_2} & O_{2,3} & \dots & O_{2,n} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & O_{n-1,n} \\ O_{n,1} & \dots & \dots & O_{n,n-1} & e^{xN_n}e^{xD_n} \end{pmatrix}, \tag{16}$$

where

$$e^{xN_i}e^{xD_i} = \begin{pmatrix} e^{-x\lambda_i} & e^{-x\lambda_i}x\lambda_i & e^{-x\lambda_i}\frac{x^2}{2}\lambda_i^2 & \dots & e^{-x\lambda_i}\frac{x^{k_i-2}}{(k_i-2)!}\lambda_i^{k_i-2} & e^{-x\lambda_i}\frac{x^{k_i-1}}{(k_i-1)!}\lambda_i^{k_i-1} \\ 0 & e^{-x\lambda_i} & e^{-x\lambda_i}x\lambda_i & \ddots & e^{-x\lambda_i}\frac{x^{k_i-3}}{(k_i-3)!}\lambda_i^{k_i-3} & e^{-x\lambda_i}\frac{x^{k_i-2}}{(k_i-2)!}\lambda_i^{k_i-2} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & e^{-x\lambda_i} & e^{-x\lambda_i}x\lambda_i & e^{-x\lambda_i}\frac{x^2}{2}\lambda_i^2 \\ 0 & 0 & \dots & 0 & e^{-x\lambda_i} & e^{-x\lambda_i}x\lambda_i \\ 0 & 0 & \dots & 0 & 0 & e^{-x\lambda_i} \end{pmatrix}, \quad (17)$$

for $i = 1, \cdots, n$.

In order to obtain the cdf of the hypoexponential distribution we only need to evaluate the sum of the coefficients of the first line of the matrix e^{xM}. Note that in the particular case of an Erlang distribution ($k_2 = k_3 = \cdots = k_n = 0$) we have $P = P^{-1} = I_{k_1}$. We then easily obtain the well-known expression $F(x) = 1 - e^{-x\lambda_1}\sum_{k=0}^{k=k_1-1}\frac{(\lambda_1 x)^k}{k!}$, for $x \geq 0$, by summing the coefficients of the first line of $e^{xN_1}e^{xD_1}$.

We have

$$Pe^{x(D+N)} = \begin{pmatrix} e^{x(N_1+D_1)} & P_{1,2}e^{x(N_2+D_2)} & P_{1,3}e^{x(N_3+D_3)} & \dots & P_{1,n}e^{x(N_n+D_n)} \\ O_{2,1} & e^{x(N_2+D_2)} & P_{2,3}e^{x(N_3+D_3)} & \dots & P_{2,n}e^{x(N_n+D_n)} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & P_{n-1,n}e^{x(N_n+D_n)} \\ O_{n,1} & \dots & \dots & O_{n,n-1} & e^{x(N_n+D_n)} \end{pmatrix}, \quad (18)$$

where

$$P_{i,j}e^{x(N_j+D_j)} =$$

$$e^{-x\lambda_j}\prod_{l=i+1}^{j-1}\left(\frac{\lambda_l}{\lambda_l-\lambda_j}\right)^{k_l}\begin{pmatrix} \left(\frac{\lambda_i}{\lambda_i-\lambda_j}\right)^{k_i} & x\lambda_j\left(\frac{\lambda_i}{\lambda_i-\lambda_j}\right)^{k_i} & \dots & \frac{x^{k_j-1}}{(k_j-1)!}\lambda_j^{k_j-1}\left(\frac{\lambda_i}{\lambda_i-\lambda_j}\right)^{k_i} \\ \left(\frac{\lambda_i}{\lambda_i-\lambda_j}\right)^{k_i-1} & x\lambda_j\left(\frac{\lambda_i}{\lambda_i-\lambda_j}\right)^{k_i-1} & \dots & \frac{x^{k_j-1}}{(k_j-1)!}\lambda_j^{k_j-1}\left(\frac{\lambda_i}{\lambda_i-\lambda_j}\right)^{k_i-1} \\ \vdots & \ddots & \ddots & \vdots \\ \left(\frac{\lambda_i}{\lambda_i-\lambda_j}\right)^{2} & x\lambda_j\left(\frac{\lambda_i}{\lambda_i-\lambda_j}\right)^{2} & \dots & \frac{x^{k_j-1}}{(k_j-1)!}\lambda_j^{k_j-1}\left(\frac{\lambda_i}{\lambda_i-\lambda_j}\right)^{2} \\ \left(\frac{\lambda_i}{\lambda_i-\lambda_j}\right)^{1} & x\lambda_j\left(\frac{\lambda_i}{\lambda_i-\lambda_j}\right)^{1} & \dots & \frac{x^{k_j-1}}{(k_j-1)!}\lambda_j^{k_j-1}\left(\frac{\lambda_i}{\lambda_i-\lambda_j}\right)^{1} \end{pmatrix},$$

for $i, j = 1, ...n$ and $i < j$.

We then have to evaluate $Pe^{x(D+N)}P^{-1} = \sum_{k=1}^{n}\binom{n}{k}(-1)^{k-1}Pe^{x(D+N)}P^{k-1}$. Note again that P can be written as $P = I_{K_n} + P - I_{K_n}$. The matrices I_{K_n} and $P - I_{K_n}$ commute, I_{K_n} is diagonal

and $P - I_{K_n}$ is nilpotent. This implies

$$\begin{aligned} Pe^{x(D+N)}P^{-1} &= \sum_{k=1}^{n} \binom{n}{k} (-1)^{k-1} Pe^{x(D+N)} (I_{K_n} + P - I_{K_n})^{k-1} \\ &= \sum_{k=1}^{n} \binom{n}{k} (-1)^{k-1} Pe^{x(D+N)} \sum_{l=0}^{k-1} \binom{k-1}{l} (P - I_{K_n})^l, \end{aligned}$$

or equivalently

$$Pe^{x(D+N)}P^{-1} = \sum_{k=1}^{n} \sum_{l=0}^{k-1} (-1)^{k-1} \binom{n}{k} \binom{k-1}{l} Pe^{x(D+N)} (P - I_{K_n})^l.$$

We have $(P - I_{K_n})^0 = I_{K_n}$, and for $1 \leq l \leq n-1$

$$(P - I_{K_n})^l = \begin{pmatrix} O_{1,1} & A_{1,2} & A_{1,3} & \dots & A_{1,n} \\ O_{2,1} & O_{2,2} & A_{2,3} & \dots & A_{2,n} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & A_{n-1,n} \\ O_{n,1} & \dots & \dots & O_{n,n-1} & O_{n,n} \end{pmatrix}, \tag{19}$$

where $A_{i,j} = O_{i,j}$ for $j - i < l$, $i, j = 1, ..., n$, otherwise $A_{i,j} = \sum_{i<a_1<a_2<\cdots<a_{l-1}<j} P_{i,a_1} P_{a_1,a_2} \cdots P_{a_{l-2},a_{l-1}} P_{a_{l-1},j}$. The matrix $P_{i,a_1} P_{a_1,a_2} \cdots P_{a_{l-2},a_{l-1}} P_{a_{l-1},j}$ is a matrix of dimension $k_i \times k_j$ where all coefficients are zero except the ones on the first column. The coefficient of line k (for $1 \leq k \leq k_i$) of the first column is equal to

$$\left(\frac{\lambda_i}{\lambda_i - \lambda_{a_1}}\right)^{k_i - k + 1} \prod_{m=i+1}^{a_1 - 1} \left(\frac{\lambda_m}{\lambda_m - \lambda_{a_1}}\right)^{k_m} \prod_{n=a_1}^{a_2 - 1} \left(\frac{\lambda_n}{\lambda_n - \lambda_{a_2}}\right)^{k_n} \cdots \prod_{r=a_{l-1}}^{j-1} \left(\frac{\lambda_r}{\lambda_r - \lambda_{a_j}}\right)^{k_r}.$$

We are only interested in the sum of the coefficients of the first line of e^{xM}. We thus evaluate the terms in the first line of $Pe^{x(D+N)}(P - I_{K_n})^l$, for $0 \leq l \leq n-1$. For $l = 0$, $Pe^{x(D+N)}(P - I_{K_n})^0 = Pe^{x(D+N)}$. Then, using Equation (18) we compute the terms in the first line of $Pe^{x(D+N)}$. For $1 \leq l \leq n-1$, we have

$$Pe^{x(D+N)}(P - I_{K_n})^l = \begin{pmatrix} O_{1,1} & B_{1,2} & B_{1,3} & \dots & B_{1,n} \\ O_{2,1} & O_{2,2} & B_{2,3} & \dots & B_{2,n} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & B_{n-1,n} \\ O_{n,1} & \dots & \dots & O_{n,n-1} & O_{n,n} \end{pmatrix}, \tag{20}$$

where $B_{i,j} = O_{i,j}$ for $j - i < l$, $i, j = 1, ..., n$, otherwise for $i = 1$ and $1 < j < n$, $B_{1,j} = \sum_{s=1}^{j-1} P_{1,s} e^{x(N_s + D_s)} \sum_{s<a_1<a_2<\cdots<a_{l-1}<j} P_{s,a_1} P_{a_1,a_2} \cdots P_{a_{l-2},a_{l-1}} P_{a_{l-1},j}$ with the convention that $P_{1,1} = I_{k_1}$. Therefore, the cdf of the hypoexponential distribution is given by

$$
\begin{aligned}
F(x) = 1 - & \\
& \sum_{k=1}^{n}\sum_{l=0}^{k-1}(-1)^{k-1}\binom{n}{k}\binom{k-1}{l}\sum_{j=1}^{n}\sum_{s=1}^{j-1}e^{-x\lambda_s}\prod_{l=1}^{s-1}\left(\frac{\lambda_l}{\lambda_l-\lambda_s}\right)^{k_s} \\
& \times \sum_{s<a_1<\cdots<a_{l-1}<j}\left(\frac{\lambda_s}{\lambda_s-\lambda_{a_1}}\right)^{k_s}\prod_{m=s+1}^{a_1-1}\left(\frac{\lambda_m}{\lambda_m-\lambda_{a_1}}\right)^{k_m}\prod_{n=a_1}^{a_2-1}\left(\frac{\lambda_n}{\lambda_n-\lambda_{a_2}}\right)^{k_n} \\
\cdots & \prod_{r=a_{l-1}}^{j-1}\left(\frac{\lambda_r}{\lambda_r-\lambda_{a_j}}\right)^{k_r}\sum_{q=0}^{k_s-1}\frac{((\lambda_s-\lambda_{a_1})x)^q}{q!},
\end{aligned}
$$

for $x \geq 0$.

3 Conclusion

This paper proposes a new approach based on linear algebraic matrix analysis for the computation of the cumulative distribution function of the sum of independent Erlang random variables. This is a simpler alternative approach to the existing ones in the literature. However, direct computation of the involved expressions may lead to numerical difficulties. As a future research, it would be interesting to develop an efficient numerical method to avoid numerical round of errors.

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